Poincaré-Melnikov-Arnold method for analytic planar maps

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Abstract. The Poincaré-Melnikov-Arnold method for planar maps gives rise to a Melnikov function defined by an infinite and (a priori) analytically uncomputable sum. Under an assumption of meromorphicity, residues theory can be applied to provide an equivalent finite sum. Moreover, the Melnikov function turns out to be an elliptic function and a general criterion about non-integrability is provided.

Several examples are presented with explicit estimates of the splitting angle. In particular, the non-integrability of non-trivial symmetric entire perturbations of elliptic billiards is proved, as well as the non-integrability of standard-like maps.

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1. Introduction

From the works by Poincaré [Poi99], Melnikov [Mel63], and Arnold [Arn64], the *Poincaré-Melnikov-Arnold method* has become the standard tool for detecting splitting of invariant manifolds for systems of ordinary differential equations close to "integrable" ones with associated separatrices. This method gives rise to an integral known as the *Melnikov function* (or Melnikov integral), whose zeros, if non-degenerate, imply the "splitting" of the former separatrices. (For a general background, see [GH83, Wig90].)

Nevertheless, the explicit computation of this function in the applications is performed via residues theory, which requires suitable meromorphic properties for the functions appearing in the Melnikov integral.

There exists a similar theory for maps [Eas84, Gam85, Gam87], and in this case the Melnikov function is not an integral anymore, but an infinite and (a priori) analytically uncomputable sum. In general, the computation of such a kind of infinite sums requires an excursion to the complex field, and in this way the first explicit computation of such an infinite sum was done in [GPB89], using the Poisson summation formula, residues theory and elliptic functions.

At a first glance, their approach seemed very specific for the examples studied therein. However, it turns out that a systematic and general theory for computing the Melnikov function can be developed in the discrete case, under adequate hypothesis of meromorphicity, like in the continuous case.

To begin with, let $F_0 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an analytic area preserving diffeomorphism, *integrable and with a separatrix to a saddle point.* Denote H, Γ and P_0, the *first integral*, the separatrix and the saddle point, respectively.

Since F_0 is area preserving, one has $\mathrm{Spec}[DF_0(P_0)] = \{\lambda, \lambda^{-1}\}$ for some real λ with $|\lambda| > 1$. Replacing the map with F_0^2 if necessary, one can assume that $\lambda > 1$.

Since F_0 is an analytic map with a separatrix to a saddle point, there exists a *natural parameterization* σ of Γ (with regard to F_0), i.e., a bijective analytic map $\sigma : \mathbb{R} \longrightarrow \Gamma$ such that:

(i) $F_0(\sigma(t)) = \sigma(t+h), \quad \forall\, t \in \mathbb{R}$,

(ii) $h = \ln \lambda$ (normalization condition).

It turns out that, maybe multiplying the first integral by a suitable constant, the above natural parameterization is a *solution* of the Hamiltonian field associated to H, i.e.,

$$\dot{\sigma} = J \cdot \nabla H \circ \sigma \quad \text{where} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Notice that this condition is equivalent to say that the time-h Hamiltonian flow associated to H *interpolates* the map on the separatrix. This fact is very useful to obtain explicitly the natural parameterization, as well as to simplify the expressions of the Melnikov function. Henceforth it will be assumed that the first integral is chosen in order to verify this interpolation condition.

Now, consider a family of analytic diffeomorphisms

$$F_\varepsilon : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad F_\varepsilon = F_0 + \varepsilon F_1 + \mathrm{O}(\varepsilon^2),$$

as a general perturbation of the above situation. Then, for ε small enough, F_ε has a saddle point P_ε "close" to P_0 and the local stable and unstable manifolds $(W^s_\varepsilon)_{\mathrm{loc}}$, $(W^u_\varepsilon)_{\mathrm{loc}}$ of P_ε are "close" to those of the unperturbed saddle point P_0.

Generically, the separatrix Γ breaks, and our aim is to compute the first order approximation on ε of the distance between the invariant manifolds along the normal directions of the separatrix.

To this end, given a natural parameterization $\sigma : \mathbb{R} \longrightarrow \Gamma$ of Γ, $\sigma^u(t,\varepsilon)$ (respectively $\sigma^s(t,\varepsilon)$) will denote the "first" intersection of W^u_ε (respectively W^s_ε) with the normal to Γ at $\sigma(t)$; in particular, $\sigma^{u,s}(t,0) = \sigma(t)$ (see Figure 1).

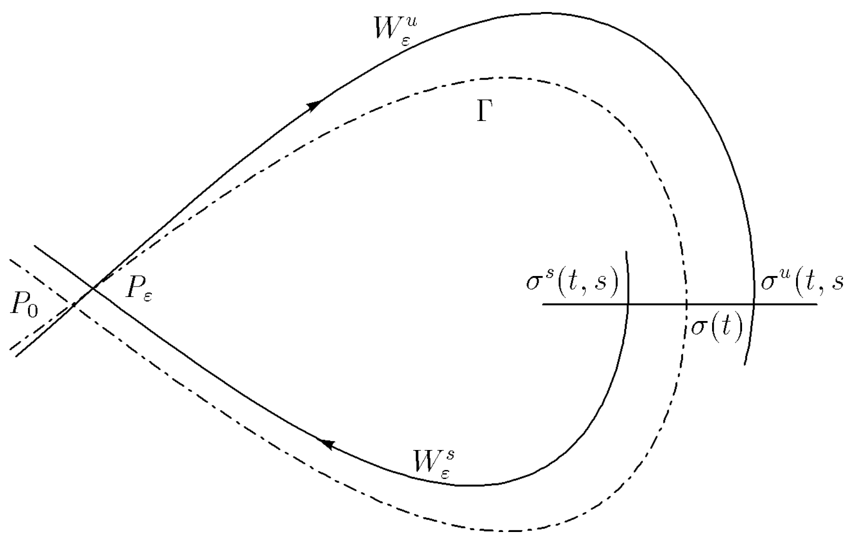

Figure 1. Perturbation of a separatrix consisting of homoclinic orbits. The dashed curve is the family of homoclinic orbits of the unperturbed map.

Following Poincaré and Arnold [Poi99, Arn64], the measure of the distance between these points is given by the difference of first integrals ("energies")

$$\Delta(t,\varepsilon) = H(\sigma^u(t,\varepsilon)) - H(\sigma^s(t,\varepsilon)) = \varepsilon M(t) + \mathrm{O}(\varepsilon^2),$$

where M is the so-called *Melnikov function*, a h-periodic function given by an infinite sum

$$M(t)=\sum_{n\in\mathbb{Z}} g(t+hn), \quad g(t)=\langle\dot\sigma(t), J\cdot F_1(\sigma(t-h))\rangle.$$

It is a commonplace to note that simple zeros of the Melnikov function give rise to transversal homoclinic points and chaotic phenomena.

A very important case takes place when F_ε is an area preserving map (a.p.m., from now on) with generating function $\mathcal{L}(x,X,\varepsilon)=\mathcal{L}_0(x,X)+\varepsilon\mathcal{L}_1(x,X)+\mathrm{O}(\varepsilon^2)$.

In this case, the Melnikov function is given by

$$M=\dot L, \quad L(t)=\sum_{n\in\mathbb{Z}} f(t+hn), \quad f(t)=\mathcal{L}_1(x(t),x(t+h)),$$

where $x(t)$ is the first component of $\sigma(t)$ and, in order to get an (absolutely) convergent sum, $\mathcal{L}_1$ is determined by $\mathcal{L}_1(x_0,x_0)=0$, if $P_0=(x_0,y_0)$. In this situation, M is actually the derivative of a h-periodic function and not only a h-periodic one. Consequently, if M is real analytic and not identically zero, it has real zeros of odd order, so the perturbed invariant curves cross and the perturbed map [Cus78] is non-integrable.

To unify notation, consider $\Sigma(t)=\sum_{n\in\mathbb{Z}} q(t+hn)$, where $q(t)$ is either $g(t)$ or $f(t)$. If $q(t)$ is a *meromorphic* function (respectively, a function with only isolated singularities), then $\Sigma(t)$ is an *elliptic* function (respectively, a double periodic function with only isolated singularities) with periods h and $T\mathrm{i}$, where usually $T=2\pi$, but if there is some symmetry in the problem, $T=\pi$. The relation beetwen elliptic functions and Melnikov functions for maps goes back to [GPB89], although until [Lev93] it is not clearly showed.

It turns out that $\Sigma(t)$ can be expressed as the following *finite* sum

$$\Sigma(t)=-\sum_{z\in\mathcal{S}(q)} \mathrm{res}\ (\chi_t q, z),$$

where $\mathcal{S}(q)=\{z\in\mathbb{C} : z \text{ is a singularity of } q,\ 0<\Im z<T\}$ and $\chi_t(z)=\chi(z-t)$, χ being the function determined (up to an additive constant) by the conditions:

(C1) χ is meromorphic on $\mathbb{C}$,

(C2) χ' is h-periodic and χ is $T\mathrm{i}$-periodic,

(C3) the set of poles of χ is $h\mathbb{Z}+T\mathrm{i}\mathbb{Z}$, being all of them simple and of residue 1.

It is worth nothing that χ can be explicitly computed in terms of the incomplete elliptic integral of the second kind.

Both from a theoretical and a practical point of view, this *summation formula* is one of the main tools of this paper, as it provides explicit computations for the Melnikov function M, assuming hypothesis of meromorphicity for the functions g or f.

The following powerful *non-integrability criterion* is easily obtained. Let F_ε be a family of analytic a.p.m. with a generating function where F_0 verifies the above-mentioned hypothesis and suppose that the function f has only isolated singularities.

Let $\mathcal{S}(f) = \{z_{s\ell}\}$, where $z_{s\ell} - z_{s'\ell'} \in h\mathbb{Z}$ iff $s = s'$ (the singularities of f have been classified modulo h), and introduce the *non-integrability coefficients* of the problem: $d_{sj} = \sum_\ell a_{-(j+1)}(f, z_{s\ell})$, where $a_{-j}(f, z_0)$ denotes the coefficient of $(z - z_0)^{-j}$ in the Laurent expansion of f around an isolated singularity z_0. Then, it turns out that if some of these non-integrability coefficients is non-zero, the Melnikov function is not identically zero and F_ε is non-integrable for ε small enough. (For the continuous case, a related criterion that also takes advantage of the structure of the singularities in the complex field can be found in [Zig82].)

The power of this criterion lies in the following two facts:

- The non-integrability coefficients can be explicitly computed, so it can be easily checked.
- This criterion detects intersections of arbitrary finite order (and not only transversal ones, like is usual in the literature).

As a first application, consider the problem of the "convex billiard table" [Bir27]: Let C be an (analytic) closed convex curve of the plane $\mathbb{R}^2$, and suppose that a material point moves inside C and collides with C according to the law "the angle of incidence is equal to the angle of reflection". Following Birkhoff, this discrete dynamical system can be modeled by an (analytic) a.p.m. in the annulus. When C is an ellipse this map is called *elliptic billiard* and is an integrable map, [Bir27].

Several authors, [Lev93, Tab93, LT93, Tab94, Lom94], have devoted their efforts to the study of perturbed elliptic billiards. All the cases where explicit computations have been performed reduce to *symmetric and reversible quartic perturbations*, i.e., the billiard in curves like

$$C_\varepsilon = \left\{ (x, y) \in \mathbb{R}^2 \ : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \varepsilon P(x, y, \varepsilon) = 1 \right\},$$

where $0 < b < a$ are constants, ε is a small perturbative parameter, C_ε is an analytic curve depending on a $\mathcal{C}^2$ way on ε, symmetric with respect to the axis of coordinates and $P_1 = P(\cdot, \cdot, 0)$ is a four degree polynomial in x, y.

As a generalization, we focus our attention on *symmetric entire perturbations of elliptic billiards*, i.e., the billiard in curves like the above-mentioned but symmetric only with respect to the origin and P_1 being an entire function.

A family of perturbed ellipses $\{C_\varepsilon\}$ will be called *trivial* if there exists a family of ellipses $\{E_\varepsilon\}$ such that $C_\varepsilon = E_\varepsilon + \mathrm{O}(\varepsilon^2)$. Using the above non-integrability criterion, it turns out that when $\{C_\varepsilon\}$ is any non-trivial symmetric entire perturbation, the billiard in C_ε is non-integrable for ε small enough. This result supports the Birkhoff's conjecture that elliptic billiards are the only integrable analytic billiards in the plane.

From a quantitative point of view, the first coefficient of the Taylor expansion of the *splitting angle* in powers of ε is easily computed in several concrete examples. Moreover, under very general perturbations, this coefficient turns out to be exponentially small in the eccentricity, when the unperturbed ellipse is near to a circle.

As a second application, consider *standard-like maps*, i.e., planar maps of the form $F(x,y) = (y, -x + g(y))$ for some function g.

The analytic standard-like maps given by the formula

$$F_0(x,y) = \left(y, -x + 2y \frac{\mu + \beta y}{1 - 2\beta y + y^2} \right), \quad -1 < \beta < 1 < \mu,$$

are integrable (see [Sur89], where several families of integrable standard-like maps are introduced), with a separatrix to the origin. For $\beta = 0$, this is the McMillan map [McM71] considered in [GPB89], where the Melnikov function was explicitly computed under the linear perturbations $F_1(x,y) = (0, ax + by)$ with a, b constants. For $a = 0$, as a consequence of the cumbersome computations of [GPB89], the non-integrability of the standard-like map $F_0 + \varepsilon F_1$ follows, for ε small enough.

