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Abstract

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The main objective of this bachelor's thesis is to introduce the fractional Laplacian operator, $(-\Delta)^s$, and study some specific problems in one dimension not yet appeared in the literature, related to periodic solutions. We begin by explaining the relation between elliptic operators and Lévy processes, as well as a brief description of their applications in physics, biology and finance. We then introduce the fractional Sobolev spaces and present some important results. Finally, we study the fractional heat and Laplace equations, both in one dimension, which involve the fractional Laplacian. In particular, we prove the existence and uniqueness of periodic solutions for fractional heat equations and semilinear elliptic fractional equations.

Contents

CHAPTER 1

Introduction

The aim of this work is to present an introduction to the fractional Laplacian operator, as well as some results of existence and uniqueness of solutions for the fractional heat equations and fractional elliptic equations. Furthermore, we prove the existence of a family of even and periodic solutions for semilinear elliptic fractional equations using the implicit function theorem on Banach spaces. These results have been recently found by *X. Cabré and J. Sola-Morales* in [**6**] and also for a more general type of elliptic problems in [**5**].

We will start our dissertation by introducing the fractional Laplace operator from a probabilistic point of view. As we will see, the fractional Laplacian represents the infinitesimal generator of a Lévy stable diffusion process. This is the reason why this operator has been widely studied in Probability.

In recent years, there has been a surge of activity focused on the use of so-called fractional diffusion operators to replace the standard Laplace operator, with the aim of further extending the theory by taking into account the presence of so-called long range interactions. These nonlocal operators can be found in an extend range of fields, such as mathematical finance and physics; and can be used to describe, for example, energy diffusion in carbon nanotubes and optimal search theory for marine predators (see, for example, [**23**] and [**8**]). These applications will be treated in more detail in chapter 3.

Let us begin by introducing the notion of nonlocal operators. Unlike traditional differential operators, where in order to compute the values of an unknown function and its derivatives of different orders, one only needs to know the values of the function in an arbitrarily small neighborhood, with nonlocal operators the opposite happens. In order to check whether a nonlocal equation (that is, equations involving nonlocal operators) holds at a point, information about the values of the function far from that point is needed. Most of the times, this is because the equation involves integrals operators. A simple example could be

$$\partial_t u(t,x) = \int_{\mathbb{R}^n} \big(u(t,x+y) - u(t,x)\big) K(y) \mathrm{d}y$$

for some kernel K.

As mentioned above, the generator of a Lévy process is a linear integro-differential operator. Indeed, the fraction Laplacian is the most canonical example of elliptic integro-differential operators

$$(-\Delta)^s u(x) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(x+y)}{|y|^{n+2s}} \mathrm{d}y, \quad s \in (0,1), \tag{1.0.1}$$

where the integral above is taken in the principal value sense. It is the infinitessimal generator of the radially symmetric and stable Lévy process of order 2s.

Nonlocal problems usually presents a similar structure to PDE problems, requiring boundary conditions and specifying the equation in a bounded domain. For example, the fractional heat equation in a bounded domain $\Omega \subset \mathbb{R}^n$ would be

$$\begin{cases} \partial_t u + (-\Delta)^s u = 0 & \text{in } \Omega, t > 0 \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, t \geq 0 \\ u(0,x) = u_0(x) & \text{in } \Omega, \text{ for } t = 0, \end{cases}$$

for some initial condition u_0.

Notice that, unlike PDE problems, the boundary conditions are prescribed outside the domain.

Let us now explain how the work will be divided. The first chapters correspond to theoretical background, necessary to introduce the main topic. The most important chapters will be chapter 5 and chapter 6.

- In chapter 2 we first present the fractional Laplacian from a very intuitive probabilistic argument, similar to what we generally use in PDE courses to present the classical Laplacian operator, with the exception that in this case long jumps are allowed. We then introduce Lévy processes and it's relation to nonlocal operators. Finally, we present another expression of the fractional Laplacian through Fourier transform.
- In chapter 3 we briefly introduce some problems in physics, ecology and finance linked with fractional diffusions. This will motivate the importance of nonlocal diffusions.
- In chapter 4 we present important results needed in this work, and some other known results which we believe should be recalled. In particular, we begin by introducing classical Sobolev spaces and some important properties. We then introduce fractional Sobolev spaces and its relation with the fractional Laplacian.
- In chapter 5 we look for periodic solutions of fractional elliptic equations in $\mathbb{R}$ and we explain how nonlinear problems can be solved once one have the semigroup for the heat equation. We compare the fractional Laplacian with other convolution operators, the so-called Hilbert-Schmidt operators. Finally, we present a short explanation on how to prove existence and uniqueness of solutions for the Laplace equation in bounded domains of $\mathbb{R}^n$, using an argument from [**19**].
- In chapter 6 we study semilinear elliptic fractional equations and we prove the existence of even and periodic solutions u_a, depending on small parameters a. To do so, we begin by introducing an extension of the concept of

differentiability on Banach spaces and some important results that we need. We then use the implicit function theorem to prove the existence of solutions $u_a := a/\lambda(a)\{\cos(\lambda(a)\cdot) + \varphi_a(\lambda(a)\cdot)\}$, for λ and φ_a both depending on a. This same proceeding has been used in [**5**] to study periodic surfaces with constant nonlocal mean curvature.

CHAPTER 2

Nonlocal operators and the fractional Laplacian

In this chapter we introduce the fractional Laplacian from a Lévy process. We begin by presenting an intuitive argument to obtain the fractional heat equation, which follows from a discretization of $\mathbb{R}^n$. After that, we give a definition of Lévy processes and their relation with nonlocal operators, thus presenting with more mathematical rigor the fractional Laplacian. Finally, we present another definition for the fractional Laplacian by its Fourier transform that we will use in chapter 4.

As mentioned in the introduction, the fractional Laplacian naturally arises from random processes with jumps. Here we give an intuitive argument taken from [**21**] that only uses basic probability.

2.1. Long jump random walks and singular integral kernels

Let $\mathcal{K} : \mathbb{R}^n \to [0, \infty)$ be even, that is $\mathcal{K}(y) = \mathcal{K}(-y)$ for any $y \in \mathbb{R}^n$, and such that

$$\sum_{k\in\mathbb{Z}^n} \mathcal{K}(k) = 1 \tag{2.1.1}$$

Give a small $h > 0$, we consider a random walk on the lattice $h\mathbb{Z}^n$. We suppose that at any unit of time τ (which may depend on h), a particle jumps from any point of $h\mathbb{Z}^n$ to any other point.

The probability for which a particle jumps from the point $hk \in h\mathbb{Z}^n$ to the point $h\widetilde{k}$ is taken to be $\mathcal{K}(k - \widetilde{k}) = \mathcal{K}(\widetilde{k} - k)$.

Note that, differently from the standard random walk, in this process the particle may experience arbitrarily long jumps, though with a small probability.

We call $u(x, t)$ the probability that our particle lies at $x \in h\mathbb{Z}^n$ at time $t \in \tau\mathbb{Z}$. Of course, $u(x, t + \tau)$ equals the sum of all the probabilities of the possible positions $x + hk$ at time t weighted by the probability of jumping from $x + hk$ to x. That is,

$$u(x, t + \tau) = \sum_{k\in\mathbb{Z}^n} \mathcal{K}(k)u(x + hk, t)$$

Therefore, recalling the normalization in (2.1.1),

$$u(x, t + \tau) - u(x, t) = \sum_{k\in\mathbb{Z}^n} \mathcal{K}(k)\big(u(x + kh, t) - u(x, t)\big) \tag{2.1.2}$$

Particularly nice asymptotics are obtained in the case in which $\tau = h^\alpha$ and $\mathcal{K}$ is a homogeneous kernel[1], say, up to normalization factors

$$\mathcal{K} = |y|^{-(n+\alpha)}, \text{ (for } y \neq 0 \text{ and, say, } \mathcal{K}(0) = 0), \tag{2.1.3}$$

with

$$\alpha \in (0, 2).$$

We observe that (2.1.1) holds (up to normalization) and

$$\frac{\mathcal{K}(k)}{\tau} = h^n \mathcal{K}(hk). \tag{2.1.4}$$

It is convenient to define

$$\psi(y, x, t) = \mathcal{K}\big(u(x+y,t) - u(x,t)\big)$$

and to use (2.1.4) to write (2.1.2) as

$$\begin{aligned} \frac{u(x,t+\tau) - u(x,t)}{\tau} &= \sum_{k\in\mathbb{Z}^n} \frac{\mathcal{K}(k)}{\tau}\big(u(x+hk,t) - u(x,t)\big) \\ &= h^n \sum_{k\in\mathbb{Z}^n} \mathcal{K}(hk)\big(u(x+hk,t) - u(x,t)\big) \\ &= h^n \sum_{k\in\mathbb{Z}^n} \psi(hk, x, t). \end{aligned} \tag{2.1.5}$$

Since the latter is just the approximating Riemann sum of

$$\int_{\mathbb{R}^n} \psi(y, x, t)\, \mathrm{d}y,$$

by sending $\tau = h^\alpha \to 0^+$ in (2.1.5), that is, by taking the continuous limit of the discrete random walk, we obtain

$$\partial_t u(x,t) = \int_{\mathbb{R}^n} \psi(y, x, t)\, \mathrm{d}y$$

that is

$$\partial_t u(x,t) = \int_{\mathbb{R}^n} \frac{u(x+y,t) - u(x,t)}{|y|^{n+\alpha}} \mathrm{d}y. \tag{2.1.6}$$

This shows that a simple random walk with possibly long jumps produces, in the limit, a singular integral with a homogeneous kernel. We remark that the integral

$$\int_{\mathbb{R}^n} \frac{u(x+y) - u(x)}{|y|^{n+\alpha}} \mathrm{d}y. \tag{2.1.7}$$

which appears in (2.1.6) has a singularity when $y = 0$.

[1]That is, $\mathcal{K}(\alpha y) = \alpha^k \mathcal{K}(y)$, with k the degree of homogeneity.

However, when $\alpha \in (0,2)$ and u is smooth and bounded, such integral is well defined as principal value, that is as

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon} \frac{u(x+y) - u(x)}{|y|^{n+\alpha}} \mathrm{d}y.$$

Indeed, $|y|^{-(n+\alpha)}$ is integrable at infinity and

$$\int_{B_1} \frac{\nabla u(x) y}{|y|^{n+\alpha}} \mathrm{d}y = 0$$

as principal value, because the function $y/|y|^{n+\alpha}$ is odd. Therefore, we may write the singular integral in (2.1.7) as principal value near 0 in the form

$$\int_{B_1} \frac{u(x+y) - u(x) - \nabla u(x) y}{|y|^{n+\alpha}} \mathrm{d}y$$

and the latter is a convergent integral near 0 because

$$\frac{|u(x+y) - u(x) - \nabla u(x) y|}{|y|^{n+\alpha}} \leq \frac{||D^2 u||_{L^\infty} |y|^2}{|y|^{n+\alpha}} = \frac{||D^2 u||_{L^\infty}}{|y|^{n-2+\alpha}}$$

which is integrable near 0. It is also interesting to write the singular integral in (2.1.7) as a weighted second order differential quotient.
For this, we observe that, substituting $\tilde{y} = -y$, we have that the integral in (2.1.7) equals to

$$\int_{\mathbb{R}^n} \frac{u(x-\tilde{y}) - u(x)}{|\tilde{y}|^{n+\alpha}} \mathrm{d}\tilde{y}. \tag{2.1.8}$$

Therefore, relabeling $\tilde{y}$ as y in (2.1.8), we have that

$$\begin{aligned} &2 \int_{\mathbb{R}^n} \frac{u(x+y) - u(x)}{|y|^{n+\alpha}} \mathrm{d}y \\ &= \int_{\mathbb{R}^n} \frac{u(x+y) - u(x)}{|y|^{n+\alpha}} \mathrm{d}y + \int_{\mathbb{R}^n} \frac{u(x-y) - u(x)}{|y|^{n+\alpha}} \mathrm{d}y \\ &= \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+\alpha}} \mathrm{d}y. \end{aligned} \tag{2.1.9}$$

The equality obtained in (2.1.9) shows that the singular integral in (2.1.7) may be written, up to factor 2, as an average of the second incremental quotient $u(x+y) + u(x-y) - 2u(x)$ against the weigth $|y|^{n+\alpha}$.
Such a representation is also useful to remove the singularity of the integral in 0, since, for smooth u, a second order Taylor expansion gives that

$$\frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+\alpha}} \leq \frac{||D^2 u||_{L^\infty}}{|y|^{n-2+\alpha}}$$

which is integrable near 0.

It is known that the singular integral in (2.1.7) is related to the fractional Laplacian $(-\Delta)^{\alpha/2}$. This relation will be outlined here below (see, in particular (2.4.1) and (2.4.2) below.)

2.2. Nonlocal operators and Lévy processes

As we have seen in the previous section, integro-differential equations arise naturally in the study of stochastic processes with jumps, and more precisely of Lévy processes.

Let us now be more precise. We have already mentioned in the introduction that the fractional Laplacian $(-\Delta)^s$ is the infinitesimal generator of a radially symmetric and stable Lévy process of order $2s$. In this section we go further in this matter.

A stochastic process is a family of random variables $(X(t), t \geq 0)$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a measurable space $(E, \mathcal{E})$. Here Ω is a set (the sample space of possible outcomes), $\mathcal{F}$ is a σ-algebra of subsets of Ω (the events), and P is a positive measure of total mass 1 on $(\Omega, \mathcal{F})$ (the probability). Each $X(t)$ is a $(\mathcal{F}, \mathcal{E})$ measurable mapping from Ω to E and should be thought of as a random observation made on E at time t. For many developments, both theoretical and applied, E is the Euclidean space $\mathbb{R}^n$. In this work, we will focus on a class of stochastic processes called Lévy processes, in honor of the great French probabilist Paul Lévy, who first studied them in the 1930s.

Roughly speaking, a Lévy process represents the random motion of a particle whose successive displacements are independent and statistically identical over different time intervals of the same length. These processes extend the concept of Brownian motion. Essentially, Lévy processes are obtained when one relaxes the assumption of stochastic continuity and jump discontinuities may appear.

Formally, a Lévy process is defined as follows:

Definition 2.2.1. A Lévy process $X = (X(t), t \geq 0)$ is a stochastic process satisfying the following properties:

(1) $X_0 = 0$ almost surely (with probability one),
(2) For any $0 \leq t_1 < t_2 < ... < t_n < \infty$, $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, ..., X_{t_n} - X_{t_{n-1}}$ are independent,
(3) For any $s < t$, $X_t - X_s$ is equal in distribution to X_{t-s}. That is the probability distribution of any increments $X_t - X_s$ depends only on the length $t - s$ of the time interval; increments on equally long time intervals are identically distributed,
(4) For any $\epsilon > 0$ and $t \geq 0$ it holds that $\lim_{h\to 0} P(|X_{t+h} - X_t| > \epsilon) = 0$, i.e. X is stochastically continuous.

