Transitive Closure of Interval-valued Fuzzy Relations

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Abstract

In this paper are introduced some concepts of interval-valued fuzzy relations and some of their properties: reflexivity, symmetry, T-transitivity, composition and local reflexivity. The existence and uniqueness of T-transitive closure of interval-valued fuzzy relations is proved. An algorithm to compute the T-transitive closure of finite interval-valued fuzzy relations is showed. Some properties and some examples is given for t-representable and t-pseudo representable generalized t-norms.

Keywords: Generalized t-norms, Interval-valued Fuzzy Relations, Interval-valued Fuzzy Sets, t-norms, t-representable t-norms, T-transitive closure.

1. Introduction

Fuzzy sets, $\mathscr{FS}$, were introduced by Zadeh in 1.965 [20]. Since then many generalizations of fuzzy sets have been proposed to model the uncertainty and the vagueness in linguistic variables replacing the unit interval by another structure such as posets or lattices [5,15,13]. One of these generalizations are type-2 fuzzy sets, $\mathscr{FS}2$, [21,22,23] were introduced by Zadeh. A Type-2 fuzzy set A on a universe of discourse X, $\mathscr{FS}2$, is a fuzzy set whose membership function is another fuzzy set on [0,1]:

$$A = \{((x,u),\mu_A(x,u)) \mid \forall x \in X, \forall u \in [0,1]\}$$

Type-2 fuzzy sets have been widely studied and applied since in many cases the uncertainty can be better expressed by a fuzzy set than by a single numeric value. The problem with type 2 fuzzy sets though is their computational complexity and the difficulty

for an expert to select the adequate fuzzy subset as membership degree of an object to a linguistic label. This is why some simplifications have been proposed, such as the use of only some families of fuzzy sets such as triangular and trapezoidal ones.

Interval-valued fuzzy sets ($\mathscr{IVFS}$) were introduced in the 60s by Grattan-Guinness [14], Jahn [16], Sambuc [19] and Zadeh [21]. They are extensions of classical fuzzy sets where the membership value between 0 and 1 is replaced by an interval in [0,1]. They easily allow to model uncertainty and vagueness because sometimes it is easier for experts to give a "membership interval" than a membership degree to objects on a universe. $\mathscr{IVFS}$ are a special case of type-2 fuzzy sets that simplifies the calculations while preserving their richness as well. The intuitionistic fuzzy sets on X ($\mathscr{IFS}$) intro-

duced by Atanassov [1] are also a extension of fuzzy sets in which each element has a membership degree, μ, and a non-membership degree, ν satisfying: $\mu + \nu \leqslant 1$.

$$A = \{(x, \mu(x), \nu(x)) \mid x \in X, \mu(x), \nu(x) \in [0,1]\}$$

The value $\pi = 1 - \mu - \nu$ is a measure of the uncertainty. Intuitionistic relations are been studied widely [3,4,9,12,11,18].

This paper is organized as follows: In section three the $\mathscr{T}$-transitive closure of interval-valued fuzzy relations is defined and it is showed that it always exists and it is unique. In section four $\mathscr{T}$-transitive closure of interval-valued fuzzy relations is studied. Traditionally, the study of conjunctions between interval-valued fuzzy sets has been reduced to be modeled with t-representable t-norms. However, not all generalized t-norms are t-representable. Moreover, some of the non t-representable t-norms sometimes satisfy even more properties than t-representable t-norms [6]. Probably the most important property a fuzzy relation can fulfil is transitivity with respect a given t-norm. Since many times the data are given by a proximity relation P (i.e.: a reflexive and symmetric but not necessarily transitive fuzzy relation), there are some methods to obtain a transitive relation close to P to replace it when transitivity is required. The most popular way to do this is calculating its transitive closure. In section four we introduce the concept of T-transitive closure for an interval-valued fuzzy relation and its expression in a finite universe for any generalized t-norm. A few methods to compute it and some examples are given. In section five an algorithm to compute the $\mathscr{T}$-transitive closure is given and some examples shown. In section six a very simple application is shown.