This result of non-integrability is generalized to the standard-like maps

$$F_\varepsilon(x,y) = \left(y, -x + 2y \frac{\mu + \beta y}{1 - 2\beta y + y^2} + \varepsilon p(y) \right), \quad -1 < \beta < 1 < \mu, \quad \varepsilon \in \mathbb{R},$$

where p is an entire function, not identically zero. Moreover, in the framework of the developed theory, the computation of the Melnikov function and estimates of splitting angles become almost trivial, when p is a polynomial.

To finish this introduction, let us point two remarks:

- The meromorphicity of f or g is only needed for the explicit computation of the Melnikov function. To prove non-integrability, it is sufficient to assume that f is a real analytic function with only isolated singularities on $\mathbb{C}$.
- The topic of splitting of separatrices consists of two subclasses: the "simple" and the "difficult" one. The first one is characterized by the following property: the splitting quantities are of finite order with respect to the small parameter, so that they may be computed by means of the usual theory of perturbations (i.e., Melnikov techniques are valid). In the second subclass the main quantities are exponentially small with respect to the small parameter (recall the case of an ellipse near to a circle). Thus, more sophisticated analytical tools are required (see for instance [FS90, DS92, GLS94]). A priori, Melnikov techniques can not be applied to this subclass. However, we feel that the "Melnikov prediction" holds in several (not all) of the "difficult" cases. This feeling is supported by numerical experiments performed with some of the maps here studied [DR95]. This topic is under current research.

The rest of the paper is devoted to the rigorous formulation and proof of the above claims and results. It is organized as follows. In section 2 the Melnikov function is introduced and its relationship with quantitative and qualitative aspects of the splitting of separatrices is given. Section 3 contains the summation formulae and the criterion of non-integrability is formulated. The final part is devoted to the study of the perturbed elliptic billiards (section 4) and standard-like maps (section 5). These sections contain the non-integrability results, as well as examples of explicit computation of Melnikov functions and estimates of splitting angles.

2. Melnikov functions

2.1. Initial setup

Let $F_0 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an analytic a.p.m. On the one hand, we suppose that F_0 has a *separatrix to a saddle point*: there exists a saddle point P_0 of F_0 such that one branch of its stable manifold, W_0^s, coincides with one branch of its unstable one, W_0^u, giving rise to a separatrix $\Gamma \subset (W_0^s \cap W_0^u) \setminus \{P_0\}$. On the other hand, we assume that F_0 is *integrable*: there exists an analytic function $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$, such that $H \circ F_0 = H$ and $\nabla H(z) \neq 0$ for all $z \in \Gamma$ (this is a non-degeneracy condition of H over Γ).

Remark 2.1. When P_0 is a hyperbolic k-periodic point we can consider the map F_0^k to get a hyperbolic fixed point.

Without loss of generality, we can assume that F_0 is orientation preserving, considering the square of the map if necessary. Thus, $\mathrm{Spec}[DF_0(P_0)] = \{\lambda, \lambda^{-1}\}$, where $\lambda > 1$. Let $h = \ln \lambda$ be the associated characteristic exponent.

First we prove the existence of natural parameterizations, as well as the existence of first integrals verifying the interpolation condition of the introduction.

Lemma 2.1. Under the above notations, let F_0 be an analytic a.p.m., integrable and with a separatrix to a saddle point.

a) Let z_0 be a point in Γ. Then there exists an unique natural parameterization σ of Γ (with regard to F_0), such that $\sigma(0) = z_0$. Moreover there exists an analytic map $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^2$ such that $\sigma(t) = \varphi(\mathrm{e}^t)$, $\forall\, t \in \mathbb{R}$.

b) There exists a constant θ such that the time-h Hamiltonian flow associated to θH interpolates the map on the separatrix.

Proof. Since P_0 is a hyperbolic fixed point of an analytic a.p.m. F_0, there exists a canonical change of variables $\Phi_0 : (x, y) \longrightarrow (\mathcal{X}, \mathcal{Y})$, analytic on a neighbourhood of P_0, that transforms F_0 in its *Birkhoff normal form* $\mathcal{F}_0 = \Phi_0 \circ F_0 \circ \Phi_0^{-1}$:

$$\mathcal{F}_0(\mathcal{X}, \mathcal{Y}) := (\mathcal{X} \cdot \mathcal{G}_0(\mathcal{X} \cdot \mathcal{Y}), \mathcal{Y}/\mathcal{G}_0(\mathcal{X} \cdot \mathcal{Y})),$$

where $\mathcal{G}_0(\mathcal{I}) = \lambda + \mathrm{O}(\mathcal{I})$. For a proof of this fact see [Mos56].

Introducing now $\mathcal{K}_0(\mathcal{I}) = h^{-1}\int_0^{\mathcal{I}} \ln \mathcal{G}_0(s)\,ds$ (i.e., $\mathcal{G}_0 = \exp(h\mathcal{K}_0')$, $\mathcal{K}_0(0) = 0$), it turns out that $\mathcal{F}_0$ is the h-time Hamiltonian flow associated to $\mathcal{H}_0(\mathcal{X},\mathcal{Y}) := \mathcal{K}_0(\mathcal{X}\cdot\mathcal{Y})$.

In particular, $\sigma(t) = \Phi_0^{-1}(\mathcal{X}_0\mathrm{e}^t, 0) =: \varphi(\mathrm{e}^t)$ satisfies $\sigma(t+h) = F_0(\sigma(t))$ for $-t$ big enough, and consequently $\forall t \in \mathbb{R}$, using the analyticity of F_0. Now $\varphi(1) = z_0$ determines uniquely $\mathcal{X}_0$, and consequently φ, and a) is proved.

On the other hand, $H_0 := \mathcal{H}_0 \circ \Phi_0$ is a (local) first integral of F_0, and thus H_0, H are functionally dependent maps. Hence, as a consequence of the non-degeneracy of H over Γ, there exists a real analytic function Θ, defined in a neighbourhood of $H(P_0)$, such that $H_0 = \Theta \circ H$. This relation allow us to extend H_0 to a neighbourhood of the separatrix Γ, since it is contained in the energy level $H^{-1}(P_0)$. Now, by analytic continuation, we get that the h-time Hamiltonian flow associated to H_0 interpolates F_0 on Γ. Finally, we observe that $\nabla H_0(z) = \Theta'(H(P_0))\nabla H(z)$, for all $z \in \Gamma$. Consequently, if we take $\theta = \Theta'(H(P_0))$, the Hamiltonian flows associated to H_0 and θH coincide on Γ, and b) follows. □

Remark 2.2. Let $X_H = J\cdot\nabla H$ be the Hamiltonian field associated to H, $A = DX_H(P_0)$, $B = DF_0(P_0)$, and θ the constant given by Lemma 2.1. Then B and $\mathrm{e}^{\theta h A}$ have the same eigenvectors and eigenvalues. Thus we can determine θ from the equality $B = \mathrm{e}^{\theta h A}$. We remark that the eigenvalues of B are $\mathrm{e}^{\pm h}$, so the eigenvalues of θA must be ± 1 and this determines θ up to the sign.

In the rest of this section it will be assumed that the first integral H is chosen in order to verify this interpolation condition. Therefore, for all $z_0 \in \Gamma$, $\sigma(t) = \Psi^t(z_0)$ is the (unique) natural parameterization of Γ such that $\sigma(0) = z_0$, where $\{\Psi^t\}_{t\in\mathbb{R}}$ is the Hamiltonian flow associated to H. Thus we can compute explicitly the natural parameterizations simply solving the Hamiltonian equations

$$\dot\sigma = J\cdot\nabla H\circ\sigma, \quad \text{where} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.1}$$

with initial conditions on the separatrix.

In this paper h and σ will be the logarithm of the eigenvalue greater than one of the saddle point and a natural parameterization of the separatrix verifying (2.1) respectively, when it has sense.

For the sake of brevity, if a map satisfies all the previous assumptions we will say that it verifies *(H)*.

2.2. Melnikov functions

Let us consider a family of analytic diffeomorphisms $F_\varepsilon = F_0 + \varepsilon F_1 + \mathrm{O}(\varepsilon^2)$, and we introduce the Melnikov function of the problem like the function $M(t)$ determined by

$$\Delta(t,\varepsilon) = H(\sigma^u(t,\varepsilon)) - H(\sigma^s(t,\varepsilon)) = \varepsilon M(t) + \mathrm{O}(\varepsilon^2), \tag{2.2}$$

where $\sigma^{u,s}$ are defined in the introduction (see Figure 1). With the foregoing notation, we now prove the following proposition.

Proposition 2.1. Suppose that F_0 verifies *(H)*. Then:

a) The Melnikov function is given by

$$M(t) = \sum_{n\in\mathbb{Z}} g(t+hn), \tag{2.3}$$

with

$$g(t) = \langle \nabla H(\sigma(t)), F_1(\sigma(t-h))\rangle = \langle \dot\sigma(t), J\cdot F_1(\sigma(t-h))\rangle. \tag{2.4}$$

b) If F_ε is an a.p.m. with generating function

$$\mathcal{L}(x,X,\varepsilon) = \mathcal{L}_0(x,X) + \varepsilon\mathcal{L}_1(x,X) + \mathrm{O}(\varepsilon^2), \tag{2.5}$$

the Melnikov function is given by

$$M = \dot L\,, \quad L(t) = \sum_{n\in\mathbb{Z}} f(t+hn)\,, \tag{2.6}$$

with

$$f(t) = \mathcal{L}_1(x(t), x(t+h))\,, \tag{2.7}$$

where $x(t)$ is the first component of $\sigma(t)$ and, in order to get an (absolutely) convergent sum, $\mathcal{L}_1$ is determined by $\mathcal{L}_1(x_0,x_0)=0$, if $P_0=(x_0,y_0)$.

Proof. These results seem to be very well known (along these lines see, for instance, [GPB89, Lev93, Lom94]), but we prefer to include the proof for the convenience of the reader. The key point is to express $M(t)$ in terms of $H\circ F_\varepsilon - H$.

a) For each fixed t we first we observe that for all $m>0$:

$$\begin{aligned}\Delta(t,\varepsilon) = {} & H(F_\varepsilon^{-m}(\sigma^u(t,\varepsilon))) - H(F_\varepsilon^{m}(\sigma^s(t,\varepsilon))) \\ & + \sum_{n=1-m}^{m} H(F_\varepsilon^{n}(\sigma^\alpha(t,\varepsilon))) - H(F_\varepsilon^{n-1}(\sigma^\alpha(t,\varepsilon))),\end{aligned}$$

where $\alpha=\alpha(n)$ is given by $\alpha=u$ if $n\le 0$, and $\alpha=s$ if $n>0$. Since $H(F_\varepsilon^{-m}(\sigma^u(t,\varepsilon))) - H(F_\varepsilon^{m}(\sigma^s(t,\varepsilon))) \longrightarrow H(P_\varepsilon)-H(P_\varepsilon)=0$ when $m\to+\infty$, we obtain by pass to the limit

$$\Delta(t,\varepsilon) = \sum_{n\in\mathbb{Z}} (H\circ F_\varepsilon - H)\left(F_\varepsilon^{n-1}(\sigma^\alpha(t,\varepsilon))\right)\,. \tag{2.8}$$

Now, since $\sigma^\alpha(t,\varepsilon)$ is an invariant curve of F_ε that is $\mathrm{O}(\varepsilon)$-close to $\sigma(t)$, it turns out that $F_\varepsilon^{n-1}(\sigma^\alpha(t,\varepsilon)) = F_0^{n-1}(\sigma(t)) + \mathrm{O}(\varepsilon) = \sigma(t+h(n-1)) + \mathrm{O}(\varepsilon)$, uniformly in n, where we have used that σ is a natural parameterization. Moreover,

$$H \circ F_\varepsilon - H = \varepsilon \langle \nabla H \circ F_0, F_1 \rangle + \mathrm{O}(\varepsilon^2)\,, \tag{2.9}$$

and putting all this together in (2.8), we obtain (2.3).

b) It is sufficient to prove that $\dot f(t) = g(t+h)$, since a shift in the index does not changes the sum. First we look for the expression of $H \circ F_\varepsilon - H$.

We introduce the notation $(X_\varepsilon, Y_\varepsilon) = F_\varepsilon(x,y) = (X_0,Y_0) + \varepsilon(X_1,Y_1) + \mathrm{O}(\varepsilon^2)$. Since $\mathcal{L}$ is the generating function of F_ε, it satisfies the equations

$$y = -\partial_1 \mathcal{L}(x, X_\varepsilon, \varepsilon), \quad Y_\varepsilon = \partial_2 \mathcal{L}(x, X_\varepsilon, \varepsilon)\,.$$

Now, by straightforward expansion in ε, it follows that

$$\begin{aligned}(H \circ F_\varepsilon - H)(x,y) &= H(X_\varepsilon, \partial_2\mathcal{L}(x,X_\varepsilon,\varepsilon)) - H(x, -\partial_1\mathcal{L}(x,X_\varepsilon,\varepsilon)) \\ &= \varepsilon\left[\partial_2 H(x,y)\partial_1\mathcal{L}_1(x,X_0) + \partial_2 H(F_0(x,y))\partial_2\mathcal{L}_1(x,X_0)\right] + \mathrm{O}(\varepsilon^2)\,,\end{aligned}$$

so, using (2.9), we get for $z=(x,y)$:

$$\langle \nabla H(F_0(z)), F_1(z)\rangle = \partial_2 H(z)\partial_1\mathcal{L}_1(x,X_0) + \partial_2 H(F_0(z))\partial_2\mathcal{L}_1(x,X_0)\,.$$

Finally, the proof is finished using this expression and (2.1):

$$\begin{aligned}\dot f(t) &= \dot x(t)\partial_1\mathcal{L}_1(x(t),x(t+h)) + \dot x(t+h)\partial_2\mathcal{L}_1(x(t),x(t+h)) \\ &= \partial_2 H(\sigma(t))\partial_1\mathcal{L}_1(x(t),x(t+h)) + \partial_2 H(\sigma(t+h))\partial_2\mathcal{L}_1(x(t),x(t+h)) \\ &= \langle \nabla H(\sigma(t+h)), F_1(\sigma(t))\rangle = g(t+h). \qquad \square\end{aligned}$$

Remark 2.3. The Melnikov function M is h-periodic. Moreover, in the a.p.m. case, it is the derivative of a h-periodic function L (called the *Melnikov potential*), and thus it has zero mean.