A Lévy process almost surely defines a semigroup $\{T_t, t \geq 0\}$ acting on functions u as follows:

$$(T_t u)(x) = \mathbb{E}[u(x + X_t)],$$

where T_0 is the identity and $T_{t+s} = T_t \circ T_s$. Since it is a semigroup, it has an infinitesimal generator given by L and defined by

$$Lu = \lim_{t \downarrow 0} \frac{\mathbb{E}[u(x + X_t)] - u(x)}{t}$$

By the Lévy-Khintchine Formula (see A.1), the infinitesimal generator of any Lévy processes is an operator of the form

$$-Lu(x) = \sum_{i,j} a_{ij}\partial_{ij}u + \sum_{j} b_j\partial_j u + \int_{\mathbb{R}^n} \big\{u(x+y) - u(x) - y\cdot\nabla u(x)\chi_{B_1}(y)\big\}\mathrm{d}\nu(y),$$

where ν is the Lévy measure[2], and satisfies $\int_{\mathbb{R}^n} \min\big\{1, |y|^2\big\} < \infty$. The first term corresponds to the *diffusion* part, the second term to the *drift* and the third term to the *jump* part. When the process has no diffusion or drift part, this operator takes the form

$$-Lu(x) = \int_{\mathbb{R}^n} \big\{u(x+y) - u(x) - y\cdot\nabla u(x)\chi_{B_1}(y)\big\}\mathrm{d}\nu(y).$$

In particular, the only (nondeterministic) continuous Lévy process is a Brownian motion with drift.

Furthermore, if one assumes the process to be symmetric, and the Lévy measure to be absolutely continuous, then L can be written as

$$Lu(x) = P.V.\int_{\mathbb{R}^n} \big\{u(x) - u(x+y)\big\}K(y)\mathrm{d}y. \tag{2.2.1}$$

Note that the term in $\nabla u(x)$ has vanished, since integrating $y\cdot\nabla u(x)$ with respect to y in the unit ball yields to zero.

A simple example is given by the following. Let us consider a bounded domain Ω in $\mathbb{R}^n$, a process X_t starting at $x\in\Omega$ and the first time τ at which the particle exists the domain. Assume now that we have a payoff function $g : \mathbb{R}^n\setminus\Omega \to \mathbb{R}$, so that when the process X_t exits Ω we get a payoff $g(X_\tau)$. Then, the expected payoff $u(x) := \mathbb{E}[g(X_\tau)]$ solves the problem

$$\begin{cases} Lu = 0 \text{ in } \Omega \\ \quad u = g \text{ in } \mathbb{R}^n\setminus\Omega. \end{cases} \tag{2.2.2}$$

Notice again, that in this problem, the boundary conditions are prescribed outside the domain. This comes from the fact that the particle may experience a long jump that sends it outside the domain.

If moreover we consider at the same time a running cost f and a final payoff g, 2.2.2 now becomes a nonhomogeneous problem of the form

$$\begin{cases} Lu = f \text{ in } \Omega \\ \quad u = g \text{ in } \mathbb{R}^n\setminus\Omega. \end{cases}$$

[2]Lévy measure describes the distribution of the jumps of the process. For example, a Poisson process of parameter $c > 0$ has the Lévy measure $c\delta(x-1)$, implying that the jump of size 1 occurs with intensity c.

We end this section by giving a definition of a class of specially relevant kind of Lévy process, the $\alpha-stable$ Lévy process. They are used, for example, in mathematical finance and signal processing among others.

Definition 2.2.2. Let $X = (X_t, t \geq 0)$ be a Lévy process on $\mathbb{R}^n$. It is called $\alpha-stable$ if

$$X_1 = \frac{1}{t^{1/\alpha}} X_t, \ \forall t \geq 0,$$

that is, there exist a property of self-similarity saying that X_b is distributed as $1/t^{\frac{1}{\alpha}} X_{bt}$ for $b, t > 0$.

Stable processes have nonlocal operators associated as infinitesimal generators (when $\alpha < 2$) of the form,

$$Lu(x) = \int_{\mathbb{R}^n} \big(u(x) - u(x+y)\big) \frac{a(y/|y|)}{|y|^{n+2s}} \mathrm{d}y, \tag{2.2.3}$$

where $s \in (0,1)$. Here, a is any symmetric nonnegative function in $\mathrm{L}^1(S^{n-1})$.

2.3. Kernels and Fourier transform

Given a "nice" (say, smooth and with fast decay, for simplicity) function u, the long jump random walk of Section 2.1 has lead us to the study of integrals of the type

$$\int_{\mathbb{R}^n} \big(u(x+y) + u(x-y) - 2u(x)\big) \mathcal{K}(y) \mathrm{d}y, \tag{2.3.1}$$

due to (2.1.9).

If we call $\mathcal{L}u$ the integral in (2.3.1), one may consider $\mathcal{L}$ a linear operator and look for its symbol in Fourier space.
That is, if $\mathcal{F}$ denotes the Fourier transform[3], one may think to write

$$\mathcal{L}u(x) = \mathcal{F}^{-1}\big(\mathcal{G}(\mathcal{F}u)\big), \tag{2.3.2}$$

for some function $\mathcal{S} : \mathbb{R}^n \to \mathbb{R}$.
We show that $\mathcal{K}$ and $\mathcal{G}$ are related as follows:

$$\mathcal{G}(\xi) = \int_{\mathbb{R}^n} (\cos(\xi \cdot y) - 1) \mathcal{K}(y) \, \mathrm{d}y, \tag{2.3.3}$$

up to normalization factors.

To check that (2.3.3) holds, one simply Fourier transforms (2.3.2) in the variable x, calling ξ the corresponding frequency variable: making use of (2.3.1) one obtains

[3]We define the Fourier transform of a function u as $(\mathcal{F}u)(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) \, \mathrm{d}x$, where $\cdot$ stands for the scalar product in $\mathbb{R}^n$. The Fourier transform $\mathcal{F}$ defines an isomorphism between the space of smooth functions with fast decay, the so-called Schwartz space $\mathcal{S}$.

$$
\begin{aligned}
\mathcal{G}(\xi)(\mathcal{F}u)(\xi) &= \mathcal{F}(\mathcal{L}u)(\xi) \\
&= \mathcal{F}\Big(\int_{\mathbb{R}^n} \big(u(x+y)+u(x-y)-2u(x)\big)\mathcal{K}(y)\,\mathrm{d}y\Big)(\xi) \\
&= \int_{\mathbb{R}^n} \Big(\mathcal{F}\big(u(x+y)+u(x-y)-2u(x)\big)(\xi)\Big)\mathcal{K}(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} (e^{i\xi\cdot y}+e^{-i\xi\cdot y}-2)\,(\mathcal{F}u)(\xi)\mathcal{K}(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} ((e^{i\xi\cdot y}+e^{-i\xi\cdot y}-2)\,\mathcal{K}(y)\,\mathrm{d}y(\mathcal{F}u)(\xi),
\end{aligned}
$$

proving (2.3.3).

2.4. The fractional Laplacian

The fractional Laplacian may be naturally introduced in the Fourier space. Indeed, one has that

$$\partial_j u = \mathcal{F}^{-1}\big(i\xi_j(\mathcal{F}u)\big)$$

and therefore

$$-\Delta u = \mathcal{F}^{-1}\big(|\xi|^2(\mathcal{F}u)\big).$$

Thus, it is natural to define, for $\alpha \in (0,2)$,

$$(-\Delta)^{\alpha/2}u = \mathcal{F}^{-1}\big(|\xi|^\alpha(\mathcal{F}u)\big). \tag{2.4.1}$$

It is known that such a fractional Laplacian may be also be represented as the principal value of singular integral, namely

$$(-\Delta)^{\alpha/2}u = C_{n,\alpha/2}P.V.\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+\alpha}}\mathrm{d}y, \tag{2.4.2}$$

where $P.V.$ stands for the principal value sense and $C_{n,\alpha/2}$ is a dimensional constant that depends on n and α, precisely given by

$$C_{n,\alpha/2} = \Big(\int_{\mathbb{R}^n}\frac{1-\cos(\zeta_1)}{|\zeta|^{n+\alpha}}\mathrm{d}\zeta\Big)^{-1}. \tag{2.4.3}$$

Notice that, by (2.1.9), one can also write (2.4.2) as

$$(-\Delta)^{\alpha/2}u = -C_{n,\alpha/2}P.V.\int_{\mathbb{R}^n}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+\alpha}}\mathrm{d}y.$$

We give here a simple proof of the equivalence between the definitions in (2.4.1) and in (2.4.2).

For this, we observe that, in the notation of (2.3.1) and (2.3.2), we may write (2.4.1) and (2.4.2) as

$$\mathcal{G}(\xi) = C_{n,\alpha/2}^{-1}|\xi|^\alpha \text{ and } \mathcal{K}(y) = -|y|^{-(n+\alpha)}.$$

Therefore, by (2.3.3), such equivalence boils down to prove that

$$C_{n,\alpha/2}^{-1}|\xi|^{\alpha} = \int_{\mathbb{R}^n} \frac{1-\cos(\xi\cdot y)}{|y|^{n+\alpha}}\mathrm{d}y. \tag{2.4.4}$$

To prove (2.4.4), first observe that, if $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{R}^n$, we have

$$\frac{1-\cos(\zeta_1)}{|\zeta|^{n+\alpha}} \le \frac{|\zeta_1|^2}{|\zeta|^{n+\alpha}} \le \frac{1}{|\zeta|^{n-2+\alpha}}$$

near $\xi = 0$. Thus,

$$\int_{\mathbb{R}^n} \frac{1-\cos(\zeta_1)}{|\zeta|^{n+\alpha}}\mathrm{d}\xi \text{ is finite and positive.} \tag{2.4.5}$$

We now consider the function $\mathcal{J} : \mathbb{R}^n \to \mathbb{R}$ defined as follows

$$\mathcal{J}(\xi) = \int_{\mathbb{R}^n} \frac{1-\cos(\xi\cdot y)}{|y|^{n+\alpha}}\mathrm{d}y.$$

We have that $\mathcal{J}$ is rotationally invariant, that is

$$\mathcal{J}(\xi) = \mathcal{J}(|\xi|e_1). \tag{2.4.6}$$

where e_1 denotes the first direction vector in $\mathbb{R}^n$. Indeed, if $n = 1$, then one easily checks that $\mathcal{J}(-\xi) = \mathcal{J}(\xi)$, proving (2.4.6) in this case.

When $n \geq 2$, we consider a rotation R for which

$$R(|\xi|e_1) = \xi$$

and we denote by R^T its transpose. We obtain, via the substitution $\tilde{y} = R^T y$,

$$\begin{aligned}
\mathcal{J}(\xi) &= \int_{\mathbb{R}^n} \frac{1-\cos\big((R(|\xi|e_1)\cdot y\big)}{|y|^{n+\alpha}}\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \frac{1-\cos\big((|\xi|e_1)\cdot(R^T y)\big)}{|y|^{n+\alpha}}\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \frac{1-\cos\big((|\xi|e_1)\cdot\tilde{y}\big)}{|\tilde{y}|^{n+\alpha}}\mathrm{d}y \\
&= \mathcal{J}(|\xi|)e_1),
\end{aligned}$$

which proves (2.4.6).

As a consequence of (2.4.6) and (2.4.5), the substitution $\zeta = |\xi|y$ gives that

$$\begin{aligned}
\mathcal{J}(\xi) &= \mathcal{J}(|\xi|e_1) \\
&= \int_{\mathbb{R}^n} \frac{1-\cos(|\xi|y_1)}{|y|^{n+\alpha}}\mathrm{d}y \\
&= \frac{1}{|\xi|^n}\int_{\mathbb{R}^n} \frac{1-\cos\zeta_1}{\big|\zeta/|\xi|\big|^{n+\alpha}}\mathrm{d}\zeta = C_{n,\alpha/2}^{-1}|\xi|^{\alpha},
\end{aligned}$$

where we recall that $C_{n,\alpha/2}^{-1}$ is equal to $\int_{\mathbb{R}^n} \frac{1-\cos(\zeta_1)}{|\zeta|^{n+\alpha}}\mathrm{d}\zeta$ by (2.4.3). Hence (2.4.4) is proved, thus so is the equivalence between (2.4.1) and (2.4.2).

We remark that, from (2.4.2), the probability density of the limit long jump random walk in (2.1.5) may be written as

$$\partial_t u = -(-\Delta)^{\alpha/2} u. \tag{2.4.7}$$

Remark 2.4.1. The classical Laplacian may be defined in $\mathbb{R}$ as the following limit

$$(-\Delta v)(x) = \lim_{|t|\to 0} 2\frac{v(x) - \frac{v(x+t)+v(x-t)}{2}}{|t|^2} \tag{2.4.8}$$

It represents the infinitesimal variation between $v(x)$ and the average of its neighbors. This definition differs a bit from that of the fractional Laplacian where, again, the integral is understood in the principal value sense

$$(-\Delta)^{\frac{\alpha}{2}} v(x) = \int_{\mathbb{R}} \frac{v(x) - v(x-t)}{|t|^{1+\alpha}}\mathrm{d}t \tag{2.4.9}$$

where $(-\Delta)^{\frac{\alpha}{2}} v(x)$ also represents the average (not infinitesimal) of what differs $v(x)$ from its neighbors. To see the difference, we make the change of variables $t = -\tilde{t}$ in (2.4.9), we obtain

$$(-\Delta)^{\frac{\alpha}{2}} v(x) = \int_{\mathbb{R}} \frac{v(x) - v(x+\tilde{t})}{|\tilde{t}|^{1+\alpha}}\mathrm{d}\tilde{t}. \tag{2.4.10}$$

Now, if we compute the average of (2.4.9) and (2.4.10), we get

$$(-\Delta)^{\frac{\alpha}{2}} v(x) = \int_{\mathbb{R}} \frac{v(x) - \frac{v(x+t)+v(x-t)}{2}}{|t|^{1+\alpha}}\mathrm{d}t,$$

expression much similar to (2.4.8). The main difference is that to compute (2.4.8) at a point $x \in \mathbb{R}$ it is sufficient to know v in a neighborhood of x. However, to compute (2.4.9) we need to know the value of v in the hole real line.

We also remark that the fractional Laplacian can be expressed as a convolution operator with kernel K. Remember that the convolution between two functions f and g is defined as

$$\big(g * f\big)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\mathrm{d}y.$$

Let $\delta u(x,y) := u(x) - u(y)$, definition (2.4.2) is then equivalent to

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = \big(K * \delta u(x,\cdot)\big)(x) = \int_{\mathbb{R}^n} \big(u(x) - u(y)\big)K(x-y)\mathrm{d}y, \tag{2.4.11}$$

where $K(x-y) := |x-y|^{-(n+\alpha)}$, again up to normalization factors.

