From clutters to matroids

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Abstract

This paper deals with the question of completing a monotone increasing family of subsets $\Gamma$ of a finite set $\Omega$ to obtain the dependent sets of a matroid. Specifically, we provide several natural processes for transforming the clutter $\Lambda$ of the inclusion-minimal subsets of the family $\Gamma$ into the set of circuits $C(M)$ of a matroid $M$ with ground set $\Omega$. In addition, by combining these processes, we prove that all the minimal matroidal completions of the family can be obtained.

Keywords: clutter, hypergraph, matroid, circuits.

1 Introduction

A monotone increasing family of subsets $\Gamma$ of a finite set $\Omega$ is a collection of subsets of $\Omega$ such that any superset of a set in the family $\Gamma$ must be in $\Gamma$. All the inclusion-minimal elements of $\Gamma$ determine a clutter $\Lambda$, that is, a collection of subsets of $\Omega$ none of which is a proper subset of another. Clutters are also known as antichains, Sperner systems or simple hypergraphs.

A wide variety of examples of monotone increasing families exist, among them the collection of the linearly dependent subsets of vectors in a vector space. A matroid $M$ is a combinatorial object that provides an axiomatic abstraction of linear dependence on a finite set $\Omega$. The minimal dependent sets of a matroid $M$ are called circuits. Therefore, the family of circuits of a matroid $M$ is a clutter.

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We say that a clutter is *matroidal* if it corresponds to the family of circuits of a matroid. Matroidal clutters, as well as "almost matroidal" clutters, play a key role in several situations. For instance, in the context of secret-sharing schemes they become a crucial issue for providing general bounds on the optimal information rate of the scheme (see [4, 8]). In the framework of algebraic combinatorics and commutative algebra, other interesting examples can be found that deal with monomial ideals and arithmetic properties of the face ring of simplicial complexes (see [1, 9]).

Since in general a clutter is far from being matroidal, it is of interest to know how it can be transformed into a matroidal one. This paper deals with the question of finding the *matroidal completions of a clutter*.

The outline of the paper is as follows. In Section 2 we recall some definitions and basic facts about clutters and matroids. Several ways to obtain matroidal completions of clutters can be found in Section 3; namely, we present the uniform completions (Proposition 3), the $I$-completions (Proposition 5), and the $T$-completions (Proposition 8). In addition, by means of the clutter transformations involved in these processes, a necessary condition for a clutter to be a matroidal port is obtained (Proposition 9). Finally, Section 4 is devoted to analyzing the minimal matroidal completions. We characterize the clutters with only one minimal element (Theorem 12), and we show how to obtain all the minimal matroidal completions of any clutter (Theorem 13).

## 2 Clutters and matroids

In this section we present the definitions and basic facts concerning families of subsets, clutters and matroids that are used in the paper. (The reader is referred to [6, 10] for general references on matroid theory).

Let $\Omega$ be a finite set. Observe that if $\Gamma$ is a monotone increasing family of subsets of $\Omega$, then the collection $\text{min}(\Gamma)$ of its inclusion-minimal elements is a clutter; while if $\Lambda$ is a clutter on $\Omega$, then the family $\Lambda^+ = \{ A \subseteq \Omega : A_0 \subseteq A \text{ for some } A_0 \in \Lambda \}$ is a monotone increasing family of subsets. Clearly $\Gamma = (\text{min}(\Gamma))^+$ and $\Lambda = \text{min}(\Lambda^+)$. So a monotone increasing family of subsets $\Gamma$ is determined uniquely by the clutter $\text{min}(\Gamma)$, while a clutter $\Lambda$ is determined uniquely by the monotone increasing family $\Lambda^+$.

Despite the foregoing, we must take into account the following lemma concerning the relationship between the inclusion and the equality of two clutters $\Lambda_1$ and $\Lambda_2$, and the inclusion and the equality of their associated monotone increasing families of subsets $\Lambda_1^+$ and $\Lambda_2^+$.

**Lemma 1.** Let $\Lambda_1, \Lambda_2$ be two clutters on a finite set $\Omega$. Then:

1. $\Lambda_1 = \Lambda_2$ if and only if $\Lambda_1^+ = \Lambda_2^+$.
2. If $\Lambda_1 \subseteq \Lambda_2$ then $\Lambda_1^+ \subseteq \Lambda_2^+$. The converse is not true.
3. $\Lambda_1^+ \subseteq \Lambda_2^+$ if and only if $\Lambda_1 \subseteq \Lambda_2^+$. 


Proof. The proofs of the statements are a straightforward consequence of the definition of $\Lambda^+$ and of the fact that $\Lambda = \min \{\Lambda^+\}$. So it is only necessary to present an example of clutters with $\Lambda_1 \not\subseteq \Lambda_2$ and $\Lambda_1^+ \subseteq \Lambda_2^+$. For instance, on the finite set $\Omega = \{1, 2, 3\}$, let us consider the clutters $\Lambda_1 = \{\{1, 2\}, \{2, 3\}\}$ and $\Lambda_2 = \{\{1, 2\}\}$. Then $\Lambda_1 \not\subseteq \Lambda_2$, while $\Lambda_1^+ = \{\{1, 2\}, \{2, 3\}, \{1, 2, 3\}\} \subseteq \{\{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} = \Lambda_2^+$. $\square$

The previous lemma leads us to consider a binary relation $\leq$ defined on the set of clutters on $\Omega$. Namely, if $\Lambda_1$ and $\Lambda_2$ are two clutters on $\Omega$, then we say that $\Lambda_1 \leq \Lambda_2$ if and only if $\Lambda_1^+ \subseteq \Lambda_2^+$. Therefore, $\Lambda_1 \leq \Lambda_2$ if and only if for all $A_1 \in \Lambda_1$ there exists $A_2 \in \Lambda_2$ such that $A_2 \subseteq A_1$. It is clear that the binary relation $\leq$ is reflexive and transitive. Besides, from statement (1) of the previous lemma, the relation $\leq$ is antisymmetric. Therefore, the binary relation $\leq$ is a partial order on the set of clutters of $\Omega$. We will use this partial order throughout the paper.

There are many interesting families of clutters that can be considered. However, because of their applications, we are interested in clutters that provide matroids.

Matroids are combinatorial objects that can be axiomatized in terms of their independent sets, bases, circuits, rank function, flats, or hyperplanes. Here we present the definition in terms of circuits. A matroid $\mathcal{M}$ is an ordered pair $\mathcal{M} = (\Omega, \mathcal{C})$ consisting of a finite set $\Omega$, called the ground set of the matroid, and a family $\mathcal{C}$ of nonempty subsets of $\Omega$ which satisfy the following three conditions:

1. $\emptyset \not\in \mathcal{C}$;
2. if $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$; and
3. if $C_1$ and $C_2$ are distinct members of $\mathcal{C}$ and $x \in C_1 \cap C_2$, then there is a member $C_3$ of $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{x\}$.

The members of the clutter $\mathcal{C}$ are the circuits of the matroid $\mathcal{M}$. We shall often write $\mathcal{C}(\mathcal{M})$ instead of $\mathcal{C}$. The dependent sets of the matroid are the supersets of the circuits, that is, the dependent sets of $\mathcal{M}$ are the members of $\mathcal{C}(\mathcal{M})^+$. Therefore, the set of dependent sets of the matroid is a monotone increasing family of subsets whose inclusion-minimal elements are its circuits. A clutter $\Lambda$ is said to be a matroidal clutter if it is the set of circuits of a matroid, that is, if there exists a matroid $\mathcal{M}_0$ such that $\mathcal{C}(\mathcal{M}_0) = \Lambda$.