The area preserving property of F_0 and the analyticity of F_ε are unnecessary to get some formulae like the previous ones, see [GPB89], but they are needed in the following theorem, so we have added these hypothesis directly. From a practical point of view, it makes no difference, since, up to our knowledge, all the integrable maps with a separatrix to a saddle point for which there exist explicitly known expressions verify these hypothesis.

The desired qualitative and quantitative information is contained in the following theorem.

Theorem 2.1. a) If M has zeros of odd order then the perturbed invariant manifolds cross at finite order, for $0 < |\varepsilon| \ll 1$.

b) In the a.p.m. case, if M is not identically zero then F_ε is non-integrable, for $0 < |\varepsilon| \ll 1$.

c) If M has a simple zero at $t = t_0$ then the associated intersection is transversal and the so-called splitting angle, $\alpha(\varepsilon)$, verifies

$$|\tan(\alpha(\varepsilon))| = \frac{|\dot{M}(t_0)\varepsilon|}{\| \dot{\sigma}(t_0) \| \cdot \| \nabla H(\sigma(t_0)) \|} + \mathrm{O}(\varepsilon^2) = \frac{|\dot{M}(t_0)\varepsilon|}{\| \dot{\sigma}(t_0) \|^2} + \mathrm{O}(\varepsilon^2)\,. \tag{2.10}$$

Proof. a) It is a direct consequence of (2.2) and the non-degeneracy condition of H over Γ.

b) $M = \dot{L}$ and L is analytic and h-periodic, so $\int_0^h M(t)dt = 0$. Thus $M \not\equiv 0$ implies that M changes the sign and has zeros of odd order. Now the non-integrability of F_ε is a consequence of a), the analyticity and the area preserving character [Cus78].

c) The second equality is obvious, since $\dot{\sigma}$ and $\nabla H \circ \sigma$ have the same norm, see (2.1). Thus we focus our attention on the first one. Let

$$v(t) = \frac{\nabla H(\sigma(t))}{\| \nabla H(\sigma(t)) \|}, \quad \mathrm{dist}(t,\varepsilon) = \langle \sigma^u(t,\varepsilon) - \sigma^s(t,\varepsilon), v(t) \rangle,$$

be the unit normal vector to Γ at $\sigma(t)$ and the (signed) distance between $\sigma^u(t,\varepsilon)$ and $\sigma^s(t,\varepsilon)$, respectively.

From Proposition 2.1 and the definition of Δ we get

$$\mathrm{dist}(t,\varepsilon) = \frac{\varepsilon M(t)}{\| \nabla H(\sigma(t)) \|} + \mathrm{O}(\varepsilon^2)\,.$$

First we suppose, momentarily, that t is an arc parameter of Γ, then:

$$|\tan(\alpha(\varepsilon))| = \left| \frac{d}{dt}[\mathrm{dist}(t,\varepsilon)]\Big|_{t=t_0} \right| + \mathrm{O}(\varepsilon^2) = \frac{|\dot{M}(t_0)\varepsilon|}{\| \nabla H(\sigma(t_0)) \|} + \mathrm{O}(\varepsilon^2),$$

where we have used that $M(t_0) = 0$ and the geometric interpretation of the derivative.

To end the proof we need only to add the normalizing factor $\| \dot{\sigma}(t_0) \|^{-1}$ to the previous formula that comes of the rule chain when t is not an arc parameter. □

Remark 2.4. The splitting angle $\alpha(\varepsilon)$ approaches $\pm\pi/2$ when $t_0 \to \pm\infty$. For a.p.m., it is better to use the area of the lobes formed by the invariant curves or the homoclinic invariant (introduced firstly in [GLT91]) since they do not depend on t_0. We have not used these a.p.m. invariants, since we have not restricted ourselves to a.p.m. perturbations.

3. Summation formulae and non-integrability

3.1. Elliptic functions

We recall that an elliptic function is a meromorphic and doubly periodic one (that is, it has two periods T_1, T_2 not zero such that $T_1/T_2 \notin \mathbb{R}$). The notations about elliptic functions are taken from [AS72].

Given the parameter $m \in [0,1]$, we recall that

$$K = K(m) := \int_0^{\pi/2} (1 - m \sin\theta)^{-1/2} d\theta\,, \quad E = E(m) := \int_0^{\pi/2} (1 - m \sin\theta)^{1/2} d\theta\,,$$

are the *complete elliptic integrals of the first and second kind* and that

$$E(u) = E(u|m) := \int_0^u \mathrm{dn}^2(v|m)\, dv$$

is the *incomplete elliptic integral of the second kind* where dn is one of the well-known *Jacobian elliptic functions.*

Moreover $K' = K'(m) := K(1-m)$, $E' = E'(m) := E(1-m)$, and we also note that if any one of the numbers m, K, K' or K'/K is given, all the rest are determined. We will not write explicitly the parameter m when no confusion is possible.

We introduce the function $\Lambda(z) := (E'/K' - 1)z + E(z + K'\mathrm{i})$. This function is meromorphic on $\mathbb{C}$, $2K'\mathrm{i}$-periodic, its derivative is $2K$-periodic and the set of its poles is $2K\mathbb{Z} + 2K'\mathrm{i}\mathbb{Z}$ being all the poles simple and of residue 1.

Indeed, the periodicities of Λ are consequence of the periodicities of E:

$$E(z + 2K) = E(z) + 2E, \quad E(z + 2K'\mathrm{i}) = E(z) + 2(K' - E')\mathrm{i},$$

and besides $\Lambda'(z) = E'/K' - 1 + \mathrm{dn}^2(z + K'\mathrm{i})$, where dn^2 is an even elliptic function, the set of its poles is $2K\mathbb{Z} + 2K'\mathrm{i}\mathbb{Z} + K'\mathrm{i}$, all they being double and of residue zero, and leading coefficient -1, so the claim about Λ is proved.

Given T, $h > 0$, we determine the parameter m by the relation

$$\frac{K'}{K} = \frac{K'(m)}{K(m)} = \frac{T}{h}, \tag{3.1}$$

and we consider the functions

$$\chi(z) = \frac{2K}{h}\Lambda\left(\frac{2Kz}{h}\right), \quad \chi_t(z) = \chi(z - t). \tag{3.2}$$

From the properties of Λ, one easily checks that:

(C1) χ is meromorphic on $\mathbb{C}$.

(C2) χ is $T\mathrm{i}$-periodic and χ' is h-periodic.

(C3) The set of poles of χ is $h\mathbb{Z} + T\mathrm{i}\mathbb{Z}$, being all of them simple and of residue 1.

The properties of χ_t are the same, except that the poles are $t + h\mathbb{Z} + T\mathrm{i}\mathbb{Z}$. Moreover, using that $E(z+2K) = E(z)+2E$ and the Legendre's relation $EK' + E'K - KK' = \pi/2$, we obtain

$$\chi(z + h) - \chi(z) = 2\pi/T. \tag{3.3}$$

Remark 3.1. Conditions *(C1)-(C3)* determine χ up to an additive constant: if χ_1 satisfies *(C1)-(C3)*, $(\chi-\chi_1)'$ is an entire doubly periodic function, and it must be a constant; thus, $\chi(z)-\chi_1(z)=az+b$, but $a=0$ due to the Ti-periodicity. In terms of the Weierstrass $\wp$-function, $\chi'(z)=-\wp(z)+\text{constant}$ since $\chi(z)=-z^{-2}+\mathrm{O}(1)$.

Now, we introduce the function that will play an important role when we compute Melnikov functions

$$\psi(t)=\chi'\left(\frac{T}{2}\mathrm{i}-t\right)-\left(\frac{2K}{h}\right)^2\left(\frac{E'}{K'}-1\right)=\left(\frac{2K}{h}\right)^2\mathrm{dn}^2\left(\frac{2Kt}{h}\middle|m\right). \quad (3.4)$$

Thus, ψ is an elliptic function of order 2, with periods h and Ti. Moreover, it is symmetric with regard to $t=0$ and $t=h/2$. Finally, we show some asymptotic expressions that will be of interest in the following sections. From approximations given in [AS72] and using relation (3.1), we have:

$$\left.\begin{array}{rcl} m &=& 16\mathrm{e}^{-\pi K'/K}[1+\mathrm{O}(\mathrm{e}^{-\pi K'/K})]=16\mathrm{e}^{-T\pi/h}[1+\mathrm{O}(\mathrm{e}^{-T\pi/h})],\\ K(m) &=& \frac{\pi}{2}+\mathrm{O}(m),\\ \mathrm{dn}(z|m) &=& 1+\mathrm{O}(m), \text{ and } \mathrm{dn}^{(j)}(z|m)=\mathrm{O}(m), \quad j\geq 1,\\ \mathrm{cn}^{(j)}(z|m) &=& \cos^{(j)}(z)+\mathrm{O}(m), \quad j\geq 0,\\ \mathrm{sn}^{(j)}(z|m) &=& \sin^{(j)}(z)+\mathrm{O}(m), \quad j\geq 0, \end{array}\right\} \quad (3.5)$$

for $0<h\ll 1$ and $z\in\mathbb{R}$. Thus, using the expressions of the derivatives of the Jacobian elliptic functions we get, for $t\in\mathbb{R}$ and $0<h\ll 1$

$$\psi^{(2j-1)}(t)=(-1)^j\left(\frac{2\pi}{h}\right)^{2j+1}2\mathrm{e}^{-T\pi/h}\left[\sin\left(\frac{2\pi t}{h}\right)+\mathrm{O}(\mathrm{e}^{-T\pi/h})\right], \quad j\geq 1. \quad (3.6)$$

Remark 3.2. It is important to bear in mind that the functions χ, χ_t and ψ are determinated by the quotient T/h, through the parameter m and the relation (3.1).

3.2. The summation formulae

The key problem, in order to compute explicitly the Melnikov function, is compute an infinite sum like $\sum_{n\in\mathbb{Z}}q(t+hn)$, where q is either the function g in (2.4) or the function f in (2.7). Our aim now is to transform this kind of infinite sums into finite ones. The idea is apply the residues theorem to $\chi_t q$, being χ_t the function defined above, first in some rectangular regions. Afterward, by a pass to the limit, the initial sum can be expressed as the sum of the residues of $-\chi_t q$ in the isolated singularities of q in a certain complex horizontal strip.

In this subsection we will assume that q is a function verifying:

(P1) q is analytic on $\mathbb{R}$ and has only isolated singulariries on $\mathbb{C}$.

(P2) q is Ti-periodic for some $T>0$.

(P3) $|q(t)| \leq A\mathrm{e}^{-c|\Re t|}$ when $|\Re t| \to \infty$ for some constants $A, c > 0$.

We denote $\mathcal{I}_T = \{z \in \mathbb{C} : 0 < \Im z < T\}$, $\mathcal{S}(q) = \{z \in \mathcal{I}_T : z$ is a singularity of $q\}$ and we write $\mathcal{S}(q) = \{z_{s\ell} : l = 1, \ldots, k_s,\ s = 1, \ldots, k\}$, where $z_{s\ell} - z_{s'\ell'} \in h\mathbb{Z} \Leftrightarrow s = s'$. (We have classified the singularities of q modulo h.) Finally, we introduce the numbers $d_{sj} = \sum_{\ell=1}^{k_s} a_{-(j+1)}(q, z_{s\ell})$, for $s = 1, \ldots, k$ and $j \geq 0$, where the notation $a_{-j}(q, z)$ has been defined in the introduction.

We are ready to give the following *summation formulae.*

Proposition 3.1 (Summation Formulae). Let $\Sigma(t) := \sum_{n\in\mathbb{Z}} q(t + hn)$. Then:

a) Σ is analytic in $\mathbb{R}$, has only isolated sigularities in $\mathbb{C}$ and is double periodic, with periods h and Ti.

b) $\Sigma(t)$ can be expressed by the following finite sum

$$\Sigma(t) = -\sum_{z\in\mathcal{S}(q)} \operatorname{res}(\chi_t q, z), \tag{3.7}$$

or equivalently, like

$$\Sigma(t) = -\sum_{z\in\mathcal{S}(q)} \sum_{j\geq 0} \frac{a_{-(j+1)}(q, z)}{j!} \chi^{(j)}(z - t). \tag{3.8}$$

c) Let $b_s \in \{z \in \mathbb{C} : |\Im z| < T/2\}$, determined modulo h by $z_{s\ell} \in b_s + T\mathrm{i}/2 + h\mathbb{Z}$ for all s, ℓ. Then

$$\Sigma'(t) = \sum_{s=1}^{k} \sum_{j\geq 0} \frac{(-1)^j}{j!} d_{sj} \psi^{(j)}(t - b_s). \tag{3.9}$$

Proof. a) Σ is obviously Ti-periodic and analytic in $\mathbb{R}$. Because of *(P3)* the sum is absolutely, unconditionally and uniformly convergent on compacts of $\mathbb{C}$ without points in the set $\mathcal{S}(q) + h\mathbb{Z} + T\mathrm{i}\mathbb{Z}$. Thus Σ is also h-periodic and has only isolated singularities, just in the above-mentioned set.

b) The hypothesis on q imply that $\mathcal{S}(q)$ is a finite set, so the sum in (3.7) is finite. Let $\mathcal{S}_\alpha(q) = \{z \in \mathbb{C} : z$ is a singularity of q, $\alpha < \Im z \leq \alpha + T\}$, for $\alpha \in \mathbb{R}$. We note that $\mathcal{S}(q) = \mathcal{S}_0(q)$. Futhermore, $\sum_{z\in\mathcal{S}_\alpha(q)} \operatorname{res}(\chi_t q, z)$ does not depend on α, since $\chi_t q$ is Ti-periodic and so, to prove the formula (3.7) it is enough to check that

$$\Sigma(t) = -\sum_{z\in\mathcal{S}_\alpha(q)} \operatorname{res}(\chi_t q, z),$$

for one value of α.

