ON THE GEOMETRY OF QUADRATIC MAPS OF THE PLANE

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Abstract. In this article we give a geometric classification of the set of quadratic maps of the plane. The fundamental step is the proof that the restriction of the map to the critical set is injective, from which it follows that there are finitely many classes of geometrically equivalent maps. In the last sections we apply this geometric knowledge to obtain some simple dynamical properties of a particular family of quadratic maps.

1. Introduction

Let Q be the set of quadratic self-mappings of the real plane endowed with the topology of coefficients. In [1] it is proved that six parameters are enough to describe an open and dense subset Q_g of Q; in addition, every map in Q_g without fixed points has trivial dynamics. This constitutes a version, for non-invertible mappings, of the well known Brouwer's theorem [2], which states that an orientation preserving homeomorphism of the plane having no fixed points has empty limit sets; on this topic see the article of J. Franks [4].

This paper is devoted to show a geometric classification of that open and dense set. We took adventage of this classification to analyze some interesting properties of a real one-parameter family of endomorphisms on the complex plane. The meaningful concept in our approach is the geometric equivalence of maps. We recall that two smooth maps $f, g : M \to N$ are *(geometrically) equivalent* if there exist smooth diffeomorphisms $\varphi : M \to M$ and $\psi : N \to N$ such that $f \circ \varphi = \psi \circ g$. A map is *stable* if it has a neighborhood consisting of equivalent maps. Clearly φ (resp. ψ) carries critical points (resp. critical values) of g to critical points (resp. critical values) of f; further, critical sets of equivalent maps are diffeomorphic.

We briefly describe some other geometric invariants that we will consider throughout this paper. If $f : M \to M$ is a smooth and proper map, then the number of preimages of every regular point is finite and constant in each connected component of the set of regular values of f. If the set of regular values of such a map f has k components and $a_1 \leq \cdots \leq a_k$ are the number of preimages in each one of these components, then we say that f has type $(a_1, \ldots, a_k)$. We also recall that generically real planar maps have only two kind ofcritical points: *folds* and *cusps*, both having simple local canonical forms. The number of cusp points, the type of the map and the absolute value of the degree of the map are invariants of geometric equivalence. Each of these invariants is sufficient to characterize the geometric equivalence classes among the endomorphisms in Q_g. This is a consequence of the following proposition whose proof is contained in lemmas 1 and 2 below.

Proposition 1. *The restriction of $G \in Q_g$ to its critical set is injective.*

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In the above referred lemmas it is also proved the existence of just two classes of geometric equivalent maps in the generic set Q_g; in addition, it can be proved that there exist finitely many classes of geometrically equivalent quadratic maps (see at the beginning of paragraph 6.1 in section 6). The proof of these lemmas rest on geometrical objects that represent the set of critical values of quadratic maps in that equivalent classes: deltoids and hypdeltoids. Deltoids, also called 3-cusped hypocycloids, were first studied by Euler in 1745 while considering optical problems, they have a simple parametrization with sine and cosine functions. Dual parametrizations with hyperbolic sine and cosine functions give rise to a geometrical object that we called hypdeltoids. The striking property of these curves in our context is that they describe the sets of critical values and its preimages, which gives an accurate geometric description of maps in Q_g:

Theorem 1. *For the open and dense set Q_g the following properties hold:*

(i) For every $G \in Q_g$, the point at ∞ is an attractor.
(ii) Every map in Q_g is geometrically stable.
(iii) There exist only two classes, Q_+ and Q_-, of geometric equivalence in Q_g.
(iv) Every $G \in Q_-$ is of type $(2,4)$, has degree ± 2 and the set of critical points is an ellipse containing exactly three cusp points.
(v) Every $G \in Q_+$ is of type $(0,2,4)$, have degree 0 and the set of critical points is a hyperbola containing exactly one cusp point.

2. The generic set Q_g

Consider the set of all real planar maps defined, for every $(x,y) \in \mathbb{R}^2$, by

$$G(x,y) = (pxy + ax + by + k_1, rx^2 + sy^2 + txy + cx + dy + k_2), \tag{1}$$

where $prs \neq 0$. It was proved in [1] that the set of maps affinely conjugated to a map of this form is open and dense in Q. Let Q_g be the set of maps G_0 satisfying:

- G_0 is affinely conjugated to a map of the form (1);
- The critical set of G_0 is either an ellipse or a hyperbola.

It is easy to see that Q_g is open and dense in Q. Additionally, note that $G_0 \in Q_g$ has an ellipse (resp. a hyperbola) as its critical set if, and only if, there is a G as in (1) with $rs < 0$ (resp. $rs > 0$) and affinely conjugated to G_0. This property splits Q_g into two disjoint subsets: Q_-, consisting of maps in Q_g whose the critical set is an ellipse, and Q_+, consisting of those maps whose critical set is a hyperbola.

Take $G_0 \in Q_-$ and G as in (1) which is affinely conjugated to G_0. After the change of variables $(X,Y) = \left(\sqrt{-rs}\,x, -sy\right)$ and an appropriate traslation, the map G is written as:

$$G(x,y) = \left(pxy + ax + by + k_1, x^2 - y^2 + txy + \frac{at-2b}{p}x + \frac{bt+2a}{p}y + k_2\right). \tag{2}$$

Let Θ_- be the family of maps in Q_- and defined as in (2). Notice that if $G \in \Theta_-$ is as above, then its critical set ℓ is given by the circle with Cartesian equation $x^2 + y^2 = (a^2 + b^2)/p^2$; obviously $a^2 + b^2 > 0$. We refer this kind of maps as the normal form for Q_-.

In analogous way, maps in Q_+ are affinely conjugated to a map of the family Θ_+ given by the normal form:

$$G(x,y) = \left(pxy + ax + by + k_1, x^2 + y^2 + txy + \frac{at-2b}{p}x + \frac{bt-2a}{p}y + k_2\right), \tag{3}$$

whose critical set ℓ is the hyperbola given by $y^2 - x^2 = (a^2 - b^2)/p^2$; note that $a^2 > b^2$, otherwise $G \notin Q_+$.

3. Deltoids and Hypdeltoids

We begin this section by recalling generic properties related to critical sets of smooth planar maps and singularities of smooth parametrized curves in $\mathbb{R}^2$.

Take a smooth map $G : \mathbb{R}^2 \to \mathbb{R}^2$. The following notions and statements were introduced by H. Whitney in [9]. The map G is said to be *good* if every point $p \in U$ is either regular or the gradient of the Jacobian matrix of G at p is non-null. If G is a good map, then its critical set ℓ is a 1-manifold. In this case, if $p \in \ell$ and φ is a parametrization of ℓ around p ($\varphi(0) = p$), then this critical point is called a *fold point of* G if $d(G \circ \varphi)/dt \neq 0$ at $t = 0$ and p is a *cusp point of* G whenever $d(G \circ \varphi)/dt = 0$ and $d^2(G \circ \varphi)/dt^2 \neq 0$ at $t = 0$; these definitions are independent of the choice of the parametrization. In that seminal article Whitney found a generic set of good maps: the set of *excellent* maps, which are characterized by the fact that the critical set is only composed by fold or cusp points. Furthermore, local normal forms for these critical points were constructed. If p is a fold point, then the map G is equivalent, in some neighborhood of p, to the map $(x, y) \mapsto (x^2, y)$ in a neighborhood of the origin; so G is locally of type $(0, 2)$. For cusp points the normal form is given by $(x, y) \mapsto (xy - x^3, y)$, which implies that cusp points are isolated and the mapping is of type $(1, 3)$ around p. It is proved in [7] that the restriction of a generic map G to any component of the complement of $G^{-1}(G(\ell))$ is a covering map whose image is a component of the complement of $G(\ell)$; see Lemma 3 in section 4. Therefore, determining the critical sets, the critical values and the preimages of the critical values is essential in the description of the geometry of a generic map.