CHAPTER 3

Applications

Fractional Laplacians appears in the study of many physical phenomenon linked with anomalous diffusion (i.e. diffusion processes modeled with a nonlocal diffusion equation). In this work, we briefly introduce some of these applications.

The motion of an individual molecule in a dense fluid does not follow a simple path. As it travels, the molecule is jostled by collisions with other molecules which prevent it from following a straight line. If the path is examined in close detail, it will be seen to be a good approximation to a random walk. Mathematically, a random walk is a series of steps, one after another, where each step is taken in a completely random direction from the one before. This kind of path was famously analyzed by *Albert Einstein* and *L. Bachelier* in a study of Brownian motion and they showed that the mean square of the distance travelled by particle following a random walk is proportional to the time elapsed. This relationship can be written as

$$< r^2 >= 6Dt + C$$

where $< r^2 >$ is the mean square distance and t is time. D and C are constants.

3.1. Carbon Nanotubes

Carbon nanotubes (CNTs) is one of the exciting nanoscale material discovered in recent years. They are described as tubular cylinders of carbon atoms that have extraordinary mechanical, electrical, thermal, optical and chemical properties. They are widely used in aerospace and defense, aviation, automotive, energy storage and electromagnetic shields.

Here we focus on single-walled nanotubes (SWNT). SWNTs are tubes of graphite that are normally caped at the ends. They only have a single cylindrical wall, hence its name.

Diameters of single-walled carbon nanotubes are typically 0.8 to 2 nanometers. In this scale and due to its density, we can picture them as a discrete set of carbon atoms connected to each other, so we end up studying the movement and interaction of each particular atom with the rest of the set.

However, compared with the mechanical and electronic properties, much less is known about the thermal properties in nanotubes. Although some researches have been done, many important and fundamental questions remain unsolved. For example,

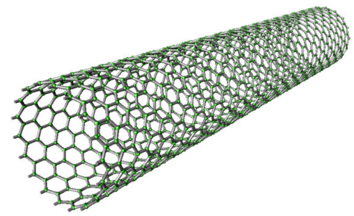

FIGURE 1. SWNT

how does the vibrational energy transport in carbon nanotubes?

It is found that energy transport ballistically at low temperature and superdiffusively at room temperature. This means that at room temperature, molecules in carbon nanotubes follows a random walk that contains occasional very long steps.

In paper [5] the above question is answered by investigating the energy diffusion in SWNTs. They find that the average energy profile spreads as

$$< \sigma^2 >= 2Dt^\alpha, (0 < \alpha \leq 2), \tag{3.1.1}$$

where $< . >$ means the ensemble average over different realizations. That is, at room temperature, the heat flows superdiffusively that leads to an anomalous thermal conductivity for SWNT and can be modeled, as shown in formula (3.1.1), with a diffusion equation where the typical laplacian Δ is replaced with a fractional laplacian $(-\Delta)^{\alpha/2}$.

Another interesting fact about carbon nanotubes is that thermal conductivity does not obey Fourier's law, namely, the thermal conductivity diverges with tube length for the tube length up to few micrometers.

3.2. Optimal search theory

Integro-differential equations also appear in Ecology. Indeed, optimal search theory predicts that predators should adopt search strategies based on long jumps (the so-called Lévy flight foraging hypothesis) where prey is sparse and distributed unpredictably, Brownian motion being more efficient only for locating abundant prey. Thus, reaction-diffusion problems with nonlocal diffusion such as

$$u_t + Lu = f(u) \text{ in } \mathbb{R}^n$$

arise naturally when studying population dynamics.

Strong support has been found for Lévy search patterns across 14 species of open-ocean predatory fish (sharks, tuna, billfish and ocean sunfish), with some individuals switching between Lévy and Brownian movement as they traversed different habitat types. They tested the spatial occurrence of these two principal patterns and found Lévy behavior to be associated with less productive waters and Brownian movements to be associated with abundant prey habitats. These results supports the contention that

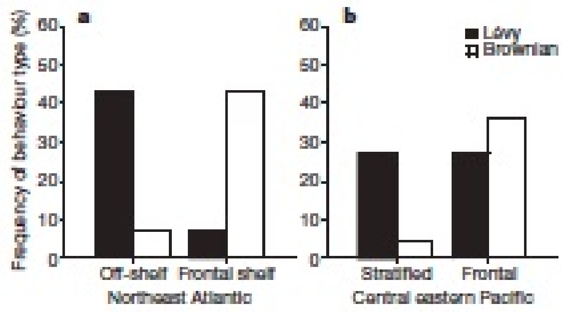

FIGURE 2. Spatial occurence of Lévy and Brownian behaviour types.

organism search strategies naturally evolved in such a way that they exploit optimal Lévy patterns.

It is proposed that organisms have therefore naturally evolved search patterns that can be modelled as optimal Lévy flights.

Informally speaking, a Lévy flight is a random walk in which the step-lengths have a probability distribution that is heavy-tailed [1](they have heavier tails than the exponential distribution).

3.3. Finance

Mathematical finance has been a very important research topic over the last century. The main reason for this is the interest in option pricing. Essentially, an option is a contract that confers upon the holder the right, but not the obligation, to purchase (or sell) a unit of a certain stock for a fixed price k on (or perhaps before) a fixed expiry date T, after which the option becomes worthless. For the option to make sense, k should be considerably less than the current price of the stock. If the stock price rises above k, the holder of the option may make a considerable profit; on the other hand, if the stock price falls dramatically, losses will be considerably less through buying options than by purchasing the stock itself.

Much of the current interest in the subject derives from Nobel-prize winning work of *F. Black, M. Scholes* and *R. Merton* in the 1970s. Underlying their analysis was a model of stock prices using geometric Brownian motion, i.e. the price $S(t)$ of a given stock at time t is

$$S(t) = S(0)\exp\Big\{\big(\mu - \frac{1}{2}\sigma^2\big)t + \sigma B(t)\Big\}.$$

[1]The tail of a distribution isn't a precisely defined term. In other words, there is not some specific place where you stop being in the middle of the distribution and start being in the tail. With that said, it is a very important concept nevertheless. The tail basically refers to the part of the distribution that is really far away from the mean. Distributions like the normal have very "skinny" tails because the density decreases like $exp(-x^2)$ for $|x|$ much greater than the mean. Other distributions like the Cauchy distribution have very "fat" tails because they decrease much more slowly.

The constant μ in $\mathbb{R}$ is the (logarithmic) expected rate of return, while $\sigma > 0$, called the volatility, is a measure of the excitability of the market.

Although very elegant, the Black-Scholes-Merton model has limitations and possible defects that have led many probabilists to query it. Indeed, empirical studies of stock prices have found evidence of heavy tails, which is incompatible with a Gaussian model, and this suggests that it might be fruitful to replace Brownian motion with a more general Lévy process.

3.3.1. **Continuous time random walks.** Continuous time random walks (CTRWs) models impose a random waiting time between particle jumps. They are used in statistical physics to model anomalous diffusion, where a cloud of particles spreads at a rate different than the classical Brownian motion. In econophysics, the CTRW model has been used to describe the movement of log-prices. Here, we give a short explain of the process.

Let $P(t)$ be the price of a financial issue at time t. Let $J_1, J_2, \dots$ denote the waiting times between trades, assumed to be non-negative and independent identically distributed (i.i.d) random variables. Also let $Y_1, Y_2, \dots$ denote the log-returns, assumed to be i.i.d. The sum $T_n = J_1 + \cdots + J_n$ represents the time of the nth trade. The log-returns are related to the price by $Y_n = \log[P(T_n)/P(T_{n-1})]$ and the log-price after n trades is $S_n = \log[P(T_n)] = Y_1 + \cdots + Y_n$. The number of trades by time $t > 0$ is $N_t = \max\{n : T_n \leq t\}$, and the log-price at time t is $\log P(t) = S_{N_t} = Y_1 + \cdots + Y_{N_t}$.

The log-price $\log P(t) = S_N$ is mathematically a random walk. Under particular hypothesis, the random walk S_n is asymptotically α-stable, and the long-time limit process assiociated $A(t)$ is an α-stable Lévy motion whose densities $p(t,x)$ solve a fractional diffusion equation

$$\frac{\partial p}{\partial t} = D\frac{\partial^\alpha p}{\partial |x|^\alpha}.$$

Just as an example, let us mention that in [**16**] *Nolan* examined the joint distribution of the German mark and the Japanese yen exchange rates, and observed that the distribution fits well in Lévy stable model. Moreover, he estimated the value of the parameter 2α to be approximately 1.51.

CHAPTER 4

Preliminaries and known results

4.1. Sobolev spaces

In this chapter we will introduce the fractional Sobolev spaces which will be used to study the fractional heat equation and the fractional Laplace equation in chapters 5 and 6. Before this, lets recall some definitions and important results from the classical Sobolev spaces.

Definition 4.1.1. Let $\mathrm{I} = (a, b)$ an interval not necessarily bounded and $p \in \mathbb{R}$ with $1 \leq p \leq \infty$. We define the Sobolev space $W^{1,p}(\mathrm{I})$ as

$$W^{1,p}(I) = \left\{u \in \mathrm{L}^p(\mathrm{I}) : \exists g \in \mathrm{L}^p(\mathrm{I}) \text{ such that } \int_{\mathrm{I}} u\varphi' = -\int_{\mathrm{I}} g\varphi \ \forall \varphi \in \mathrm{C}_c^\infty(\mathrm{I})\right\} \tag{4.1.1}$$

With this definition, the Sobolev spaces admit a natural norm,

$$||u||_{W^{1,p}} = \left(||u||_{L^p}^p + ||u'||_{L^p}^p\right)^{\frac{1}{p}}.$$

Equipped with the norm $||\cdot||_{W^{1,p}}$, $W^{1,p}$ becomes a Banach space.

From now on, we will write

$$H^1(I) = W^{1,2}(I).$$

For $u \in W^{1,p}(I)$ one notes $u' = g$.

The definition above can be generalized to any integer m:

Definition 4.1.2. Let I be an interval in $\mathbb{R}$, let $m \geq 2$ be an integer and $1 \leq p \leq \infty$. The Sobolev space $W^{m,p}(I)$ is defined recursively as

$$W^{m,p}(I) = \left\{u \in W^{m-1,p}(I) : u' \in W^{m-1,p}(I)\right\},$$

with the natural norm

$$||u||_{W^{m,p}} = \left(\sum_{i=0}^{m} ||u^{(i)}||_{L^p}^p\right)^{\frac{1}{p}}$$

Theorem 4.1.3. *$H^1(I)$ is a Hilbert space with the inner product*

$$(u, v)_{H^1} = (u, v)_{L^2} + (u', v')_{L^2};$$

with the associate norm

$$||u||_{H^1} = (||u||_{L^2}^2 + ||u'||_{L^2}^2)^{1/2}. \tag{4.1.2}$$

The space of infinitely differentiable functions with compact support is dense in $W^{1,p}(I)$:

Theorem 4.1.4. *Let $u \in W^{1,p}(I)$; $1 \le p < \infty$. There exists a sequence (u_n) in $C_c^\infty(\mathbb{R})$ such that $u_{n|I} \to u$ in $W^{1,p}(I)$.*

Proof. See Theorem VIII.6 in [**3**]. □

A classical result of Sobolev embedding is the following:

Proposition 4.1.5. *There exist a constant C such that*

$$||u||_{L^\infty(I)} \le C||u||_{W^{1,p}(I)} \quad \forall u \in W^{1,p}(I),\ \forall\, 1 \le p \le \infty,$$

in other words $W^{1,p}(I)$ is continously embedded into L^∞ for all $1 \le p \le \infty$.

Moreover, when I is bounded we have the following embedding

$$W^{1,p}(I) \subset L^{p'}(I),\ \text{for } \frac{1}{p'} + \frac{1}{p} = 1.$$

In order to prove the proposition above, we need the following result (see Theorem VIII.2 in [**3**]).

Theorem 4.1.6. *Let $u \in W^{1,p}(I)$; then there exist a function $\tilde{u} \in C(\overline{I})$ such that*

$$u = \tilde{u} \qquad \text{a.e. in } I.$$

and

$$\tilde{u}(x) - \tilde{u}(y) = \int_x^y u'(t)\mathrm{d}t \quad \forall x, y \in \overline{I}. \tag{4.1.3}$$

Remark 4.1.7. To give a meaning to $u(x)$, $\forall x \in \overline{I}$, u will be replaced by it's representative $\tilde{u}$. Note that Theorem 4.1.6 states that u admits a unique continuous representative $\tilde{u}$ i.e. there exists a continuous function which belongs to the equivalence class of u by the relation $u \sim v$ if $u = v$ almost everywhere.

Proof. (Proposition 4.1.5) Consider first a bounded interval $I = [a, b]$. From the mean formula there exists $x_0 \in [a, b]$ such that

$$\begin{aligned} |u(x_0)| = \frac{1}{b-a}\int_a^b |u(x)|\mathrm{d}t &\le \Big(\frac{1}{b-a}\Big)|b-a|^{1-\frac{1}{p}}||u||_{L^p([a,b])} \\ &\le |b-a|^{-\frac{1}{p}}||u||_{L^p(I)}, \end{aligned}$$

where in the first inequality we have used Hölder's inequality. Now, from (4.1.4) and Hölder's inequality, we have

$$\begin{aligned} |u(x) - u(y)| &\le \Big(\int_x^y 1^{p'}\mathrm{d}t\Big)^{\frac{1}{p'}}\Big(\int_x^y |u'|^p\mathrm{d}t\Big)^{\frac{1}{p}} \\ &\le |x-y|^{1-\frac{1}{p}}||u'||_{L^p(I)} \\ &\le |x-y|^{1-\frac{1}{p}}||u||_{W^{1,p}(I)}. \end{aligned}$$

It follows that for all $x \in [a, b]$,

$$\begin{aligned}
|u(x)| &\leq |u(x) - u(x_0)| + |u(x_0)| \\
&\leq |x - x_0|^{1-\frac{1}{p}} ||u||_{W^{1,p}(I)} + |b - a|^{-\frac{1}{p}} ||u||_{L^p(I)} \\
&\leq \left(|b - a|^{1-\frac{1}{p}} + |b - a|^{-\frac{1}{p}}\right) ||u||_{W^{1,p}(I)} \\
&\leq C||u||_{W^{1,p}(\mathbb{R})}.
\end{aligned}$$

Since we can take $x \in \mathbb{R}$ arbitrary (and consider, for example, $I = [x - r, x + r]$ for any $r > 0$), the inequality above implies

$$\sup_{\mathbb{R}} |u| \leq C||u||_{W^{1,p}(\mathbb{R})}.$$

The second statement comes easily from the inclusions in $L^p(X)$, for a finite-measure space X. $\square$

To end with this section, we remember Poincaré's inequality:

Corollary 4.1.8 (Poincaré's inequality)**.** *We suppose that I is bounded. Then there exist a constant C such that*

$$||u||_{L^p} \leq C||u'||_{L^p} \qquad \forall u \in W_0^{1,p}(I). \tag{4.1.4}$$

Here, $W_0^{1,p}(I) = \Big\{u \in W^{1,p}(I) : u = 0 \text{ on } \partial I \Big\}$. It is a closed subspace of $W^{1,p}(I)$ and therefore a Banach space with the induced norm.