We choose $\alpha \in [-T/2, 0)$ such that q has no singularities with imaginary part α, and we consider the rectangle of vertexes $M_+ + \alpha\mathrm{i}$, $M_+ + (\alpha + T)\mathrm{i}$, $M_- + (\alpha + T)\mathrm{i}$ and

$M_- + \alpha\mathrm{i}$, where $M_\pm = t \pm (N+1/2)h$, $N \in \mathbb{N}$. If N is big enough, $\chi_t p$ is analytic on the border C_N of the rectangle and has only isolated singularities on its interior R_N, so the residues theorem gives

$$\frac{1}{2\pi i}\oint_{C_N} \chi_t q = \sum\nolimits_N \operatorname{res}(\chi_t q, z),$$

where $\sum_N$ indicates sum over the singularities $\{z = t + hn\} \cup \mathcal{S}_\alpha(q)$ of $\chi_t q$ in R_N. Since $\chi_t p$ is $T\mathrm{i}$-periodic the horizontal integrals cancel and, on the other hand, the vertical ones vanish when N tends to infinity, using *(P3)* and *(C2)*. Thus the sum of residues of $\chi_t q$ in $\{z \in \mathbb{C};\ \alpha < \Im z \leq \alpha + T\}$ is zero and since χ_t, q have no common singularities by *(C3)* and *(P1)*, we get

$$\Sigma(t) = \sum_{n\in\mathbb{Z}} q(t+hn) = \sum_{n\in\mathbb{Z}} \operatorname{res}(\chi_t q, t+hn) = -\sum_{z\in\mathcal{S}_\alpha(q)} \operatorname{res}(\chi_t q, z).$$

Finally, we note that $\operatorname{res}(\chi_t q, z) = \sum_{j\geq 0} \frac{1}{j!} a_{-(j+1)}(q,z)\chi^{(j)}(z-t)$, for $z \in \mathcal{S}(q)$, and this proves (3.8).

c) First, we observe that q' verifies *(P1)-(P3)*. The properties *(P1)-(P2)* are obvious and *(P3)* is a consequence of Cauchy's inequalities. Thus, applying the summation formula (3.7) to q' instead of q and using the h-periodicity of χ', we get

$$\begin{aligned}\Sigma'(t) &= -\sum_{s,\ell} \operatorname{res}(\chi_t q', z_{s\ell}) = \sum_{s,\ell} \operatorname{res}(\chi_t' q, z_{s\ell}) = \\ &= \sum_{s,\ell,j} \frac{1}{j!} a_{-(j+1)}(q, z_{s\ell})\chi^{(j+1)}\left(b_s + T\mathrm{i}/2 - t\right).\end{aligned}$$

Now, notice that $\sum_{z\in\mathcal{S}(q)} \operatorname{res}(\chi_t q, z)$ does not change when replacing χ by χ+constant (see remark 3.1), so it follows that $\sum_{z\in\mathcal{S}(q)} \operatorname{res}(q,z) = 0$. In consequence, we obtain

$$\sum_{s,\ell} a_{-1}(q, z_{s\ell})\chi'\left(T\mathrm{i}/2 + b_s - t\right) = \sum_{s,\ell} a_{-1}(q, z_{s\ell})\psi(t - b_s).$$

Moreover, it turns out that $\chi^{(j+1)}\left(b_s + T\mathrm{i}/2 - t\right) = (-1)^j \psi^{(j)}(t - b_s)$, for all $j \geq 1$.

Finally, c) follows from all the previous formulae. □

If q is meromorphic on $\mathbb{C}$, Σ is elliptic. Moreover, in this case the sums in (3.8) are finite. Thus, from a computational point of view, this is the interesting case, because then $\Sigma(t)$ can be explicitly computed.

Now, we will give necessary and sufficient conditions (in terms of the principal parts of q in its singularities), so that the sum $\Sigma(t)$ should be identically constant.

Lemma 3.1. Let $\Sigma(t) = \sum_{n\in\mathbb{Z}} q(t+hn)$. Then $\Sigma \equiv$ constant if and only if Σ has no singularities, or equivalently, if and only if $d_{sj} = 0$, for all s, j.

Proof. Using the Liouville's Theorem and from the double periodicity of Σ, we deduce that Σ is constant if and only if all its singularities are removable ones (i.e., with principal part identically zero).

The set of singularities of Σ is $\{z_{s1} : s = 1, \ldots, k\} + h\mathbb{Z} + T\mathrm{i}\mathbb{Z}$. Let $z_* \in z_{s1} + h\mathbb{Z} + T\mathrm{i}\mathbb{Z}$ an arbitrary singularity of Σ. Directly from the definition of Σ, we see that the principal part of Σ in z_* is the sum of the principal parts of q over the points of the set $\{z_{s\ell} : \ell = 1, \ldots, k_s\}$. Thus, all the singularities of Σ are removable ones if and only if $d_{sj} = 0$, for all $s = 1, \ldots, k$ and $j \geq 0$. □

3.3. The hypothesis of isolated singularities

Formulae (3.7), (3.8) and (3.9) give a way to compute the Melnikov function if either the function g in (2.4) or the function f in (2.7) verify *(P1)-(P3)*. Here we show that if some of these functions has only isolated singularities on $\mathbb{C}$, it automatically verifies *(P1)-(P3)* with $T = 2\pi$. Consequently, the Melnikov function has only isolated singularities on $\mathbb{C}$ and is doubly periodic with periods h and $2\pi\mathrm{i}$. Of course, when f or g are meromorphic functions, the Melnikov function is an elliptic one.

Lemma 3.2. Let $\{F_\varepsilon\}_{\varepsilon\in\mathbb{R}}$ be a family of analytic diffeomorphisms where F_0 verifies *(H)*. Moreover, assume that the function g defined in (2.4) (respectively, f defined in (2.7)) has only isolated singularities on $\mathbb{C}$. Then g (respectively, f) verifies *(P1)-(P3)* with $T = 2\pi$.

Proof. The proof is the same in both cases. Thus we prove only one case, for instance for f.

(P1) is obvious. Using Lemma 2.1 and the definition of f is clear that there exists a function $\mathcal{F} : \mathbb{R} \longrightarrow \mathbb{R}$ analytic on $\mathbb{R}$ such that $f(t) = \mathcal{F}(\mathrm{e}^t)$ for all $t \in \mathbb{R}$. Thus f can be expressed as a power series in the variable $s = \mathrm{e}^t$ if $|s|$ is small enough, or equivalently, if $-\Re t$ is big enough. This proves that $f(t) = f(t + 2\pi i)$ for $-\Re t$ big enough and, by an argument of analytic continuation, *(P2)*, with $T = 2\pi$, is proved. Therefore $\mathcal{F}$ can be extended to a function with only isolated singularities by the relation $\mathcal{F}(s) = f(t)$.

Using that $\lim_{|\Re t|\to\infty} \sigma(t) = P_0$ it is easy to see, from the definition of f, that $\lim_{|\Re t|\to\infty} f(t) = 0$. Thus the relation $\mathcal{F}(s) = f(t)$ implies that $\mathcal{F}$ can be considered as function with only isolated singularities over the Riemann Sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, that vanishes at $s = 0$ and $s = \infty$. (We note that if f is a meromorphic function, then $\mathcal{F}$ is a rational one.) To complete the proof of *(P3)* we need only to apply the mean value theorem to $\mathcal{F}$ at $s = 0$ and $s = \infty$ and the result follows, again from the relation $\mathcal{F}(s) = f(t)$. □

Remark 3.3. If there is some symmetry in the problem g and f can be πi-periodic and in consequence the same happens to the Melnikov function. Thus, in practical cases, we will use the summation formulae with $T = 2\pi$ or $T = \pi$.

3.4. Non-integrability Criterion

In order to simplify the computations in the examples of sections 4 and 5, we are going to compute the Melnikov function M given by (2.6) and (2.7), if f has only isolated singularities or, equivalently, if f satisfies *(P1)-(P3)* with $T = 2\pi$ or $T = \pi$. Moreover, a non-integrability criterion is given.

Let $\{F_\varepsilon\}_{\varepsilon\in\mathbb{R}}$ be a family of analytic a.p.m. with generating function (2.5) where F_0 verifies *(H)*. Moreover, assume that the function f in (2.7) has only isolated singularities. By Lemma 3.2, f verifies *(P1)-(P3)* with $T = 2\pi$ or $T = \pi$.

Let $\mathcal{S}(f) = \{z_{s\ell} : l = 1, \ldots, k_s,\ s = 1, \ldots, k\}$ be the singularities of f in the complex strip $\mathcal{I}_T$ classified modulo h, like in subsection 3.2. Finally, let be $b_s \in \mathbb{C}$ such that $|\Im b_s| < T/2$ and $z_{s\ell} \in b_s + T\mathrm{i}/2 + h\mathbb{Z}$ for all s and ℓ, like in Proposition 3.1.

Theorem 3.1 (Non-integrability Criterion). With this notations and assumptions, the Melnikov function is given by:

$$M(t) = \sum_{s=1}^{k} \sum_{j\geq 0} \frac{(-1)^j}{j!} d_{sj} \psi^{(j)}(t - b_s), \tag{3.10}$$

where d_{sj} are the so-called non-integrability coefficients

$$d_{sj} = \sum_{\ell=1}^{k_s} a_{-(j+1)}(f, z_{s\ell}), \quad s = 1, \ldots, k, \quad j \geq 0. \tag{3.11}$$

Finally, if some of the non-integrability coefficients is non-zero, F_ε is non-integrable, for $0 < |\varepsilon| \ll 1$.

Proof. The first part follows from the summation formula (3.9). The non-integrability follows from Theorem 2.1 and Lemma 3.1. □

4. Perturbed elliptic billiards

4.1. Convex Billiards

Consider the problem of the "convex billiard table" [Bir27]: Let C be an (analytic) closed convex curve of the plane $\mathbb{R}^2$, parameterized by $\gamma : \mathbb{T} \longrightarrow C$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and C is traveled counterclockwise. Suppose that a material point moves inside C and collides with C according to the law "the angle of incidence is equal to the angle of reflection". In the annulus $\mathbb{A} = \{(\varphi, v) \in \mathbb{T} \times \mathbb{R} : |v| < |\dot\gamma(\varphi)|\}$, the coordinate φ is the parameter on C and $v = |\dot\gamma(\varphi)| \cos\vartheta$, where $\vartheta \in (0, \pi)$ is the angle of incidence-reflection of the material point. In this way, we obtain a map $T : \mathbb{A} \longrightarrow \mathbb{A}$ that models the billiard (see Figure 2).

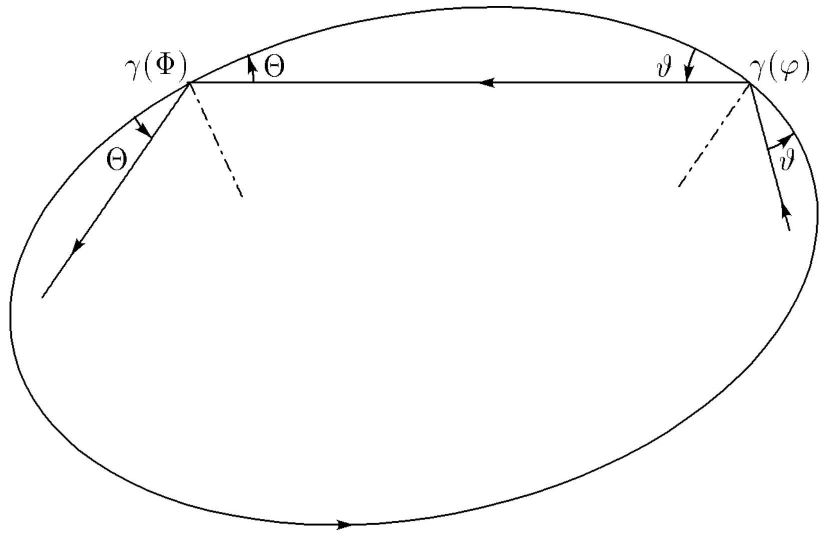

Figure 2. $T(\varphi, v) = (\Phi, V)$, where $v = |\dot\gamma(\varphi)| \cos\vartheta$ and $V = |\dot\gamma(\Phi)| \cos\Theta$.

The function $S : \{(\varphi, \Phi) \in \mathbb{T}^2 : \varphi \neq \Phi\} \longrightarrow \mathbb{R}$ defined by $S(\varphi, \Phi) = |\gamma(\varphi) - \gamma(\Phi)|$ is a generating function of T:

$$\frac{\partial S}{\partial \varphi}(\varphi, \Phi) = \frac{\langle \gamma(\varphi) - \gamma(\Phi), \dot\gamma(\varphi)\rangle}{|\gamma(\varphi) - \gamma(\Phi)|} = -|\dot\gamma(\varphi)| \cos\vartheta = -v,$$
$$\frac{\partial S}{\partial \Phi}(\varphi, \Phi) = \frac{\langle \gamma(\Phi) - \gamma(\varphi), \dot\gamma(\Phi)\rangle}{|\gamma(\varphi) - \gamma(\Phi)|} = |\dot\gamma(\Phi)| \cos\Theta = V.$$

Thus T is an a.p.m. and (φ, v) are canonical conjugated coordinates.