Additionally to the notion of cusp point as critical point of smooth maps we deal with cusp points as singularities of plane smooth curves. In order to recall this notion we consider a parametrized smooth curve $\gamma(t) = (x(t), y(t))$, where t is varying in an open interval. Take a singular point p on this curve, that is, $p = (x(t_0), y(t_0))$ for some $t_0 \in I$, and $x'(t_0) = y'(t_0) = 0$. Hence it holds that

$$x(t) = x_0 + a(t - t_0)^2 + b(t - t_0)^3 + R_1(t), \text{ and}$$
$$y(t) = y_0 + c(t - t_0)^2 + d(t - t_0)^3 + R_2(t),$$

where $R_i(t)/(t - t_0)^3 \to 0$ when $t \to t_0$, $i = 1, 2$. Assuming $a^2 + c^2 > 0$, the curve γ is tangent to the line through p with slope c/a if $a \neq 0$, and it is tangent to the vertical line $x = x_0$ at that point when $a = 0$. Observe that this assumption implies that near p the curve is injective and this singularity is isolated. The singular point p is said to be an *ordinary cusp* (or simply a cusp) on γ when $ad - bc \neq 0$. It is easy to see that under this open condition on the derivatives of second and third order, the two branches of γ near p, that is $\{\gamma(t) : t < t_0\}$ and $\{\gamma(t) : t > t_0\}$ with $|t - t_0|$ small, are located in different sides of the tangent line. The same notion of cusp point on simple and piecewise regular curves is introduced in [3].

3.1. Deltoids and maps in Θ_-.

Definition 1. For $\alpha \in [0, 2\pi)$, the regular α-*deltoid* (*deltoid*, for short) is the parametrized smooth closed curve Δ_α given by:

$$\Delta_\alpha(\omega) = (\sin(2\omega) + 2\sin(\omega + \alpha), \cos(2\omega) - 2\cos(\omega + \alpha)), \quad \omega \in [0, 2\pi). \tag{4}$$

Notice that $\Delta_\alpha(\omega) = ie^{-2i\omega} - 2ie^{i(\omega+\alpha)}$. In this way it is easy to verify that:
• The function $\Delta_\alpha : [0, 2\pi) \to \mathbb{R}^2$ is injective; hence Δ_α is a closed simple curve.
• Only at $\omega = (\pi - \alpha)/3, (3\pi - \alpha)/3, (5\pi - \alpha)/3$ it holds $d\Delta_\alpha(\omega)/dw = 0$. That is, Δ_α has three singularities. Since $d^2\Delta_\alpha(\omega)/dw^2 \neq 0$ for all $\omega \in [0, 2\pi)$ and the imaginary part of the product $d^2\Delta_\alpha(\omega)/dw^2 . \overline{d^3\Delta_\alpha(\omega)/dw^3}$ is non-null at the values where $d\Delta_\alpha(\omega)/dw = 0$, then that three singularities are cusp on the deltoid Δ_α. Here $\overline{z}$ denotes the conjugate of the complex numbre z.

An implicit Cartesian equation of Δ_α can be obtained by eliminating the variable ω in the equations $x = \sin(2\omega) + 2\sin(\omega + \alpha)$ and $y = \cos(2\omega) - 2\cos(\omega + \alpha)$. Indeed, with the proceduce described in [8, p. 206] one arrives to $D_\alpha(x, y) = 0$, where

$$D_\alpha(x, y) = (x^2 + y^2)(x^2 + y^2 + 18) + 8x(3y^2 - x^2)\sin(2\alpha) + 8y(3x^2 - y^2)\cos(2\alpha) - 27.$$

It is simple to check that for all $\alpha \in [0, 2\pi)$ the function D_α satisfies

$$D_\alpha = D_0 \circ J \circ R_{-2\alpha/3}, \tag{5}$$

where J is the reflection with respect to the vertical axis and $R_{-2\alpha/3}$ is the rotation by angle $-2\alpha/3$.

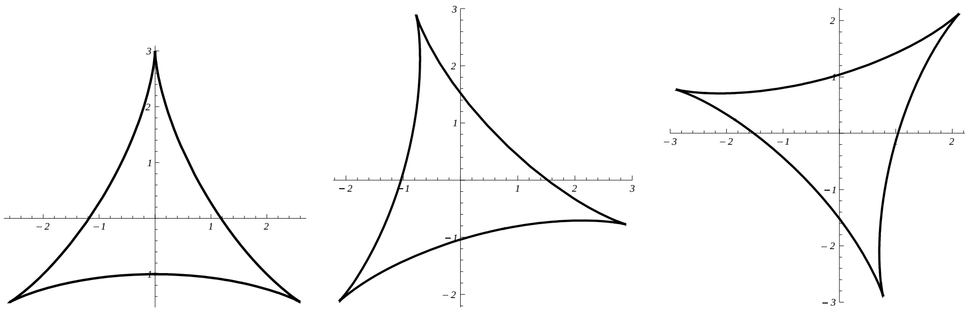

FIGURE 1. Regular deltoids with $\alpha = 0, \frac{\pi}{8}, \frac{13\pi}{8}$.

Take $G \in \Theta_-$ as in equation (2). Recall that its critical set ℓ is the ellipse given by $x^2 + y^2 = \rho^2/p^2$, where $\rho = \sqrt{a^2 + b^2}$. Fix $\alpha \in [0, 2\pi)$ such that

$$(a, b) = \rho(\cos(\alpha), \sin(\alpha)), \tag{6}$$

and parametrize ℓ as

$$\ell(\omega) = \frac{\rho}{p}(\sin(\omega), \cos(\omega)),\ \omega \in [0, 2\pi). \tag{7}$$

With these considerations $G(\ell(\omega))$ is written, for all $\omega \in [0, 2\pi)$, as:

$$G(\ell(\omega)) = A\left(\frac{\rho^2}{p^2}\Delta_\alpha(\omega)\right), \tag{8}$$

where α and ρ are given by (6) and A is the affine bijection:

$$A(x, y) = \frac{1}{2}((p, t)x + (0, -2)y) + (k_1, k_2). \tag{9}$$

Notice that (8) and the existence of the three cusp points on Δ_α imply that G has only three critical point of cusp type: $\ell((\pi - \alpha)/3), \ell((3\pi - \alpha)/3)$ and $\ell((5\pi - \alpha)/3)$. It is also concluded that the restriction of G to ℓ is an injective function.