Since the set of circuits of a matroid is a clutter on the ground set of the matroid, we can consider the partial order induced by $\leq$ on the set of matroids with ground set $\Omega$. Thereby, if $\mathcal{M}_1$ and $\mathcal{M}_2$ are two matroids with ground set $\Omega$, then we say that $\mathcal{M}_1 \leq \mathcal{M}_2$ if and only if $\mathcal{C}(\mathcal{M}_1) \leq \mathcal{C}(\mathcal{M}_2)$ where $\mathcal{C}(\mathcal{M}_i)$ is the clutter of the circuits of $\mathcal{M}_i$. So, $\mathcal{M}_1 \leq \mathcal{M}_2$ if and only if every circuit of $\mathcal{M}_1$ contains a circuit of $\mathcal{M}_2$. In matroid theory this is equivalent to saying that the identity map on $\Omega$ is a weak map from the matroid $\mathcal{M}_1$ to the matroid $\mathcal{M}_2$ (see [6, Proposition 7.3.11]).
3 Matroidal completions of a clutter

The set of circuits of a matroid is a clutter, but there are clutters on a finite set \( \Omega \) that are not the set of circuits of a matroid with ground set \( \Omega \). So, a natural question that arises at this point is to determine how to complete a clutter \( \Lambda \) to obtain a matroid; that is to say, to transform the clutter \( \Lambda \) into a matroidal clutter.

In order to look for matroidal completions, it is important to take into account the dependent sets of the matroid instead of the circuits. This is due to the fact that, as the following example shows, there exist clutters \( \Lambda \) such that \( \Lambda \not\subseteq C(M) \) for any matroid \( M \).

**Example 2.** Let \( n \geq 4 \). On the finite set \( \Omega = \{1, 2, 3, \ldots, n\} \), we consider the clutter \( \Lambda = \{\{1, 2\}, \{1, 3\}, \{2, 3, \ldots, n\}\} \). Observe that \( (\{1, 2\} \cup \{1, 3\}) \setminus \{1\} = \{2, 3\} \not\subseteq \{2, 3, \ldots, n\} \).

Hence it follows that \( \Lambda \not\subseteq C(M) \) for any matroid \( M \).

The above example leads us to the following definition. Let \( \Lambda \) be a clutter on a finite set \( \Omega \), and let \( M \) be a matroid with ground set \( \Omega \). We say that the matroid \( M \) is a matroidal completion of the clutter \( \Lambda \) if \( \Lambda \subseteq C(M) + \). In other words, \( M \) is a matroidal completion of \( \Lambda \) if and only if every subset \( A \in \Lambda \) is a dependent set in \( M \). From Lemma 1 we get that \( M \) is a matroidal completion of \( \Lambda \) if and only if \( \Lambda \subseteq C(M) \).

We will write \( \Lambda \subseteq M \) instead of \( \Lambda \subseteq C(M) \). The set of all the matroidal completions of a clutter \( \Lambda \) is denoted by \( \text{Mat}(\Lambda) \), that is \( \text{Mat}(\Lambda) = \{M : \Lambda \subseteq C(M)\} \).

Observe that if \( \emptyset \in \Lambda \) then \( \text{Mat}(\Lambda) = \emptyset \). So, from now on, throughout the paper we assume that \( \emptyset \not\in \Lambda \) if \( \Lambda \) is a clutter. As is shown in the next subsection, this assumption guarantees that \( \text{Mat}(\Lambda) \neq \emptyset \) for all clutters.

The aim of this section is to provide three methods in order to obtain matroidal completions of \( \Lambda \); that is, to obtain matroids \( M \) in \( \text{Mat}(\Lambda) \). By combining these methods, the minimal matroidal completions will be studied in Section 4.

3.1 Uniform completion

The following proposition states that the family of uniform matroids provides matroidal completions of clutters. Recall that if \( \Omega \) is a finite set of size \( |\Omega| = \omega \) and if \( m \leq \omega \) is a non-negative integer, then the uniform matroid of rank \( m \) on \( \Omega \) is the matroid \( U_{m,\omega} \) with ground set \( \Omega \) and set of circuits \( C(U_{m,\omega}) = \{C \subseteq \Omega : |C| = m + 1\} \) if \( m < \omega \) and \( C(U_{m,\omega}) = \emptyset \) if \( m = \omega \).

**Proposition 3.** Let \( \Lambda \) be a clutter on a finite set \( \Omega \) of size \( |\Omega| = \omega \). Then, \( U_{m,\omega} \in \text{Mat}(\Lambda) \) if and only if \( m \leq s - 1 \) where \( s = \min\{|A| : A \in \Lambda\} \).

**Proof.** The dependent sets of the uniform matroid \( U_{m,\omega} \) are those subsets \( X \subseteq \Omega \) with \( |X| \geq m + 1 \). Therefore, \( \Lambda \subseteq C(U_{m,\omega})^+ \) if and only if \( |A| \geq m + 1 \) for all \( A \in \Lambda \); that is, if and only if \( s \geq m + 1 \).

It is clear that \( U_{m_1,\omega} \leq U_{m_2,\omega} \) if and only if \( m_1 \geq m_2 \). Hence, the uniform matroids in \( \text{Mat}(\Lambda) \) form a chain \( U_{s-1,\omega} \leq \cdots \leq U_{0,\omega} \leq U_{0,\omega} \) whose minimal element is \( U_{s-1,\omega} \). We will say that \( U_{s-1,\omega} \) is the uniform completion of \( \Lambda \) and it is denoted by \( U(\Lambda) \).
The following example shows that, in general, there are matroids in \( \text{Mat}(\Lambda) \) that are not uniform matroids. Moreover, from the example, it follows that in general the uniform completion \( \mathcal{U}(\Lambda) \) is not a minimal matroidal completion of \( \Lambda \).

**Example 4.** On the finite set \( \Omega = \{1, 2, 3, 4, 5\} \), we consider the clutter \( \Lambda = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}, \{2, 3, 5\}\} \). We have that \( \Lambda \subseteq \mathcal{C}(\mathcal{M})^{+} \) where \( \mathcal{M} \) is the matroid with set of circuits \( \mathcal{C}(\mathcal{M}) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\} \). So \( \mathcal{M} \) is a matroidal completion of \( \Lambda \) which is not a uniform matroid. Observe that the uniform matroids in \( \text{Mat}(\Lambda) \) are \( \mathcal{U}_{1,4} \) and \( \mathcal{U}_{0,4} \), and here \( \mathcal{M} \preceq \mathcal{U}(\Lambda) = \mathcal{U}_{1,4} \preceq \mathcal{U}_{0,4} \).

### 3.2 Completion with intersections: \( I \)-transformations

In this subsection we prove that it is possible to transform a clutter \( \Lambda \) into a matroidal clutter by adding intersections of suitable subsets of \( \Lambda \). To present our result we need to introduce some previous definitions.

Let \( \Lambda \) be a clutter on a finite set \( \Omega \). For a subset \( X \subseteq \Omega \), we denote by \( I_{\Lambda}(X) \) the intersection of the subsets \( A \) in \( \Lambda \) contained in \( X \), (this intersection is the one involved in the characterization of the set of circuits in connected matroids, see [6, Theorem 4.3.2]). We say that a clutter \( \Lambda' \) is an \( I \)-transformation of the clutter \( \Lambda \) if \( \Lambda' = \min \{\Lambda \cup \{A_{1} \cap A_{2}\}\} \) where \( A_{1}, A_{2} \in \Lambda \) are two different subsets with \( I_{\Lambda}(A_{1} \cup A_{2}) \neq \emptyset \).

**Proposition 5.** Let \( \Lambda \) be a clutter on a finite set \( \Omega \). If \( \Lambda \) is not matroidal then there exists a chain of clutters \( \Lambda = \Lambda_{0} \preceq \Lambda_{1} \preceq \cdots \preceq \Lambda_{r} \) such that the clutter \( \Lambda_{i} \) is an \( I \)-intersection of \( \Lambda_{i-1} \) for \( i \geq 1 \) and the clutter \( \Lambda_{r} \) is a matroidal clutter.

