

A METHOD TO APPROXIMATE THE STEEPEST DESCENT DIRECTION OF THE O-D MATRIX ADJUSTMENT PROBLEM

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ABSTRACT

This paper presents a computational method to solve a recently presented scheme to approximate gradients of the upper level objective function of a demand adjustment problem. The approximation scheme is not based on the proportions of utilization of the O-D flows at the links, avoiding its problems. The computational method is based on the use of the concept of partial linearization and in the simplicial decomposition method with nonlinear columns generation.

1 Introduction

In the last years the OD matrix adjustment models from traffic counts have been formulated as bilevel programming (BLP) problems and in the algorithmic field several methods have been developed specifically that, although can be viewed as heuristics, adopt approaches common to the general methods in optimization. The methods of Spiess in [12], Yang in [14], Chen in [1], Codina and Barceló in [3] for the case of the deterministic user equilibrium are representatives of this approach. These methods require the calculation of a descent direction at each step and, optionally, a line search in order to complete an iteration. The calculation of the descent direction can be carried out by means of the link use proportions (Spiess in [12] and Chen in [1]) or using the sensibility analysis of Tobin et al. in [13] (Yang in [14]). They all assumed explicitly or implicitly, differentiability at the point where the descent direction is going to be evaluated. The method of evaluating the link use proportions by means of an additional equilibrium assignment is computationally efficient although does not guarantee that the evaluated direction is of descent. Recently in [3], Codina and Barceló suggested an approach for the BLP matrix adjustment problem based on a method for nondifferentiable optimization and developed an approximation to the Clarke's generalized gradient of the upper level function based on a proximal point approach that requires the solution of a variable demand assignment problem. In this paper a solution algorithm for the approximation of the steepest direction is presented and evaluated, along with a test that detects whether a given direction is of descent for the upper level function of the demand adjustment problem. The solution algorithm is based on the simplicial decomposition with nonlinear column generation (SDN) of Larsson *et al.* and on the partial linearization of Patriksson [11].

2 The steepest descent approximation approach

If the general asymmetric assignment map $v^*(g)$ presents uniqueness, then the BLP formulation takes the form:

$$\text{Min}_{g \geq 0} F(g) = z_1 R_1(v^*(g)) + z_2 R_2(g) \quad (1)$$

Usually $R_1(v) = \frac{1}{2}(v_1 - \hat{v}_1)^\top U (v_1 - \hat{v}_1)$ and $R_2(g) = \frac{1}{2}(g - \bar{g})^\top B (g - \bar{g})$ or an entropy function and $z_1 > 0$, $z_2 \geq 0$. The upper level objective function $R_1(v^*(g))$ of a demand adjustment problem (1) verifies the Lipschitz condition and, accordingly to Clarke [2], Proposition 2.1.2, the generalized gradient of Clarke $\partial F(\check{g})$ at a point \check{g} is well defined and is a non empty, convex and compact set. In [3] Codina and Barceló show how a consistent approximation to an element of the generalized gradient of Clarke can be made to the upper level objective function $F(g) = R_1(v^*(g))$ at a specified point \check{g} , by approximating the following proximal point problem:

$$\text{Min}_g R_1(v^*(g)) + \frac{\rho}{2} \|g - \check{g}\|_2^2 \quad (2)$$

If g^* is the solution of problem (2), then for ρ high enough, an approximation of an element of $\partial F(\check{g})$ is given by $-\rho(g^* - \check{g})$.