2. Preliminaries

Definition 2.1 [7] *Let L be the lattice of intervals in [0,1] that satisfies:*

1. *$L = \{[x_1, x_2] \in [0,1]^2 \text{ with } x_1 \leqslant x_2\}$.*

2. *$[x_1, x_2] \leqslant_L [y_1, y_2]$ if and only if $x_1 \leqslant y_1$ and $x_2 \leqslant y_2$*

Also by definition:

$$[x_1, x_2] <_L [y_1, y_2] \Leftrightarrow x_1 < y_1, x_2 \leqslant y_2 \text{ or}$$
$$x_1 \leqslant y_1, x_2 < y_2$$
$$[x_1, x_2] =_L [y_1, y_2] \Leftrightarrow x_1 = y_1, x_2 = y_2.$$

$0_L =_L [0,0]$ and $1_L =_L [1,1]$ are the smallest and the greatest elements in L respectively.

L is a complete lattice and the supremum and infimum are defined as follows.

Definition 2.2 [6] *Let $\{[v_i, w_i]\}$ be a set of intervals on L. Then*

1. *$\wedge \{[v_i, w_i]\} \equiv [\inf\{v_i\}, \inf\{w_i\}]$*

2. *$\vee \{[v_i, w_i]\} \equiv [\sup\{v_i\}, \sup\{w_i\}]$*

Definition 2.3 [7] *An interval-valued fuzzy set A on a universe X can be represented by the mapping:*
$$A = \{(a, [x_1, x_2]) \mid a \in X, [x_1, x_2] \in L\}$$

Definition 2.4 [7] *Let X be a universe and A and B two interval-valued fuzzy sets. The equality between A and B is defined as: $A =_L B$ if and only if $A(a) =_L B(a) \; \forall a \in X$.*

Definition 2.5 [7] *Let X be a universe and A and B two interval-valued fuzzy sets. The inclusion of A in to B is defined as: $A \subseteq_L B$ if and only if $A(a) \leqslant_L B(a)$ $\forall a \in X$.*

Definition 2.6 [7] *A negation function for interval-valued fuzzy sets $\mathscr{N}$ is a decreasing function, $\mathscr{N} : L \to L$, that satisfies:*

1. *$\mathscr{N}(0_L) =_L 1_L$*

2. *$\mathscr{N}(1_L) =_L 0_L$*

If $\mathscr{N}(\mathscr{N}([x_1, x_2])) =_L [x_1, x_2]$ for all $[x_1, x_2]$ in L then $\mathscr{N}$ is called an involutive negation.

Definition 2.7 *A strong negation function for interval-valued fuzzy sets, $\mathscr{N}$, is a involutive function, $\mathscr{N} : L \to L$, that satisfies:*

1. *$\mathscr{N}(0_L) =_L 1_L$*

2. *$\mathscr{N}(1_L) =_L 0_L$*

Example 2.1 *Let $\mathscr{N}$ be the involutive mapping defined by :*

$$\mathscr{N} : L \to L$$
$$\mathscr{N}([x_1, x_2]) =_L [1 - x_2, 1 - x_1]$$

Then $\mathcal{N}$ is a negation operator for interval-valued fuzzy sets. It is trivial to prove that: $\mathcal{N}(0_L) =_L 1_L$, $\mathcal{N}(1_L) =_L 0_L$ and $\mathcal{N}(\mathcal{N}([x_1,x_2])) =_L [x_1,x_2]$.

T-norms are generalized to the lattice L in a straightforward way.

Definition 2.8 [7] *A t-norm on L function $\mathcal{T}$ is a monotone increasing, symmetric and associative operator, $\mathcal{T} : L^2 \to L$, that satisfies: $\mathcal{T}(1_L, [x_1,x_2]) =_L [x_1,x_2]$ for all $[x_1,x_2]$ in L.*

Due to monotony, it is easy to show:
$$\mathcal{T}(Joint\{[v_i,w_i]\}, [y_1,y_2]) \geqslant_L$$
$$Joint\{\mathcal{T}([v_i,w_i], [y_1,y_2])\}$$
$$\mathcal{T}(Meet\{[v_i,w_i]\}, [y_1,y_2]) \leqslant_L$$
$$Meet\{\mathcal{T}([v_i,w_i], [y_1,y_2])\}$$
Due to the associativity of $\mathcal{T}$ the conjunction of three or more intervals can be defined inductively as:
$$\mathcal{T}(a, \mathcal{T}(b,c)) =_L \mathcal{T}(\mathcal{T}(a,b), c) =_L a \triangle b \triangle c$$
$$\text{where } \triangle =_L \mathcal{T}.$$
where $a =_L [a_1,a_2]$, $b =_L [b_1,b_2]$ and $c =_L [c_1,c_2]$.