**Proof.** First of all notice that if \( A_{1}, A_{2} \in \Lambda \) are different then, \( I_{\Lambda}(A_{1} \cup A_{2}) = \emptyset \) if and only if for all \( x \in A_{1} \cap A_{2} \) there exists \( A_{3} \in \Lambda \) with \( A_{3} \subseteq (A_{1} \cup A_{2}) \setminus \{x\} \). Therefore we get that a clutter \( \Lambda \) is a matroidal clutter if and only if \( I_{\Lambda}(A_{1} \cup A_{2}) = \emptyset \) for any two different \( A_{1}, A_{2} \in \Lambda \).

The proof of the proposition follows from this equivalence. Indeed, if \( \Lambda \) is not a matroidal clutter, then we get that there exist two different subsets \( A_{1}, A_{2} \in \Lambda \) with \( I_{\Lambda}(A_{1} \cup A_{2}) \neq \emptyset \). So we can consider the clutter \( \Lambda_{1} = \min \{\Lambda \cup \{A_{1} \cap A_{2}\}\} \), which is an \( I \)-intersection of \( \Lambda \). Clearly \( \Lambda \preceq \Lambda_{1} \) because \( A_{1} \) and \( A_{2} \) are different. We now proceed in the same way with the clutter \( \Lambda_{1} \), and so on. \( \square \)

Therefore, by means of \( I \)-transformations we can transform a clutter \( \Lambda \) into a matroidal clutter. The matroids obtained in this way will be called \( I \)-matroidal completions of \( \Lambda \). Next we present some examples to show that it no general result exists concerning the comparison between two different \( I \)-matroidal completions of a clutter.

**Example 6.** On the finite set \( \Omega = \{1, 2, 3, 4, 5\} \), we consider the clutters \( \Lambda_{1}, \Lambda_{2} \) and \( \Lambda_{3} \) where \( \Lambda_{1} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{3, 4, 5\}\} \), \( \Lambda_{2} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 5\}, \{2, 5\}\} \) and \( \Lambda_{3} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{4, 5\}\} \). It is clear that \( \Lambda_{1} \) has only one \( I \)-matroidal completion, the matroid \( \mathcal{M}_{1} \) with set of circuits \( \mathcal{C}(\mathcal{M}_{1}) = \{\{1, 2\}, \{3, 4, 5\}\} \). The two \( I \)-matroidal completions of the clutter \( \Lambda_{2} \) are the matroids \( \mathcal{M}_{2,1} \) and \( \mathcal{M}_{2,2} \) with set
of circuits $C(M_{2,1}) = \{\{1,2\}, \{1,5\}, \{2,5\}\}$ and $C(M_{2,2}) = \{\{1,2\}, \{5\}\}$. The two $I$-matroidal completions of $\Lambda_3$ are the matroids $M_{3,1}$ and $M_{3,2}$ with set of circuits $C(M_{3,1}) = \{\{1,2\}, \{4,5\}\}$ and $C(M_{3,2}) = \{\{1,2,3\}, \{4\}\}$. Therefore the $I$-matroidal completions of $\Lambda_3$ are not comparable, that is, $M_{3,1} \not\leq M_{3,2}$ and $M_{3,2} \not\leq M_{3,1}$.

### 3.3 Completion with unions: $T$-transformations

The aim of this subsection is to present some natural ways to obtain matroidal completions of a clutter $\Lambda$, that is, to obtain matroids in $\text{Mat}(\Lambda)$. Unlike in the previous subsection, here we proceed in a recursive way by adding, in each step of the process, some slight modifications of the union of two distinct elements of the clutter. Our result is stated in Proposition 8, and by using these matroidal completions a necessary condition for matroid ports is presented in Proposition 9. Let us start by defining the two elementary transformations involved in the recursive process.

Let $\Lambda$ be a clutter on a finite set $\Omega$. We define the elementary transformations $T^{(1)}(\Lambda)$ and $T^{(2)}(\Lambda)$ of $\Lambda$ as the clutters:

- $T^{(1)}(\Lambda) = \min (\Lambda \cup \{(A_1 \cup A_2) \setminus \{x\} \text{ where } A_1, A_2 \in \Lambda \text{ are different and } x \in A_1 \cap A_2\})$; that is, in the first elementary transformation $T^{(1)}(\Lambda)$, we consider the minimal elements of the family obtained by adding to $\Lambda$ those subsets that arise from the circuit condition.

- $T^{(2)}(\Lambda) = \min (\Lambda \cup \{(A_1 \cup A_2) \setminus I_\Lambda(A_1 \cup A_2), \text{ where } A_1, A_2 \in \Lambda \text{ are different}\})$; that is, in the second elementary transformation $T^{(2)}(\Lambda)$, we add to $\Lambda$ the subsets obtained from the union after removing the intersections $I_\Lambda(X)$ defined in the previous subsection.

Since $T^{(1)}(\Lambda)$ and $T^{(2)}(\Lambda)$ are clutters, we can apply the elementary transformations again. Hence, for $(i_1, i_2) \in \{1,2\} \times \{1,2\}$ we can consider the clutter $T^{(i_2)}(T^{(i_1)}(\Lambda))$. At this point we proceed in a recursive way. Let $r \geq 2$ be a non-negative integer and let $(i_1, \ldots, i_r) \in \{1,2\}^r$ be an $r$-tuple. Then we define the clutter $T^{(i_1, \ldots, i_r)}(\Lambda)$ by the recursion formula $T^{(i_1, \ldots, i_r)}(\Lambda) = T^{(i_r)}(T^{(i_1, \ldots, i_{r-1})}(\Lambda))$; that is, $T^{(i_1, \ldots, i_r)}(\Lambda)$ is the $i_r$ elementary transformation of $T^{(i_1, \ldots, i_{r-1})}(\Lambda)$.

In this way we obtain the following tree diagram of $T$-transformations of the clutter $\Lambda$, where we write $(i_1, \ldots, i_r)$ instead of $T^{(i_1, \ldots, i_r)}(\Lambda)$:
We will say that a clutter $\Lambda'$ is a $\mathcal{T}$-transformation of $\Lambda$ if it is obtained from $\Lambda$ in this way, that is, if $\Lambda' = T^{(i_1, \ldots, i_r)}(\Lambda)$ for some $r$-tuple $(i_1, \ldots, i_r)$. The next lemma points out the relationship between two $\mathcal{T}$-transformations, that is, between two clutters of the above diagram. The first statement of the lemma deals with the relationship between clutters in each branch of the diagram, whereas the last two statements deal with the comparison of clutters in a same row of the diagram, that is, the $2^r$ possible clutters $T^{(i_1, \ldots, i_r)}(\Lambda)$.

**Lemma 7.** Let $\Lambda$ be a clutter on a finite set $\Omega$. Let $\mathcal{I} = (i_1, i_2, i_3, \ldots, i_s, i_{s+1}, \ldots)$ be a sequence with $i_s \in \{1, 2\}$, and let $(j_1, \ldots, j_r), (k_1, \ldots, k_r) \in \{1, 2\}^r$ be two $r$-tuples. Then, the following statements hold:

1. $\Lambda \leq T^{(i_1)}(\Lambda) \leq T^{(i_1, i_2)}(\Lambda) \leq \cdots \leq T^{(i_1, \ldots, i_s)}(\Lambda) \leq T^{(i_1, \ldots, i_s, i_{s+1})}(\Lambda) \leq \cdots$, and there exists $r \geq 1$ such that $T^{(i_1, \ldots, i_r)}(\Lambda) = T^{(i_1, \ldots, i_r, i_{r+1})}(\Lambda)$.

2. If $(j_1, \ldots, j_{r-1}) = (k_1, \ldots, k_{r-1})$ and $j_r \leq k_r$, then $T^{(j_1, \ldots, j_r)}(\Lambda) \leq T^{(k_1, \ldots, k_r)}(\Lambda)$. In general this inequality is not an equality.

3. If $(j_1, \ldots, j_r) \neq (k_1, \ldots, k_r)$ then, in general, there is no relationship between the clutters $T^{(j_1, \ldots, j_r)}(\Lambda)$ and $T^{(k_1, \ldots, k_r)}(\Lambda)$.