Proof. Let $I = [a, b]$. For $u \in W_0^{1,p}(I)$ we have

$$|u(x)| = |u(x) - u(a)| = \Big|\int_a^x u'(t)\mathrm{d}t\Big| \leq ||u'||_{L^1(I)}.$$

Therefore, $||u||_{L^\infty(I)} \leq ||u'||_{L^1(I)}$. Now, using Hölder's inequality,

$$||u'||_{L^1(I)} \leq C||u'||_{L^p(I)}, \text{ for } C = |x - a|^{1-\frac{1}{p}}.$$

Integrating over I, one gets

$$\int_a^b |u(x)|^p \mathrm{d}x \leq \Big(\int_a^b |x - a|^{p-1}\mathrm{d}x\Big)||u'||^p_{L^p(I)}$$

and the inequality comes directly. $\square$

4.2. Fourier Series

We will give another definition for space H^1. But first, we recall that if $\varphi \in \mathrm{L}^2(-\pi,\pi)$ then φ has a Fourier expansion

$$\varphi = \sum_{m=-\infty}^{\infty} a_m e^{imt}, \quad a_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(t)e^{-imt}\mathrm{d}t,$$

where the convergence of the infinite sum is in $\mathrm{L}^2(-\pi.\pi)$, i.e.,

$$\lim_{N\to\infty} ||\varphi - \sum_{-N}^{N} a_m e^{imt}||_{\mathrm{L}^2} = 0,$$

Thus we see that $\{e^{-imt}\}$ form an orthonormal basis in $\mathrm{L}^2(-\pi,\pi)$.

Parseval's identity states that

$$\sum_{-\infty}^{\infty} |a_m|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\varphi(t)|^2\mathrm{d}t = \frac{1}{2\pi}||\varphi||^2_{L^2(-\pi,\pi)}.$$

Remark 4.2.1. We are now interested in Sobolev spaces on the cirlce. To this end we need to ensure that the point $t = 0$ is identified with $t = 2\pi$. This is accomplished by requiring that a function f is in $C[0, 2\pi]$ means that f is continuous on $[0, 2\pi]$ and periodic of period 2π, so that $f(0) = f(2\pi)$.

4.3. Sobolev spaces on the circle

We now focus in the case where $p = 2$ and $I = (0, 2\pi)$. The so-called Sobolev spaces on the circle.

For $s \in \mathbb{R}^+$ the Sobolev spaces $H^s(0, 2\pi)$ are subspaces of $L^2(0, 2\pi)$ which are determined by certain decay in the Fourier coefficients.

Definition 4.3.1. Let $0 < s < \infty$. Then by $H^s_{per}(0, 2\pi)$ we denote

$$H^s_{per}(0,2\pi) = \big\{\varphi \in \mathrm{L}^2_{per}(0,2\pi) : \sum_{-\infty}^{\infty}(1+m^2)^s|a_m|^2 < \infty\big\}$$

where $\{a_m\}$ are the Fourier coefficients of φ and the subindex $\{per\}$ stands for the periodicity of the functions, with period 2π.

We can also define an inner product so that the space above is a Hilbert space.

Theorem 4.3.2. *$H^s_{per}(0, 2\pi)$ is a Hilbert space with the inner product*

$$\langle\varphi,\phi\rangle_s = \sum_{-\infty}^{\infty}(1+m^2)^s a_m\overline{b_m}$$

for $\varphi, \phi \in H^s_{per}(0, 2\pi)$ with Fourier coefficients $\{a_m\}, \{b_m\}$ respectively.

PROOF. That $H^s_{per}(0,2\pi)$ is a linear space and that $\langle\cdot,\cdot\rangle_s$ is an inner product is easy to prove. We show that $\langle\cdot,\cdot\rangle_s$ is well defined and that $H^s_{per}(0,2\pi)$ is complete.

We can use the usual Cauchy-Schwartz inequality to show that $\langle\cdot,\cdot\rangle_s$ is well defined as follows:

$$\Big|\sum_{-\infty}^{\infty}(1+m^2)^s a_m\overline{b_m}\Big|^2 \le \Big(\sum_{-\infty}^{\infty}(1+m^2)^s|a_m|^2\Big)\Big(\sum_{-\infty}^{\infty}(1+m^2)^s|b_m|^2\Big),$$

which implies that $\langle\varphi,\phi\rangle$ exists for every $\varphi,\phi\in H^s_{per}(0,2\pi)$.

To see that $H^s_{per}(0,2\pi)$ is complete, let $\{\varphi_j\}_{j\in\mathbb{N}}$ be a Cauchy sequence, i.e. for every $\epsilon>0$ there exists an V_ϵ such that

$$||\varphi_j-\varphi_m||_{H^s(0,2\pi)}<\epsilon,\quad j,m>N_\epsilon$$

or what is the same thing

$$\sum_{k=-\infty}^{\infty}(1+m^2)^s|a_{k,j}-a_{k,m}|^2<\epsilon^2,\quad \forall m,j>N_\epsilon. \tag{4.3.1}$$

From this we can see that for each integers M_1,M_2 greater than zero, the following holds

$$\sum_{k=-M_1}^{M_2}(1+m^2)^s|a_{k,j}-a_{k,m}|^2<\epsilon^2,\forall m,j>N_\epsilon.$$

Now since $\mathbb{C}$ is complete we see that there is a sequence $\{a_k\}$ such that $a_{k,j}\to a_k$ as $j\to\infty$. Now passing to the limit in (4.3.1) as m goes to infinity we have

$$\sum_{k=-M_1}^{M_2}(1+m^2)^s|a_{k,j}-a_k|^2<\epsilon^2,\forall j>N_\epsilon.$$

And from this it follows that

$$\varphi=\sum_{m=-\infty}^{\infty}a_m e^{imt}\in H^s_{per}(0,2\pi)$$

since

$$\begin{aligned}\sum_{m=-\infty}^{\infty}(1+m^2)|a_m|^2 &\le \sum_{m=-\infty}^{\infty}(1+m^2)|a_m-a_{m,j}+a_{m,j}|^2\\ &\le 2\sum_{m=-\infty}^{\infty}(1+m^2)|a_m-a_{m,j}|^2+2\sum_{m=-\infty}^{\infty}(1+m^2)|a_{m,j}|^2<\infty.\end{aligned} \tag{4.3.2}$$

Also this shows that $||\varphi-\varphi_j||_{H^s(0,2\pi)}\to 0$ as $j\to\infty$.

□

It is easy to show the equivalence between Definitions 4.1.1 and 4.3.1 as we show in next theorem.

Theorem 4.3.3. *Let* $u = \sum_{m\in\mathbb{Z}} a_m e^{imx}$. *Then,*

$$\frac{\partial u}{\partial x} \in L^2(0,2\pi) \iff \sum_{m\in\mathbb{Z}} |a_m|^2|m|^2 < \infty.$$

Furthermore, we get

$$u \in H^s_{per}(0,2\pi) \iff \sum_{m\in\mathbb{Z}} |a_m|^2(1+|m|^2)^s < \infty.$$

PROOF. (Here we only give a proof for the case $s = 1$). Let us assume $\frac{\partial u}{\partial x} \in \mathrm{L}^2(0,2\pi)$. This means

$$\int_0^{2\pi} \frac{\partial u}{\partial x}\varphi(x)\mathrm{d}x = -\int_0^{2\pi} \frac{\partial \varphi}{\partial x}u(x)\mathrm{d}x$$

for every $\varphi \in C_0^\infty(0,2\pi)$. Now, we get

$$\begin{aligned}\int_0^{2\pi} \frac{\partial \varphi}{\partial x}u(x)\mathrm{d}x &= \sum_{m\in\mathbb{Z}} a_m \int_0^{2\pi} e^{imx}\frac{\partial \varphi}{\partial x}\mathrm{d}x \\ &= -\sum_{m\in\mathbb{Z}} a_m im \int_0^{2\pi} e^{imx}\varphi \,\mathrm{d}x.\end{aligned}$$

Let b_m be the Fourier coefficients of $\frac{\partial u}{\partial x}$. Then, we obtain

$$\sum_{m\in\mathbb{Z}} \int_0^{2\pi} e^{imx}\varphi \,\mathrm{d}x = \sum_{m\in\mathbb{Z}} a_m im \int_0^{2\pi} e^{imx}\varphi \,\mathrm{d}x.$$

Let us choose a sequence of functions in $C_c^\infty(0,2\pi)$ which converges in $\mathrm{L}^2(0,2\pi)$ to e^{imx} (since $C_c^\infty(\mathrm{I})$ is dense in $L^2(\mathrm{I})$). By this sequence, we get

$$b_m = a_m im.$$

This implies that

$$\sum_{m\in\mathbb{Z}} |a_m|^2 m^2 < \infty.$$

Finally, using (4.1.2) and Parseval's identity, we get

$$||u||^2_{H^1} = ||u||^2_{\mathrm{L}^2} + ||u'||^2_{\mathrm{L}^2} = \sum_{m\in\mathbb{Z}} |a_m|^2(1+|m|^2) < \infty.$$

To prove the result for any integer s, we use the same process as above for every weak derivative of u. □

4.4. Fractional Sobolev spaces

Let I be an interval in $\mathbb{R}$. For any real $s > 0$ and for any $p \in [1, \infty)$, we want to define the fractional Sobolev spaces $\mathrm{W}^{s,p}(\mathrm{I})$.

We start by fixing the fractional exponent s in $(0, 1)$. For any $p \in [1, \infty)$, we define $\mathrm{W}^{s,p}(\mathrm{I})$ as follows

$$\mathrm{W}^{s,p}(\mathrm{I}) := \Big\{ u \in \mathrm{L}^p(\mathrm{I}) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{1}{p}+s}} \in \mathrm{L}^p(\mathrm{I} \times \mathrm{I}) \Big\}; \tag{4.4.1}$$

i.e, an intermediary Banach space between $\mathrm{L}^p(\mathrm{I})$ and $\mathrm{W}^{1,p}(\mathrm{I})$, endowed with the natural norm

$$||u||_{\mathrm{W}^{s,p}}(\mathrm{I}) := \Big(\int_{\mathrm{I}} |u|^p \mathrm{d}x + \int_{\mathrm{I}} \int_{\mathrm{I}} \frac{|u(x) - u(y)|^p}{|x-y|^{1+sp}} \mathrm{d}x\mathrm{d}y \Big)^{\frac{1}{p}}, \tag{4.4.2}$$

where the term

$$\Big(\int_{\mathrm{I}} \int_{\mathrm{I}} \frac{|u(x) - u(y)|^p}{|x-y|^{1+sp}} \mathrm{d}x\mathrm{d}y \Big)^{\frac{1}{p}} \tag{4.4.3}$$

is the so-called *Gagliardo* (*semi*)*norm* of u.

It is worth noticing that, as in the classical case with s being an integer, the space $\mathrm{W}^{s',p}$ is continuously embedded in $\mathrm{W}^{s,p}$ when $s \leq s'$, as next result points out.

Proposition 4.4.1. *Let $p \in [1, \infty)$ and $0 < s \leq s' \leq 1$. Let I be an interval in $\mathbb{R}$ and $u : I \to \mathbb{R}$ be a measurable function. Then*

$$||u||_{W^{s,p}(I)} \leq C ||u||_{W^{s',p}(I)}$$

for some suitable positive constant $C \geq 1$. In particular,

$$W^{s',p}(I) \subseteq W^{s,p}(I).$$

Proof. First,

$$\begin{aligned}\int_I \int_{I \cap \{|x-y| \geq 1\}} \frac{|u(x)|^p}{|x-y|^{1+sp}} \mathrm{d}x\mathrm{d}y &\leq \int_I \Big(\int_{|z| \geq 1} \frac{1}{|z|^{1+sp}} \mathrm{d}z \Big) |u(x)|^p \mathrm{d}x \\ &\leq C ||u||^p_{\mathrm{L}^p(I)},\end{aligned}$$

where we have used the fact that the kernel $1/|z|^{1+sp}$ is integrable since $1 + sp > 1$. Taking into account the above estimate, it follows

$$\begin{aligned}\int_I \int_{I \cap \{|x-y| \geq 1\}} \frac{|u(x) - u(y)|^p}{|x-y|^{1+sp}} \mathrm{d}x\mathrm{d}y &\leq 2^{p-1} \int_I \int_{I \cap \{|x-y| \geq 1\}} \frac{|u(x)|^p + |u(y)|^p}{|x-y|^{1+sp}} \mathrm{d}x\mathrm{d}y \\ &\leq 2^p C ||u||^p_{\mathrm{L}^p(\Omega)}.\end{aligned} \tag{4.4.4}$$

On the other hand,

$$\int_I \int_{I \cap \{|x-y| \geq 1\}} \frac{|u(x) - u(y)|^p}{|x-y|^{1+sp}} \mathrm{d}x\mathrm{d}y \leq \int_I \int_{I \cap \{|x-y| \geq 1\}} \frac{|u(x) - u(y)|^p}{|x-y|^{1+s'p}} \tag{4.4.5}$$

Thus, combining (4) with (5), we get

$$\int_I \int_{I\cap\{|x-y|\geq 1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{1+sp}} \leq 2^p C||u||^p_{\mathrm{L}^p(I)} + \int_I \int_{I\cap\{|x-y|\geq 1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{1+s'p}}$$

and so

$$\begin{aligned} ||u||^p_{\mathrm{W}^{s,p}(I)} &\leq (2^p C+1)||u||^p_{\mathrm{L}^p(I)} + \int_I \int_{I\cap\{|x-y|\geq 1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{1+s'p}} \\ &\leq C||u||^p_{\mathrm{W}^{s',p}(I)}, \end{aligned}$$

which gives the desired estimate. □

The space of smooth functions with compact support is also dense in $W^{s,p}(I)$:

Proposition 4.4.2. *Let $p \in [1,\infty), s \in (0,1)$ and I be a bounded open interval in $\mathbb{R}$. Then for any $u \in W^{s,p}(I)$, there exists a sequence $u_n \in C_c^\infty(\mathbb{R})$ such that $u_n \to u$ as $n \to \infty$ in $W^{s,p}(I)$, i.e.,*

$$\lim_{n\to\infty} ||u_n - u||_{W^{s,p}(I)} = 0.$$

It is a natural question to ask if the same embeddings from Sobolev spaces also hold for fractional orders. We can find a complete proof of the next two theorems in [**15**].