Now we will analize the preimage under G of $G(\ell)$. First, it is clear that (x, y) belongs to $G^{-1}(G(\ell))$ if and only if $\frac{p^2}{\rho^2} A^{-1} G(x, y) \in \Delta_\alpha$; that is,

$$\frac{p^2}{\rho^2}\left(2xy + \frac{2a}{p}x + \frac{2b}{p}y, -x^2 + y^2 + \frac{2b}{p}x - \frac{2a}{p}y\right) \in D_\alpha^{-1}(0).$$

Introducing the change of variable $(X, Y) = \frac{p}{\rho}(x, y)$, using (6) and defining

$$H(X, Y) = (2XY + 2X\cos(\alpha) + 2Y\sin(\alpha), -X^2 + Y^2 + 2X\sin(\alpha) - 2Y\cos(\alpha)),$$

it follows that $(x, y) \in G^{-1}(G(\ell))$ if and only if $(D_\alpha \circ H)(X, Y) = 0$. A straightforward calculation leads to the identity

$$(D_\alpha \circ H \circ R_{-\alpha/3})(X, Y) = (X^2 + Y^2 - 1)^2 D_\alpha(X, Y).$$

From (5) one obtains $D_\alpha \circ R_{\alpha/3} = D_{\alpha/2}$, consequently

$$(D_\alpha \circ H)(X, Y) = (X^2 + Y^2 - 1)^2 D_{\alpha/2}(X, Y).$$

This implies that $G^{-1}(G(\ell)) = \ell \cup \widetilde{\ell}$, where $\widetilde{\ell}$ is the deltoid obtained as the homothetic transformation with scale ρ/p of the deltoid $\Delta_{\alpha/2}$. From this fact one can verify that ℓ is contained in the closure of the bounded component of the complement of $\widetilde{\ell}$; moreover, ℓ and $\widetilde{\ell}$ are tangent at the three cusp points in ℓ.

The following lemma summarizes the preceding discussion.

Lemma 1. *If $G \in \Theta_-$, then its critical set ℓ is a circle having exactly three cusp points, the restriction of G to ℓ is injective, the set $G(\ell)$ is a deltoid and $G^{-1}(G(\ell))$ is the union of ℓ and another deltoid $\widetilde{\ell}$ which is tangent to ℓ at the three critical points of cusp type.*

3.2. **Hypdeltoids and maps in Θ_+.**

Now we will proceed in very similar way as above to analize the geometry of the set of critical values of maps in Θ_+.

Definition 2. Given $\alpha \in \mathbb{R}$, the regular α-*hypdeltoid* (*hypdeltoid*, for short) is the pair of parametrized curves $\Lambda_\alpha^\pm$ given by:

$$\Lambda_\alpha^\pm(\omega) = (\sinh(2\omega) \pm 2\sinh(\omega + \alpha), -\cosh(2\omega) \pm 2\cosh(\omega + \alpha)),\ \omega \in \mathbb{R}. \tag{10}$$

For $i = 1, 2$ and $\sigma = \pm$ we denote by $\varphi_i^\sigma(\omega)$ the ith-coordinate of $\Lambda_\alpha^\sigma(\omega)$. Suppose that for $\omega, \omega' \in \mathbb{R}$ are satisfied $\varphi_1^+(\omega) = \varphi_1^-(\omega')$ and $\varphi_2^+(\omega) = \varphi_2^-(\omega')$. This implies that $-\cosh(3\omega + \alpha) = \cosh(3\omega' + \alpha)$, which occurs when $\omega = \omega' = -\frac{\alpha}{3}$; but $\varphi_2^+(-\frac{\alpha}{3}) \neq \varphi_2^-(-\frac{\alpha}{3})$. Thus, the branches Λ_α^- and Λ_α^+ are disjoint. On the other hand, since φ_1^+ is a function onto $\mathbb{R}$ and $d\varphi_1^+(\omega)/d\omega \neq 0$, it follows that Λ_α^+ is an embbeding of $\mathbb{R}$; indeed, it is the graph of a smooth function. With respect to the branch Λ_α^-, it is easy to see that $d\varphi_1^-(\omega)/d\omega = d\varphi_2^-(\omega)/d\omega$ if and only if $\omega = -\frac{\alpha}{3}$. Hence, on Λ_α^- there is only one singularity; moreover, by analyzing the values of the second and third derivatives of φ_i^- at $\omega = -\frac{\alpha}{3}$ we conclude that this singularity is a cusp point. Furthermore, as φ_1^- is a function onto $\mathbb{R}$, $d^2\varphi_2^-(\omega)/d\omega^2 < 0$ and $d\varphi_1^-(\omega)/d\omega = 0$ exactly at $\omega = -\frac{\alpha}{3}$ and $\omega = \alpha$, then the function $\omega \mapsto \Lambda_\alpha^-(\omega)$ is injective, and the branch Λ_α^- is topological immersion of $\mathbb{R}$.

Now we will obtain a Cartesian equation for $\Lambda_\alpha^\pm$. First we introduce

$$x = \sinh(2\omega) \pm 2\sinh(\omega) \text{ and } y = -\cosh(2\omega) \pm 2\cosh(\omega). \tag{11}$$

Since $y^2 - x^2 = 5 \mp 4\cosh(3\omega)$, the equation on the right side of (11) implies that $u = \frac{1}{2}(\sqrt{3 - 2y} \pm 1)$, by setting $u = \cosh(\omega)$. But $\cosh(3\omega) = 4u^3 - 3u$, then

$$(y^2 - x^2)(y^2 - x^2 + 18) + 8y(3x^2 + y^2) - 27 = 0$$

is an implicit Cartesian equation of $\Lambda_0^{\pm}$. Thanks to this Cartesian representation and the identity

$$\Lambda_\alpha^{\pm}(\omega) = \varphi(\omega) + B_\alpha(\Lambda_0^{\pm}(\omega) - \varphi(\omega)),$$

where $\varphi(\omega) = (\sinh(2\omega), -\cosh(2\omega))$ and B_α is the linear map given by the matrix $\begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix}$, we get (after a tedious computation) that the zero set of

$$\begin{aligned} H_\alpha(x,y) = (x^2-y^2)(x^2-y^2-18) - 8x(x^2+3y^2)\sinh(2\alpha) \\ + 8y(3x^2+y^2)\cosh(2\alpha) - 27 \end{aligned}$$

is a the Cartesian description of $\Lambda_\alpha^{\pm}$. It is simple to check that $H_\alpha = H_0 \circ S_\alpha$, where S_α is the linear isomorphisms given by $\begin{pmatrix} \cosh(2\alpha/3) & -\sinh(2\alpha/3) \\ -\sinh(2\alpha/3) & \cosh(2\alpha/3) \end{pmatrix}$.

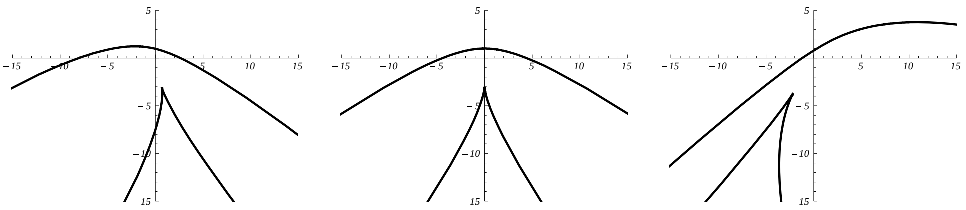

FIGURE 2. Hypdeltoids with $\alpha = -\frac{1}{3}, 0, 1$.