**Proof.** From the definitions of the two elementary transformations it follows that if $\Lambda_0$ is a clutter on $\Omega$ then $\Lambda_0 \leq T^{(1)}(\Lambda_0)$ and $\Lambda_0 \leq T^{(2)}(\Lambda_0)$. Therefore, we have that the iteration of the elementary transformations provides a monotone increasing sequence of clutters $\Lambda \leq T^{(i_1)}(\Lambda) \leq T^{(i_1, i_2)}(\Lambda) \leq \cdots \leq T^{(i_1, \ldots, i_s)}(\Lambda) \leq T^{(i_1, \ldots, i_s, i_{s+1})}(\Lambda) \leq \cdots$. The proof of statement (1) is concluded by noticing that there are only a finite number of clutters in a finite set.

Next let us prove statement (2). It is necessary to prove the inequality $T^{(i_1, \ldots, i_{r-1}, 1)}(\Lambda) \leq T^{(i_1, \ldots, i_{r-1}, 2)}(\Lambda)$ and, in addition, we must also show that in general this inequality is not an equality. Clearly, to do this it is enough to prove that if $\Lambda_0$ is a clutter on the finite set $\Omega$ then $T^{(1)}(\Lambda_0) \leq T^{(2)}(\Lambda_0)$ and, in addition, we must show that there are clutters $\Lambda_0$ of $\Omega$ with $T^{(1)}(\Lambda_0) \neq T^{(2)}(\Lambda_0)$.
First let us show that $\mathcal{T}(1)(\Lambda_0) \leq \mathcal{T}(2)(\Lambda_0)$; that is, we must demonstrate that if $X \in \mathcal{T}(1)(\Lambda_0)$, then there exists $X' \in \mathcal{T}(2)(\Lambda_0)$ such that $X' \subseteq X$. So let $X \in \mathcal{T}(1)(\Lambda_0) = \min \{ \Lambda_0 \cup \{ (A_1 \cup A_2) \setminus \{ x \} \text{ where } A_1, A_2 \in \Lambda_0 \text{ are different and } x \in A_1 \cap A_2 \} \}$. If $X = A \in \Lambda_0$, then from the definition of $\mathcal{T}(2)(\Lambda_0)$, it follows that there exists $X' \in \mathcal{T}(2)(\Lambda_0)$ with $X' \subseteq X$, as we wanted to prove. Therefore, we may assume now that $X = (A_1 \cup A_2) \setminus \{ x \}$ where $A_1$ and $A_2$ are two distinct members of $\Lambda_0$ and where $x \in A_1 \cap A_2$. At this point we distinguish two cases. First suppose that $x \in I_{\Lambda_0}(A_1 \cup A_2)$. In such a case we have that $(A_1 \cup A_2) \setminus I_{\Lambda_0}(A_1 \cup A_2) \subseteq (A_1 \cup A_2) \setminus \{ x \}$ and from the definition of $\mathcal{T}(2)(\Lambda_0)$ we get that there exists $X' \in \mathcal{T}(2)(\Lambda_0)$ with $X' \subseteq (A_1 \cup A_2) \setminus I_{\Lambda_0}(A_1 \cup A_2)$. Hence, if $x \in I_{\Lambda_0}(A_1 \cup A_2)$, then there exists $X' \in \mathcal{T}(2)(\Lambda_0)$ with $X' \subseteq X$. Now assume that $x \notin I_{\Lambda_0}(A_1 \cup A_2)$. Then from the definition of $I_{\Lambda_0}(A_1 \cup A_2)$ we get that there exists $A_3 \subseteq \Lambda_0$ such that $A_3 \subseteq (A_1 \cup A_2) \setminus \{ x \}$. Since $A_3 \in \Lambda_0$, there exists $X' \in \mathcal{T}(2)(\Lambda_0)$ with $X' \subseteq A_3$. Therefore, if $x \notin I_{\Lambda_0}(A_1 \cup A_2)$, there exists $X' \in \mathcal{T}(2)(\Lambda_0)$ with $X' \subseteq X$. Thus, in both cases we conclude that there exists $X' \in \mathcal{T}(2)(\Lambda_0)$ such that $X' \subseteq X$, that is, $\mathcal{T}(1)(\Lambda_0) \leq \mathcal{T}(2)(\Lambda_0)$, as we wanted to prove.

To finish the proof of statement (2) we must show that there are clutters $\Lambda_0$ with $\mathcal{T}(1)(\Lambda_0) \neq \mathcal{T}(2)(\Lambda_0)$. Let us consider the clutter $\Lambda_0 = \{ \{ 1, 2, 3 \}, \{ 1, 2, 4, \ldots , n \} \}$ of the finite set $\Omega = \{ 1, 2, 3, 4, \ldots , n \}$ where $n \geq 4$. Then we have that $\mathcal{T}(1)(\Lambda_0) = \{ \{ 1, 2, 3 \}, \{ 1, 2, 4, \ldots , n \}, \{ 1, 3, 4, \ldots , n \}, \{ 2, 3, 4, \ldots , n \} \}$, whereas $\mathcal{T}(2)(\Lambda_0) = \{ \{ 1, 2, 3 \}, \{ 1, 2, 4, \ldots , n \}, \{ 3, 4, \ldots , n \} \}$. So in this case we have that $\mathcal{T}(1)(\Lambda_0) \nsubseteq \mathcal{T}(2)(\Lambda_0)$. This completes the proof of statement (2).

To finish, let us prove statement (3); namely, we are going to show that in general neither the lexicographic order nor the reverse lexicographic order between $r$-tuples is preserved by applying $\mathcal{T}$-transformations and, moreover, we prove that there exist clutters $\Lambda$ for which the $\mathcal{T}$-transformations $\mathcal{T}^{(i_1, \ldots, i_r)}(\Lambda)$ and $\mathcal{T}^{(j_1, \ldots, j_r)}(\Lambda)$ are not comparable. On the finite set $\Omega = \{ 1, 2, 3, 4, 5 \}$, we consider the clutters $\Lambda_1 = \{ \{ 1, 2 \}, \{ 1, 3 \}, \{ 3, 4 \}, \{ 4, 5 \} \}$, $\Lambda_2 = \{ \{ 1, 2 \}, \{ 1, 4 \}, \{ 3, 5 \} \}$ and $\Lambda_3 = \{ \{ 1, 2 \}, \{ 2, 3 \}, \{ 3, 4 \} \}$. For the clutter $\Lambda_1$ we have that $\mathcal{T}^{(1,2)}(\Lambda_1) = \{ \{ 1, 5 \}, \{ 4, 5 \}, \{ 1, 2, 3 \}, \{ 1, 2, 4 \}, \{ 1, 3, 4 \}, \{ 2, 3, 4 \}, \{ 2, 3, 5 \} \}$ and that $\mathcal{T}^{(2,1)}(\Lambda_1) = \{ \{ 1, 4 \}, \{ 1, 5 \}, \{ 4, 5 \}, \{ 1, 2, 3 \}, \{ 2, 3, 4 \}, \{ 2, 3, 5 \} \}$, and therefore we have the inequality $\mathcal{T}^{(1,2)}(\Lambda_1) \leq \mathcal{T}^{(2,1)}(\Lambda_1)$. However, since the transformations (1, 2) and (2, 1) of the clutter $\Lambda_2$ are $\mathcal{T}^{(1,2)}(\Lambda_2) = \{ \{ 2, 4 \}, \{ 3, 5 \}, \{ 1, 2, 3 \}, \{ 1, 2, 5 \}, \{ 1, 3, 4 \}, \{ 1, 4, 5 \} \}$ and $\mathcal{T}^{(2,1)}(\Lambda_2) = \{ \{ 3, 5 \}, \{ 1, 2, 3 \}, \{ 1, 2, 4 \}, \{ 1, 2, 5 \}, \{ 1, 3, 4 \}, \{ 1, 4, 5 \}, \{ 2, 3, 4 \}, \{ 2, 4, 5 \} \}$, then $\mathcal{T}^{(2,1)}(\Lambda_2) \leq \mathcal{T}^{(1,2)}(\Lambda_2)$. On the other hand, the $\mathcal{T}$-transformations (1, 2) and (2, 1) of the clutter $\Lambda_3$ are $\mathcal{T}^{(1,2)}(\Lambda_3) = \{ \{ 1, 5 \}, \{ 1, 2, 3 \}, \{ 1, 2, 4 \}, \{ 1, 3, 4 \}, \{ 2, 3, 4 \}, \{ 2, 3, 5 \}, \{ 2, 4, 5 \}, \{ 3, 4, 5 \} \}$ and $\mathcal{T}^{(2,1)}(\Lambda_3) = \{ \{ 1, 4 \}, \{ 2, 5 \}, \{ 1, 2, 3 \}, \{ 1, 3, 5 \}, \{ 2, 3, 4 \}, \{ 3, 4, 5 \} \}$, and so we have that $\mathcal{T}^{(1,2)}(\Lambda_3) \nsubseteq \mathcal{T}^{(2,1)}(\Lambda_3)$ and that $\mathcal{T}^{(2,1)}(\Lambda_3) \nsubseteq \mathcal{T}^{(1,2)}(\Lambda_3)$. This completes the proof of the lemma.