For a non negative O-D matrix g let $\mathcal{V}(g)$ the set of feasible link flows and let Ω_V denote the cone of points (v, g) so that $g \geq 0$ and $v \in \mathcal{V}(g)$. Let $G(v, g)$ denote the gap function $\text{Min}_{v' \in \mathcal{V}(g)} c(v)^\top (v' - v)$ for a general inelastic demand traffic assignment problem. Problem (2) can then be reformulated as:

$$\begin{aligned} \text{Min}_{(v, g) \in \Omega_V} \quad & R_1(v) + \frac{\rho}{2} \|g - \check{g}\|_2^2 \\ \text{s.t.} \quad & G(v, g) = 0 \quad (EQ) \end{aligned} \quad (3)$$

It must be noted that the function $-G(v, g)$ is always nonnegative on Ω_V and that $G(v^*(g), g) = 0$. Therefore partial penalization can be applied for the constraint (EQ) in problem (3) with a penalty parameter $1/z$, yielding the the following approximated problem (4):

$$\text{Min}_{(v, g) \in \Omega_V} \quad \psi(v, g) = -G(v, g) + zR(v) + \frac{z\rho}{2} \|g - \check{g}\|_2^2 \quad (4)$$

For the fixed demand additive and separable traffic assignment problem, the term $-G(v, g)$ in (4) could be replaced by $T(v) - V(g)$, where $V(g)$ is the minimum value of the objective function $T(v) = \sum_{a \in \mathcal{A}} \int_0^{v_a} c_a(x) dx$ on $\mathcal{V}(g)$. For this case, in [3] Codina and Barceló give the conditions under which problem (4) presents uniqueness of solutions.

For problem (4), with $-G(v, g)$ substituted by $T(v) - V(g)$, it is possible to apply the partial linearization scheme of Patrikson [11]. Chen in [1] proves that the function $V(g)$ is differentiable and Codina in [4] proves that $t(g)$, the OD travel times at equilibrium, are Lipschitz continuous functions of the demand g , completing a result of Hall in [9]. Therefore it is possible to state that $\nabla V(g) = t(g)$. If the partial linearization scheme of Patriksson in [11] is applied on ψ , at the term $V(g)$, then at iteration ℓ -th the subproblem step at a point $(\mathbf{v}^\ell, \mathbf{g}^\ell)$, would consist of solving:

$$\text{Min}_{(v, g) \in \Omega_V} \quad \phi^\ell(v, g) = T(v) - g^\top t^\ell + zR(v) + \frac{z\rho}{2} \|g - \check{g}\|_2^2 + K \quad (5)$$

where $t^\ell = t(\mathbf{g}^\ell)$ and $K = \mathbf{g}^{\ell \top} t^\ell - V(\mathbf{g}^\ell)$ is a constant.

Variable demand structure of the subproblems. The subproblem (5) can be rewritten as:

$$\text{Min}_{(v, g) \in \Omega_V} \quad \phi^\ell(v, g) = \sum_{a \in \mathcal{A}} \int_0^{v_a} s_a(x) dx - \sum_{w \in \mathcal{W}} \int_0^{g_w} (t_w^{(\ell)} - z\rho(x - \check{g}_w)) dx + K' \quad (6)$$

This problem presents the structure of an elastic demand assignment problem. The link cost functions $s_a(v_a)$ are defined as $s_a(v_a) = c_a(v_a) + zu_a(v_a - \hat{v}_a)$ if link $a \in \tilde{\mathcal{A}}$ and $s_a(v_a) = c_a(v_a)$ otherwise. The excess demand transformation of Gartner in [8], converts an elastic demand assignment problem into a fixed demand one by modifying the network structure of the problem. Given an OD matrix $\bar{g} > g^*$, an artificial link is added from each origin node to each destination node absorbing the excess demand f_w for the OD pair $w \in W$, so that $g_w + f_w = \bar{g}_w$. In the case of problem (4), the cost for the added link for OD pair $w \in W$ would be $t_w^\ell + z\rho(f_w - \hat{f}_w)$. Combined with the SDN algorithm of Larsson *et al.* in [10], this device will be used in order to solve problem (5).