This map has no fixed points but is geometrically clear that it has periodic orbits of period 2. In these orbits the angle of incidence-reflection is $\pi/2$ and thus $v = 0$.

Suppose now that C is symmetric with regard to a point (without loss of generality we can assume that this point is the origin, see remark 4.2). Then it is possible to work with a parameterization γ of C such that $\gamma(\varphi + \pi) = -\gamma(\varphi)$ and the 2-periodic orbits are of the form $(\varphi_0, 0)$, $(\varphi_0 + \pi, 0)$, that is, two opposite points over C. Let $R : \mathbb{A} \longrightarrow \mathbb{A}$ be the involution $R(\varphi, v) = (\varphi + \pi, v)$, then T and R commute and it is a commonplace to use this symmetry to convert the 2-periodic points into fixed points. Concretely, we define a new map $F : \mathbb{A} \longrightarrow \mathbb{A}$ by $F = R \circ T$. Since $F^2 = T^2$, the dynamics of F and T are equivalent. Moreover, F is an a.p.m. and its generating function, using that $\gamma(\Phi + \pi) = -\gamma(\Phi)$, is

$$\mathcal{L}(\varphi, \Phi) = S(\varphi, \Phi + \pi) = |\gamma(\varphi) + \gamma(\Phi)|\,. \tag{4.1}$$

Remark 4.1. We can consider the variable φ defined modulo π in the symmetric case. This idea goes back to [Tab93, Tab94].

Remark 4.2. Let C and C' two closed convex curves such that one is the image of the other by a similarity. Then the two associated a.p.m. have an equivalent dynamics since the angle of incidence-reflection remains unchanged by the similarity.

4.2. Elliptic Billiards

The simplest examples of convex curves are the elipses. It is clear that the case of a circunference is very degenerated for a billiard, since it consists only of 2-periodic orbits. So, let us consider now a non-circular ellipse

$$C_0 = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\} = \{\gamma_0(\varphi) = (a \cos \varphi, b \sin \varphi) : \varphi \in \mathbb{T}\},$$

with $a^2 \neq b^2$. Without loss of generality we can assume that $a^2 - b^2 = 1$ (we change the ellipse using a similarity, if necessary). Thus $a > 1$, $b > 0$ and the foci of the ellipse are $(\pm 1, 0)$.

Let us recall that a caustic is a smooth curve with the following property: if at least one of the segments (or its prolongation) of the polygonal trajectory of the point is tangent to the curve, then all the other segments (or theirs prolongations) are tangent to the curve. It is a very well-known fact that all the orbits of an elliptic billiard have a caustic, and actually the caustics are just the family of confocal conics to C_0 (little Poncelet's theorem, [KT91]).

This property indicates the integrability of elliptic billiards since the existence of caustics reflects some stability in the system. In fact, it is not difficult to obtain an explicit expression for a first integral of the elliptic billiard in (φ, v) coordinates, under the assumption $a^2 - b^2 = 1$. In [LT93] the following first integral is given in (φ, ϑ) coordinates

$$I(\varphi, \vartheta) = a^2 \cos^2 \vartheta + \cos^2 \varphi \sin^2 \vartheta = a^2 \cos^2 \vartheta - \cos^2 \varphi \cos^2 \vartheta - \sin^2 \varphi + 1.$$

Moreover, using that $a^2 - b^2 = 1$, we get

$$v^2 = |\dot{\gamma}_0(\varphi)|^2 \cos^2 \vartheta = a^2 \cos^2 \vartheta - \cos^2 \varphi \cos^2 \vartheta,$$

and the integral I in (φ, v) coordinates becomes $I(\varphi, v) = v^2 - \sin^2 \varphi + 1$. This integral can be found in [Lom94] in a slightly different manner. As a consequence, the curves $\{I = c\}_{0<c<b^2+1}$ are invariant for T_0 and F_0 where $T_0 : \mathbb{A} \longrightarrow \mathbb{A}$ is the analytic a.p.m. associated to C_0 and $F_0 = R \circ T_0$. In connection with the little Poncelet's theorem, the caustics of the points on one of these invariant curves are: a confocal hyperbola if $0 < c < 1$, a confocal ellipse if $1 < c < b^2 + 1$, and the foci $(\pm 1, 0)$ when $c = 1$. Obviously, the foci are not smooth curves, but if some segment of the trajectory goes through a focus then the same happens to all the other segments.

Besides, the points $(0, 0)$ and $(\pi, 0)$ form a 2-periodic orbit for T_0 that corresponds to the vertexes $(\pm a, 0)$ of the ellipse, and hence $(0, 0)$ is a fixed point for F_0.

Let $R^* : \mathbb{A} \longrightarrow \mathbb{A}$ be the involution given by $R^*(\varphi, v) = (\pi - \varphi, v)$, then $F_0^{-1} = R^* \circ F_0 \circ R^*$ and thus F_0 is reversible.

The dynamics of F_0 is drawn in Figure 3 where the resemblance with the phase portrait of a pendulum shows up clearly.

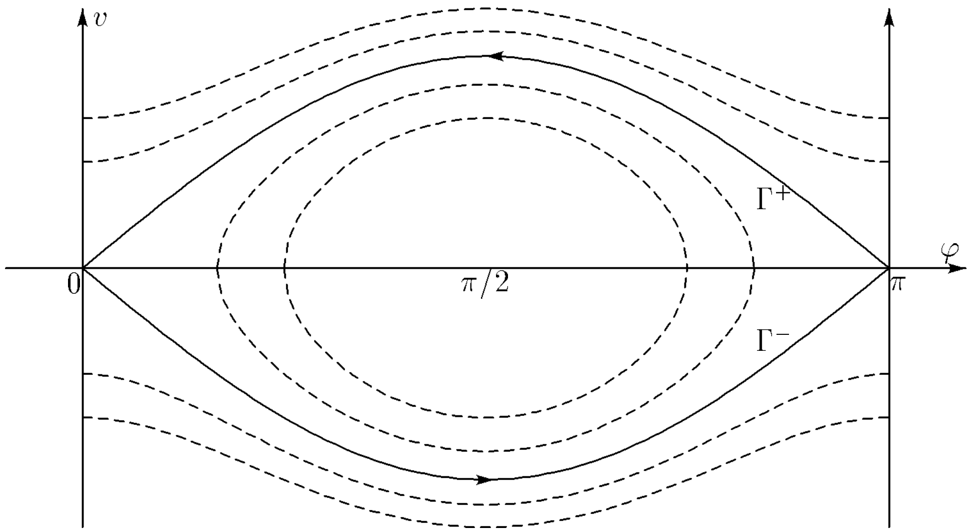

Figure 3. Phase portrait of F_0; $\Gamma^{\pm}$ are the separatrices of F_0.

The main properties of F_0 are listed in the following Lemma.

Lemma 4.1. a) $P_0 = (0,0)$ is a saddle point of F_0 and Spec $[DF_0(P_0)] = \{\lambda, \lambda^{-1}\}$, with $\lambda = (a+1)(a-1)^{-1} > 1$. Moreover, if $h = \ln\lambda$ the following expressions hold

$$a = \coth(h/2), \quad b = \operatorname{cosech}(h/2). \tag{4.2}$$

b) $\Gamma^{\pm} = \{(\varphi, \pm\sin\varphi) : 0 < \varphi < \pi\}$ are the separatrices of F_0.

c) The time-h Hamiltonian flow associated to

$$H(\varphi, v) = -\frac{1}{2} I(\varphi, v) = (\sin^2\varphi - v^2 - 1)/2$$

interpolates F_0 on the separatrices.

d) If $\sigma^{\pm}(t) = (\varphi(\pm t), \pm v(t))$, where $\varphi(t) = \arccos(\tanh t) = \arcsin(\operatorname{sech} t)$ and $v(t) = \operatorname{sech} t$, then $\sigma^{\pm}$ are natural parameterizations of $\Gamma^{\pm}$ (with regard to F_0).

e) Let be $\Phi(t) = \varphi(t+h)$. Then

$$b\frac{\sin\varphi(t) + \sin\Phi(t)}{|\gamma_0(\varphi(t)) + \gamma_0(\Phi(t))|} = \operatorname{sech}(t + h/2). \tag{4.3}$$

Proof. a) We know that P_0 is fixed by F_0. Let

$$\mathcal{L}_0(\varphi,\Phi) = |\gamma_0(\varphi)+\gamma_0(\Phi)| = 2a + [(a^2-1)\varphi\Phi - (a^2+1)(\varphi^2+\Phi^2)/2]/2a + \mathrm{O}_3(\varphi,\Phi)$$

be the generating function of $F_0 : (\varphi, v) \mapsto (\Phi, V)$, where we have used that $a^2 - b^2 = 1$. From the implicit equations of F_0 generated by $\mathcal{L}_0$ we get

$$\mathrm{trace}[DF_0(P_0)] = \partial_1\Phi(0,0) + \partial_2 V(0,0) = -[\partial_{11}\mathcal{L}_0(0,0) + \partial_{22}\mathcal{L}_0(0,0)]/\partial_{12}\mathcal{L}_0(0,0),$$

and a straightforward calculus yields $\mathrm{trace}[DF_0(P_0)] = 2(a^2+1)/(a^2-1)$. Moreover, $\det[DF_0] \equiv 1$. Thus $\lambda = (a+1)(a-1)^{-1} > 1$ is an eigenvalue of $DF_0(P_0)$. From $\mathrm{e}^h = \lambda$, one gets $a = \coth(h/2)$ and $b = \sqrt{a^2-1} = \mathrm{cosech}(h/2)$.

b) This is a direct consequence of the conservation of the first integral I.

c) Using Lemma 2.1 there exists a constant θ such that $H = \theta I$ verifies c). We need only to check that $\theta = -\frac{1}{2}$. Let $X_I = J \cdot \nabla I$ be the Hamiltonian field associated to I, then

$$A = DX_I(P_0) = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

Thus $|\theta| = \frac{1}{2}$, according to remark 2.2. Finally, the sign is determined in order to get the right sense over the separatrices, see again Figure 3.

d) It is enough to prove it for Γ^+ by symmetry. We observe that σ^+ is a solution of the Hamiltonian equations associated to H and $\sigma^+(0) \in \Gamma^+$. Thus d) is an immediate consequence of c).

e) Let be $(\varphi, \sin\varphi) \in \Gamma^+$ and $(\Phi, \sin\Phi) = F_0(\varphi, \sin\varphi)$. The points $\gamma_0(\varphi)$, $-\gamma_0(\Phi) = \gamma_0(\Phi - \pi)$ and the focus $(-1,0)$ are aligned, since $(\Phi-\pi, \sin(\Phi-\pi)) = T_0(\varphi, \sin\varphi)$ and the foci are the "caustic" of the points in Γ^+. Moreover, the vectors $\gamma_0(\varphi)+\gamma_0(\Phi)$ and $\gamma_0(\varphi)+(1,0)$ are parallel with the same sense (see Figure 4), and hence

$$\frac{\gamma_0(\varphi)+\gamma_0(\Phi)}{|\gamma_0(\varphi)+\gamma_0(\Phi)|} = \frac{\gamma_0(\varphi)+(1,0)}{|\gamma_0(\varphi)+(1,0)|}. \tag{4.4}$$

Since $\gamma_0(\varphi) = (a\cos\varphi, b\sin\varphi)$, using (4.2), and the expressions of $\varphi(t)$ in d), we obtain

$$\gamma_0(\varphi) + (1,0) = \frac{1}{\sinh h/2 \cdot \cosh t}(\sinh(t+h/2), 1),$$

and thus

$$\frac{\gamma_0(\varphi)+(1,0)}{|\gamma_0(\varphi)+(1,0)|} = (\tanh(t+h/2), \mathrm{sech}(t+h/2)).$$

Now, the second components of (4.4) give (4.3). □

At this stage, it is important to point out that the map F_0 verifies *(H)* and we have explicit expressions for the natural parameterizations of the separatrices with regard to

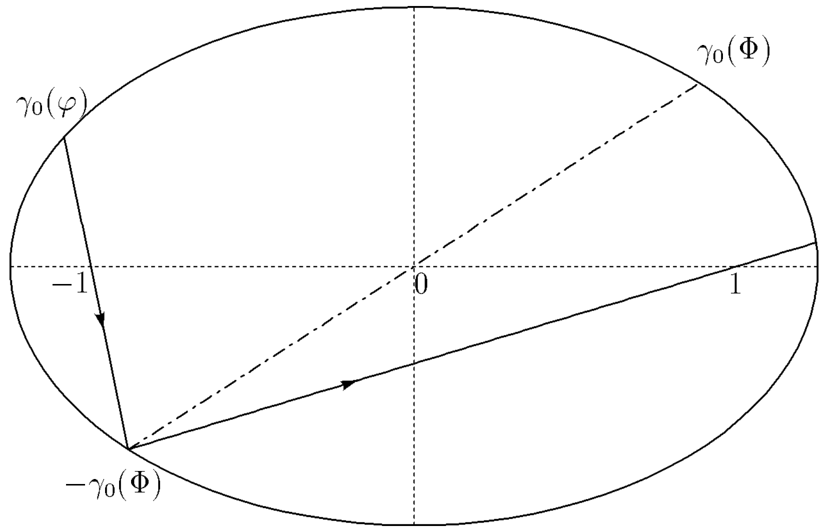

Figure 4. $(\varphi, \sin\varphi) \in \Gamma^+$, $(\Phi, \sin\Phi) = F_0(\varphi, \sin\varphi)$.

F_0. Thus a complete computation of the Melnikov functions, angles of splitting, etc. and a deep study about non-integrability, using the tools developed in sections 2 and 3, is possible for a huge class of perturbations. Due to the symmetry between Γ^+ and Γ^-, we restrict our study to $\Gamma = \Gamma^+$, $\sigma(t) = \sigma^+(t) = (\varphi(t), v(t))$.