Example 2.2 *Let Inf_L be defined as follows:*
$$Inf_L([x_1,x_2], [y_1,y_2]) = Meet\{[x_1,x_2], [y_1,y_2]\}$$
It easy to prove that Inf_L is a t-norm on L.
$\underline{T}([x_1,x_2], [y_1,y_2])$ and $\overline{T}([x_1,x_2], [y_1,y_2])$ will denote the lower and the higher values of $T([x_1,x_2], [y_1,y_2])$.

Definition 2.9 [2] *Let $\{x_i\}$ in $[0,1]$. A t-norm T in $([0,1], \leqslant)$ is left-continuous if it satisfies:*
$$T(Sup \; x_i, y) = Sup \; T(x_i, y)$$
Right-continuity can be defined in a similar way. This property is also called sup-preserving.

Definition 2.10 [7] *A t-norm on L operator $\mathcal{T}$ is t-representable in L if there are two t-norms: T_1 and T_2 (T_1, T_2, in $([0,1], \leqslant)$) that satisfy:*
$$\mathcal{T}([x_1,x_2], [y_1,y_2]) =_L [T_1(x_1,y_1), T_2(x_2,y_2)]$$
where $T_1(v,w) \leqslant T_2(v,w) \; \forall v,w \in [0,1]$.

Let $x =_L [x_1,x_2]$ and $y =_L [y_1,y_2]$ be two intervals on L:

Example 2.3 $\mathcal{T} =_L [\min(x_1,y_1), \min(x_2,y_2)]$ *is t-representable in $([0,1], \leqslant)$. Note that $\min$ is the highest t-norm.*

Example 2.4 *The following product t-norm on L $\mathcal{T}$ is t-representable:*

$$\mathcal{T}([x_1,x_2], [y_1,y_2]) =_L [x_1 * y_1, x_2 * y_2]$$

Example 2.5 *There are two generalizations of the Lukasiewicz t-norm* [6]:

- $T_w([x_1,x_2], [y_1,y_2]) =_L$
  $[\max(0, x_1 + y_1 - 1), \max(0, x_2 + y_2 - 1)]$
- $T_W([x_1,x_2], [y_1,y_2]) =_L$
  $[\max(0, x_1 + y_1 - 1), \max(0, x_1 + y_2 - 1, x_2 + y_1 - 1)]$

Note that T_w is t-representable but T_W is not.

Definition 2.11 [10] *A t-norm on L operator $\mathcal{T}$ is pseudo-t-representable in $\mathcal{L}^I$ if there exists a t-norm T in $([0,1], \leqslant)$ that satisfies:*
$$\mathcal{T}([x_1,x_2], [y_1,y_2]) =_L$$
$$[T(x_1,y_1), \max\{T(x_1,y_2), T(x_2,y_1)\}]$$
The t-norm T is called the representant of $\mathcal{T}$.

Example 2.6 *It is showed some examples of pseudo-t-representable t-norms on L:*

T	$\mathcal{T}$
$\min(x,y)$	$[\min(x_1,y_1), \max(\min(x_1,y_2), \min(x_2,y_1))]$
$x*y$	$[x_1*y_1, \max(x_1*y_2, x_2*y_1)]$
$\max(0, x+y-1)$	$[\max(0, x_1+y_1-1), \max(0, x_1+y_2-1, x_2+y_1-1)]$

Definition 2.12 [7] *A t-conorm on L $\mathcal{S}$ is an increasing, commutative, and associative operator $\mathcal{S} : L^2 \to L$, that satisfies: $\mathcal{S}(0_L, [x_1,x_2]) =_L [x_1,x_2]$.*

Due to the associativity of $\mathcal{S}$ we can write:
$$\mathcal{S}(a, \mathcal{S}(b,c)) =_L \mathcal{S}(\mathcal{S}(a,b), c) =_L a \triangledown b \triangledown c$$
$$\text{where } \triangledown =_L \mathcal{S}.$$

Example 2.7 *Let Sup_L be defined as follows:*
$$Sup_L([x_1,x_2], [y_1,y_2]) = Joint\{[x_1,x_2], [y_1,y_2]\}$$
It easy to prove that Sup_L is a t-conorm on L.