2.1 Application of the SDN Algorithm to solve the PL subproblem (6)

For high values of the parameter ρ , the solution g^* of problem (4) is very close to the current point \check{g} at which the steepest descent direction of $F(g)$ is going to be approximated. This implies that very few quadratic approximations of ϕ^ℓ will be required in order to solve (5) using the SDN algorithm. Let $\bar{g} > \check{g}$ be an excess demand. The excess demand transformation of Gartner defines a polytope $\Omega_{\mathbf{HF}}(\bar{g})$ on the path flows space (h, f) and another polytope $\Omega_{\mathbf{VF}}(\bar{g})$ on the link flows space (v, f) of the transformed network. Clearly, if $(v, \bar{g} - g) \in \Omega_{\mathbf{VF}}(\bar{g})$ and $\lambda = \text{Min}\{\bar{g}_w/g_w \mid g_w > 0, w \in W\}$, then $(\lambda v, \bar{g} - \lambda g) \in \Omega_{\mathbf{VF}}(\bar{g})$.

Let now $\xi(v, g; \tilde{v})$ a quadratic approximation to ϕ^ℓ at a point $(\tilde{v}, \tilde{g}) \in \Omega_{\mathbf{V}}$ and $\theta(v, g; v^k)$ a partial linearization of $\xi(v, g; \tilde{v})$ with respect to v at v^k . Also, let $\chi(v, f; \tilde{v})$ a quadratic approximation to $\bar{\phi}^\ell(v, f)$ at $(\tilde{v}, \tilde{f}) \in \Omega_{\mathbf{VF}}(\bar{g})$.

Then an algorithm to solve $\text{Min}_{(v,g) \in \Omega_{\mathbf{V}}} \xi(v, g; \tilde{v})$ is:

- **Initialization** $X_0 = \{(0, \bar{g})\}$, $Y_0 = \{(\tilde{\lambda}\tilde{v}, \bar{g} - \tilde{\lambda}\tilde{g})\}$, $Z_0 = X_0 \cup Y_0$; $k = 0$

- **At iteration k-th:**

- **Subproblem.** Solve $\text{Min}_{(v,g) \in \Omega_{\mathbf{V}}} \theta(v, g; v^k) \rightarrow \hat{x} = (\hat{v}, \hat{g})$.

$$\hat{\lambda} = \text{Min}_{w \in W} \left\{ \frac{\bar{g}_w}{\hat{g}_w} \mid \hat{g}_w > 0 \right\}; \text{ New column: } \hat{y}^k = (\hat{\lambda}\hat{v}, \bar{g} - \hat{\lambda}\hat{g})$$

$$Y_{k+1} = Y_k \cup \{\hat{y}^k\}, \quad Z_{k+1} = Y_{k+1} \cup X_0.$$

- **RMP.** Solve $\text{Min}_{(v,f) \in \text{Hull}(Z_{k+1})} \chi(v, f; \tilde{v}) \rightarrow (v^{k+1}, f^{k+1})$.

Remove columns with null barycentric coordinate in Z_{k+1} , if they exist. or the column with the smallest one if Z_{k+1} has reached its maximum size. In this case replace (v^k, f^k) by (v^{k+1}, f^{k+1}) in Z_{k+1} or add (v^{k+1}, f^{k+1}) if $(v^k, f^k) \notin Z_{k+1}$; $k \leftarrow k + 1$.

3 A simple test to evaluate a direction

It is assumed now that the assignment map $v^*(g)$ is a singleton and that defining a function of the O-D matrix g . In [6], Dafermos and Nagurney present sufficient conditions under which $v^*(g)$ is a Lipschitz continuous function of g . Now, if $v^*(\cdot)$ is differentiable at g , let $d_v(g, d_g)$ be the direction followed by the equilibrium flows on the link flows space corresponding to the direction d_g at the point g . This is, $d_v(g, d_g)$ can be defined as:

$$d_v(g, d_g) = \left. \frac{dv^*(g + \alpha d_g)}{d\alpha} \right|_{\alpha=0} \quad (7)$$

It can be shown that $d_v(g, -g) = -v^*(g)$. Let now \check{d}_g be a direction that is supposed to be an approximation to the steepest descent direction of $F(g) = R(v^*(g))$ at \check{g} . Let $\check{d}_v = d_v(\check{g}, \check{d}_g)$. An approximation to the proximal point problem objective function (1) is then given by $\mathcal{R}(\lambda; \check{d}_g) = R_1(\check{v} + \lambda\check{d}_v) + \frac{\rho}{2}\lambda^2\|\check{d}_g\|_2^2$.