4.3. Non-integrability of symmetric entire billiards

Let $\{C_\varepsilon\}$ be an arbitrary family of perturbations of the ellipse C_0, consisting of analytic curves depending on a $\mathcal{C}^2$ way on ε and symmetric with regard to a point O_ε. Let us denote by $Q_\varepsilon^\pm$ the two furthest (and opposite) points over C_ε with $Q_0^\pm = (\pm a, 0)$. Using a similarity that takes O_ε and $Q_\varepsilon^\pm$ to $(0,0)$ and $(\pm a, 0)$ respectively, the initial family can be put in the following form

$$C'_\varepsilon = \left\{ (x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \varepsilon P(x,y,\varepsilon) = 1 \right\}, \tag{4.5}$$

where:

I) P is analytic in x, y and at least $\mathcal{C}^1$ in ε,

II) $P(x,y,\varepsilon) = P(-x,-y,\varepsilon)$,

III) $P(a,0,\varepsilon) = \partial_y P(a,0,\varepsilon) = 0$,

or equivalently, like

$$C'_\varepsilon = \{\gamma(\varphi,\varepsilon) = (a\cos\varphi, \sin\varphi\,[b + \varepsilon\eta(\varphi,\varepsilon)]) : \varphi \in \mathbb{T}\}, \tag{4.6}$$

where:

i) η is analytic in φ and at least $\mathcal{C}^1$ in ε,

ii) η is π-periodic in φ.

Remark 4.3. This kind of perturbations preserves the reversibility R^* of the system if and only if η is even in φ. In this particular case C'_ε has two axial symmetries (given by the two axis) and the Melnikov function is odd. This has been the standard case studied until now [Tab93, Tab94].

From III), it follows that $P(a\cos\varphi, b\sin\varphi, \varepsilon) = p(a\cos\varphi, b\sin\varphi, \varepsilon)\sin^2\varphi$, with p satisfying also I), II). It is easy to check that, in first order in ε, the relation between η and P is given by $P(a\cos\varphi, b\sin\varphi, 0) = -2b\eta(\varphi, 0)\sin^2\varphi$. Thus, if $P(\cdot,\cdot,0)$ is an entire function (respectively, a polynomial) in the x, y variables, $\eta_1 := \eta(\cdot, 0)$ is an entire function (respectively, a trigonometric polynomial). It is clear that if $\eta_1 =$ constant, C'_ε is, in first order, a family of ellipses.

Definition 4.1. Let $\{C_\varepsilon\}$ be a perturbation of the ellipse C_0. We say that $\{C_\varepsilon\}$ is a *non-trivial symmetric entire* (respectively, *polynomial*) *perturbation* of the ellipse when it can be put, using similarities, in the form (4.6) and moreover, $\eta_1 := \eta(\cdot, 0)$ is a non-constant entire function (respectively, polynomial).

Let T_ε be the map in the annulus associated to the billiard in C_ε, where the perturbation considered is a symmetric entire one and let $F_\varepsilon = R \circ T_\varepsilon$. If ε is small enough, C_ε is an analytic convex closed curve, and thus $\{F_\varepsilon\}_{|\varepsilon|\ll 1}$ is a family of analytic a.p.m. with generating function $\mathcal{L}(\varphi, \Phi, \varepsilon) = |\gamma(\varphi, \varepsilon) + \gamma(\Phi, \varepsilon)|$, see (4.1), and

$$\mathcal{L}_1(\varphi, \Phi) = \frac{\partial \mathcal{L}}{\partial \varepsilon}(\varphi, \Phi, 0) = b\frac{\sin\varphi + \sin\Phi}{|\gamma_0(\varphi) + \gamma_0(\Phi)|}[\sin\varphi\, \eta_1(\varphi) + \sin\Phi\, \eta_1(\Phi)].$$

Now using formula (4.3) we find the following expression for the function f in (2.7)

$$f = v_{h/2}\cdot(\delta + \delta_h), \quad v = \mathrm{sech}, \quad \delta = v\cdot(\eta_1\circ\varphi),$$

where, henceforth, given a function v and a number x, v_x stands for the function $v_x(t) = v(t + x)$.

Since η_1 is an entire π-periodic function, there exists an *even* function ζ analytic in $\mathbb{C}\backslash\{0\}$, such that $\eta_1(\varphi) = \zeta(\mathrm{e}^{\mathrm{i}\varphi})$. Moreover, using Lemma 4.1, $\mathrm{e}^{\mathrm{i}\varphi(t)} = (\mathrm{i} + \sinh t)\,\mathrm{sech}\, t$, so the following properties of the function $\eta_1\circ\varphi$ are easily obtained:

a) For all η_1 entire and π-periodic, the function $\eta_1\circ\varphi$ has only isolated singularities in $\mathbb{C}$. Its singularities are the points of the set $\pi\mathrm{i}/2 + \pi\mathrm{i}\mathbb{Z}$, since just in these points $\mathrm{e}^{\mathrm{i}\varphi(t)}$ reaches the values 0 and ∞.

b) A singularity of $\eta_1 \circ \varphi$ is removable if and only if η_1 (i.e., ζ) is constant.

c) Moreover, $\eta_1 \circ \varphi$ is symmetric with regard to these singularities.

By a), f has only isolated singularities for any symmetric entire perturbation, and a study about non-integrability can be performed. However, before to begin this and further studies, it is very convenient to arrange the sum $\sum_{n\in\mathbb{Z}} f(t+hn)$ and express the Melnikov potential L as $L(t) = \sum_{n\in\mathbb{Z}} q(t+hn)$, where

$$q = [v_{h/2} + v_{-h/2}]\delta = (2a/b)v_{h/2} \cdot v_{-h/2} \cdot (\eta_1 \circ \varphi). \tag{4.7}$$

(Relations (4.2) and the adittion formulae for the hyperbolic cosinus have been used to obtain the second equality.)

Theorem 4.1. Let $\{C_\varepsilon\}$ be any non-trivial symmetric entire perturbation of an ellipse. Then the billiard in C_ε is non-integrable for $0 < |\varepsilon| \ll 1$.

Proof. It is enough to prove that F_ε is non-integrable, since the dynamics of the symmetric billiard is equivalent to the dynamics of F_ε.

Since q also satisfies properties *(P1)-(P3)*, Theorem 3.1 can be applied to $L(t) = \sum_{n\in\mathbb{Z}} q(t+hn)$. By this Theorem, it is enough to prove that there exists *some* non-integrability coefficient not zero. Looking at the expression (4.7) and using property a) of $\eta_1 \circ \varphi$, the only possible singularities of q with $\Im t = \pi/2$ are $z_{11} := \pi \mathrm{i}/2$, $z_{21} := z_{11} - h/2$ and $z_{22} := z_{11} + h/2$. In particular, we note that q is analytic in $z_{11} + hn$ for all integer $n \neq 0$. Thus, the non-integrability coefficients associated to the singularity z_{11} are $d_{1j} = a_{-(j+1)}(q, z_{11})$, $j \geq 0$.

Using the fact that $v_{h/2}$ and $v_{-h/2}$ are analytic and not zero in z_{11} together with property b) of $\eta_1 \circ \varphi$, it turns out that z_{11} is a non-removable singularity of q. Consequently, the non-integrability coefficients d_{1j} can not be all zero. $\square$

Remark 4.4. The same proof works for the point $3\pi\mathrm{i}/2$ (instead of $\pi\mathrm{i}/2$). The assumption of entire function on η_1 has only been used to ensure that for $t_p = \pi\mathrm{i}/2$ or $t_p = 3\pi\mathrm{i}/2$, $\eta_1 \circ \varphi$ has an isolated singular point at t_p but is analytic on $t_p + hn$ for $n \neq 0$.

4.4. Reversible polynomial examples

4.4.1. The general case In order to perform explicit computations of Melnikov functions we must restrict ourselves to symmetric *polynomial* perturbations. Moreover, following [Tab93, Tab94], we focus our attention on *reversible* perturbations. Therefore, η_1 is an even (see remark 4.3) and π-periodic trigonometrical polynomial that we can write in the following way $\eta_1(\varphi) = \sum_{n=0}^{N} c_n \sin^{2n}(\varphi)$. Now, using that $v(t) = \sin(\varphi(t)) = \mathrm{sech}(t)$, $\eta_1(\varphi(t)) = \sum_{n=0}^{N} c_n \,\mathrm{sech}^{2n}(t)$. Thus $q = (2a/b)v_{h/2} \cdot v_{-h/2} \cdot (\eta_1 \circ \varphi)$ is πi-periodic (i.e., $T = \pi$) and has exactly three poles in $\mathcal{I}_\pi$. These poles are $z_{11} = \pi\mathrm{i}/2$ (of order $2N$) and $z_{21} = z_{11} + h/2$, $z_{22} = z_{11} - h/2$ (simple ones).

The non-integrability coefficients (3.11) of the problem, that may be different from zero, are

$$d_{1j} = a_{-(j+1)}(q, z_{11}), \quad (j = 0, \ldots, 2N-1), \qquad d_{20} = \mathrm{res}(q, z_{21}) + \mathrm{res}(q, z_{22}).$$

By property c) of $\eta_1 \circ \varphi$, it is easy to check that q is symmetric with regard to $z_{11} = (z_{21} + z_{22})/2$, hence $d_{1j} = 0$, for odd j, and $d_{20} = 0$. Moreover, because of the symmetry of v with regard to z_{11}, the even coefficients in the Taylor expansion of the functions $v_{h/2}$ and $v_{-h/2}$ around z_{11} are equal. Thus, $a_{-j}(v_{h/2} \cdot \delta, z_{11}) = a_{-j}(v_{-h/2} \cdot \delta, z_{11})$, for all even j, and

$$d_{1,2j-1} = 2a_{-2j}(v_{h/2} \cdot \delta, z_{11}), \quad j = 1, \ldots, N.$$

Consequently, using the formulae (3.10) and (3.11) one gets the Melnikov function

$$M(t) = -2 \sum_{j=1}^{N} \frac{a_{-2j}(v_{h/2} \cdot \delta, \pi\mathrm{i}/2)}{(2j-1)!} \psi^{(2j-1)}(t), \tag{4.8}$$

where the parameter of the elliptic functions has been determined by relation (3.1) with $T = \pi$, see remark 3.2. (For the notations about elliptic functions and the definition of ψ we refer to subsection 3.1.) We note that $t = 0$ and $t = h/2$ are zeros of M, because of the symmetries of ψ.

This formula allows to compute the Melnikov function in a finite number of steps. We need only to compute the numbers $a_{-2j}(v_{h/2} \cdot \delta, \pi\mathrm{i}/2)$, $j = 1, \ldots, N$, in each concrete case. For instance, it is easy to compute $a_{-2N}(v_{h/2} \cdot \delta, \pi\mathrm{i}/2) = (-1)^N abc_N$.

4.4.2. A particular case For $N = 1$ formula (4.8) reads

$$M(t) = 2abc_1\psi'(t) = -4abc_1 m \left(\frac{2K}{h}\right)^3 (\mathrm{dn} \cdot \mathrm{sn} \cdot \mathrm{cn}) \left(\frac{2Kt}{h}\Big| m\right).$$

This particular case ($\eta_1(\varphi) = c_0 \sin\varphi + c_1 \sin^3\varphi$) is already studied in [Lev93, Tab93, LT93, Tab94, Lom94]. It corresponds to symmetric and reversible *quartic* perturbations of the ellipse (see the introduction for the definition). In this case M has just two (simple) zeros in the period $[0, h)$: $t = 0$ and $t = h/2$, thus (see Theorem 2.1) there exist exactly two (transversal) primary homoclinic orbits for the perturbed billiard and the splitting angle $\alpha(\varepsilon)$ of the intersection near $\sigma(0) = (\pi/2, 1)$, using the formula (2.10), verifies $|\tan[\alpha(\varepsilon)]| = \mathcal{A}(h)|\varepsilon| + \mathrm{O}(\varepsilon^2)$, where

$$\mathcal{A}(h) = \frac{|M'(0)|}{\|\dot\sigma(0)\|^2} = |M'(0)| = 4ab|c_1| m \left(\frac{2K}{h}\right)^4.$$

4.4.3. Ellipse close to a circle: a particular case Let $\rho = \sqrt{a^2-b^2}/a = 1/a = \tanh(h/2)$ be the eccentricity of the ellipse. If the ellipse is close to the circle (i.e., if ρ is close to zero), then h is also close to zero and the formulae (3.5), with $T=\pi$, give:

$$\left.\begin{aligned} M(t) &= -32\pi^3 abc_1 h^{-3}\mathrm{e}^{-\pi^2/h}\left[\sin\left(\tfrac{2\pi t}{h}\right)+\mathrm{O}(\mathrm{e}^{-\pi^2/h})\right], \\ \mathcal{A}(h) &= 64\pi^4 ab|c_1|h^{-4}\mathrm{e}^{-\pi^2/h}\left[1+\mathrm{O}(\mathrm{e}^{-\pi^2/h})\right]. \end{aligned}\right\} \tag{4.9}$$

The results can be expressed in terms of ρ instead of h, but we refer to the works [Tab93, LT93, Tab94] for the sake of brevity.

Remark 4.5. It is worth mentioning that we have to assume, h is fixed (although small) and $\varepsilon \to 0$. When these two parameters are dependent by a potential relation like $\varepsilon = \varepsilon(h) = h^p$, $p>0$, and $h \to 0$, then one is confronted with the difficult problem of justifying the asymptotic “Melnikov prediction” $|\tan[\alpha(\varepsilon)]| \sim h^p\mathcal{A}(h)$, as we have already mentioned in the introduction.