Now let us consider the “stable” value $\mathcal{T}^*_{\mathcal{I}}(\Lambda)$ of the monotone increasing sequence of clutters $\Lambda \leq \mathcal{T}^{(i_1)}(\Lambda) \leq \cdots \leq \mathcal{T}^{(i_1, \ldots, i_r)}(\Lambda) \leq \mathcal{T}^{(i_1, \ldots, i_r, i_{r+1})}(\Lambda) \leq \cdots$ obtained from $\Lambda$ by using the sequence $\mathcal{I} = (i_1, i_2, \ldots , i_r, i_{r+1}, \ldots )$ where $i_{\ell} \in \{ 1, 2 \}$; that is, if
\[ r_0 = \min \{ r : \mathcal{T}^{(i_1, \ldots, i_r)}(\Lambda) = \mathcal{T}^{(i_1, \ldots, i_{r+1})}(\Lambda) \}, \]

\[
\begin{align*}
\mathcal{T}_s^{(2)}(\Lambda) &= \Lambda & \text{if } \mathcal{T}^{(i_1)}(\Lambda) &= \Lambda, \\
\mathcal{T}_s^{(2)}(\Lambda) &= \mathcal{T}^{(i_1, \ldots, i_r)}(\Lambda) & \text{if } \mathcal{T}^{(i_1)}(\Lambda) \neq \Lambda.
\end{align*}
\]

The main result of this subsection is the following proposition, which states that for any sequence \( I \) the clutter \( T_s^{(2)}(\Lambda) \) defines a matroid.

**Proposition 8.** Let \( \Lambda \) be a clutter on a finite set \( \Omega \). Let \( I = (i_1, i_2, i_3, \ldots) \) be a sequence where \( i_\ell \in \{1, 2\} \). Then, the clutter \( T_s^{(2)}(\Lambda) \) is a matroidal clutter with \( \Lambda \leq T_s^{(2)}(\Lambda) \) if \( \Lambda \) is a clutter on a finite set \( \Omega \). Furthermore, if \( r_0 = \min \{ r : \mathcal{T}^{(i_1, \ldots, i_r)}(\Lambda) = \mathcal{T}^{(i_1, \ldots, i_{r+1})}(\Lambda) \} \), then for \( s \leq r_0 - 1 \) the clutter \( T^{(i_1, \ldots, i_s)}(\Lambda) \) is not matroidal.

**Proof.** From Lemma 7 we get that \( \Lambda \leq T^{(i_1, \ldots, i_\ell)}(\Lambda) \) for any \( \ell \)-tuple \( (i_1, \ldots, i_\ell) \). Therefore, \( \Lambda \leq T_s^{(2)}(\Lambda) \). Recall that by definition \( T^{(i_1, \ldots, i_\ell)}(\Lambda) = T^{(i_1)}(T^{(i_2, \ldots, i_{\ell-1})}(\Lambda)) \). Therefore, to prove the proposition it is enough to demonstrate that if \( \Lambda_0 \) is a clutter on \( \Omega \), then \( \Lambda_0 = T^{(1)}(\Lambda_0) \) if and only if \( \Lambda_0 \) is a matroidal clutter, and that \( \Lambda_0 = T^{(2)}(\Lambda_0) \) if and only if \( \Lambda_0 \) is a matroidal clutter. In other words, we must prove that \( \Lambda_0 = T^{(2)}(\Lambda_0) \) if and only if \( \Lambda_0 = T^{(1)}(\Lambda_0) \), if and only if \( \Lambda_0 \) is a matroidal clutter.

First let us show that if \( \Lambda_0 = T^{(2)}(\Lambda_0) \) then \( \Lambda_0 = T^{(1)}(\Lambda_0) \). By applying statements (1) and (2) of Lemma 7 we get that \( \Lambda_0 \leq T^{(1)}(\Lambda_0) \) and that \( T^{(1)}(\Lambda_0) \leq T^{(2)}(\Lambda_0) \). Hence, if \( \Lambda_0 = T^{(2)}(\Lambda_0) \), then \( \Lambda_0 \leq T^{(1)}(\Lambda_0) \leq T^{(2)}(\Lambda_0) = \Lambda_0 \) and so \( \Lambda_0 = T^{(1)}(\Lambda_0) \).

Now let us show that if \( \Lambda_0 = T^{(1)}(\Lambda_0) \) then \( \Lambda_0 \) is a matroidal clutter; that is, we must demonstrate that \( \Lambda_0 \) satisfies the conditions of the set of circuits of a matroid. So let \( A_1, A_2 \in \Lambda_0 \) be different and let \( x \in A_1 \cap A_2 \). Since \( \Lambda_0 = T^{(1)}(\Lambda_0) \), there exists \( A_3 \in \Lambda_0 \) such that \( A_3 \subseteq (A_1 \cup A_2) \setminus \{x\} \). Therefore, the circuit conditions are fulfilled.

Finally it is necessary to demonstrate that if \( \Lambda_0 \) is a matroidal clutter then \( \Lambda_0 = T^{(2)}(\Lambda_0) \). Recall that in the proof of Proposition 5 it was stated that a clutter \( \Lambda \) is a matroidal clutter if and only if \( I_\Lambda(A_1 \cup A_2) = \emptyset \) for any two different \( A_1, A_2 \in \Lambda \). Therefore, if the clutter \( \Lambda_0 \) is a matroidal clutter, then \( I_\Lambda(\Lambda_0) = \emptyset \) if \( A_1, A_2 \in \Lambda_0 \) are different, and so \( T^{(2)}(\Lambda_0) = \Lambda_0 \). This completes the proof of the proposition.

In some way, the stable value of the above proposition indicates how far \( \Lambda \) is from being a matroid. For instance, from the above proposition it follows that a clutter \( \Lambda \) is the set of circuits of a matroid with ground set \( \Omega \) if and only if there exists a sequence \( I \) such that \( T_s^{(2)}(\Lambda) = \Lambda \).

A matroid is said to be **connected** if for every pair of distinct elements of the ground set, there is a circuit containing both. A clutter \( \Lambda \) of a finite set \( \Omega \) is said to be a **matroidal port** if it corresponds to the set of circuits of a connected matroid containing a fixed point, that is, if there exists a connected matroid \( \mathcal{N} \) with ground set \( \Omega \cup \{\omega_0\} \), where \( \omega_0 \not\in \Omega \), such that \( \Lambda = \{C \setminus \{\omega_0\} : \omega_0 \in C \in \mathcal{C}(\mathcal{N})\} \). In such a case it is said that the clutter \( \Lambda \) is the **port** of the connected matroid \( \mathcal{N} \) at the point \( \omega_0 \).

Matroid ports were introduced by Lehman [2] to solve the Shannon switching game. There are several characterizations of these combinatorial objects which range from ex-

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cluding minors \([3, 7]\) to optimal information rates in secret-sharing schemes [4]. In addition, a combinatorial necessary condition for a clutter to be a matroid port was stated in [5]. Here we present a necessary condition in terms of clutter transformations.