Take $G \in \Theta_+$ as in (3), recall that its critical set ℓ is given by the equation $y^2 - x^2 = \rho^2/p^2$, where $\rho = \sqrt{a^2-b^2}$ and $|a| > |b|$. Consider $\alpha \in \mathbb{R}$ such that the coefficients a and b in (3) satisfy

$$(a,b) = \rho(\cosh\alpha, \sinh\alpha). \tag{12}$$

We parametrize the branches $\ell_\pm$ of ℓ by

$$\ell_\pm(\omega) = \frac{\rho}{p}(\sinh(\omega), \pm\cosh(\omega)),\ \omega \in \mathbb{R}.$$

Then the image by G of $\ell_\pm(\omega)$ is expressed as $G(\ell_\pm(\omega)) = A\left(\frac{\rho^2}{p^2}\Lambda_\alpha^{\pm}(\omega)\right)$, where A is defined in (9). This expression allows to conclude that:

- The map G restricted to each branch $\ell_\pm(\omega)$ is an injective function.
- There is only one cusp point in the critical set of G, which belongs to ℓ_-. The remaining critical points are all of the fold type.
- A point $(x,y) \in G^{-1}(G(\ell))$ if and only if $\frac{p^2}{\rho^2}A^{-1}G(x,y) \in \Lambda_\alpha^{\pm}$, that is

$$\frac{p^2}{\rho^2}\left(2xy + \frac{2a}{p}x + \frac{2b}{p}y, -x^2 - y^2 + \frac{2b}{p}x + \frac{2a}{p}y\right) \in H_\alpha^{-1}(0).$$

Making $(X,Y) = \frac{p}{\rho}(x,y)$, equation (12) implies that $(x,y) \in G^{-1}(G(\ell))$ if and only if $(H_\alpha \circ h)(X,Y) = 0$, where

$$h(X,Y) = (2XY + 2\cosh(\alpha)X + 2\sinh(\alpha)Y, -X^2 - Y^2 + 2\sinh(\alpha)X + 2\cosh(\alpha)Y).$$

It can be checked that for all $X, Y \in \mathbb{R}$ it holds

$$(H_\alpha \circ h)(X,Y) = (1 + X^2 - Y^2)^2 H_{\alpha/2}(X, -Y).$$

Thus, the zero set of the polynomial $(H_\alpha \circ h)(X,Y)$ is the union of the hyperbola $1+X^2-Y^2=0$ and the hypdeltoid $H_{\alpha/2}(X,-Y)=0$. Therefore, $G^{-1}(G(\ell))$ is the union of the hyperbola ℓ and the hypdeltoid obtained as the homethetic transformation with scale ρ/p of the reflection respect to the horizontal axis of the hypdeltoid $\Lambda^{\pm}_{\alpha/2}$.

We summarize the precedent exposition in the following lemma.

Lemma 2. *If $G \in \Theta_+$, then its critical set ℓ is a hyperbola containing only one cusp point, the mapping G is injective when it is restricted to ℓ, the set of critical values $G(\ell)$ is a hypdeltoid and its preimage is the union of ℓ and a hypdeltoid $\widetilde{\ell}$.*

4. Proof of Theorem 1

As all the statements in Theorem 1 are invariant under affine conjugation, we only consider generic quadratic maps.

Proof of part (i) of Theorem 1. Take a generic map G as in equation (1), that is

$$G(x,y) = (pxy + ax + by + k_1, rx^2 + sy^2 + txy + cx + dy + k_2),$$

with $prs \neq 0$, by simplicity we assume $p > 0$. Let $|(x,y)| = \max\{|x|,|y|\}$. We show that there exists $K_0 > 0$ depending only on G such that, for $K > K_0$ the condition $|(x,y)| > K$ implies $|G(x,y)| > 2K$. So it is clear that ∞ is an attracting fixed point for G. Indeed, assume that $|(x,y)| > K$ and $|x| \geq |y|$. If $|pxy + ax + by + k_1| < 2K$ and K is large enough, then:

$$\begin{aligned} |y| &< \frac{2K + |ax + k_1|}{|px + b|} \leq \frac{2K + |a||x| + |k_1|}{p|x| - |b|} \\ &\leq \frac{2K + |k_1|}{pK - |b|} + \frac{|a|}{p - |b|/K} \leq \frac{3}{p} + \frac{2|a|}{p} \leq \frac{3 + 2|a|}{p}. \end{aligned}$$

This inequality implies that:

$$|G(x,y)| \geq |r|x^2 - \left(\left|\frac{t(3+2|a|)}{p} + c\right|\right)|x| - \frac{|s|(3+2|a|)^2}{p^2} - \frac{|d|(3+2|a|)}{p} - k_2;$$

since $r \neq 0$ it follows that $|G(x,y)| > 2K$ if K is sufficiently large and $|x| > K$. The proof for the case $|y| \geq |x|$ is similar. □

For the proof of the other parts of theorem 1 we will use the following result, which can be found in [7].

Lemma 3. *Let G be a smooth proper map on a manifold M. The restriction of G to any component of the complement of $G^{-1}(G(\ell))$ is a covering map whose image is a component of the complement of $G(\ell)$.*

The proof of this lemma is based on the fact that every point y in $G(C)$ has finitely many preimages, where C is a component of the complement of $G^{-1}(G(\ell))$. Then the result holds even if the set of critical points is not bounded. Note that part (i) proved above implies that the restriction of G to a component of $G^{-1}(G(\ell))$ is a proper map.

Proof of part (iv) of Theorem 1. Let G be a map in Θ_- as in (2). Denote by c_1, c_2 and c_3 the cusp points of G. Besides the injectivity of G when restricted to ℓ, Lemma 1 describes the way as the sets ℓ, $G(\ell)$ and $G^{-1}(G(\ell)) = \ell \cup \widetilde{\ell}$ are displayed, just as figure 3 shows.

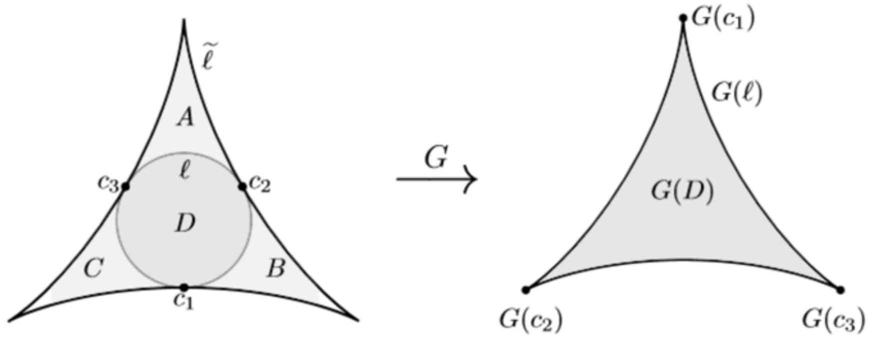

FIGURE 3. Critical set, critical values and its preimage for maps in Θ_-.

Note that the regions A, B, C and D are topological discs and constitute the bounded components of the complement of $G^{-1}(G(\ell))$. It follows from Lemma 3 that the restriction of G to each of these regions is a homeomorphism onto the bounded component of the complement of $G(\ell)$.