4.4.4. Ellipse close to a circle: the general case Coming back to a general N, we give a generalization of the asymptotic expressions (4.9), that is, for ellipses close to a circle.

Since $\delta(t) = v(t)\eta_1(\varphi(t)) = \sum_{n=0}^N c_n \operatorname{sech}^{2n+1}(t)$, we can rewrite the Melnikov function (4.8) like

$$M(t) = -2\sum_{n=1}^{N}\sum_{j=1}^{n}[c_n B_{n,j}(t,h)/(2j-1)!], \tag{4.10}$$

where $B_{n,j}(t,h) = a_{-2j}(v_{h/2}\cdot\operatorname{sech}^{2n+1}, z_{11})\psi^{(2j-1)}(t)$. To get the dominant term of (4.10), we must study the order in h of $B_{n,j}(t,h)$, for $j=1,\dots,n$ and $n=1,\dots,N$.

First, we split the function $v_{h/2}$ in the principal $v^{\mathrm{p}}_{h/2}$ and regular $v^{\mathrm{r}}_{h/2}(=v_{h/2}-v^{\mathrm{p}}_{h/2})$ parts around its singularity $z_{22} = z_{11} - h/2$. It turns out that $v^{\mathrm{p}}_{h/2}(t) = -\mathrm{i}(t-z_{22})^{-1}$. From the Cauchy inequalities, the coefficients in the Taylor expansion of $v^{\mathrm{r}}_{h/2}$ around z_{11} are $\mathrm{O}(1)$, since $v^{\mathrm{r}}_{h/2}$ is uniformly (when h is small) bounded in a ball of fixed radius centered at z_{11}. Thus,

$$a_\ell(v_{h/2}, z_{11}) = a_\ell(v^{\mathrm{p}}_{h/2}, z_{11}) + a_\ell(v^{\mathrm{r}}_{h/2}, z_{11}) = (-2/h)^{\ell+1}\mathrm{i} + \mathrm{O}(1), \quad \forall \ell \geq 1.$$

Besides, the principal part of $\operatorname{sech}^{2n+1}$ around z_{11} is $\mathrm{O}(1)$ and, in particular, $a_{-(2n+1)}(\operatorname{sech}^{2n+1}, z_{11}) = (-1)^{n+1}\mathrm{i}$. Finally, we use the asymptotic expression (3.6), taking into account that $T=\pi$, and we deduce that

$$\begin{aligned} B_{n,j}(t,h) &= \psi^{(2j-1)}(t)\sum_{\ell=0}^{n-j} a_{2\ell+1}(v_{h/2}, z_{11})\cdot a_{-(2j+2\ell+1)}(\operatorname{sech}^{2n+1}, z_{11}) \\ &= (-1)^{n+j}2^{2n+4}\pi^{2j+1}h^{-(2n+3)}\mathrm{e}^{-\pi^2/h}[\sin(2\pi t/h)+\mathrm{O}(h^2)], \end{aligned}$$

so the dominant terms of (4.10) are attained when $n = N$, and the general asymptotic expressions that we were looking for are:

$$M(t) = (-1)^{N+1} 2^{2N+5} c_N \left\{ \sum_{j=1}^{N} \left[(-1)^j \frac{\pi^{2j+1}}{(2j-1)!} \right] \right\} h^{-(2N+3)} \mathrm{e}^{-\pi^2/h} \left[\sin\left(\frac{2\pi t}{h}\right) + \mathrm{O}(h^2) \right],$$

$$\mathcal{A}(h) = |M'(0)| = 2^{2N+6} |c_N| \left| \sum_{j=1}^{N} \left[(-1)^j \frac{\pi^{2j+2}}{(2j-1)!} \right] \right| h^{-(2N+4)} \mathrm{e}^{-\pi^2/h} [1 + \mathrm{O}(h^2)].$$

We note that the sum $\sum_{j=1}^{N} [(-1)^j \pi^{2j+1}/(2j-1)!]$ never is zero (π is a transcendental number), so M has exactly two zeros in the period $[0, h)$: $t = 0$ and $t = h/2$, if h is enough small. Therefore, for $0 < h \ll 1$, there are exactly two (transversal) primary homoclinic orbits. Moreover, the splitting angle $\alpha(\varepsilon)$ admits the asymptotic approximations, when h is small but *fixed*, $|\tan[\alpha(\varepsilon)]| = \mathcal{A}(h)|\varepsilon| + \mathrm{O}(\varepsilon^2)$, as before. The remark 4.5 also holds in this case.

5. Standard-like maps

5.1. *Integrable standard-like maps with separatrices*

A planar map is called a standard-like map if it has the form $F(x, y) = (y, -x + g(y))$ for some function g. If g is odd, F is $\mathcal{R}$-reversible, where $\mathcal{R}$ is the involution $\mathcal{R}(x, y) = (y, x)$. When g is analytic on $\mathbb{R}$, F is an analytic a.p.m. and its generating function is

$$\mathcal{L}(x, X) = -xX + \int^X g(s)\, ds\,. \tag{5.1}$$

If g is entire, the same happens to F, and therefore it has no separatrices [Laz88]. Suris, weakening the regularity of g, gives three families of integrable standard-like maps in [Sur89]. The first integrals of these three families are, respectively, polynomials of degree four in x and y, functions involving exponential terms, and functions with trigonometric terms. For the sake of brevity we focus our attention on the first case, but exactly the same study can be carried out for the other two ones.

It is easy to see that all the maps of the polynomial family with a separatrix to the origin can be written, after rescaling, normalizations, etc., like

$$F_0(x, y) = \left(y, -x + 2y \frac{\mu + \beta y}{1 - 2\beta y + y^2} \right), \quad -1 < \beta < 1 < \mu, \tag{5.2}$$

and the corresponding first integral given by Suris is

$$I(x, y) = \frac{1}{2}[x^2 - 2\mu xy + y^2 - 2\beta xy(x + y) + x^2 y^2].$$

The map (5.2) with $\beta = 0$ is called McMillan map and is studied in [GPB89] under a linear perturbation. The map (5.2) has two separatrices $\Gamma^\pm = \Gamma^\pm_{\mu,\beta}$ contained in the energy level $\{I = 0\}$, as shown in Figure 5.

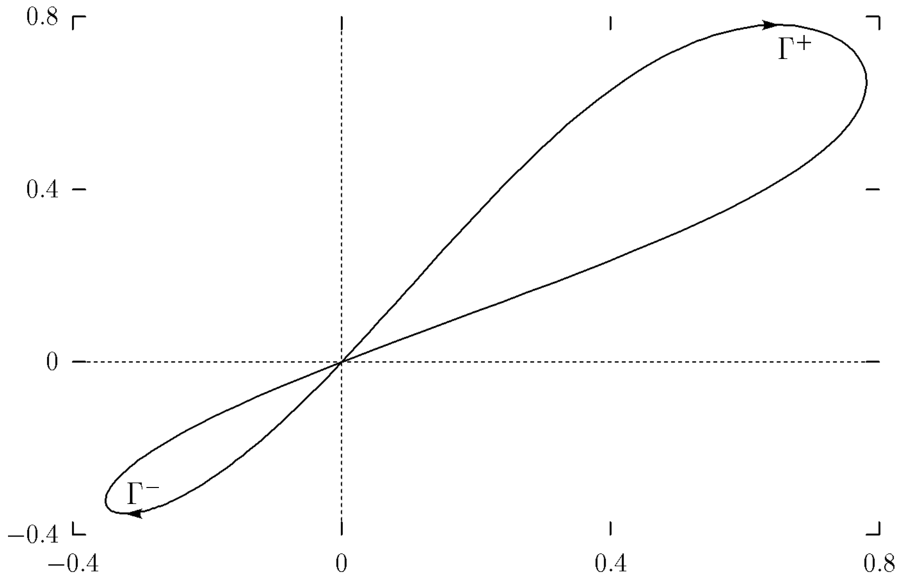

Figure 5. The separatrices $\Gamma^{\pm}$ of F_0 for $\beta = 0.1$ and $\mu = \cosh(0.5)$.

Denoting $P_0 = (0,0)$, the eigenvalue greater than 1 of $DF_0(P_0)$ is $\lambda = \mu + \sqrt{\mu^2 - 1}$, so if $h = \ln \lambda$, as usual, then $\cosh h = \mu$. Using Lemma 2.1, there exists a constant θ such that the time-h Hamiltonian flow associated to $H = \theta I$ interpolates the map on the separatrices. Let $X_I = J \cdot \nabla I$ be the Hamiltonian field associated to I; then

$$A = DX_I(P_0) = \begin{pmatrix} -\mu & 1 \\ -1 & \mu \end{pmatrix}, \quad B = DF_0(P_0) = \begin{pmatrix} 0 & 1 \\ -1 & 2\mu \end{pmatrix}.$$

Thus relation $A = \mathrm{e}^{\theta h B}$ gives $\theta = (\mu^2 - 1)^{-1/2}$ (see remark 2.2). Therefore, the first integral that we will use is

$$H(x,y) = \frac{1}{2\sqrt{\mu^2 - 1}}[x^2 - 2\mu xy + y^2 - 2\beta xy(x+y) + x^2y^2].$$

Now, solving the Hamiltonian equations associated to H in the energy level $\{H = 0\}$, we obtain the natural parameterizations of the separatrices $\Gamma^{\pm}$

$$\Gamma^{\pm} = \left\{\sigma^{\pm}(t) = (x^{\pm}(t-h), x^{\pm}(t)) : t \in \mathbb{R}\right\},$$

where

$$x^{\pm}(t) = \frac{\pm c}{\Delta \cosh t \mp b}, \; a = \beta^2 - 1, \; b = \beta(\mu + 1), \; c = \mu^2 - 1, \; \Delta = \sqrt{b^2 - ac}. \tag{5.3}$$

We note that $\Gamma^-_{\mu,\beta} = -\Gamma^+_{\mu,-\beta}$ (in particular the case $\beta = 0$ is symmetric since then $\Gamma^- = -\Gamma^+$), so we study only $\Gamma = \Gamma^+$, $\sigma = \sigma^+$ and $x = x^+$.

In this situation the function g in (2.4) has only isolated singularities (respectively, is a meromorphic function) for all entire (respectively, vectorial polynomial) perturbation

F_1. In consequence, for a family of analytic diffeomorphisms like $F_\varepsilon = F_0 + \varepsilon F_1 + \mathrm{O}(\varepsilon^2)$, with F_1 a vectorial polynomial in x and y, it is possible to compute explicitly the Melnikov function. Moreover, if F_ε is an a.p.m. and F_1 is an entire function, then Theorem 3.1 can be applied directly to study the non-integrability.

5.2. *Non-integrable standard-like maps*

We consider now the family of standard-like maps

$$F_\varepsilon(x,y) = \left(y, -x + 2y\frac{\mu+\beta y}{1-2\beta y+y^2} + \varepsilon p(y)\right), \quad -1<\beta<1<\mu, \quad \varepsilon \in \mathbb{R}. \tag{5.4}$$

Let δ be the primitive of p such that $\delta(0) = 0$. Using equation (5.1), the generating function $\mathcal{L}(x,X,\varepsilon)$ of F_ε has the form (2.5) with $\mathcal{L}_1(x,X) = \delta(X)$. Thus $f(t) = \mathcal{L}_1(x(t-h),x(t)) = \delta(x(t))$ has only isolated singularities for any entire function δ and hence $\{F_\varepsilon\}$ verifies the hypothesis of Theorem 3.1 if p is an entire function. This allows us to prove the following result.

Theorem 5.1. If p is an entire function not identically zero, the map (5.4) is non-integrable for $0 < |\varepsilon| \ll 1$.

Proof. It is sufficient to see that the non-integrability coefficients of the problem can not be all zero.

The poles of x in $\mathcal{I}_{2\pi}$ are $\mathrm{i}t_p^\pm$, where $t_p^- \in (0,\pi)$ and $t_p^+ \in (\pi, 2\pi)$ are determined by $\cos t_p^\pm = b/\Delta \in (-1,1)$. Since $f = \delta \circ x$ and δ is an entire function, $\mathrm{i}t_p^\pm$ are exactly the singularities of f in $\mathcal{I}_{2\pi}$. Consequently, there are not different singularities whose difference is a multiple of h (in fact a real number), and thus each one of the sums in (3.11) have only one term. Hence, the coefficients of non-integrability (3.11) are all zero if and only if f is analytic on $\mathcal{I}_{2\pi}$ or, equivalently (using the 2π-periodicity of f), if and only if f is an entire function. But f cannot be an entire function, since δ is a *non-constant* entire function. This finishes the proof. $\square$

Remark 5.1. In order to apply the non-integrability criterion, we simply need that $f = \delta \circ x$ have only isolated singularities in $\mathbb{C}$. Thus, it is not absolutely necessary that δ would be an entire function, although it is the simplest case to study, since then the singularities are easily found.

5.3. *Examples*

5.3.1. Reversible polynomial standard-like perturbations To show the simplicity of the explicit computations, we focus our attention on $\mathcal{R}$-reversible and polynomial standard-like perturbations. Due to the reversibility, $\beta = 0$, p is odd, and these maps have a primary homoclinic point on the bisectrix of the first quadrant. We give expressions for the splitting angle at this point.