Theorem 4.4.3. *Let $s \in (0,1)$ and $p \in [1,\infty)$ such that $sp < 1$. Then there exists a positive constant C such that, for any $f \in W^{s,p}(I)$, we have*

$$||f||_{L^q(I)} \leq C||f||_{W^{s,p}(I)}$$

for any $q \in [p,p^*]$; i.e., the space $W^{s,p}(I)$ is continuously embedded in $L^q(I)$ for any $q \in [p,p^*]$, where p^* is the so-called "fractional critical exponent" and it is equal to $p/(1-sp)$.

We note that when $sp \to n$ the critical exponent p^* goes to ∞ and so it is not surprising that, in this case, if f is in $W^{s,p}$ then f belongs to L^q for any q, as stated in the following theorem.

Theorem 4.4.4. *Let $s \in (0,1)$ and $p \in [1,\infty)$ such that $sp = 1$. Then there exists a positive constant C such that, for any $f \in W^{s,p}(I)$, we have*

$$||f||_{L^q(I)} \leq C||f||_{W^{s,p}(I)}$$

for any $q \in [p,\infty)$; i.e., the space $W^{s,p}(I)$ is continuously embedded in $L^q(I)$ for any $q \in [p,\infty)$.
If, in addition, I is bounded, then the space $W^{s,p}(I)$ is continuously embedded in $L^q(I)$ for any $q \in [1,\infty)$.

For the next result, we give an alternative definition of the space $H^s(\mathbb{R}) = W^{s,2}(\mathbb{R})$. Precisely, we may define

$$\hat{H}^s(\mathbb{R}) = \Big\{ u \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^2 \mathrm{d}\xi < \infty \Big\} \tag{4.4.6}$$

The equivalence of both definitions will be demonstrated in Proposition 4.4.6.

Theorem 4.4.5. *Let $s \in (0, 1)$. Then the space $H^s(\mathbb{R})$ is continuously embedded in $L^\infty(\mathbb{R})$ for $s > \frac{1}{2}$.*

For the following proof we will use the inverse Fourier transform of a function defined as $\mathcal{F}^{-1} = (2\pi)^{-1}\check{\mathcal{F}}$, i.e.

$$u(x) = (2\pi)^{-1} \int_{\mathbb{R}} e^{ix\xi} \hat{u}(\xi) \mathrm{d}\xi.$$

PROOF.

$$\max_{\mathbb{R}} |u| \le \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{u}(\xi)| \mathrm{d}\xi \le \frac{1}{2\pi} \Big(\int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{(1+\xi^2)^s} \Big)^{\frac{1}{2}} \cdot \Big(\int_{-\infty}^{\infty} (1+\xi^2)^s |\hat{u}(\xi)|^2 \mathrm{d}\xi \Big)^{\frac{1}{2}},$$

where $\hat{u}$ denotes the Fourier transform of u. Hence follows the required estimate

$$\max_{\mathbb{R}} |u(x)| \le C ||u||_{H^s(\mathbb{R})}.$$

□

Proposition 4.4.6. *Let $s \in (0, 1)$. Then the fractional Sobolev space $H^s(\mathbb{R})$ defined in (4.4.1) with $p = 2$ (that is , $H^s(\mathbb{R}) = W^{s,2}(\mathbb{R})$) coincides with $\hat{H}^s(\mathbb{R})$ defined in (4.4.6). In particular, for any $u \in H^s(\mathbb{R})$*

$$[u]^2_{H^s(\mathbb{R})} = 2C^{-1}_{1,s} \int_{\mathbb{R}} |\xi|^{2s} |\mathcal{F}u(\xi)|^s \mathrm{d}\xi$$

where $C_{1,s}$ is the normalization constant defined in (2.4.3).

PROOF. For every fixed $i \in \mathbb{R}$, by changing of variable choosing $z = x - y$, we get

$$\begin{aligned}
\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}} \mathrm{d}x \Big) \mathrm{d}y &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x+x) - u(y)|^2}{|z|^{1+2s}} \mathrm{d}z \mathrm{d}y \\
&= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \Big| \frac{u(z+y) - u(y)}{|z|^{1/2+s}} \Big|^2 \mathrm{d}y \Big) \mathrm{d}z \\
&= \int_{\mathbb{R}} \Big|\Big| \frac{u(z+\cdot) - u(\cdot)}{|z|^{n/2+s}} \Big|\Big|^2_{L^2(\mathbb{R})} \mathrm{d}z \\
&= \int_{\mathbb{R}} \Big|\Big| \mathcal{F}\Big(\frac{u(z+\cdot) - u(\cdot)}{|z|^{n/2+s}} \Big) \Big|\Big|^2_{L^2(\mathbb{R})} \mathrm{d}z,
\end{aligned}$$

where Plancherel Formula (see A.4) has been used.

Now, using (2.4.4) we obtain

$$\begin{aligned}\int_{\mathbb{R}}\Big|\Big|\mathcal{F}\Big(\frac{u(z+\cdot)-u(\cdot)}{|z|^{n/2+s}}\Big)\Big|\Big|^2_{L^2(\mathbb{R})}\mathrm{d}z &= \int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|e^{i\xi\cdot z}-1|^2}{|z|^{1+2s}}|\mathcal{F}u(\xi)|^2\mathrm{d}\xi\mathrm{d}z\\ &= 2\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(1-\cos(\xi\cdot z))}{|z|^{1+2s}}|\mathcal{F}u(\xi)|^2\mathrm{d}z\mathrm{d}y\\ &= 2C_{1,s}^{-1}\int_{\mathbb{R}}|\xi|^{2s}|\mathcal{F}u(\xi)|^2\mathrm{d}z\mathrm{d}y.\end{aligned}$$

This completes the proof. □

The space H^s is strictly related to the fractional Laplacian operator as follows:

Proposition 4.4.7. *Let $s\in(0,1)$ and let $u\in H^s(\mathbb{R})$. Then,*

$$[u]^2_{H^s(\mathbb{R})} = 2C_{1,s}^{-1}\big|\big|(-\Delta)^{\frac{s}{2}}u\big|\big|^2_{L^2(\mathbb{R})}. \tag{4.4.7}$$

Proof. The equality in (4.4.7) plainly follows from (2.4.1) and Proposition 4.4.6. Indeed,

$$\begin{aligned}\big|\big|(-\Delta)^{\frac{s}{2}}u\big|\big|^2_{L^2(\mathbb{R})} = \big|\big|\mathcal{F}(-\Delta)^{\frac{s}{2}}u\big|\big|^2_{L^2(\mathbb{R})} &= \big|\big||\xi|^s\mathcal{F}u\big|\big|^2_{\mathrm{L}^2(\mathbb{R})}\\ &= \frac{1}{2}C_{1,s}[u]^2_{H^s(\mathbb{R})}.\end{aligned}$$

where the first identity comes from the fact that the Fourier transform is an unitary operator, i.e. preserves the inner product on the Hilbert space $\mathrm{L}^2(\mathbb{R})$ (see Plancherel Theorem A.4). □

It follows the next relation

$$||u||^2_{H^s(\mathbb{R})} = ||u||^2_{L^2(\mathbb{R})} + \frac{2}{C_{1,s}}||(-\Delta)^{\frac{s}{2}}u||^2_{L^2(\mathbb{R})}.$$

So we have that the fractional Laplacian $(-\Delta)^{\frac{s}{2}}$ send continuously functions from H^s into L^2.

In order to finish the introduction of fractional Sobolev spaces, we present the Sobolev inequality for fractional order s and arbitrary dimension n. We do not provide a proof, since its demonstration is much more complex than the one dimensional case, and it is not the aim of this work to study in detail the Sobolev spaces.

Proposition 4.4.8 (Fractional Sobolev inequality)**.** *Let $u\in H^s(\mathbb{R}^n)$, and let $p\in[1,\frac{n}{s})$. Then*

$$\Big(\int_{\mathbb{R}^n}|u(x)|^{\frac{np}{n-ps}}\mathrm{d}x\Big)^{\frac{n-ps}{np}} \le C\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\mathrm{d}y\mathrm{d}x\Big)^{\frac{1}{p}}, \tag{4.4.8}$$

where C is a constant that depends only on n,p and $s\in(0,1)$.

CHAPTER 5

Fractional heat equation

5.1. Homogeneous heat equation

We are now able to study the following problem. Let $s \in (0,1)$ and $(-\Delta)^s u$ denotes the fractional Laplacian. We define F as the space of functions in $H^{2s}(0,\pi)$ which are the restriction of an even and 2π periodic function on $\mathbb{R}$

$$F := \Big\{ u \in \mathrm{H}^{2s}_{per}(0,\pi) : u \text{ is even} \Big\}. \tag{5.1.1}$$

We want to find $u \in F$ solution of the homogeneous heat equation with initial condition:

$$\begin{cases} \partial_t u + (-\Delta)^s u = 0 & x \in \mathbb{R}, t > 0, \\ u(x,0) = g(x) & x \in \mathbb{R}. \end{cases} \tag{5.1.2}$$

To find u as above, we will proceed as follows:

(1) We check that $(-\Delta)^s u$ is well defined for $u \in F$.
(2) For k=0,1, 2, ..., $\varphi_k(x) = \cos(k \cdot x)$ are eigenfunctions of $(-\Delta)^s$ in F.
(3) We look for the semigroup of (5.1.2).
(4) Using Duhamel's principle, we find a solution for the non-homogeneous equation.
(5) Finally, we see how to proceed to prove the existence and uniqueness of solution for the nonlinear problem using Banach fixed-point theorem.

The first point comes from Proposition 4.4.7. To prove second point, we use the following result from [**5**].

Lemma 5.1.1. *The functions $e_k(s) = \cos(ks)$, $k = 0,1,2,3\ldots$, are all eigenfunctions of $(-\Delta)^s$ in the space F of even and periodic $L^2(0,\pi)$ functions.*

Proof. In the principal value sense, we have

$$\int_{\mathbb{R}} \frac{\cos(ks) - \cos(ks - kt)}{|t|^{1+2s}} \mathrm{d}t = \int_{\mathbb{R}} \frac{\cos(ks) - \cos(ks)\cos(kt) - \sin(ks)\sin(kt)}{|t|^{1+2s}} \mathrm{d}t,$$

and the last integral, in the principal value sense, is equal to

$$\Big(\int_{\mathbb{R}} \frac{1 - \cos(kt)}{|t|^{1+2s}} \mathrm{d}t \Big) \cos(ks), \tag{5.1.3}$$

where $\lambda_k = \int_{\mathbb{R}} \frac{1-\cos(kt)}{|t|^{1+2s}} \mathrm{d}t$ are the eigenvalues associated to eigenfunctions $\cos(kx)$. $\square$

Remark 5.1.2. Eigenvalues of $(-\Delta)^s$ can be also written as

$$\lambda_k = ak^{2s}, \tag{5.1.4}$$

with

$$a = \text{constant} = \int_{\mathbb{R}} \frac{1-\cos(y)}{|y|^{1+2s}} \mathrm{d}y.$$

This can easily be seen using the change of variables $kt = y$ in (5.1.2):

$$\lambda_k = \int_{\mathbb{R}} \frac{1-\cos(kt)}{|t|^{1+2s}} \mathrm{d}t = \int_{\mathbb{R}} \frac{1-\cos(y)}{\frac{|y|^{1+2s}}{k^{1+2s}}} \frac{\mathrm{d}y}{k} = ak^{2s} > 0.$$

Now, let $g \in F$ with $\{b_k\}$ its Fourier coefficients. Since eigenfunctions $\cos(kx)$ form an orthogonal basis of F, the semigroup of equation (5.1.2) is given by

$$(\mathrm{T}_t g)(x) = \sum_{k=0}^{\infty} b_k e^{-ak^{2s}t} \cos(kx), \tag{5.1.5}$$

and we have that $\forall t > 0$, the serie converges $\forall x \in \mathbb{R}$ in L^2.

The proceeding to find the semigroup above is the same one usually performs for the classical heat equation: separation of variables. We begin assuming that a solution of the problem can be written as $u(t,x) = A(t)X(x)$. Equation (5.1.2) now becomes

$$A'(t)X(x) + A(t)(-\Delta)^s X(x) = 0.$$

Because functions A and X depends on different variables, it implies that

$$\begin{cases} -(-\Delta)^s X(x) = -\lambda X(x) \\ \qquad\qquad A'(t) = -\lambda A(t), \end{cases}$$

for some constant $\lambda \in \mathbb{R}$. Using Lemma 5.1.1, we know that the only solution for the first expression are $X(x) = \cos(kx)$ and $\lambda = \lambda_k$ for some integer k. Second equation is easily solved, being $A(t) = A(0)e^{-\lambda_k t}$. A general solution will be written as a linear combination of all the possible solution, that is

$$u(t,x) = \sum_{k \geq 0} b_k e^{-\lambda_k t} \cos(kx),$$

where the coefficients b_k correspond to the Fourier coefficients of the initial condition $g(x)$. Fixing t=0,

$$g(x) = u(0,x) = \sum_{k \geq 0} b_k \cos(kx)$$

which is the Fourier expansion of g (since g is an even function).

Now we need to check that the semigroup $\mathrm{T}_t g$ is defined in F. Using the definition of fractional Sobolev spaces given by its Fourier coefficients, we have

$$\mathrm{T}_t g \in F \iff \mathrm{T}_t g \text{ is even and } \sum_{k=0}^{\infty} k^{4s} |c_k|^2 < \infty,$$

where $c_k = b_k e^{-ak^{2s}t}$, for $t > 0$. This comes immediately since $g \in F$ and $e^{-ak^{2s}t} < 1, \ \forall t > 0$.

We remark that $u(\cdot, t) \to g(x)$ as $t \downarrow 0$ in $\mathrm{L}^2(0,\pi)$. Using Parseval's identity,

$$||u(\cdot,t) - g||^2_{\mathrm{L}^2(0,\pi)} = \Big|\Big| \sum_{k\geq 0} b_k \cos(kx)(e^{-ak^{2s}t} - 1)\Big|\Big|^2_{\mathrm{L}^2(0,\pi)} = \sum_{k\geq 0} b_k^2 (e^{-ak^{2s}t} - 1)^2.$$

Each term of the last series goes to 0 as $t \downarrow 0$, and they can also be bounded by the terms of a convergent series $b_k^2(e^{-ak^{2s}t} - 1)^2 < b_k^2$, since $\sum b_k^2 < \infty$. Therefore $||u(\cdot,t) - g||^2_{\mathrm{L}^2(0,\pi)} \to 0$ as $t \downarrow 0$.

Finally, the partial derivative with respect time t is also in $\mathrm{L}^2(0,\pi)$:

$$u_t = \sum_{k=0}^{\infty} \tilde{c}_k e^{-ak^{2s}t} \cos(kx) \text{ with } \tilde{c}_k = -b_k a k^{2s} \text{ and } \sum_{k=0}^{\infty} |\tilde{c}_k|^2 = a^2 \sum_{k=0}^{\infty} |b_k|^2 k^{4s} < \infty.$$

So that $u(t,x) = (\mathrm{T}_t g)(x)$ is a solution of (5.1.2) with the L^2-norm.

