CONFINEMENT AROUND THE VERTICAL FAMILY
OF PERIODIC ORBITS CLOSE TO $L_4$ AND $L_5$

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Abstract

We consider the spatial restricted three-body problem (RTBP) for values of the mass parameter close to $\mu_{\text{Routh}}$. We analyse the dynamics around the vertical periodic orbits which are born at the equilateral equilibrium points $L_4$ and $L_5$, and we describe the confinement around the complex-unstable periodic orbits due to the 2D stable and unstable invariant manifolds.

Key words and expressions: periodic orbits - stable and unstable manifolds - invariant tori - Restricted problem.

1 Introduction

Our frame of work is the spatial restricted three-body problem as a particular example of a Hamiltonian system with three degrees of freedom. We consider the equilateral point $L_4$ (or $L_5$) and the Lyapunov family of vertical periodic orbits which are born at the equilibrium point. Our aim is to describe the dynamics of this family when we increase the mass parameter. Since the point $L_4$ changes its (linear) stability at the particular value $\mu_{\text{Routh}}$, we expect a different dynamical behaviour of the periodic orbits when crossing this value of the mass parameter. Actually, this kind of phenomenon (change of stability from a center to a complex saddle) has been analyzed from an analytical point of view (see [12], [5], [1]), [13]) as well as in some particular problems (the elliptic RTBP in [11], some galactic dynamics models in [9], [4], [15] or 4D symplectic mappings in [3], [8], [10] and [14]).

In this communication, we briefly recall the RTBP in section 2, we describe the vertical families of periodic orbits in section 3, and finally we analyse the dynamics close to them in section 4.
2 The 3D restricted three-body problem

We consider a system of three bodies in an inertial (also called sidereal) reference system: two bodies (called primaries) of masses $1 - \mu$ and $\mu$ (in suitable units), describing circular orbits around their common center of mass (the origin of coordinates) in a plane, and a particle of infinitesimal mass which moves in the 3D space under the gravitational effect of the primaries but has negligible effect on their motion. The 3D restricted three-body problem (RTBP) consists of describing the motion of the particle and its evolution along the time. It is well known that the equations of motion in a rotating canonical system of coordinates (that fixes the big and small primaries at points $(\mu, 0, 0)$ and $(\mu - 1, 0, 0)$ respectively) are obtained from a Hamiltonian system of three degrees of freedom, whose Hamiltonian reads as (see Szebehely [17])

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + yp - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2}$$

with the position given by $(x, y, z)$, the momenta by $(p_x, p_y, p_z)$ and where $\mu \in [0, 1/2]$ is the mass parameter and $r_1^2 = (x - \mu)^2 + y^2 + z^2$, $r_2^2 = (x - \mu + 1)^2 + y^2 + z^2$ are the distances between the particle and the big and small primaries respectively. It is also well known that the equations of motion have five equilibrium points: $L_1$, $L_2$ and $L_3$ (the collinear points), located on the $x$ axis, and $L_4$ and $L_5$ (the equilateral ones), located on the points $(\mu - \frac{1}{2}, \pm \frac{\sqrt{3}}{2}, 0)$. Since the dynamics close to $L_4$ is similar to the one near $L_5$, we will focus our attention on $L_4$.

3 The Lyapunov family of vertical periodic orbits

The characteristic exponents of the equilibrium point $L_4$ are (see [17]) $\lambda_{1,2} = \pm ai$, $\lambda_{3,4} = \pm bi$ and $\lambda_{5,6} = \pm i (a > 0, b > 0)$, if $\mu < \mu_{Routh} = \mu_R = 0.0385...$; for $\mu = \mu_R$, $\lambda_{1,2} = \lambda_{3,4} = \pm ci (c > 0)$ and $\lambda_{5,6} = \pm i$, and, finally if $\mu_R < \mu \leq 1/2$, $\lambda_{1,2}$, $\lambda_{3,4}$ are a quadruplet located on the complex plane outside the imaginary axis and $\lambda_{5,6} = \pm i$. Therefore the linear dynamics around $L_4$ on the $z$ axis is described by a harmonic oscillator with frequency 1, for any value of the mass parameter.

On one hand, the Lyapunov center theorem (see [16]) states, under generic hypothesis, the existence of a family of periodic orbits of the nonlinear system, generalizing the linear oscillations to the full (nonlinear) system. This family is called, from now on, the vertical family of periodic orbits of $L_4$, and is born at $L_4$. On the other hand, close to the equilibrium point, the vertical family has the same stability as the point, but it may change as far as the family evolves (see [6]).

Concerning stability of periodic orbits, Broucke (see [2]) listed 6 kinds of instability according to the position of the eigenvalues of the monodromy matrix of a periodic orbit, or equivalently, depending on the stability parameters $\alpha$ and $\beta$ of the characteristic
Figure 1: Projection \((x, y, z)\) of periodic orbits (complex unstable, critical and stable one) of the vertical family for \(\mu = 0.05\).
Figure 2: Broucke’s diagram. The marked points correspond to six selected orbits of the vertical family for $\mu = 0.04$: 1 and 2 are complex-unstable, 3 is the critical one, 4 is stable and 5, 6 are even-semi unstable.

polynomial of the monodromy matrix

$$(s - 1)^2(s^4 + \alpha s^3 + \beta s^2 + \alpha s + 1) = 0.$$  

In Figure 1 we show, as an example, the vertical family of periodic orbits computed for $\mu = 0.05 > \mu_R$: we plot three selected periodic orbits of the family; we can see how the amplitude of the orbit on the z axis grows with $p_z$. On the other hand, we plot Broucke’s diagram in Figure 2 with the stable/unstable character of the vertical family computed.