Since the perturbation is polynomial and odd, we write $p(y) = \sum_{n=1}^{N} c_n y^{2n-1}$, so $\delta(y) = \sum_{n=1}^{N} c_n y^{2n}/2n$. Using that $\beta = 0$ in (5.3), we get $x(t) = \sinh(h)\,\mathrm{sech}(t)$. Thus $f(t) = \delta(x(t))$ is πi-periodic (i.e., $T = \pi$) and has only a pole $z_{11} = \pi\mathrm{i}/2$ (of order $2N$) in $\mathcal{I}_\pi$. Moreover, $a_{-j}(f, \pi\mathrm{i}/2) = 0$ for odd j, since f is symmetric with regard to $\pi\mathrm{i}/2$. Now formulae (3.10) and (3.11) give the Melnikov function

$$M(t) = -\sum_{j=1}^{N} \frac{a_{-2j}(f, \pi\mathrm{i}/2)}{(2j-1)!}\psi^{(2j-1)}(t), \tag{5.5}$$

where the parameter of the elliptic functions have been determined by relation (3.1) with $T = \pi$, see remark 3.2. We note that $t = 0$ and $t = h/2$ are zeros of M, because of the symmetries of ψ. This formula allows to compute the Melnikov function in a finite number of steps. We need only to compute the numbers $a_{-2j}(f, \pi\mathrm{i}/2)$, $j = 1, \ldots, N$, in each concrete case. For instance, it is easy to compute $a_{-2N}(f, \pi\mathrm{i}/2) = (-1)^N c_N \sinh^{2N}(h)/2N$, and in particular, for $p(y) = y$ ($N = 1$ and $c_1 = 1$) we get

$$M(t) = \frac{\sinh^2 h}{2}\psi'(t) = -\left(\frac{2K}{h}\right)^3 m \sinh^2 h\,(\mathrm{dn}\cdot\mathrm{cn}\cdot\mathrm{sn})\left(\frac{2Kt}{h}\middle| m\right).$$

This particular case is already studied in [GPB89]. Our result coincides with the one given there except for a multiplicative factor (due to the difference between the first integrals used) and a sign (due to the different sense used in the natural parameterization, since M is odd). In this case M has just two (simple) zeros in the period $[0, h)$: $t = 0$ and $t = h/2$, thus there exist exactly two transversal primary homoclinic orbits for the map (5.4) with $\beta = 0$ and $p(y) = y$, if $|\varepsilon|$ is non-zero but small enough. The zero $t = h/2$ is related with the homoclinic point on the bisectrix of the first quadrant near $\sigma(h/2)$ and using (2.10), the splitting angle $\alpha(\varepsilon)$ in this point verifies that $|\tan[\alpha(\varepsilon)]| = \mathcal{A}(h)|\varepsilon| + \mathrm{O}(\varepsilon^2)$, where

$$\mathcal{A}(h) = \frac{|M'(h/2)|}{\|\,\dot\sigma(h/2)\,\|^2} = \left(\frac{2K}{h}\right)^4 \frac{\cosh^4(h/2)}{2\sinh^2(h/2)} m\,(1-m).$$

5.3.2. Weakly hyperbolic examples When the origin is a *weakly* hyperbolic fixed point for the unperturbed map (i.e., when $0 < h \ll 1$), formulae (3.5) applied with $T = \pi$ give:

$$M(t) = -\,8\pi^3 h^{-3}\sinh^2 h \cdot \mathrm{e}^{-\pi^2/h}\left[\sin\left(\frac{2\pi t}{h}\right) + \mathrm{O}(\mathrm{e}^{-\pi^2/h})\right],$$
$$\mathcal{A}(h) = \left(\frac{2\pi}{h}\right)^4 \frac{\cosh^4(h/2)}{2\sinh^2(h/2)}\mathrm{e}^{-\pi^2/h}\left[1 + \mathrm{O}(\mathrm{e}^{-\pi^2/h})\right].$$

Coming back to a general N, we can give a generalization of the previous asymptotic expressions. Since $x(t) = \sinh(h)\,\mathrm{sech}(t)$, the principal part of $x^{2n}(t)$ around $\pi\mathrm{i}/2$ is

$O(h^{2n})$ and, in particular, $a_{-2n}(x^{2n}, \pi\mathrm{i}/2) = (-1)^n \sinh^{2n}(h)$. From these results and formula (3.6) also applied with $T = \pi$, it is easy to get the dominant term of the Melnikov function (5.5), and the general asymptotic expressions are:

$$M(t) = -2\left[\sum_{n=1}^{N} \frac{c_n (2\pi)^{2n+1}}{(2n)!}\right] h^{-1} \mathrm{e}^{-\pi^2/h}\left[\sin\left(\frac{2\pi t}{h}\right) + O(h^2)\right],$$
$$\mathcal{A}(h) = \frac{|M'(h/2)|}{\| \dot{\sigma}(h/2) \|^2} = 4\left|\sum_{n=1}^{N} \frac{c_n (2\pi)^{2n+2}}{(2n)!}\right| h^{-6} \mathrm{e}^{-\pi^2/h}\left[1 + O(h^2)\right].$$

The error $O(h^2)$ in these last formulae is bigger than the error $O(\mathrm{e}^{-\pi^2/h})$ in the former ones. However, it is posible to obtain formulae with exponentially small error in any case, but it involves a cumbersome computations if N is large. For instance, when $p(y) = y^3$, it is not difficult to derive the following formula for $\mathcal{A}(h)$

$$\mathcal{A}(h) = \left(\frac{2\pi}{h}\right)^4 \left[1 + \left(\frac{\pi}{h}\right)^2\right] \frac{\sinh^2(h)\cosh^4(h/2)}{6\sinh^2(h/2)} \mathrm{e}^{-\pi^2/h}\left[1 + O(\mathrm{e}^{-\pi^2/h})\right].$$

As in the billiard, the discussion or remark 4.5 is still valid.

5.3.3. A dissipative example We have seen several examples where the Melnikov function is exponentially small in $h = \ln\lambda$, when $h \to 0$. This is a typical phenomenon for area preserving perturbations, but if the conservative character is destroyed by the perturbation then this kind of phenomena, in general, does not take place. As a sample of this claim we choose the following case (studied in [GPB89])

$$F_\varepsilon(x,y) = \left(y, -x + \frac{2\mu y}{1+y^2} + \varepsilon x\right), \quad \mu > 1. \tag{5.6}$$

The Jacobian of (5.6) is $J(\varepsilon) = 1 - \varepsilon$, so (5.6) becomes dissipative for $\varepsilon > 0$ and we can not expect to find a generating function of it. In spite of this, an explicit computation of the Melnikov function is still possible using the formulae (2.3) and (2.4), and we obtain

$$M(t) = \sum_{n\in\mathbb{Z}} g(t + hn), \quad g(t) = \dot{x}(t)x(t-h), \quad x(t) = \sinh(h)\,\mathrm{sech}(t).$$

First, we note that g is πi-periodic (i.e., $T = \pi$) and we study the complex singularities of g in $\mathcal{I}_\pi$, being $\pi\mathrm{i}/2$ (a double pole) and $h + \pi\mathrm{i}/2$ (a simple pole) with

$$a_{-1}(g, h + \pi\mathrm{i}/2) = -a_{-1}(g, \pi\mathrm{i}/2) = \cosh h, \quad a_{-2}(g, \pi\mathrm{i}/2) = -\sinh h.$$

Now, using equation (3.3) with $T = \pi$ and the summation formula (3.8), we get the Melnikov function

$$\begin{aligned} M(t) &= -\cosh h[\chi(h + \pi\mathrm{i}/2 - t) - \chi(\pi\mathrm{i}/2 - t)] + \sinh h \cdot \chi'(\pi\mathrm{i}/2 - t) \\ &= \left(\frac{2K}{h}\right)^2 \sinh h\, \mathrm{dn}^2\left(\frac{2Kt}{h}\right) - 2\cosh h + \frac{2\sinh h}{h}\left(1 - \frac{2KE}{h}\right). \end{aligned}$$

This result coincides with the one given in [GPB89] except for a multiplicative factor, as before, and a small mistake in the final formula (B.12) of the cited reference. The sign is the same since M is even.

If h is small enough, M has no real zeros, and consequently the perturbed invariant curves do not intersect, for fixed small h and small enough ε.

We split M into mean and oscillatory parts: $M(t) = M_{\mathrm{mean}} + M_{\mathrm{oscill}}(t)$. It is easy to obtain their expressions:

$$
\begin{aligned}
M_{\mathrm{mean}} &= h^{-1}\int_0^h M(t)\,dt = 2\left(\frac{\sinh h}{h} - \cosh h\right), \\
M_{\mathrm{oscill}}(t) = M(t) - M_{\mathrm{mean}} &= \left(\frac{2K}{h}\right)^2 \sinh h\left[\mathrm{dn}^2\left(\frac{2Kt}{h}\right) - \frac{E}{K}\right].
\end{aligned}
$$

It is not hard to verify that M_{oscill} is exponentially small in h but M_{mean} is not, thus we have given a example of a Melnikov function not exponentially small in h.

Remark 5.2. In fact, under the usual hypothesis of meromorphicity, the oscillatory part is *always* exponentially small in h. The same happens to the Melnikov function in the a.p.m. case, since then its mean is zero.

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References

[Arn64] V.I. Arnold. Instability of dynamical systems with several degrees of freedom. *Sov. Math. Dokl.*, 5:581–585, 1964.

[AS72] M. Abramowitz and I. Stegun. *Handbook of mathematical functions.* Dover, 1972.

[Bir27] G.D. Birkhoff. *Dynamical systems.* A.M.S. Publications, 1927.

[Cus78] R. Cushman. Examples of nonintegrable analytic Hamiltonian vectorfields with no small divisors. *Trans. Amer. Math. Soc.*, 238(1):45–55, 1978.

[DS92] A. Delshams and T. M. Seara. An asymptotic expression for the splitting of separatrices of the rapidly forced pendulum. *Commun. Math. Phys*, 150:433–463, 1992.

[DR95] A. Delshams and R. Ramírez-Ros. Melnikov methods and exponentially small splitting of separatrices for planar maps: A experimental study. Preprint in progress, 1995.

[Eas84] R.W. Easton. Computing the dependence on a parameter of a family of unstable manifolds: Generalized Melnikov formulas. *Nonlinear Analysis Theory, Methods and Applic.*, 8:1–4, 1984.

[FS90] E. Fontich and C. Simó. The splitting of separatrices for analytic diffeomorphisms. *Ergod. Th. & Dynam. Sys.*, 10:295–318, 1990.

[Gam85] J.M. Gambaudo. Perturbation of a Hopf bifurcation by an external time-periodic forcing. *Jour. Diff. Eq.*, 57:172–199, 1985.

[Gam87] J.M. Gambaudo. Perturbation de l' "application temps τ" d'un champ de vecteurs integrable de $\mathbf{R}^2$. *C.R. Acad. Sci. Paris*, 297:245–248, 1987.

[GH83] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, volume 42 of *Applied Mathematical Sciences*. Springer-Verlag, New York, Heidelberg, Berlin, 1983.

[GLT91] V.G. Gelfreich, V.F. Lazutkin, and M.B. Tabanov. Exponentially small splittings in Hamiltonian systems. *Chaos*, 1(2):137–142, 1991.

[GLS94] V.G. Gelfreich, V.F. Lazutkin, and N.V. Svanidze. A refined formula for the separatrix splitting for the standard map. *Physica D*, 71:82–101, 1994.

[GPB89] M.L. Glasser, V.G. Papageorgiou, and T.C. Bountis. Melnikov's function for two-dimensional mappings. *SIAM Jour. Appl. Math.*, 49(3):692–703, 1989.

[KT91] V.V. Kozlov and D.V. Treshchev. *Billiards: A genetic introduction to the dynamics of systems with impacts*, volume 89 of *Trans. Math. Mon.* A.M.S., 1991.

[Laz88] V.F. Lazutkin. Splitting of complex separatrices. *Func. Anal. and Appl.*, 22:154–156, 1988.

[Lev93] P. Levallois. *Non-integrabilité des billiards definis par certaines perturbations algébraques d'une ellipse et du flot géodésique de certaines perturbations algébraques d'un ellipsoide.* PhD thesis, Univ. Paris VII, 1993.

[Lom94] H.E. Lomelí. Transversal heteroclinic orbits for perturbed billiards. Preprint University of Minnesota, 1994.

[LT93] P. Levallois and M.B. Tabanov. Séparation des séparatrices du billiard elliptique pour une perturbation algébrique et symétrique de l'ellipse. *C. R. Acad. Sci. Paris*, 316:589–592, Série I,1993.

[McM71] E.M. McMillan. A problem in the stability of periodic systems. In *Topics in modern physics*, pages 219–244. Colorado Assoc. Univ. Press, 1971.

[Mel63] V.K. Melnikov. On the stability of the center for time periodic perturbations. *Trans. Moscow Math. Soc.*, 12:3–56, 1963.

[Mos56] J. Moser. The analytic invariants of an area-preserving mapping near a hyperbolic fixed point. *Comm. Pure Appl. Math.*, 9:673–692, 1956.

[Poi99] H. Poincaré. *Les méthodes nouvelles de la mécanique céleste*, volume 1, 2, 3. Gauthier-Villars, Paris, 1892-1899.

[Sur89] Yu B. Suris. Integrable mappings of the standard type. *Func. Anal. and Appl.*, 23:74–76, 1989.

[Tab93] M.B. Tabanov. Splitting of separatrices for Birkhoff's billiard under some symmetrical perturbation of an ellipse. *Prépublications Math. de l'U.R.A. 212*, (65), 1993.

[Tab94] M.B. Tabanov. Separatrices splitting for Birkhoff's billiard in symmetric convex domain, closed to an ellipse. *Chaos*, 4(4):595–606, 1994.

[Wig90] S. Wiggins. *Global bifurcations and chaos: Analytical methods*, volume 73 of *Applied Mathematical Sciencies*. Springer-Verlag, 1990.

[Zig82] S.L. Ziglin. Splitting of separatrices, branching of solutions and nonexistence of an integral in the dynamics of a solid body. *Trans. Moscow Math. Soc.*, 41:282–298, 1982.

List of Figures