Claim: *Every point in the unbounded component of the complement of $G(\ell)$ has two preimages.*

Note that the first coordinate of $G(x, y)$ can be made $pxy + by + u$ by a translation in the second coordinate. The preimage of a point (u, v) (with $v > 0$ large enough) satisfies $y = 0$ or $x = -b/p$. Substituting $x = -b/p$ in the second coordinate of G, and assuming v large, there exist two solutions for y. On the other hand, substituting $y = 0$ in the second coordinate of $G(x, y)$, and taking v large, it comes that two solutions for x exist because $v > 0$. Hence, from Lemma 3, the restriction of G to the unbounded component E of the complement of $G^{-1}(G(\ell))$ is a two-to-one covering of the unbounded component of the complement of $G(\ell)$.

It remains to calculate the degree of G. As in the claim, take (u, v) with $v > 0$ large enough and having preimages $(x_\pm, 0)$. The determinant of DG at these points has the same sign of $-v$, so the degree is -2. □

Proof of part (v) of Theorem 1. Consider a map $G \in \Theta_+$ whose critical set is the hyperbola ℓ. Let ℓ_1 and ℓ_2 be the branches of ℓ; we assume that ℓ_1 contains the unique cusp point c_1 of G, the remaining critical points of G are fold points. As $G(\ell_1)$ and $G(\ell_2)$ are the branches of the hypdeltoid $G(\ell)$ (see Lemma 2), it follows that the set of regular values has three components: Y_0, Y_2 and Y_4. As in the proof of part (iv) above, one can take now a point of the form $(u, -v)$ with v large enough to conclude that there are points with no preimages. It follows immediately that the degree of G is zero in this case. Moreover, by the normal form at cusp points (see the remarks at the beginning of Section 3) there exists a basis of neighborhoods of a cusp whose images are open. Therefore, exactly one of the three regions contained in the complement of $G(\ell)$ has no preimage under G; this component will be denoted by Y_0. It follows also that the boundary of Y_0 is equal to $G(\ell_2)$, because there cannot be images of cusps points in the boundary of Y_0. Then denote by Y_2 the other region having $G(\ell_2)$ in its boundary. Points in Y_2 have two preimages because passing through $G(\ell_2)$ from Y_0 to Y_2 means an increasing in two units of the number of preimages; this follows by using the normal form at fold points. With similar arguments using the normal form at cusp points it follows that every point in the region Y_4 whose boundary is $G(\ell_1)$, has four preimages (ℓ_1 contains the cusp point).

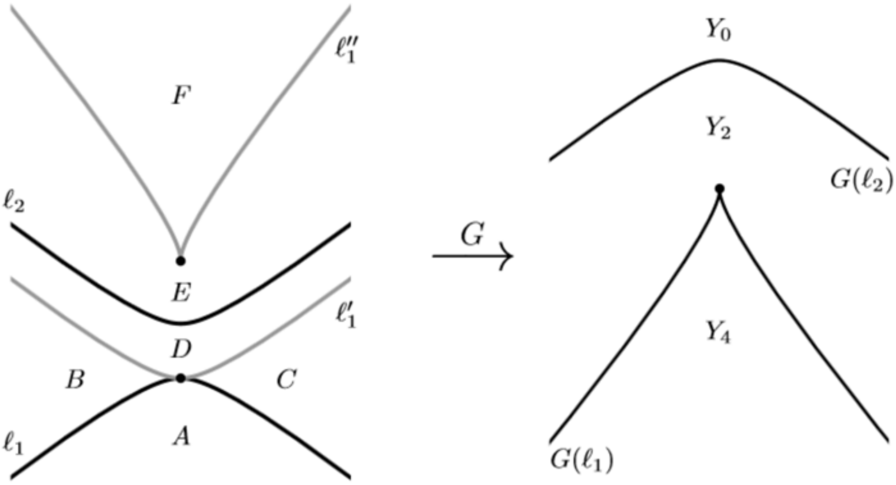

FIGURE 4. Critical set, critical values and its preimage for maps in Θ_+.

It remains to describe the set $G^{-1}(G(\ell))$. Note that $G^{-1}(G(\ell_2))$ has only one component, the contrary assumption would imply that points in Y_0 have preimages. It follows that the hypdeltoid $\widetilde{\ell} = \ell_1' \cup \ell_1''$ in the preimage of $G(\ell)$ (see Lemma 2) is contained in $G^{-1}(G(\ell_1))$. Then $G^{-1}(G(\ell_1)) = \ell_1 \cup \ell_1' \cup \ell_1''$. Now we want to determine the location of the branches of $\widetilde{\ell}$. First note that as ℓ_1 has a cusp, then using again normal forms, it comes that one of the branches of $\widetilde{\ell}$, say ℓ_1', is tangent to ℓ_1 at c_1. Now take a simple curve γ joining $G(\ell_1) \setminus \{G(c_1)\}$ to $G(\ell_2)$ and whose interior is contained in Y_2. It is claimed that $G^{-1}(\gamma)$ satisfies the following properties:

(1) Its interior is a simple arc, denoted from now on as γ'.
(2) One of the extreme points of γ' belongs to ℓ_1', the other one belongs to ℓ_1''.
(3) The preimage of $\gamma \cap Y_2$ does not intersect ℓ_1.

Proof of these assertions: (1) Note that the two preimages of points in Y_2 are located at different sides of ℓ_2. This is because ℓ_2 only contains fold points. Recall from the normal form at a fold type critical point that the preimage of a simple curve intersecting ℓ_2 at just one point is a simple curve.
(2) and (3). Note that ℓ_1' cannot intersect ℓ_2 because their images are disjoint, recall that $G|_\ell$ is injective. The same thing occurs with ℓ_1'' and ℓ_2. Hence γ' cannot have both extreme points in the same component of the preimage of $G(\ell)$. So, to complete the proof of both (2) and (3) it remains to show that its end points cannot belong to ℓ_1. Assume that one of the extreme points of γ' belongs to ℓ_1. As ℓ_1 is a set whose points (excepting c_1) are critical points of fold type, then points in $\gamma \cap Y_2$ would have preimages at both sides of ℓ_1, but then these points would have more than two preimages contradicting the definition of Y_2.

Supported on these arguments we conclude that the complement of $G^{-1}(G(\ell))$ is the union of six regions: A, B, C, D, E and F. The restrictions of G to A, B, C and F are homeomorphisms onto the region Y_4, while the restrictions of G to D and E are homeomorphisms onto the region Y_2; see figure 4. $\square$

Proof of part (ii) of Theorem 1. For the proof of this parte one must find, for any perturbation $\widetilde{G}$ of G, diffeomorphisms φ and ψ such that $\psi \circ G = \widetilde{G} \circ \varphi$. Consider first a map $G \in \Theta_-$. It was shown above that this map has the geometrical structure described in figure 3. What must be shown now is that any perturbation $\widetilde{G}$ of G has the same geometrical structure, that is, the set of critical points of G and $\widetilde{G}$ must be diffeomorphic, as well as the sets of critical values; moreover, the number of components of the complement of $G^{-1}(G(\ell))$ must remain unchanged after perturbation. Indeed, assume that we have proved that for $\widetilde{G}$ the following picture is realized:

(1) The set of critical points ℓ of $\widetilde{G}$ is diffeomorphic to a circle and contains exactly three cusps.
(2) The image $\widetilde{G}(\ell)$ of ℓ is a simple closed curve of class C^1 except at the image of the cusp points of $\widetilde{G}$.
(3) The preimage of $\widetilde{G}(\ell)$ is equal to the union of ℓ and another simple closed curve δ which is of class C^1 except at three points. Moreover, ℓ is contained in the bounded component of the complement of δ except at the cusps of ℓ, where a tangency between ℓ and δ occurs.
(4) The complement of the preimage of $\widetilde{G}(\ell)$ is equal to the union of five regions, the map $\widetilde{G}$ is injective in each one of the four bounded regions, and it is two-to-one in the unbounded one.