In fact, our aim is to describe the dynamics close to the vertical family of periodic orbits when $\mu \in (\mu_R - \delta, \mu_R + \delta)$, $\delta > 0$. We show in Figure 3 the stability behaviour of any vertical family of periodic orbits when varying the mass parameter in a neighbourhood of $\mu_R$: for moderate values of $p_z$ ($p_z < 0.8$), and $\mu > \mu_R$ ($\mu$ close to $\mu_R$), the vertical periodic orbits are, as $p_z$ grows starting from 0, complex unstable (four eigenvalues of the monodromy matrix form a quadruple in the complex plane outside the unit circle) as expected, there is a critical orbit (two multiple eigenvalues on the unit circle) and they become (linearly) stable (four eigenvalues on the unit circle) for higher values of $p_z$.

4 Dynamics close to the vertical periodic orbits

In this section we describe the dynamics around the vertical periodic orbits when varying $\mu$ and $p_z$ (or equivalently the Hamiltonian). For this purpose, we consider a Poincaré section ($z = 0, p_z > 0$ in this case), that will be 4-dimensional for each fixed value of the Hamiltonian, and the corresponding Poincaré map.

On one hand, it is clear from Figure 3 that, for a fixed $\mu < \mu_R$ and as we increase $p_z$, the vertical periodic orbit of the Lyapunov family is stable, and there will be families of 2D
Figure 3: Change of stability for the vertical family of periodic orbits of $L_4$. The parameters are $\mu$ and the value of $p_z$ when $z = 0$. 1 corresponds to stable orbits while 2 to complex-unstable ones.

Figure 4: Projection $(x, y)$. Left. 3D torus around a stable vertical periodic orbit. Right. Initial conditions close to a complex-unstable orbit. 10000 consequents by the Poincaré map.
and 3D tori - KAM tori - near it (see [6], [7]). Of course, if we consider an initial condition on a KAM torus, the motion will remain confined for ever; otherwise, the phenomenon of Arnold diffusion can appear since the hamiltonian of the RTBP is non integrable.

On the other hand, if we fix a value $\mu > \mu_R$ and for moderate values of $p_z$, we have either stable or complex-unstable vertical periodic orbits when varying $p_z$ (see Figure 3):

(i) If $p_z$ is big enough, the vertical periodic orbit will be stable and therefore the dynamics will be as the one just described. For example, for $\mu = 0.04$, we consider a stable vertical periodic orbit, with $p_z = 0.22877$, and we compute a KAM torus around it; in Figure 4 (left) we plot the $(x, y)$ projection of 10000 consequents (by the Poincaré map) of an initial condition on a 3D torus and also the periodic orbit which becomes a point on the surface of section.

(ii) Of course, we expect a different behaviour around a complex-unstable vertical periodic orbit, that is, when decreasing $p_z$ below the critical value (for the critical orbit). For example, we fix $\mu = 0.04$ and we consider a CU vertical periodic orbit with $p_z = 0.209297$, close to the critical one ($p_z = 0.216366$). We have plotted the 10000 consequents, by the Poincaré map, of an initial condition very close to the one of the periodic orbit. As Figure 4 (right) shows, the consequents are confined for a very long time in a rather big region (we remark the different size of the region of confinement between the KAM torus -left- and the one around the CU orbit). The explanation of such confinement remains on the behaviour of the invariant manifolds. As it is well known (see [13]), the fixed point (associated with a complex-unstable periodic orbit, through the Poincaré section) has two 2D invariant manifolds (the stable and the unstable ones). For $\mu = 0.04$, we consider the mentioned CU vertical periodic orbit and we have plotted the 100, 2000 and 10000 consequents of an initial condition on the unstable manifold. As Figure 5 shows, the unstable manifold is very close to the stable one and the wandering of the consequents confines the region already observed in Figure 4 (right).
Figure 6: Projection ($x, y$). The confinement is less powerful as $p_z$ decreases to 0: 800 consequents by the Poincaré map. Initial condition close to the one of a complex-unstable orbit.

As $p_z$ decreases, the confinement is less powerful and the region of confinement grows considerably; for example, we show the 800 consequents of an initial condition on the unstable manifold of the fixed point corresponding to a CU periodic orbit with $p_z = 0.019938$. We also point out the growing size of the region of confinement.

Finally, we remark that an interesting and intricate dynamics related to the transition from stability to complex instability is also expected in our case. An analytical study of such transition by means of normal forms, carried out in [12], reveals the Hopf bifurcation of stable 2D tori around the CU orbits. However, this subject of research will be treated in the next future.

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References