Furthermore, the solution $u(t,x)$ is the only solution of (5.1.2). To see this, consider two functions u_1, u_2 solutions of the problem. We define $v := u_1 - u_2$; v is a solution of the fractional heat equation with zero initial condition. We proceed with the energy method,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} ||v||^2_{\mathrm{L}^2(0,\pi)} &= \int_0^{\pi} v(t,x) v_t(t,x) \ \mathrm{d}x \\ &= (v, v_t)_{L^2(0,\pi)} = (v, -(-\Delta)^s v)_{L^2(0,\pi)} \\ &= -C_{1,s} \int_0^{\pi} \int_{\mathbb{R}} \frac{(v(t,x) - v(t,y))^2}{|x-y|^{1+2s}} \mathrm{d}x\mathrm{d}y \leq 0. \end{aligned}$$

So that v has decreasing L^2 norm, its initial value being 0, therefore v is 0 in L^2.

5.2. Non-homogeneous heat equation

We are now going to study the non-homogeneous fractional heat equation. The problem now becomes

$$\begin{cases} \partial_t u + (-\Delta)^s u = f(t,x) & x \in \mathbb{R}, t > 0, \\ u(x,0) = g(x) & x \in \mathbb{R}. \end{cases} \tag{5.2.1}$$

where $f(t,x)$ is the non-homogeneous term whose regularity will be treated later. Intuitively, the non-homogeneous term represents the rate of probability for a particle to appear at a certain location, at a certain moment.

In order to find a solution to the previous problem, we will apply the same approach used for differential operators: Duhamel's principle.

Using Duhamel's formula and the semigroup of the homogeneous equation, a solution of $\partial_t u + (-\Delta)^s u = f(t,x)$ is given by

$$u(t,x) = \underbrace{T_t g(x)}_{u_1(t,x)} + \underbrace{\int_0^t \mathrm{d}\xi \big(\overbrace{\mathrm{T}_{t-\xi} f(\xi,\cdot)}^{u_\xi(t,\cdot)}\big)(x)}_{u_2(t,x)}. \tag{5.2.2}$$

where the term in the second integral is a solution of the problem

$$\begin{cases} \partial_t u_\xi + (-\Delta)^s u_\xi = 0 & x \in \mathbb{R}, t > \xi \\ u_\xi = f(\xi, x) & x \in \mathbb{R}, t = \xi \end{cases} \tag{5.2.3}$$

And so the second term verifies the following problem

$$\begin{cases} \partial_t u_2 + (-\Delta)^s u_2 = f(t,x) & x \in \mathbb{R}, t > 0 \\ u_2(0,x) = 0 & x \in \mathbb{R}, t > 0. \end{cases} \Downarrow \tag{5.2.4}$$

To check that u above is indeed a solution for the non homogeneous problem, we only need to verify that the second term u_2 is a solution of (5.2.4), since we already know that u_1 solves the homogeneous problem.

The initial condition is easy to check,

$$u_2(0,x) = \int_0^0 u_\xi(t,x) d\xi = 0, \quad \forall x \in \mathbb{R}. \tag{5.2.5}$$

Then notice that

$$\partial_t u_2(t,x) = u_t(t,x) + \int_0^t \partial_t u_\xi(t,x) \mathrm{d}\xi \tag{5.2.6}$$

and

$$\begin{aligned} (-\Delta)^s u_2(t,x) &= (-\Delta)^s \int_0^t u_\xi(t,x) \mathrm{d}\xi \\ &= C_{1,s} \int_{\mathbb{R}} \int_0^t \frac{u_\xi(t,x) - u_\xi(t,y)}{|x-y|^{1+2s}} \mathrm{d}\xi \mathrm{d}y \\ &= \int_0^t (-\Delta)^s u_\xi(t,x) \mathrm{d}\xi. \end{aligned}$$

Where we have used Fubini's theorem to switch integrals.

Adding the previous two expression one gets

$$\begin{aligned}\partial_t u_2(t,x) + (-\Delta)^s u_2(t,x) &= u_t(t,x) + \int_0^t \big(\partial_t u_\xi(t,x) + (-\Delta)^s u_\xi(t,x)\big)\mathrm{d}\xi \\ &= u_t(t,x) = f(t,x).\end{aligned}$$

Therefore, u_2 is a solution of (5.2.4).

We will end this section with a regularity result.

Proposition 5.2.1. *If $f \in L^\infty\big((0,T)\times\mathbb{R}\big)$ for any $T>0$, then problem* (5.1.2) *admits a unique solution.*

Proof. We only need to study the second term of solution (5.2.2). Let $u_2(t,x)$ be the second term,

$$\begin{aligned}||u_2(t,\cdot)||_{\mathrm{L}^2(0,\pi)} = \Big|\Big|\int_0^t \mathrm{d}\xi\big(T_{t-\xi}f(\xi,\cdot)\big)(x)\Big|\Big|_{\mathrm{L}^2(0,\pi)} &\le \int_0^t ||\big(T_{t-\xi}f(\xi,\cdot)\big)(x)||_{\mathrm{L}^2(0,\pi)}\mathrm{d}\xi \\ &\le \int_0^t ||\big(T_0 f(\xi,\cdot)\big)(x)||_{\mathrm{L}^2(0,\pi)}\mathrm{d}\xi \\ &= \int_0^t ||f(\xi,x)||_{\mathrm{L}^2(0,\pi)}\mathrm{d}\xi \\ &\le t\sqrt{\pi}||f||_{\mathrm{L}^\infty(\mathbb{R}\times[0,t])} < \infty,\end{aligned}$$

where it has been used the decrease of the solution of problem (5.1.2) with the L^2-norm, so that $u_2 \in \mathrm{L}^2(0,\pi)$.

To end with the proof, we need to verify $u_2(\cdot,t)\to 0$ as $t\downarrow 0$ in $\mathrm{L}^2(0,\pi)$, which follows exactly the same way as we did for the solution of problem (5.1.2). The uniqueness follows immediately from the energy method. ☐

5.3. Nonlinear parabolic equations

Now that we have found the semigroup for the fractional heat equation (5.1.2), we can give a glance to the nonlinear heat equation where the non-homogeneous term f is now a function of u,

$$\begin{cases}\partial_t u + (-\Delta)^s u = f(u) & x\in\mathbb{R}, t>0 \\ u(x,0) = g(x) & x\in\mathbb{R}\end{cases} \tag{5.3.1}$$

We can show existence and uniqueness for the problem above using Banach fixed-point theorem (at least for t in a small neighborhood of 0). Let us first remember what the theorem states.

Theorem 5.3.1 (Banach fixed-point)**.** *Let (X,d) be a non-empty complete metric space with a contraction mapping $V: X\to X$. Then V admits a unique fixed-point x^* in X.*

The idea is to reformulate the problem as a fixed-point problem using Duhamel's formula. For this, let's define the space X of L^∞ functions in $t \in [0,T]$, $T \in \mathbb{R}$, valued in F, i.e.

$$\begin{aligned} u : [0,T] &\to F \\ t &\mapsto u(t) : (-\pi,\pi) \to \mathbb{R} \\ &\qquad\qquad\qquad x \mapsto u(t)(x) = u(t,x). \end{aligned}$$

Our problem has now become:

$$\text{Find } u = N(u), \quad u \in \overline{B}_r(0) \subset X \text{ for an appropriate } r > 0, \tag{5.3.2}$$

where

$$\begin{aligned} N : X &\to X \\ u &\mapsto N(u)\ (t) := T_t g + \int_0^t T_{t-\xi} f(u(\xi,\cdot))\mathrm{d}\xi. \end{aligned}$$

The space X is a Banach space with the ∞-norm. Moreover $N(u) \in X$ since $T_t : F \to F$.

Now we would have to see that, for $T > 0$ sufficiently small, $N(\overline{B}_r(0)) \subset \overline{B}_r(0)$ and N is a contraction mapping, i.e.

$$||N(u) - N(v)|| \leq \mu||u - v||, \quad \forall u, v \in \overline{B}_r(0), \text{ with } \mu < 1.$$

With these ingredients, we could apply Banach fixed-point theorem and prove existence and uniqueness of solutions.

Note that the choice of space X is not unique, it depends on the problem and on the regularity we want to give to the solutions. Note also that the argument above is for T sufficiently small. Then we should discuss what to do in order to prove existence for all T in $\mathbb{R}$; for example, if T does not depend on the initial conditions.

Before moving to the next section, it is interesting to compare the above problem with nonlinear problems where the fractional operator is replaced with a Hilbert-Schmidt operator.

Definition 5.3.2. Let H be a Hilbert space and $P \in \mathscr{L}(H,H)$. We say that P is a Hilbert-Schmidt operator if there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}^*}$ of H such that

$$\sum_{n=1}^{+\infty} ||Pe_n||^2 < +\infty. \tag{5.3.3}$$

Remark 5.3.3. Condition (5.3.3) is intrinsic to the operator P. That is, if (5.3.3) is verified for an orthonormal basis of H, then for all orthonormal basis $(f_n)_{n\in\mathbb{N}^*}$ of H, we also have $\sum_{n=1}^{\infty} ||Pf_n||^2 < \infty$.

An example of a Hilbert-Schmidt operator would be the following:

$$(Kf)(t) = \int_a^b k(t,s) \cdot f(s)\mathrm{d}s, \tag{5.3.4}$$

where $f \in L^2([a,b])$ and k is a continuous and symmetric kernel satisfying

$$\int_a^b \int_a^b |k(t,s)|^2 \mathrm{d}t \ \mathrm{d}s < \infty.$$

Proposition 5.3.4. *Operator K in (5.3.4) is an element of $\mathscr{L}(L^2([a,b]), L^2([a,b]))$.*

PROOF. For all $t \in [a,b]$ fixed, we note $k_t : s \mapsto k(t,s)$.

In terms of the scalar product in $L^2([a,b])$, we can write

$$(Kf)(t) = \langle k_t, \overline{f} \rangle. \tag{5.3.5}$$

We consider an orthonormal basis $(e_n)_{n\in\mathbb{N}^*}$ of $L^2([a,b])$. According to (5.3.5), we have

$$||Ke_n||^2 = \int_a^b \langle k_t, \overline{f} \rangle^2 \mathrm{d}t.$$

Where, using Fubini-Tonelli's theorem,

$$\begin{aligned}\sum_{n=1}^{\infty} ||Ke_n||^2 &= \sum_{n=1}^{\infty} \int_a^b \langle k_t, \overline{e_n} \rangle^2 \mathrm{d}t \\ &= \int_a^b \Big(\sum_{n=1}^{\infty} \langle k_t, \overline{e_n} \rangle^2 \Big) \mathrm{d}t.\end{aligned}$$

But, from Parseval's identity,

$$\sum_{n=1}^{\infty} |\langle k_t, \overline{e_n} \rangle|^2 = ||k_t||^2.$$

It follows

$$\sum_{n=1}^{\infty} ||Ke_n||^2 = \int_a^b ||k_t||^2 \mathrm{d}t = \int_a^b \int_a^b |k(t,s)|^2 \mathrm{d}t \ \mathrm{d}s < \infty.$$

□

The nonlinear problem is

$$\begin{cases} u_t - Ku = f(u) & x \in [a,b], t > 0 \\ u(0,x) = g(x) & x \in [a,b], \ g \in L^2([a,b]). \end{cases} \tag{5.3.6}$$

We begin by looking at the equivalent linear and homogeneous problem

$$\begin{cases} u_t - Ku \ = 0 & x \in [a,b], t > 0 \\ u(0,x) = g(x) & x \in [a,b], \ g \in L^2([a,b]). \end{cases} \tag{5.3.7}$$

It is known that operator K admits an orthonormal basis of eigenfunctions ϕ. Thus we can find a semigroup for (5.3.7) just as we did in previous section for (5.1.2). And then, using Duhamel's principle, we could obtain a solution for the non homogeneous problem.

Now, to prove existence of solutions for problem (5.3.6) we do as follows.

We write $w(t) = u(t, \cdot)$, regarded as a function of x, and take Y to be a space of functions on $[a, b]$ (in this case we would take $Y = L^2([a, b])$). We define G as

$$Gv = Kv + f(v) \text{ for } v \in Y. \tag{5.3.8}$$

The problem is now equivalent to find $w \in X$, for $X = \{w : w(t) \in Y\}$ a Banach space (for example C^0), solution of

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} w(t) = G[w(t)], \quad t > 0 \\ \quad w(0) = g. \end{cases}$$

In this case, we can directly apply Picard's theorem in Banach spaces (see A.4) to prove existence and unicity of solutions in $[0, T] \times [a, b]$, for a certain $T > 0$.

Remark 5.3.5. Note that $(-\Delta)^s$ is also of type (5.3.4), with $k(x, y) = |x - y|^{-(n+2s)}$. But contrary to the above operator, this definition is not integrable and thus we can not apply the same proceeding ($(-\Delta)^s$ does not send reasonable Banach spaces into themselves).

Remark 5.3.6. In the case of the Hilbert-Schmidt operators, Picard's theorem allow us to solve a much more general type of nonlinear problems. For example, taking G in (5.3.8) to be Lipschitz but nonlinear on Kv:

$$\frac{\mathrm{d}}{\mathrm{d}t} w(t) = \varphi(Kw),$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ can be any globally Lipschitz function.

This cannot be done with the fractional Laplacian, at least with the methods outlined here.

5.4. Dirichlet problem for the fractional Laplacian

In this Section we briefly explain how to prove existence of weak solutions to the Laplace equation of fractional order using variational methods. We define the Dirichlet problem for the fractional Laplacian in a bounded domain $\Omega \in \mathbb{R}^n$ as

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ \qquad u = g & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{5.4.1}$$

As we will see, the existence and uniqueness of weak solution follows from the Riesz-Fréchet theorem once one has the appropriate ingredients.

To do so, first we need to define the concept of weak solution for the fractional Laplace equation. In the classical Laplacian, a weak solution is obtained multiplying both sides of the equation by an arbitrary function $v \in \mathrm{H}_0^1$ and integrating in the domain of the problem. It is possible to perform a similar reasoning for the fractional Laplacian.

We begin by introducing the functional space where we are going to work; the space of $\mathrm{H}^s(\Omega)$ functions with null boundary or exterior conditions.

Definition 5.4.1. Let $u \in H^s(\Omega)$. We define the subspace $H_0^s(\Omega)$ as

$$H_0^s = \big\{u \in H^s(\mathbb{R}^n) : u \equiv 0 \ in\ \mathbb{R}^n \setminus \Omega\big\}, \tag{5.4.2}$$

with the inner product

$$\big(u, v\big)_{H_0^s(\Omega)} := \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y. \tag{5.4.3}$$

Essentially, the only assumption which is needed in order to prove the existence of solutions is the fractional Poincaré inequality.