With these properties at hand, one can easily construct the diffeomorphisms making the equivalence. Begin with a diffeomorphism φ carrying the closure of the bounded component D of the complement of $\ell(G)$ to the closure of the bounded component $\widetilde{D}$ of the complement of $\ell(\widetilde{G})$. By property (1) above it is obvious that this can be done with the additional assumption that φ carries cusps to cusps. Denote by c_1, c_2, c_3 the cusps of G and by $c_i' = \varphi(c_i)$, $i = 1, 2, 3$. Properties (2) and (3) above imply that the bounded components of the complement of $\widetilde{G}^{-1}(\widetilde{G}(\ell))$ are four: $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ and $\widetilde{D}$, they are labeled as in figure 3; that is, $\widetilde{A}$ is the region containing in its boundary c_2' and c_3', $\widetilde{B}$ is the component containing in its boundary c_2' and c_1', $\widetilde{C}$ is the component containing c_1' and c_3', and $\widetilde{D}$ as described above. Use corresponding notations (A, B, C, D) for the components of the complement of the preimage of $G(\ell(G))$. We proceed to extend φ as follows: for a point $x \in A$, there exists a unique point in $y \in D$ such that $G(y) = G(x)$. Then define $\varphi(x)$ as the unique point in $\widetilde{x} \in \widetilde{A}$ such that $\widetilde{G}(\widetilde{x}) = \widetilde{G}(\varphi(y))$. Note that φ was defined in A as

$$\varphi = (\widetilde{G}|_{\widetilde{A}})^{-1} \circ \widetilde{G}|_{\widetilde{D}} \circ \varphi \circ (G|_D)^{-1} \circ G|_A.$$

Similar extension to B and C. Thus, φ is defined in the closure of the union $A \cup B \cup C \cup D$. Observe that the equation above implies that φ is differentiable in the union of the interiors of these regions. It is also is smooth in the boundary of D. In common boundaries it is well defined because the common boundaries are critical points and φ satisfies the symmetric property:

$$G(x) = G(y) \text{ implies } \widetilde{G}(\varphi(x)) = \widetilde{G}(\varphi(y)). \tag{13}$$

This formula also implies the smoothness of φ at those boundaries. It remains to define φ in E, the unbounded component of the complement of $G^{-1}(G(\ell(G)))$. That is, φ must be any diffeomorphism from E onto $\widetilde{E}$ with prescribed boundary values, and such that the symmetric property holds. To construct this, let L be an unbounded simple line starting at $G(c_1)$, and note that the preimage of L under G has two components: L_1 and

L_2, recall that L is simple and $G|_E$ is a covering of the annulus. Note that $G^{-1}(G(c_1))$ has two preimages, one of which is c_1, and assume that L_1 has c_1 as its unique extreme point. Let $\widetilde{L}_i$ $(i = 1, 2)$ be equally constructed for $\widetilde{G}$; as above c'_1 is the extreme point of $\widetilde{L}_1$. Let φ be any diffeomorphism from L_1 to $\widetilde{L}_1$. Then extend φ to L_2, making as before: for $x_2 \in L_2$ there is a unique x_1 in L_1 such that $G(x_2) = x_1$, hence one can define $\varphi(x_2)$ as the unique point $\widetilde{x}_2$ in $\widetilde{L}_2$ such that $\widetilde{G}(\widetilde{x}_2) = \widetilde{G}(\varphi(x_1))$. Observe that φ was defined twice at the points $G^{-1}(G(c_1))$, but both definitions coincide. As E is an annulus, it follows that $E \setminus (L_1 \cup L_2)$ equals the union of two connected components V_1 and V_2, and that G is injective in each V_i. Define $\widetilde{V}_1$ and $\widetilde{V}_2$ in similar way. Note that V_1 and $\widetilde{V}_1$ are half-planes and that φ was already defined as a diffeomorphism from the boundary of V_1 onto the boundary of $\widetilde{V}_1$. It is easy then to extend φ to a diffeomorphism from V_1 onto $\widetilde{V}_1$. Making the same trick as above, extend φ to the whole L_2.

At this point, we have constructed a diffeomorphism φ from the plane onto itself satisfying the symmetry property (13). To define ψ we proceed as follows: let y be any point in the plane and choose any x such that $G(x) = y$. Then define $\psi(y) = \widetilde{G}(\varphi(y))$. This definition does not depend on the choice of x by the symmetry of φ. It is smooth since it is locally a composition of diffeomorphisms at any point $y \notin G(\ell_1)$, and every point in $G(\ell)$ has a preimage that is not critical. Clearly $\widetilde{G} \circ \varphi = \psi \circ G$. This finishes the construction of the equivalence between G and $\widetilde{G}$.

Now it remains to prove that the properties (1) to (4) are satisfied for a perturbation $\widetilde{G}$ of G. A strong C^∞ neighborhood of G is given by a function $\epsilon : \mathbb{R}^2 \to \mathbb{R}^+$ and defined as the class of maps $\widetilde{G}$ of the plane such that every derivative of $\widetilde{G}$ at a point z is at a distance less than $\epsilon(z)$ from the corresponding derivative of G at z. If $\widetilde{G}$ is a C^0 strong perturbation of G, then $\widetilde{G}$ has an attractor at ∞, from which it follows that it is a proper map and has degree two. Moreover, given a neighborhood U of the critical set of G, there exists a C^1 strong neighborhood of G such that the set of critical points of any $\widetilde{G}$ in that neighborhood is contained in U. This is also easy to prove since critical points are determined by a C^1 condition: the Jacobian equal to zero. Furthermore, if the perturbation is of class C^3, and the initial map G is generic, then the critical set of the perturbation is C^1 close to that of G, and the type of the critical points is preserved. That is, properties (1) and (2) are immediate application of the generic conditions imposed on the maps G under consideration. Then property (3) follows from the fact that the map is two-to-one in the un bounded component of the complement of the preimage of the critical set, and finally this implies property (4).