Let λ_d be the step length minimizing $\mathcal{R}(\lambda; \check{d}_g)$ and let λ_g be the step length minimizing $\mathcal{R}(\lambda; -\check{g}) = R_1((1 - \lambda)\check{v}) + \frac{\rho}{2}\lambda^2\|\check{g}\|_2^2$. Then, if $R(v^*(g))$ is differentiable at \check{g} , it can be proved that for ρ taken high enough the vectors $A - C \equiv -\lambda_g\check{g}$ and $A - B \equiv -(\lambda_g\check{g} + \lambda_d\check{d}_g)$ are approximately orthogonal if \check{d}_g is parallel to the steepest descent direction of $R(v^*(g))$ at \check{g} . There follows that, if φ is the angle between $A - C$ and $A - B$, then $\cos\varphi \approx 0$ and a good measure of the descent properties of a direction \check{d}_g must be given by:

$$\cos\varphi = \text{sign}(\lambda_g) \frac{\lambda_g\|\check{g}\|_2^2 + \lambda_d\check{g}^\top\check{d}_g}{\|\check{g}\|_2 \cdot \|\lambda_g\check{g} + \lambda_d\check{d}_g\|_2} \quad (8)$$

This expression can be used advantageously to evaluate a direction \check{d}_g at a given point \check{g} calculated by means of the link use proportions as in the methods of Spiess in [12] or Yang in [14] detecting if it will be an acceptable approximation to the steepest descent or provides an ascent direction (as, for example in a test network shown by Codina and Barceló in [3]). It must be noticed that it does not depend explicitly on \check{d}_v . The approximation $d_v(\check{g}, \check{d}_g) \approx P^\top\check{d}_g$ can be made in order to approximately evaluate λ_d . A better choice is to use $\check{d}_v \approx v^*(\check{g} + \alpha d_g) - v^*(\check{g})$, for some little α , taking advantage of the paths and proportions used for $v^*(\check{g})$.

4 Some numerical results

In order to evaluate the descent approximation method and test described in the previous sections a test network has been used consisting of nine nodes (1 to 9) and two O-D pairs : (1 \rightarrow 2) with demand g_1 and (3 \rightarrow 4) with demand g_2 . Network parameters are shown on table 1. The volume delay function is $c(v) = t_0(1 + 10(\frac{v}{\gamma})^4)$.

Link No.	(i, j)	t_0	γ	$\tilde{v} = v^*(400, 400)$
1	(1, 5)	.1	200	400
2	(3, 8)	.1	200	400
3	(5, 6)	.2	100	188.26
4	(5, 8)	.7	100	0
5	(5, 9)	.5	100	431.36
6	(6, 7)	1.0	100	368.64
7	(7, 2)	.1	200	400
8	(7, 9)	.1	100	180.38
9	(8, 5)	1.0	100	219.62
10	(8, 6)	1.0	100	180.38
11	(9, 4)	.08	200	400
12	(9, 7)	.7	100	211.74

Table 1. Parameters for the test network.

The following figure 1 shows two upper level objective functions $R_1(v^*(g))$ using as count set the links (4, 8) and (8, 12) respectively. Using these test functions the SDN algorithm has

been used in order to solve the subproblem step $\text{Min}_{(v,g) \in \Omega_V} \phi^\ell(v, g)$ and to approximate the direction of steepest descent. A set of ten points \check{g} in the demand space has been selected to evaluate this method. It must be noticed that in all cases after the first iteration of the SDN algorithm, a descent direction \check{d}_g has been obtained. This direction has been compared to an approximation of the steepest descent one calculated by finite differences at \check{g} . The angle β between \check{d}_g and ∇R and the results of the test provided by $\cos \varphi$ are shown in table 2. Notice that $\cos \varphi$ is not reliable (*) when \check{g} and \check{d}_g are almost parallel.