**Proposition 9.** Let \(\Lambda\) be a clutter on a finite set \(\Omega\). Assume that \(\Lambda\) is a matroid port. Then, \(T^{(2)}(\Lambda) = T^{(i_1)}(\Lambda)\) for any sequence \(I = (i_1, i_2, \ldots)\) with \(i_\ell \in \{1, 2\}\) and \(i_1 = 2\).

**Proof.** By assumption, \(\Lambda\) is a matroid port, so there exists a connected matroid \(N\) with ground set \(\Omega \cup \{\omega_0\}\), where \(\omega_0 \not\in \Omega\), such that \(\Lambda = \{C \setminus \{\omega_0\} : \omega_0 \in C \in C(N)\}\). Let us denote \(\Lambda = \{A_1, \ldots, A_r\}\). On one hand, since the clutter \(\Lambda\) is the port of the matroid \(N\) at the point \(w_0\), the circuits of \(N\) containing the point \(w_0\) are \(\{A_1 \cup \{w_0\}, \ldots, A_r \cup \{w_0\}\}\).

On the other hand, by applying [6, Theorem 4.3.2] it follows that the circuits of \(N\) not containing \(w_0\) are the minimal elements of the form \((A_1 \cup A_2) \setminus I_\Lambda(A_1 \cup A_2)\), where \(A_1, A_2 \in \Lambda\) are distinct. Therefore we have that \(C(N) = \{A_1 \cup \{w_0\}, \ldots, A_r \cup \{w_0\}\} \cup \min\{(A_1 \cup A_2) \setminus I_\Lambda(A_1 \cup A_2)\}, \text{ where } A_1, A_2 \in \Lambda\) are distinct\).

At this point let us consider the matroid \(N/\{\omega_0\}\) obtained by the contraction of the subset \(\{w_0\}\) from the matroid \(N\) (see [6, page 104] for the definition of contraction). From [6, Proposition 3.1.11] we know that the circuits of the matroid \(N/\{\omega_0\}\) are the minimal non-empty members of \(\{C \setminus \{\omega_0\}, \text{ where } C \in C(N)\}\). Hence it follows that \(C(N/\{\omega_0\}) = \min(\Lambda \cup \{A_1 \cup A_2 \setminus I_\Lambda(A_1 \cup A_2)\}, \text{ where } A_1, A_2 \in \Lambda\) are different\); that is, we get that \(C(N/\{\omega_0\}) = T^{(2)}(\Lambda)\). In particular, this equality implies that the clutter \(T^{(2)}(\Lambda)\) is a matroidal clutter. So, by applying Proposition 8 to the clutter \(T^{(2)}(\Lambda)\) it follows that \(T^{(2)}(\Lambda) = T^{(2)}(\Lambda)\) for any sequence \(J = (j_1, j_2, \ldots)\) with \(j_\ell \in \{1, 2\}\).

Therefore we conclude that \(T^{(2)}(\Lambda) = T^{(i_1)}(\Lambda)\) for any sequence \(I = (i_1, i_2, \ldots)\) with \(i_\ell \in \{1, 2\}\) and \(i_1 = 2\), as we wished to prove. \(\square\)

The clutters in the following example show that the necessary condition of the above proposition is not sufficient, and that there is no analogous result if we use the first elementary transformation instead of the second one.

**Example 10.** Let us consider the clutter \(\Lambda = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}\}\) of the finite set \(\Omega = \{1, 2, 3, 4\}\). As shown in [7], this clutter is not a matroid port. However, it is not hard to check that \(T^{(2)}(\Lambda) = T^{(2)}(\Lambda) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}\) for any sequence \(I = (i_1, i_2, \ldots)\). Therefore, the necessary condition of Proposition 9 is not sufficient. Now, in order to prove that the proposition does not work with the first elementary \(T\)-transformation, we consider the clutter \(\Lambda = \{\{1, 2, 3\}, \{2, 3, 4, 5\}, \{2, 3, 5, 6\}\}\) of the finite set \(\Omega = \{1, 2, 3, 4, 5, 6\}\). Observe that \(\Lambda = \{C \setminus \{7\} : 7 \in C \in C(N)\}\), where \(N\) is the connected matroid with ground set \(\Omega \cup \{7\}\) and set of circuits \(C(N) = \{\{1, 2, 3, 7\}, \{2, 3, 4, 5, 7\}, \{2, 3, 5, 6, 7\}, \{1, 4, 5\}, \{1, 5, 6\}, \{4, 6\}\}\). Therefore, the clutter \(\Lambda\) is a matroidal clutter. However, it is straightforward to check that the first elementary transformation of the clutter \(\Lambda\) is \(T^{(1)}(\Lambda) = \{\{1, 2, 3\}, \{1, 2, 4, 5\}, \{1, 2, 5, 6\}, \{1, 3, 4, 5\}, \{1, 3, 5, 6\}, \{2, 3, 4, 5\}, \{2, 3, 4, 6\}, \{2, 3, 5, 6\}, \{2, 4, 5, 6\}, \{3, 4, 5, 6\}\). Observe that \(T^{(1)}(\Lambda)\) is not a matroidal-clutter because \(\{\{1, 2, 3\} \cap \{2, 3, 4, 6\}\} \setminus \{3\} = \{1, 2, 4, 6\} \not\in T^{(1)}(\Lambda)\). Therefore, by applying Proposition 8 it follows that \(T^{(2)}(\Lambda) \neq T^{(i_1)}(\Lambda)\) for any sequence \(I = (i_1, i_2, \ldots)\) with \(i_\ell \in \{1, 2\}\) and \(i_1 = 1\).
To conclude this subsection, we focus our attention on the matroid implicit on Proposition 8. Let us denote by $\mathcal{M}(\Lambda)$ the unique matroid with ground set $\Omega$ and set of circuits $\mathcal{C}(\mathcal{M}(\Lambda)) = T(\Lambda)$. Since $\Lambda \leq \mathcal{T}(\Lambda)$, the matroid $\mathcal{M}(\Lambda)$ is a matroidal completion of $\Lambda$; that is, $\mathcal{M}(\Lambda) \in \text{Mat}(\Lambda)$. The matroids obtained in this way will be called $\mathcal{T}$-matroidal completions of $\Lambda$.

Let us show that there exists no general result concerning the comparison between two different $\mathcal{T}$-matroidal completions of a clutter. The three clutters in the following example illustrate this fact.

**Example 11.** On the finite set $\Omega = \{1, 2, 3, 4, 5\}$ let us consider the clutters $\Lambda_1, \Lambda_2$ and $\Lambda_3$ where $\Lambda_1 = \{\{1, 2, 3\}, \{2, 3, 4, 5\}\}$, $\Lambda_2 = \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}\}$ and $\Lambda_3 = \{\{1, 2, 3\}, \{1, 2, 4\}, \{3, 4, 5\}\}$. It is a straightforward calculation to check that the clutter $\Lambda_1$ has only two $\mathcal{T}$-matroidal completions $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,2}$, while for $i = 2, 3$ the clutter $\Lambda_i$ has exactly three $\mathcal{T}$-matroidal completions $\mathcal{M}_{i,1}$, $\mathcal{M}_{i,2}$ and $\mathcal{M}_{i,3}$. Namely, if $\mathcal{I} = (i_1, i_2, i_3, \ldots)$ is a sequence with $i_\ell \in \{1, 2\}$, then:

- the $\mathcal{T}$-matroidal completions of the clutter $\Lambda_1$ are $\mathcal{M}^{(\mathcal{T})}(\Lambda_1) = \mathcal{M}_{1,1}$ if $i_1 = 1$ while $\mathcal{M}^{(\mathcal{T})}(\Lambda_1) = \mathcal{M}_{1,2}$ if $i_1 \neq 1$, where $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,2}$ are the matroids with ground set $\Omega$ and circuits $\mathcal{C}(\mathcal{M}_{1,1}) = \{\{1, 2, 3\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}\}$ and $\mathcal{C}(\mathcal{M}_{1,2}) = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 3, 4, 5\}\}$.