The proof for $G \in \Theta_+$ is similar and will be omitted. □

Proof of part (iii) of Theorem 1. Until now it was proved that generic quadratic maps belong to one of two classes of geometric equivalence. It follows that no other class may contain an open set. □

5. A one-parameter family

In this section we analyze some global aspects of the dynamics of the one-parameter family $f_\mu(z) = z^2 - \mu\bar{z}$, where $\mu \in \mathbb{R}$, $z \in \mathbb{C}$ and $\bar{z}$ denotes the conjugated of the complex number z. Several properties about the dynamics of this family are exposed in [5] and [6]. If $I : \mathbb{C} \to \mathbb{R}^2$ is given by $I(x + iy) = (y, -x)$ and $G_\mu = I \circ f_\mu \circ I^{-1}$, then

$$G_\mu(x, y) = (-2xy + \mu x, x^2 - y^2 - \mu y). \tag{14}$$

Observe that for every μ the map G_μ belongs to Θ_- and verifies the symmetries: $G_\mu \circ J = J \circ G_\mu$ and $R_\alpha \circ G_\mu \circ R_\alpha = G_\mu$, where $J(x,y) = (-x,y)$ and R_α is the rotation of angle $\alpha = \frac{2\pi}{3}$.

The map G_2 has very interesting features: it is a two-dimensional analogue of the map $x \to -x^2 - 2x$ on the interval $[-3,1]$. The next theorem, which was proved in [5], emphasizes the importance of this map.

Theorem 2. *The following properties are satisfied by the mapping G_2:*

(i) The basin of attraction of ∞ is the unbounded component of the complement of $G_2(\ell)$, where ℓ is the critical set of G_2.

(ii) The restriction of G_2 to the complement of this basin is conjugated to a Baker-like map.

Let T be an equilateral triangle; joining the middle points of the sides of T one obtains an equilateral triangle T'. The Baker-like map mentioned in the statement above is obtained as follows (see [5]): first, carry T into T' by means of four affine maps with singularities at the sides of T', then apply a symmetry with respect to one of the sides of T' and finally multiply by two, to fit again on T. The map obtained has a fixed vertex while the other two are two-periodic. Observe that the map is expanding except for the singularities, each point in the interior of T has four preimages, while the restriction to the boundary is two-to-one. Moreover, this map preserves the Lebesgue measure.

Denote by B_μ the basin of attraction of ∞ for the map G_μ and by ∂B_μ its boundary. The results stated in the theorem above imply that the deltoid $\widetilde{\ell}$, closure of $G_2^{-1}(G_2(\ell)) \setminus \ell$, is equal to $G_2(\ell)$ and also equal to ∂B_2, while the restriction of G_2 to the complement of B_2 preserves a smooth measure. Note that the restriction of G_2 to ∂B_2 is conjugated to the circle map $z \to z^2$. The map G_2 is highly unstable, the bifurcations of the dynamics around G_2 depends on the relative positions of the critical points and the basin of attraction of ∞. The study of the boundary of B_2 is determinant in the global behavior of the perturbations of G_2.

We will just consider perturbations of G_2 within the family G_μ. The main goal in this section is to prove the next theorem:

Theorem 3. *The family G_μ has the following two properties:*

(i) If $\mu < 2$, then B_μ is simply connected (if considered as $B_\mu \cup \{\infty\}$).

(ii) The complement of B_μ is a Cantor set for every μ large.

Since G_μ belongs to Θ_-, the following parameter values are obtained from (2): $p = -2$, $a = \mu$, $b = t = k_1 = k_2 = 0$. So, $\alpha = 0$, $\rho = \mu$ and $A = -Id$; see (6), (7) and (9). Keeping the meaning of ℓ and $\widetilde{\ell}$ for the mapping G_μ, it follows that $G_\mu(\ell)$ and $\widetilde{\ell}$ are parametrized, respectively, by $\omega \mapsto -\frac{\mu^2}{4}\Delta_0(\omega)$ and $\omega \mapsto -\frac{\mu}{2}\Delta_0(\omega)$, with $\omega \in [0, 2\pi)$. From these facts it is easy to conclude that:

- If $\mu = 2$, then $\widetilde{\ell} = G_\mu(\ell)$.
- If $\mu < 2$, then $G_\mu(\ell) \subset \operatorname{int} \widetilde{\ell}$.
- If $\mu > 2$, then $G_\mu(\ell) \subset \operatorname{ext} \widetilde{\ell}$.

Here $\operatorname{int} \gamma$ and $\operatorname{ext} \gamma$ denote, respectively, the bounded and unbounded regions provided by the complement of the plane simple closed curve γ.

Lemma 4. *If L is a simple closed curve such that $G_\mu(\ell) \subset \operatorname{int} L$, then $G_\mu^{-1}(L)$ is also a simple closed curve and $G_\mu : G_\mu^{-1}(L) \to L$ is two-to-one. Moreover, if $L_1 = G_\mu^{-1}(L) \subset$*

$\overline{\operatorname{ext} L}$, then $L_2 = G_\mu^{-1}(L_1) \subset \overline{\operatorname{ext} L_1}$ and $G_\mu(L) \subset \overline{\operatorname{int} L}$. On the other hand, if L is a simple closed curve and $L \subset \operatorname{int} G_\mu(\ell)$, then L_1 is the disjoint union of four simple closed curves.

Proof. The first assertion is an immediate consequence of part (iv) in Theorem 1. Next assume by contradiction that $L_1 \subset \overline{\operatorname{ext} L}$ and suppose that there exists a point $x \in \operatorname{ext} L_2 \cap \operatorname{int} L_1$. Observe that if $x \in \operatorname{ext} L_2$, then $G_\mu(x) \in \operatorname{ext} L_1$. But if $x \in \operatorname{int} L_1$, then $G_\mu(x) \in \operatorname{int} G_\mu(L_1) = \operatorname{int} L \subset \operatorname{int} L_1$, which is absurd; thus $L_2 \subset \overline{\operatorname{ext} L_1}$. To prove that $G_\mu(L) \subset \overline{\operatorname{int} L}$, note that $G_\mu^{-1}(L) \subset \overline{\operatorname{ext} L}$ implies that $L \subset G_\mu(\overline{\operatorname{ext} L}) = \overline{\operatorname{ext} G_\mu(L)}$, therefore $G_\mu(L) \subset \overline{\operatorname{int} L}$. The last assertion is also direct consequence of Theorem 1. $\square$

Proof of part (i) of Theorem 3. As exhibited above, the deltoids $G(\ell)$ and $\tilde{\ell}$ coincide for $\mu = 2$. The vertices of this deltoid are the fixed point $r_2 = (0, -3)$ and a two-periodic orbit $\{p_2, q_2\}$. Denote by r_μ, p_μ, q_μ the analytic continuation of these points. The circle C through these three points is centered at the origin and has radius $1 + \mu$. Since $\mu < 2$ we have

$$G_\mu(\ell) \subset \operatorname{int} \widetilde{\ell} \subset \operatorname{int} C. \tag{15}$$

Consider the function $\chi : \mathbb{C} \to \mathbb{R}$ defined by $\chi(z) = |z|$ for all $z \in \mathbb{C}$. Observe that for every $z \in \operatorname{ext} C$, $\chi(f_\mu(z)) - \chi(z) \geq |z|^2 - (\mu + 1)|z| > 0$. This says that χ is a Lyapunov function for the restriction of f_μ to $\operatorname{ext} C$, therefore this set is contained in the basin of ∞ for this map.