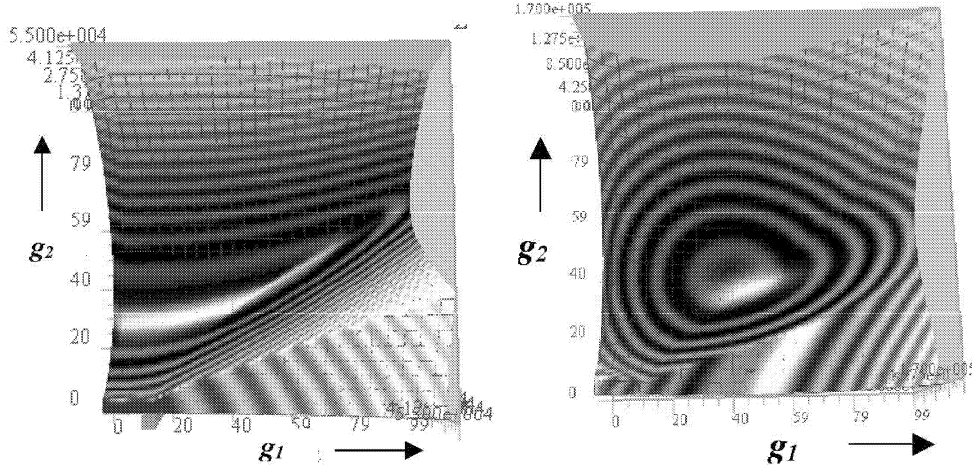


Figure 1: The upper level test functions $R(v) = \frac{1}{2}(\frac{3}{2}(v_4 - \tilde{v}_4)^2 + (v_8 - \tilde{v}_8)^2)$ (left) and $R(v) = \frac{1}{2}(\frac{3}{2}(v_8 - \tilde{v}_8)^2 + (v_{12} - \tilde{v}_{12})^2)$ (right)

\check{g}_1	\check{g}_2	d_{g_1}	d_{g_2}	$\cos \beta$	$\cos \varphi$
664	477	-0.017346	0.017675	0.9452	-0.0731
800	800	0.005472	-0.005712	0.8280	0.5384
800	100	-0.002158	-0.000352	0.8551	* ≈ 1.00
100	800	0.006739	-0.006741	0.7095	0.6972
100	100	-1.535680	1.703043	0.8145	0.4483
500	100	-0.005024	-0.000920	0.8869	* -0.9296
100	500	0.008502	-0.008526	0.7142	0.6933
600	300	-0.162064	0.170001	0.9162	-0.2401
500	500	0.005565	-0.005842	0.8290	0.5353
500	550	0.006913	-0.007206	0.8197	0.5528
664	477	0.000685	-0.001817	0.0857	0.9987
800	800	0.004627	-0.005930	0.4542	0.8273
800	100	-0.008720	0.009169	0.9797	0.2707
100	800	0.019543	-0.014161	0.7998	0.4923
100	100	-1.299988	1.876096	0.4781	0.7045
500	100	-0.001308	0.001311	0.9818	0.1999
100	500	0.049666	-0.030341	0.8951	-0.0203
600	300	-0.111881	0.117776	0.9042	0.3020
500	500	0.004781	-0.006075	0.4520	0.8318
500	550	0.006899	-0.008233	0.5813	0.7862

Table 2. Application of the SDN algorithm with 1 iteration.

5 Conclusions

The use of a combination of the SDN algorithm and of the partial linearization scheme shall be shown to solve an elastic demand assignment problem that arises in using a consistent method for the approximation of the steepest descent direction of a demand adjustment problem recently shown by Codina and Barceló in [3] at a point \check{g} . This method requires slightly more computational effort than the ones of Spiess in [12] or Yang in [14] based on the link use proportions. Also, a simple test is shown in order to evaluate a direction as a valid descent direction which in general provides good results.

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