- the $\mathcal{T}$-matroidal completions of $\Lambda_2$ are $\mathcal{M}^{(\mathcal{T})}(\Lambda_2) = \mathcal{M}_{2,1} = \mathcal{U}_{2,5}$ the uniform matroid if $\mathcal{I} = (1, 1, i_3, \ldots)$; while if $\mathcal{I} = (1, 2, i_3, \ldots)$ then $\mathcal{M}^{(\mathcal{T})}(\Lambda_2) = \mathcal{M}_{2,2}$ is the matroid with set of circuits $\mathcal{C}(\mathcal{M}_{2,2}) = \{\{1, 5\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\}$ whereas if $\mathcal{I} = (2, i_2, i_3, \ldots)$ then $\mathcal{M}^{(\mathcal{T})}(\Lambda_2) = \mathcal{M}_{2,3}$ is the matroid with set of circuits $\mathcal{C}(\mathcal{M}_{2,3}) = \{\{1, 4\}, \{2, 5\}, \{1, 2, 3\}, \{1, 3, 5\}, \{2, 3, 4\}, \{3, 4, 5\}\}$.

- the $\mathcal{T}$-matroidal completions of $\Lambda_3$ are $\mathcal{M}^{(\mathcal{T})}(\Lambda_3) = \mathcal{M}_{3,1} = \mathcal{U}_{2,5}$ the uniform matroid if $\mathcal{I} = (1, 1, i_3, \ldots)$; while if $\mathcal{I} = (1, 2, i_3, \ldots)$ then $\mathcal{M}^{(\mathcal{T})}(\Lambda_3) = \mathcal{M}_{3,2}$ is the matroid with circuits $\mathcal{C}(\mathcal{M}_{3,2}) = \{\{1, 2\}, \{1, 5\}, \{2, 5\}, \{1, 3, 4\}, \{2, 3, 4\}, \{3, 4, 5\}\}$ whereas if $\mathcal{I} = (2, i_2, i_3, \ldots)$ then $\mathcal{M}^{(\mathcal{T})}(\Lambda_3) = \mathcal{M}_{3,3}$ is the matroid with set of circuits $\mathcal{C}(\mathcal{M}_{3,3}) = \{\{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\}$.

Then we have that the $\mathcal{T}$-matroidal completions of $\Lambda_1$ form a chain $\mathcal{M}_{1,1} \leq \mathcal{M}_{1,2}$, while there are $\mathcal{T}$-matroidal completions of the clutters $\Lambda_2$ and $\Lambda_3$ that are not comparable. To be precise, for the clutter $\Lambda_2$ we have $\mathcal{M}_{2,1} \leq \mathcal{M}_{2,2}$ and $\mathcal{M}_{2,1} \leq \mathcal{M}_{2,3}$, but $\mathcal{M}_{2,2} \not\leq \mathcal{M}_{2,3}$ and $\mathcal{M}_{2,3} \not\leq \mathcal{M}_{2,2}$, while for the clutter $\Lambda_3$ we have $\mathcal{M}_{3,1} \leq \mathcal{M}_{3,2}$, but $\mathcal{M}_{3,1} \not\leq \mathcal{M}_{3,3}$ and $\mathcal{M}_{3,3} \not\leq \mathcal{M}_{3,1}$ for $i = 1, 2$. Observe that the clutters $\Lambda_1$ and $\Lambda_2$ have only one minimal $\mathcal{T}$-matroidal completion, but the clutter $\Lambda_3$ has two minimal $\mathcal{T}$-matroidal completions.

## 4 Minimal matroidal completions of a clutter

The set $\text{Mat}(\Lambda)$ of all the matroidal completions of a clutter $\Lambda$ is a non-empty partially ordered set, the *poset of matroids of the clutter $\Lambda$*. Therefore, the minimal elements of
this poset will be the minimal matroidal completions of the clutter. In this section we present two results concerning minimal matroidal completions. The first one (Theorem 12) deals with the number of minimal matroidal completions of a clutter, while in the second (Theorem 13) we focus our attention on how the minimal matroidal completions can be obtained.

In general, the poset \( \text{Mat}(\Lambda) \) is not a totally ordered set (see Example 11). Therefore, we do not know how many minimal elements this poset has. Our first result states that the non-matroidal clutters have at least two minimal matroidal completions.

**Theorem 12.** Let \( \Lambda \) be a clutter. Then, the poset \( (\text{Mat}(\Lambda), \leq) \) has a unique minimal element if and only if \( \Lambda \) is a matroidal clutter.

**Proof.** If \( \Lambda \) is a matroidal clutter, then there exists a matroid \( \mathcal{M}_0 \) such that \( \mathcal{C}(\mathcal{M}_0) = \Lambda \), and hence \( \min(\text{Mat}(\Lambda)) = \{\mathcal{M}_0\} \). Let us show that the converse is true. So let \( \Lambda \) be a clutter and assume that there exists a matroid \( \mathcal{M} \) such that \( \min(\text{Mat}(\Lambda)) = \{\mathcal{M}\} \). In such a case, it is necessary to demonstrate that \( \Lambda \) is a matroidal clutter.

To do this we consider the blocker \( b(\Lambda) \) of the clutter \( \Lambda \). The blocker of the clutter \( \Lambda \) is defined as the clutter \( b(\Lambda) = \min\{B \subseteq \Omega : B \cap A \neq \emptyset \text{ for all } A \in \Lambda\} \). It is well known that \( b(b(\Lambda)) = \Lambda \) (see for instance [6, Proposition 2.1.12]). Thus, if \( X \) is a subset of \( \Omega \) such that \( X \cap B \neq \emptyset \) for all \( B \in b(\Lambda) \), then \( X \in \Lambda^+ \).

Let us denote \( b(\Lambda) = \{B_1, \ldots, B_s\} \). For \( 1 \leq i \leq s \) let us consider the matroid \( \mathcal{M}_{B_i} \) with ground set \( \Omega \) and set of circuits \( \mathcal{C}(\mathcal{M}_{B_i}) = \{x : x \in B_i\} \). Since \( B_i \in b(\Lambda) \), then \( A \cap B_i \neq \emptyset \) for all \( A \in \Lambda \). Thus, \( \Lambda \leq \mathcal{M}_{B_i} \), and therefore \( \mathcal{M} \leq \mathcal{M}_{B_i} \) because we are assuming \( \min(\text{Mat}(\Lambda)) = \{\mathcal{M}\} \). Let \( C \in \mathcal{C}(\mathcal{M}) \) be a circuit of the matroid \( \mathcal{M} \). Since \( \mathcal{M} \leq \mathcal{M}_{B_i} \), there exists a circuit \( C_i \in \mathcal{C}(\mathcal{M}_{B_i}) \) such that \( C_i \subseteq C \), and so \( C \cap B_i \neq \emptyset \). Therefore, if \( C \in \mathcal{C}(\mathcal{M}) \) then \( C \cap B_i \neq \emptyset \) for \( i = 1, \ldots, s \). Hence, it follows that \( C \in \Lambda^+ \) because \( b(\Lambda) = \{B_1, \ldots, B_s\} \). Therefore we have that \( \mathcal{C}(\mathcal{M}) \subseteq \Lambda^+ \), and thus \( \mathcal{C}(\mathcal{M}) \subseteq \Lambda \) (see Lemma 1). But the matroid \( \mathcal{M} \) is a matroidal completion of \( \Lambda \), so \( \Lambda \leq \mathcal{C}(\mathcal{M}) \).

Therefore \( \Lambda = \mathcal{C}(\mathcal{M}) \), as we wished to prove. \( \square \)

The following result concerns non-matroidal clutters; namely, it states that any minimal matroidal completion of the clutter can be obtained by combining the transformations of the previous section.