We claim that $\{G_\mu^{-n}(\operatorname{ext} C)\}_{n \geq 0}$ is an increasing sequence of simply connected sets. Note that the map I, defined at the beginnig of this section, leaves C invariant; hence $\operatorname{ext} C$ is also invariant under G_μ and contained in B_μ. It follows from Lemma 4 that $G_\mu^{-1}(C) \subset \operatorname{int} C$. On the other hand, equation (15) and the same lemma imply that $G_\mu^{-1}(C)$ is a simple closed curve. Joining these two facts we have that

$$G_\mu^{-1}(\operatorname{ext} C) = \operatorname{ext} G_\mu^{-1}(C) \supset \operatorname{ext} C.$$

As $G_\mu(\ell) \subset \operatorname{int} C$, Lemma 4 also implies that $\widetilde{\ell} \subset \operatorname{int} G_\mu^{-1}(C)$. It follows that $G_\mu(\ell)$ is also contained in $\operatorname{int} G_\mu^{-1}(C)$. Hence, the preceding argument implies that $G_\mu^{-2}(C)$ is a simple closed curve, which obviously is contained in $\operatorname{int} G_\mu^{-1}(C)$. Thus by a recursive discourse, the claim follows by induction. Finally, since the basin of ∞ satisfies $B_\mu = \bigcup_{n \geq 0} G_\mu^{-n}(C)$, the proof of this part of the theorem is complete. $\square$

Proof of part (ii) of Theorem 3. A simple calculation shows that $G_\mu(\ell) \subset \operatorname{int} C$ for all $\mu > 2(1 + \sqrt{2})$; so, every critical value, and hence every critical point, belongs to B_μ. In this case the complement B_μ^c of B_μ satisfies $B_\mu^c = \bigcap_{n \geq 0} G_\mu^{-n}(\operatorname{int} \widetilde{\ell})$. Then, by standard arguments one can prove that B_μ^c has uncountably many components, but to show that it is a Cantor set we need to make μ larger. Indeed, for μ sufficiently large, it will be showed that the distance between the critical set ℓ and the preimage of $\widetilde{\ell}$ is large and the differential at these points expands any vector at a constant rate. Note that $G_\mu^{-1}(\widetilde{\ell})$ has four connected components, denoted by K_μ^i $(i = 0, 1, 2, 3)$. One of these components, say K_μ^0, is contained in $\operatorname{int} \ell$. By calculating the vertices of K_μ^0 one can see that for every $R > 4$ there exists $\mu(R)$ such that K_μ^0 is contained in the disc of center 0 and radius R, for every $\mu > \mu(R)$. Using the symmetries of the mapping, the same property holds for every K_μ^i. Fix any $R > 4$. Since $DG_\mu(x, y) = \begin{pmatrix} -2y + \mu & -2x \\ 2x & -2y - \mu \end{pmatrix}$, it follows that for all (x, y) in K_μ^0, $DG_\mu(x, y)$ expands uniformly any nonzero vector for every $\mu > \mu(R)$. On the other hand, let K^1_μ be the component contained in the exterior of ℓ and intersecting the vertical axis. For every $(x, y) \in K^1_\mu$, it holds that $|x| < R$ and $-\frac{\mu}{2} - y \sim \mu$. Then the expansion of DG_μ at points in K^1_μ can be equally obtained. Since the regions K^2_μ and K^3_μ can be obtained from K^1_μ by special rotations, the result follows using the symmetries of the map G_μ. □

6. Conclusions and questions.

In this final section we discuss some problems related to the topics of this article.

6.1. **The geometry of critical sets.** In the wide world of planar quadratic maps there are finitely many classes of geometrically equivalent maps. In the generic set Q_g such a classification was possible mainly by two reasons: first, the mechanism created to understand the preimages of the deltoids or hypdeltoids; second, because these maps are injective when restricted to its critical sets. However, for nongeneric quadratic maps these restrictions are not necessarily injective, but one can directly check the assertion in each one of the parts of the decomposition established in sections 7 to 9 in [1], where the nongeneric quadratic maps of plane were classified.

It is easy to see that even within the class of generic maps of the plane having just one component of critical points, there exist infinitely many nonequivalent maps. The next example illustrates this claim.

Example 1. Consider the one-parameter family of plane endomorphisms:

$$F_\mu(x, y) = (x^3 - 3xy, y + f_\mu(x)).$$

The critical set of F_μ is the curve $y = x^2 + xf'_\mu(x)$. These maps have nondegenerate critical points, a cusp point occurs at every (x, y) in the critical set such that $2x+2f'_\mu(x)+xf''_\mu(x) = 0$. For example, by choosing the function f_μ so that $xf''_\mu(x)+2f'_\mu(x) = \mu \sin x$, there exist values of μ for which the number of cusps is arbitrarily large. Then there exist infinitely many different classes of geometric equivalence within that family.

The problem of classifying under geometric equivalence degree three polynomial endomorphisms is not possible with similar techniques. At this point, we like to pose the following question: *Are there finitely many equivalence classes of generic polynomials of a given degree?*

6.2. **Dynamics of plane maps.** As was said in the introduction, it is known that generic quadratic maps of the plane having no fixed points, must have empty limit sets. The question arising is if the bifurcation giving rise to the appearance of a first fixed point occurs in the boundary of the basin of attraction of ∞. That is, if f_μ is a one-parameter family of generic maps, and f_0 is the first map having fixed points, then we ask wether there exists an interval $[0, \mu_0]$ such that f_μ has a fixed point in the boundary of the basin of ∞. This problem seems to be very difficult, and a positive answer would give a new element for understanding globally the dynamics of these maps. More generally, we state the following open question: *Does a generic quadratic map of the plane (having fixed points) necessarily have a fixed point in the boundary of the basin of* ∞*?*

6.3. **Dynamics of the family $G_\mu(z) = z^2 - \mu\bar{z}$ and its perturbations.** Some questions that would be interesting to answer concern also with the boundary of the basin of ∞. It is natural to ask if the fact that every critical point is contained in the basin of ∞ implies that the nonwandering set is a hyperbolic set, however this seems to be very difficult to prove. Being less ambitious, one can ask if at least for the one parameter family under consideration, it holds that for parameters μ little larger than two, the complement of the basin is an expanding Cantor set.

There is another interesting question concerning the basin of ∞. It is clear that the complement of the basin is a forward invariant set. It was sometimes conjectured (for this family and also for others families of endomorphisms of the plane appearing in diverse models) that, as some numerical experiments have shown, there is a unique attractor in the complement of the basin of ∞. In other words, it is asked if the plane is subdivided into three sets: the basin of ∞, the basin of another attractor and the boundary of both sets; see [6] and references therein.

We want to state another problem. Observe that the restriction of G_2 to the boundary of the basin of ∞ is a degree two map isotopic to the map $z \to z^2$ in the unit circle. Moreover, this map is a local homeomorphism, but has three degenerate critical points, whose images are periodic repellers. It is an interesting problem to solve if this invariant curve has some kind of persistence. When $\mu > 2$ the set of critical points is contained in the basin of ∞, and it is impossible for the curve to persist. But consider the case where $\mu < 2$. In this case the situation is different because the set of critical points does not intersect the closure of the basin of ∞. We finish this section with two questions: *Is it true when $\mu < 2$ that an invariant curve persists in the boundary of the basin of ∞?* If the answer to this question is yes, *what can be said about the dynamics of the restriction of G_μ to the invariant curve?*

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