Proposition 5.4.2. *(Fractional Poincaré Inequality) Let $\Omega \subset \mathbb{R}^n$ be any bounded domain, and let $u \in H_0^s(\Omega)$. Then*

$$\int_\Omega u^2 \leq C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big|u(x)-u(y)\big|^2}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y \tag{5.4.4}$$

Proof. We begin by recalling generalized Hölder's inequality. Let f_1 and f_2 be measurables functions, p and q such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then we have

$$||f_1 \cdot f_2||_{L^p} \leq ||f_1||_{L^q} \cdot ||f_2||_{L^r}. \tag{5.4.5}$$

Now using fractional Sobolev inequality and taking $f_1 = u$ and $f_2 = 1$, $p = 2$ and $q = \frac{2n}{n-2s}$ in (4.4.8) ends with the proof. $\square$

Once one has the Poincaré inequality (5.4.4), then it follows that the space $H_0^1(\Omega)$ is a Hilbert space.

Now, we can give a definition to the notion of weak solutions. But first, let us prove that the fractional Laplacian is self-adjoint in $H_0^s(\Omega)$.

Lemma 5.4.3. *Given u and v regular enough functions with compact support, then*

$$\big((-\Delta)^s u, v\big)_{L^2(\mathbb{R}^n)} = \frac{C_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big(\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y = \big(u, (-\Delta)^s v\big)_{L^2\mathbb{R}^n}.$$

Proof. Simply write

$$\big((-\Delta)^s u, v)\big)_{L^2(\mathbb{R}^n)} = C_{n,s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} v(x)\frac{\big(u(x)-u(y)\big)}{|x-y|^{1+2s}}\mathrm{d}y\mathrm{d}x,$$
$$\big((-\Delta)^s u, v\big)_{L^2(\mathbb{R}^n)} = C_{n,s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} v(y)\frac{\big(u(y)-u(x)\big)}{|y-x|^{1+2s}}\mathrm{d}x\mathrm{d}y,$$

and add both expressions. $\square$

Then, the weak formulation of (5.4.1) is just:

Definition 5.4.4. Let $u \in H_0^s(\Omega)$. We say that u is a weak solution of problem (5.4.1) if, for all $v \in H_0^s(\Omega)$ we have

$$\frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}} = \int_{\Omega} fv. \tag{5.4.6}$$

Theorem 5.4.5. *(Riesz-Fréchet) Let H be a Hilbert space, we consider a mapping δ defined as*

$$\delta \begin{cases} H & \to H' \\ x & \mapsto \delta(x) : h \mapsto (h|x) \end{cases}$$

where H' is the dual space of H and $(h|x)$ represents the form $h \in H'$ acting on $x \in H$. It is an isometric anti-linear bijection.

In other words, let L be a continuous linear form of H. Then there exists a unique $u \in H$ such that $L(v) = (u, v)_H$ for all $v \in H$.

The existence and uniqueness of weak solution follows immediately from the Riesz-Fréchet theorem.

CHAPTER 6

Periodic solutions of semilinear fractional elliptic equations

In this chapter we proceed to prove the existence of even and periodic functions u solutions of the problem

$$\begin{cases} (-\Delta)^s u = f(u) \text{ in } \mathbb{R}, \\ \quad f(0) = 0 \text{ and } f'(0) = b > 0. \end{cases} \tag{6.0.1}$$

for any positive real number b. In particular, we prove the existence of a family of solutions $u_a : \mathbb{R} \to \mathbb{R}$ parametrized by a small parameter $a \in (-\nu, \nu)$. In our proof, we will apply the implicit function theorem at $a = 0$, $\lambda = \lambda^*$, and $\varphi_a = 0$ -taking φ_a orthogonal to $\cos(\cdot)$ in $L^2(0, \pi)$, and see that the solutions are of the form

$$u_a(x) = \frac{a}{\lambda(a)}\{\cos(\lambda(a)x) + \varphi_a(\lambda(a)x)\}. \tag{6.0.2}$$

This will give us λ and φ_a as functions of a.

Furthermore, we will prove that u_a has minimal period $2\pi/\lambda(a)$ if $a \neq 0$, and $u_a \neq u_{a'}$ for $a \neq a'$.

In order to do this, we start by pointing out some definitions and results needed, such as the notion of Fréchet derivative and the implicit function theorem on Banach spaces.

6.1. Differentiability on Banach spaces

We are very familiar with the concept of differentiation for real valued functions $\phi : \mathbb{R} \to \mathbb{R}$. In order to study the above problem, we need to work with a more extended definition of differentiation, particularly to any Banach space.

Definition 6.1.1. Let V and W be Banach spaces, and $U \subset V$ be an open subset of V. A function $\phi : U \to W$ is called Fréchet differentiable at $x \in U$ if there exists a bounded linear operator $A : V \to W$ such that

$$\lim_{h \to 0} \frac{\big|\big|\phi(x+h) - \phi(x) - Ah\big|\big|_W}{||h||_V} = 0. \tag{6.1.1}$$

If there exists such an operator, it is unique and we write $D\phi(x) = A$ and call it the Fréchet derivative of ϕ at x.

Definition 6.1.2. A function ϕ that is Fréchet differentiable for any point of U is said to be continuously Fréchet differentiable if the function

$$D\phi : U \to B(V, W)$$
$$x \mapsto D\phi(x)$$

is continuous. Note that this is not the same as requiring that the map $D\phi(x) : V \to W$ be continuous for each value of x (which is assumed).

The concept of directional derivative in differential calculus can also be generalized to any function between locally convex topological vectors spaces such as Banach spaces. Like the Fréchet derivative on a Banach space, the Gâteaux differential is often used to formalize the functional derivative commonly used in the calculus of variations and physics.

Definition 6.1.3. Let V and W be Banach spaces, $\phi : V \to W$ be a function between them and let $h \neq 0$ and x be vectors in V. The Gâteaux differential $d_h\phi$ is defined

$$d_h\phi = \lim_{\epsilon \to 0} \frac{\phi(x + \epsilon h) - \phi(x)}{\epsilon}. \tag{6.1.2}$$

If the limit exists for all $h \in W$, then one says that ϕ is Gâteaux differentiable at x.

Remark 6.1.4. For any Banach space V with dimension ≥ 2 there exists infinite Gâteaux differentials (one for each direction h).

The Fréchet derivative $D\phi$ of $\phi : V \to U$ is defined implicitly by

$$\phi(x + k) = \phi(x) + (D\phi)k + o(||k||), \text{ for } k \in V. \tag{6.1.3}$$

To establish the relationship to the Gâteaux differential, take $k = \epsilon h$ and write

$$\phi(x + \epsilon h) = \phi(x) + \epsilon(D\phi)h + ho(\epsilon). \tag{6.1.4}$$

In the limit $\epsilon \to 0$, we have $(D\phi)h = d_h\phi$. Then, if $d_h\phi$ has the form Ah, we can identify $D\phi = A$. However, not every Gâteaux differentiable function is Fréchet differentiable.

Here we cite two theorems of existence and uniqueness of the Fréchet derivative, both of them without proof.

Theorem 6.1.5. *The Fréchet derivative exists at $x = a$ if and only if all Gâteaux differentials are continuous functions of x at $x = a$.*

Proof. See, for example, Munkres or Spivak (for $\mathbb{R}^n$) or Cheney (for any normed vector space). □

Theorem 6.1.6. *If it exists for a function ϕ at a point x, the Fréchet derivative is unique.*

Proof. Assume otherwise, then construct a contradiction. See, for example, Munkres or Spivak (for $\mathbb{R}^n$) or Cheney (for any normed vector space). □

Remark 6.1.7. The rules of classical differentiation, such as the product and the chain rule, also work for Fréchet derivatives.

We now present a generalization of the Implicit function theorem for Banach spaces.

Theorem 6.1.8 (Implicit function theorem)**.** *Let X, Y and Z be Banach spaces. Let the mapping $\phi : X \times Y \to Z$ be continuously Fréchet differentiable. If $(x_0, y_0) \in X \times Y$, $\phi(x_0, y_0) = 0$, and $y \mapsto D\phi(x_0, y_0)(0, y)$ is a Banach space isomorphism from Y onto Z, then there exists neighborhoods U of x_0 and V of y_0 and a Fréchet differentiable function $g : U \to V$ such that $\phi(x, g(x)) = 0$ and $\phi(x, y) = 0$ if and only if $y = g(x)$, for all $(x, y) \in U \times V$.*

To end with this section, we present a fundamental result which states that if a continuous linear operator between Banach spaces is surjective then it is an open map. More precisely,

Theorem 6.1.9 (Open mapping theorem)**.** *Let X and Y be Banach spaces and let $A : X \to Y$ be a surjective continuous linear operator. Then A is an open map, i.e. if U is an open set in X, then $A(U)$ is open in Y.*

The open mapping theorem has several important consequences. Here we mention one of them:

Theorem 6.1.10 (Inverse mapping theorem)**.** *Let $A : X \to Y$ be a bijective continuous linear operator between Banach spaces. Then there exists a continuous inverse operator $A^{-1} : Y \to X$.*

6.2. The semilinear problem

With these results in mind, we can finally strike problem (6.0.1). Remember that we want to show the existence of even and periodic solutions u. Any even and periodic function u on $\mathbb{R}$ can be written in form

$$u(x) := u_a(x) = \frac{a}{\lambda}\{\cos(\lambda x) + \varphi_a(\lambda x)\}, \text{ for a certain period } \lambda = \lambda(a), \tag{6.2.1}$$

with $\varphi_a \in \langle \cos(\cdot)\rangle^{\perp}$. Notice that, aside from the additive constant, the most dominant term in a Fourier series is the first (this is why we split the function u_a in a $\cos(\cdot)$ and a function φ_a orthogonal to $\cos(\cdot)$ in $L^2(0, \pi)$).

Note that the functions $u_a = u_a(x)$ have a period that may change with a. It will be very convenient, in order to apply the implicit function theorem, to work with functions all with the same period 2π. Thus, we rescale the variable $x = x_1$ and $u(x) = x_2$ and see how equation (6.0.1) changes after rescaling.

We define the variable $y = (x/\lambda) \cdot 2\pi$ and the function $v_a(y) = u_a(x)$. The new function $v_a = v_a(y)$ is even and 2π-periodic

$$v(y + 2\pi) = u(\frac{y}{2\pi}\lambda + \lambda) = v(y).$$

Taking $z = \frac{\zeta}{2\pi}\lambda$, the fractional Laplacian $(-\Delta)^s v$ is then equivalent to

$$\begin{aligned}
(-\Delta)^s v(y) &= \int_{\mathbb{R}} \frac{v(y) - v(\zeta)}{|y-\zeta|^{1+2s}} \mathrm{d}\zeta \\
&= \int_{\mathbb{R}} \frac{u(x) - u(\frac{\zeta}{2\pi}\lambda)}{|y-\zeta|^{1+2s}} \mathrm{d}\zeta \\
&= \int_{\mathbb{R}} \frac{u(x) - u(z)}{\left|(x-z)\frac{2\pi}{\lambda}\right|^{1+2s}} \frac{2\pi \mathrm{d}z}{\lambda} \\
&= \left(\frac{\lambda}{2\pi}\right)^{2s} \cdot \int_{\mathbb{R}} \frac{u(x) - u(z)}{|x-z|^{1+2s}} \mathrm{d}z \\
&= \left(\frac{\lambda}{2\pi}\right)^{2s} \cdot (-\Delta)^s u(x).
\end{aligned}$$

Therefore, we must look for functions $v = (v_a)^\lambda$ of the form

$$v(y) = a\cos(y) + a\varphi(y), \tag{6.2.2}$$

with φ even, 2π-periodic, and orthogonal to $\cos(\cdot)$ in $L^2(0,\pi)$, where v are solutions of

$$(-\Delta)^s v = \left(\frac{\lambda}{2\pi}\right)^{2s} \cdot f(v). \tag{6.2.3}$$

To have the semilinear operator to be invertible and be able to use directly the implicit function theorem, it is necessary to divide equation (6.0.1) by a and work with the new operator

$$\Phi(v) = \frac{1}{a}\Big((-\Delta)^s v - \left(\frac{\lambda}{2\pi}\right)^{2s} f(v)\Big). \tag{6.2.4}$$

Equivalently, taking $v = v_a$ and expressing $f(v) = bv + g(v)v$ for some continuous function g with $g(0) = 0$,

$$\begin{aligned}
\Phi(v) := \Phi(a,\lambda,\varphi) &= (-\Delta)^s \cos(y) + (-\Delta)^s \varphi(y) \\
&\quad - \left(\frac{\lambda}{2\pi}\right)^{2s} \Big\{ b\big(\cos(y) + \varphi(y)\big) + g\big(a(\cos(y) + \varphi(y))\big)\big(\cos(y) + \varphi(y)\big) \Big\}.
\end{aligned} \tag{6.2.5}$$

We need to solve the nonlinear equation

$$\Phi(a,\lambda,\varphi) = 0. \tag{6.2.6}$$

Let us introduce the functional spaces in which we work. We consider the space $X := F \cap \langle \cos(\cdot) \rangle^\perp$, F defined in (5.1.1), and the space $L^2(0,\pi)$.

Let us suppose f and g to be $C^\infty(\mathbb{R})$ and bounded, and such that all derivatives of different orders are bounded, that is

$$D^j f, D^j g \in L^\infty(\mathbb{R}) \quad \forall j \in \mathbb{N}. \tag{6.2.7}$$

Then, we have $g \circ \varphi \in L^2(0,\pi)$, and it's Fréchet derivative $g'(\varphi)\psi$ is also in $L^2(0,\pi)$ for $\psi \in H^{2s}(0,\pi) \subset L^2(0,\pi)$ since g and g' are bounded.

Remark 6.2.1. For example, if we consider $f(u) = b\sin(u)$ we have that $f(u) = bu + ug(u)$, where g is the series expansion of the function $b\big(\cos(\cdot) - 1\big)$, i.e.

$$g(u) = b\sum_{n=1}^{\infty}(-1)^n \frac{u^{2n}}{(2n)!}.$$

Both f and g are $C^\infty(\mathbb{R})$, bounded, and with bounded derivatives of all orders, and g satisfies $g(0) = 0$.

We have

$$\Phi : (-\nu,\nu) \times \mathbb{R} \times X \to L^2(0,\pi). \tag{6.2.8}$$

The next task is to study the operator Φ at $a = 0, \lambda = \lambda^*$ and $\varphi = 0$. Let us begin by computing the value of Φ at this point:

$$\Phi(0,\lambda^*,0) = \big\{\lambda_1 - b\Big(\frac{\lambda^*}{2\pi}\Big)^{2s}\big\}\cos(\cdot), \tag{6.2.9}$$

where λ_1 is the eigenvalue of $(-\Delta)^s$ associated to $\cos(\cdot)$.