**Theorem 13.** Let \( \Lambda \) be a non-matroidal clutter on a finite set \( \Omega \) and let \( \mathcal{M} \) be a minimal element of the poset of matroids \( (\text{Mat}(\Lambda), \leq) \). Then there is a monotone increasing sequence of clutters \( \Lambda = \Lambda_0 \preceq \Lambda_1 \preceq \cdots \preceq \Lambda_r = \mathcal{C}(\mathcal{M}) \) such that for \( i \geq 1 \), either \( \Lambda_i \) is an I-transformation of \( \Lambda_{i-1} \) or \( \Lambda_i \) is a T-transformation of \( \Lambda_{i-1} \).

**Proof.** It suffices to prove that if \( \Lambda' \) is a non-matroidal clutter on \( \Omega \), and if \( \mathcal{N} \) is a matroidal completion of \( \Lambda' \), then either there exists an I-transformation \( \Lambda'_1 \) of \( \Lambda \) such that \( \Lambda'_1 \preceq \mathcal{N} \), or there exists a T-transformation \( \Lambda'_1 \) of \( \Lambda' \) such that \( \Lambda'_1 \preceq \mathcal{N} \).

So, let \( \Lambda' \) be a non-matroidal clutter on \( \Omega \) and let \( \mathcal{N} \) be a matroidal completion of \( \Lambda' \). Let us assume that there exists no T-transformation \( \Lambda'_1 \) of \( \Lambda' \) with \( \Lambda'_1 \preceq \mathcal{N} \). In such a case, we must demonstrate that there exists an I-transformation \( \Lambda'_1 \) of \( \Lambda' \) such that \( \Lambda'_1 \preceq \mathcal{N} \).
Therefore, we conclude that we get that \( x \in (A_1 \cup A_2) \setminus \{ x \} \in T^{(1)}(A) \). On one hand, we have that \( A_i \in \Lambda' \subseteq \mathcal{N} \) for \( i = 1, 2 \). On the other, \( C \not\subseteq X = (A_1 \cup A_2) \setminus \{ x \} \) if \( C \in \mathcal{C}(\mathcal{N}) \). Therefore, for \( i = 1, 2 \) there exists a circuit \( C_i \in \mathcal{C}(\mathcal{N}) \) with \( C_i \subseteq A_i \) and such that \( x \in C_i \). At this point, notice that if \( C_1 \neq C_2 \), then there exists \( C \in \mathcal{C}(\mathcal{N}) \) such that \( C \subseteq (C_1 \cup C_2) \setminus \{ x \} \), and so there exists \( C \in \mathcal{C}(\mathcal{N}) \) such that \( C \not\subseteq (A_1 \cup A_2) \setminus \{ x \} = X \), a contradiction. Therefore, we conclude that \( C_1 = C_2 \). Let us denote \( C_0 = C_1 = C_2 \). Then we have that \( C_0 \subseteq A_1 \cap A_2 \). Therefore, \( \Lambda' = \min (\Lambda' \cup \{ A_1 \cap A_2 \}) \leq \mathcal{N} \). Now the proof of our claim will be completed by showing that \( I_{\Lambda'}(A_1 \cup A_2) \neq 0 \). If \( I_{\Lambda'}(A_1 \cup A_2) = 0 \), then there exists \( A_3 \in \Lambda' \) with \( A_3 \subseteq A_1 \cup A_2 \) and such that \( x \not\in A_3 \). Thus, \( A_3 \subseteq (A_1 \cup A_2) \setminus \{ x \} \), which is a contradiction because \( A_3 \in \Lambda' \) and \( x \in T^{(1)}(A) \setminus \Lambda' \). This completes the proof of our claim, and thereby the proof of the theorem.

To conclude we provide two examples. In the first one the clutter has two minimal matroidal completions and they are obtained by \( I \)-transformations or by \( T \)-transformations, while in the second the clutter has four minimal matroidal completions, one of which is obtained by combining both kinds of transformations.

**Example 14.** First let us consider the clutter \( \Lambda = \{ \{ 1, 2, 3 \}, \{ 1, 2, 4 \} \} \) of the finite set \( \Omega = \{ 1, 2, 3, 4 \} \). In this case, the \( I \)-transformation of \( \Lambda \) is \( \Lambda_1 = \{ \{ 1, 2 \} \} \), the first elementary transformation of \( \Lambda \) is \( \Lambda_2 = \{ \{ 1, 2, 3 \}, \{ 1, 2, 4 \}, \{ 1, 3, 4 \}, \{ 2, 3, 4 \} \} \), and the second elementary transformation is \( \Lambda_3 = \{ \{ 1, 2, 3 \}, \{ 1, 2, 4 \}, \{ 3, 4 \} \} \). Observe that \( \Lambda_1, \Lambda_2 \) and \( \Lambda_3 \) are matroidal clutters. Therefore, from Theorem 13 it follows that the minimal matroidal completions of \( \Lambda \) are the minimal elements of \( \{ \Lambda_1, \Lambda_2, \Lambda_3 \} \). In this case, \( \Lambda_2 \leq \Lambda_3 \), and so \( \min (\text{Mat}(\Lambda)) = \{ M_1, M_2 \} \) where \( M_i \) is the matroid with set of circuits \( \Lambda_i \).

**Example 15.** Finally, on the finite set \( \Omega = \{ 1, 2, 3, 4, 5 \} \), we consider the clutter \( \Lambda = \{ \{ 1, 2, 3 \}, \{ 1, 2, 4 \}, \{ 1, 5 \}, \{ 4, 5 \} \} \). In such a case, eleven matroidal clutters \( \Lambda_1, \ldots, \Lambda_{11} \) can be obtained by using or by combining \( I \)-transformations and \( T \)-transformations. Namely, by using only \( I \)-transformations we obtain the matroidal clutters \( \Lambda_1 = \{ \{ 5 \}, \{ 1, 2 \} \} \) and \( \Lambda_2 = \{ \{ 1 \}, \{ 4, 5 \} \} \). The matroidal clutters obtained by using only \( T \)-transformations are the clutters \( \Lambda_3 = \{ \{ 1, 4 \}, \{ 1, 5 \}, \{ 4, 5 \}, \{ 1, 2, 3 \}, \{ 2, 3, 4 \} \} \) and \( \Lambda_4 = \{ \{ 1, 3 \}, \{ 1, 4 \}, \{ 1, 5 \}, \{ 3, 4 \}, \{ 3, 5 \}, \{ 4, 5 \} \} \), whereas the matroidal clutters obtained by combining the \( I \)-transformations and the \( T \)-transformations are the clutters \( \Lambda_5 = \{ \{ 1 \}, \{ 5 \} \} \), \( \Lambda_6 = \{ \{ 1 \}, \{ 2, 4 \}, \{ 2, 5 \}, \{ 4, 5 \} \} \), \( \Lambda_7 = \{ \{ 5 \}, \{ 1, 2 \}, \{ 1, 4 \}, \{ 2, 4 \} \} \), \( \Lambda_8 = \{ \{ 1, 2 \}, \{ 1, 4 \}, \{ 1, 5 \}, \{ 2, 4 \}, \{ 2, 5 \}, \{ 4, 5 \} \} \), \( \Lambda_9 = \{ \{ 5 \}, \{ 1, 2, 3 \}, \{ 1, 2, 4 \}, \{ 1, 3, 4 \}, \{ 2, 3, 4 \} \} \), \( \Lambda_{10} = \{ \{ 5 \}, \{ 3, 4 \}, \{ 1, 2, 3 \}, \{ 1, 2, 4 \} \} \), and \( \Lambda_{11} = \{ \{ 4 \}, \{ 1, 5 \}, \{ 1, 2, 3 \}, \{ 2, 3, 5 \} \} \). Therefore, if we denote by \( M_i \) the matroid with set of circuits \( \Lambda_i \), then by applying Theorem 13 we get that the set of minimal matroidal completions of \( \Lambda \) is \( \min (\text{Mat}(\Lambda)) = \min \{ M_1, \ldots, M_{11} \} = \{ M_1, M_2, M_3, M_9 \} \).
References