In order to have equation (6.2.6) to be 0, we need to take $\lambda^* = 2\pi \cdot (\lambda_1/b)^{1-2s}$. Thus, we have

$$\Phi(0,\lambda^*,0) = \big\{\lambda_1 - b\Big(\frac{\lambda^*}{2\pi}\Big)^{2s}\big\}\cos(\cdot) = 0. \tag{6.2.10}$$

Let us check that Φ satisfies the hypothesis of the implicit function theorem, that is, $\Phi(a,\lambda,\varphi)$ is continuously Fréchet differentiable and $D_{(\lambda,\varphi)}\Phi(0,\lambda^*,0)$ defines an isomorphism from $\mathbb{R} \times X$ into $L^2(0,\pi)$.

Proposition 6.2.2. *We have that*

$$D_\lambda \Phi(0,\lambda^*,0) = \gamma\cos(\cdot) \tag{6.2.11}$$

for some constant $\gamma > 0$. On the other hand, for all $\psi \in \langle\cos(\cdot)\rangle^\perp$,

$$L\psi := D_\varphi(0,\lambda^*,0)\psi = (-\Delta)^s\psi - b\Big(\frac{\lambda^*}{2\pi}\Big)^{2s}\psi. \tag{6.2.12}$$

PROOF. We start by calculating the partial derivative of Φ with respect to λ at $(0,\lambda^*,0)$:

$$\partial_\lambda \Phi(0,\lambda^*,0) = b\frac{2s}{2\pi}\Big(\frac{\lambda}{2\pi}\Big)^{2s-1}_{\big|\lambda=\lambda^*} \cos(\cdot) = b\Big(\frac{s}{\pi}\Big)\cos(\cdot). \tag{6.2.13}$$

To find the partial derivate of Φ with respect to φ at $(0,\lambda^*,0)$, we use Definition 6.1.3:

$$\begin{aligned} d_h\Phi(0,\lambda^*,0) &= \lim_{\epsilon\to 0}\frac{\Phi(0,\lambda^*,\epsilon h)-\Phi(0,\lambda^*,0)}{\epsilon} \\ &= \lim_{\epsilon\to 0}\frac{\epsilon(-\Delta)^s h - \epsilon b\Big(\frac{\lambda^*}{2\pi}\Big)^{2s} h}{\epsilon} = (-\Delta)^s h - b\Big(\frac{\lambda^*}{2\pi}\Big)^{2s} h. \end{aligned}$$

Hence

$$\partial_\varphi \Phi(0,\lambda^*,0)\cdot\psi = \Big((-\Delta)^s - b\Big(\frac{\lambda^*}{2\pi}\Big)^{2s} I\Big)\psi =: L\psi, \tag{6.2.14}$$

where I stands for the identity operator. $\square$

Proposition 6.2.3. *The operator $\Phi = \Phi(a,\lambda,\varphi) : (-\nu,\nu)\times\mathbb{R}\times X \to L^2(0,\pi)$ is continuously Fréchet differentiable. Moreover, the linear operator*

$$(D_\lambda\Phi, D\varphi\Phi)(0,\lambda^*,0) : \mathbb{R}\times X \to L^2(0,\pi)$$

is continuous and invertible.

PROOF. Let $g\in C^\infty(\mathbb{R})$ such that (6.2.7) holds. Let us consider the partial derivatives of Φ with respect a, λ and φ:

$$\begin{aligned} \partial_a\Phi(a,\lambda,\varphi) &= -\Big(\frac{\lambda}{2\pi}\Big)^{2s} g'\big(a(\cos(y)+\varphi(y))\big)\big(\cos(y)+\varphi(y)\big)^2, \\ \partial_\lambda\Phi(a,\lambda,\varphi) &= -\frac{2s}{2\pi}\Big(\frac{\lambda}{2\pi}\Big)^{2s-1}\Big\{b\big(\cos(y)+\varphi(y)\big) \\ &\qquad + g\big(a(\cos(y)+\varphi(y))\big)\big(\cos(y)+\varphi(y)\big)\Big\}, \\ \partial_\varphi\Phi(a,\lambda,\varphi)\cdot\psi &= (-\Delta)^s\psi(y) - \Big(\frac{\lambda}{2\pi}\Big)^{2s}\Big\{b\psi(y) + g'\big(a(\cos(y)+\psi(y))\big)a\psi\Big\}. \end{aligned}$$

We already know that $(-\Delta)^s$ is continuous from H^{2s} into L^2 (see Proposition 4.4.7). It follows that $\Phi(a,\lambda,\varphi)$ is C^1 on $(-\nu,\nu)\times\mathbb{R}\times X$.

In order to prove the second statement, we use the following result:

Proposition 6.2.4. *L is a bounded linear operator from X into $L^2(0,\pi)$.*

PROOF. For simplicity, we write $r = b\Big(\frac{\lambda^*}{2\pi}\Big)^{2s}$. Let $\Psi\in X$, we can express Ψ using it's Fourier expansion

$$\Psi = b_0 + \sum_{k>1} b_k\cos(ky).$$

Thus,

$$L\Psi = c \cdot \big\{b_0 \cdot 0 + \sum_{k>1} b_k k^{2s}\cos(ky)\big\} - r \cdot \big\{b_0 + \sum_{k>1} b_k \cos(ky)\big\},$$

where we've used the expression of the eigenvalues λ_k given in (5.1.4).

Using Parseval's identity,

$$\begin{aligned} ||L\Psi||_{L^2(0,\pi)} &= \sum_{k\geq 0, k\neq 1} \big(b_k(ck^{2s}-r)\big)^2 \\ &= c^2 \sum_{k\geq 0,k\neq 1} b_k^2 k^{4s} + r^2 \sum_{k\geq 0,k\neq 1} b_k^2 - 2cr \sum_{k\geq 0,k\neq 1} b_k^2 k^{2s} \qquad (6.2.15) \\ &= c^2||\Psi||_{H^{2s}(0,\pi)} + r^2||\Psi||_{L^2(0,\pi)} - 2cr||\Psi||_{H^s(0,\pi)}. \end{aligned}$$

Now, using Theorem 4.4.5 and Propostion 4.4.1, for $2s > 1/2$ we obtain

$$||L\Psi||_{L^2(0,\pi)} \leq M||\Psi||_{H^2(0,\pi)} \qquad (6.2.16)$$

for some constant M, so that the operator L is bounded. Moreover, since L is linear, it is continuous from X into $L^2(0,\pi)$. □

We observe the following:

$$\begin{aligned} L1 &= -b\Big(\frac{\lambda^*}{2\pi}\Big)^{2s}, \\ L\cos(\cdot) &= \big\{\lambda_1 - b\Big(\frac{\lambda^*}{2\pi}\Big)^{2s}\big\}\cos(\cdot) = 0 \text{ from (6.2.10)}, \\ L\cos(2\cdot) &= (\lambda_2 - b\Big(\frac{\lambda^*}{2\pi}\Big)^{2s})\cos(2\cdot), \\ L\cos(3\cdot) &= (\lambda_3 - b\Big(\frac{\lambda^*}{2\pi}\Big)^{2s})\cos(3\cdot), \\ &\vdots \end{aligned}$$

We denote the standard projections by

$$\Pi_1 : L^2(0,\pi) \to L^2(0,\pi) \cap \langle\cos(\cdot)\rangle \quad \Pi_2 : L^2(0,\pi) \to L^2(0,\pi) \cap \langle\cos(\cdot)\rangle^{\perp}.$$

From (6.2.13), we deduce that

$$D_\lambda(\Pi_1\Phi)(0,\lambda^*,0)(y) = b\Big(\frac{s}{\pi}\Big)\cos(\cdot) =: \gamma\cos(\cdot),$$

and also that

$$D_\lambda(\Pi_2\Phi)(0,\lambda^*,0)(y) \equiv 0.$$

Let $\psi \in \langle\cos(\cdot)\rangle^{\perp}$ (i.e. a linear combination of $(1, \cos(2\cdot), \cos(3\dot{)}, \dots))$, from the observation above we have

$$L\cos(ky) := D_\varphi\Phi(0,\lambda^*,0)\cos(ky) = \Big\{(\lambda_k - b\Big(\frac{\lambda^*}{2\pi}\Big)^{2s}\Big\}\cos(ky),$$

for $k \neq 1$ and 0 for $k = 1$, so that $D_\varphi(\Pi_2\Phi)(0,\lambda^*,0)\psi(y)$ is a bijection from $L^2(0,\pi) \cap \langle\cos(\cdot)\rangle^\perp$ into himself.

We have obtained the following triangular matrix

$$\begin{pmatrix} D_\lambda(\Pi_1\Phi)(0,\lambda^*,0) & D_\varphi(\Pi_1\Phi)(0,\lambda^*,0)\psi \\ D_\lambda(\Pi_2\Phi)(0,\lambda^*,0) & D_\varphi(\Pi_2\Phi)(0,\lambda^*,0)\psi \end{pmatrix} = \begin{pmatrix} \gamma\cos(\cdot) & D_\varphi(\Pi_1\Phi)(0,\lambda^*,0)\psi \\ 0 & L\psi \end{pmatrix},$$

where the diagonal elements generate all $L^2(0,\pi)$. This shows that $D_{(\lambda,\varphi)}\Phi(0,\lambda^*,0)$ is indeed a bijection between $\mathbb{R}\times X$ and $L^2(0,\pi)$.

Finally, using Theorem 6.1.10 we see that $D_{(\lambda,\varphi)}\Phi(0,\lambda^*,0)$ is invertible, with continuous inverse, and thus it defines a Banach space isomorphism from $\mathbb{R}\times X$ into $L^2(0,\pi)$. This ends with the demonstration of Proposition 6.2.3. $\square$

Applying the implicit function theorem, there exists open subsets $U \subset \mathbb{R}$ and $V \subset X$, with $(\lambda^*,0) \in U \times V$, and two Fréchet differentiable functions $\lambda(a)$ and $\varphi(a)$ such that $\Phi(a,\lambda(a),\varphi(a)) = 0$ for all $(a,\lambda,\varphi) \in (-\nu,\nu)\times U \times V$.

We have proven the existence of functions $u = u_a$ solutions of (6.0.1), where u_a are of the form

$$u_a = \frac{a}{\lambda}\{\cos(\lambda x) + \varphi_a(\lambda x)\},$$

for λ and v_a functions of a.

It remain to prove that the minimal period of u_a is $2\pi/\lambda(a)$ if $a \neq 0$, and that $u_a \neq u_{a'}$, if $a \neq a'$. Let us start from the first statement. Clearly, it is equivalent to prove that, after the rescaling, the function

$$v_a(x) = a\{\cos(x) + \varphi_a(x)\}$$

with $a \neq 0$ and φ_a orthogonal to $\cos(\cdot)$ in $L^2(0,\pi)$, has minimal period 2π. This is an easy task, by expressing $v_a(x)$ as a Fourier series $a_0 + \sum_{k=2}^{\infty} a_k\cos(kx)$. Now, if T is the minimal period of u, we must have

$$\begin{aligned} &\cos(x) + \varphi_a(x) = \cos(x+T) + \varphi_a(x+T) \\ &= \cos(x)\cos(T) - \sin(x)\sin(T) + a_0 + \sum_{k=2}^{\infty} a_k\{\cos(kx)\cos(kT) - \sin(kx)\sin(kT)\}. \end{aligned}$$

Multiplying the first and the last terms in the above expression by $\cos(x)$ and integrating in $(0,2\pi)$, we deduce that $\cos(T) = 1$. Hence the minimal period is $T = 2\pi$. Finally, from this we can easily deduce that $u_a \neq u_{a'}$ if $a \neq a'$. Indeed, if $u_a \equiv u_{a'}$ then their minimal periods would agree, and thus $\lambda(a) = \lambda(a')$. Now, this leads to $a = a'$, since $u_a(x) = \frac{a}{\lambda(a)}\{\cos(\lambda(a)x) + \varphi_a(\lambda(a)x)\}$ and $\varphi_a(\sigma)$ is orthogonal to $\cos(\sigma)$ in $L^2(0,\pi)$.

APPENDIX A

Theorem A.1 (Lévy-Khintchine formula)**.** Let $X = (X_t, t \geq 0)$ be a Lévy process, then the characteristic function $\psi_t(z)$ is

$$\psi_t(z) = exp\Big\{ia \cdot z + \frac{1}{2}Q(z) + \int_{\mathbb{R}^n} \big(1 - e^{iz\cdot x} + iz \cdot x 1_{|x|<1}\big)\nu \mathrm{d}x\Big\}, \tag{A.0.1}$$

for $a \in \mathbb{R}^n$, Q a quadratic form on $\mathbb{R}^n$, and ν a so-called Lévy measure satisfying

$$\int \big(1 \wedge |x|^2\big)\nu \mathrm{d}x < \infty. \tag{A.0.2}$$

Remark A.2. This looks a bit arbitrary, so let's explain what each of these terms 'means'.

(1) $ia \cdot z$ comes from a drift of $-a$. Note that a deterministic linear function is a Lévy process.

(2) $\frac{1}{2}Q(z)$ comes from a Brownian part $\sqrt{Q}B_t$.

The rest corresponds to the jump part of the process. The reason why there is an indicator function floating around is that we have to think about two regimes separately, namely large and small jumps.

Theorem A.3 (Picard's theorem on Banach spaces)**.** Let E be a Banach space and let $F : E \to E$ be locally Lipschitz. For $u : [t_0 - \epsilon, t_0 + \epsilon] \subset \mathbb{R} \to E$, consider the initial value problem

$$u'(t) = F(t, u(t)), \quad u(t_0) = u_0, \quad t \in [t_0 - \epsilon, t_0 + \epsilon], \tag{A.0.3}$$

Then, for some value $\epsilon > 0$, there exists a unique solution $u \in C^1([t_0 - \epsilon, t_0 + \epsilon]; E)$ to the initial value problem.

Theorem A.4 (Plancherel's Theorem)**.** Let $E(t)$ be a function that is sufficiently smooth and that decays sufficiently quickly near infinity so that its integrals exist. Further, let $E(t)$ and E_v be Fourier transform pairs so that

$$E(t) \equiv \int_{-\infty}^{\infty} E_v e^{-2\pi ivt} \mathrm{d}v$$
$$\overline{E}(t) \equiv \int_{-\infty}^{\infty} \overline{E}_{v'} e^{2\pi iv't} \mathrm{d}v',$$

where $\overline{z}$ denotes the complex conjugate.
Then

$$\int_{-\infty}^{\infty} |E(t)|^2 \mathrm{d}t = \int_{-\infty}^{\infty} |E_v|^2 \mathrm{d}v,$$

i.e. the Fourier transform in a unitary transformation with respect to the L^2-norm.

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