In this paper the following definitions are proposed.

Definition 2.13 *Let $\{[v_i,w_i]\}$ be in L. A t-norm on L operator $\mathcal{T}$ is sup-preserving if and only if:*
$$\mathcal{T}(Sup_L\{[v_i,w_i]\}, [y_1,y_2]) =_L$$
$$Sup_L\{\mathcal{T}([v_i,w_i], [y_1,y_2])\}$$

Definition 2.14 *Given a t-norm on L $\mathcal{T}$ and a generalized negation $\mathcal{N}$, the operator:*
$$\mathcal{T}_{\mathcal{N}}^* =_L \mathcal{N}(\mathcal{T}(\mathcal{N}([x_1,x_2]), \mathcal{N}([y_1,y_2])))$$
is a t-conorm on L called dual t-conorm of $\mathcal{T}$ with respect to $\mathcal{N}$.

A t-norm, a negation and the dual t-conorm of $\mathcal{T}$ with respect to $\mathcal{N}$ is called a De Morgan triplet.

3. On the existence of $\mathscr{T}$-transitive closure of interval-valued fuzzy relations

Definition 3.1 *An interval-valued fuzzy relation $R : X^2 \to L$ is a generalized $\mathscr{T}$-indistinguishability if it is reflexive, symmetric and $\mathscr{T}$-transitive.*

Definition 3.2 *Let L be the lattice of intervals in $[0,1]$ and let P be a property. Let $R : X^2 \to L$ be a relation on a finite universe X. The P closure of R is the relation $R^{\mathscr{P}} : X \times X \to L$ that satisfies:*

1. *$R^{\mathscr{P}}$ satisfies P.*

2. *$R \subseteq_L R^{\mathscr{P}}$.*

3. *If $R \subseteq_L R'$ and R' satisfies P then $R^{\mathscr{P}} \subseteq_L R'$.*

Lemma 3.1 *Let R be an interval-valued fuzzy relation in a universe X and let $\mathscr{T}$ be an arbitrary t-norm on L. Then the $\mathscr{T}$-transitive closure of R always exists.*

Proof. In similar way than [2] let *Meet* the greatest lower bound of any subset of elements on L. Consider the set Ω_R of $\mathscr{T}$-transitive fuzzy relations containing R. Let us define the fuzzy relation $S_R^*(a,b) =_L Meet_{S \in \Omega_R}\{S(a,b)\}$. We will prove that S_R^* is $\mathscr{T}$-transitive. Due to monotony of generalized $\mathscr{T}$-norms it is obtained:
$\mathscr{T}(S_R^*(a,b), S_R^*(b,c)) =$
$\mathscr{T}(Meet_{S_1 \in \Omega_R}\{S_1(a,b)\}, Meet_{S_2 \in \Omega_R}\{S_2(b,c)\})$
$\leqslant Meet_{S_1 \in \Omega_R} Meet_{S_2 \in \Omega_R}\{\mathscr{T}(S_1(a,b), S_2(b,c))\}$
$\leqslant Meet_{S \in \Omega_R}\{\mathscr{T}(S(a,b), S(b,c))\} = S_R^*(a,c)\square$

Lemma 3.2 *If R be an interval-valued relation then $\mathscr{T}$-transitive closure is unique.*

Proof. Let S_1 and S_2 be two relations. If S_1 and S_2 are transitive closures of R then according to definition 3.2: $S_1 \subseteq_L S_2$ and $S_2 \subseteq_L S_1$, consequently $S_1 =_L S_2$ $\qquad\square$

4. $\mathscr{T}$-transitive closure of interval-valued fuzzy relations

The calculation of $\mathscr{T}$-transitive closure of interval-valued fuzzy relations via $sup - \mathscr{T}$ is showed. The $sup - \mathscr{T}$ product is associative if $\mathscr{T}$ is sup-preserving and the universe is finite. Moreover, it is proved that the $sup - \mathscr{T}$ product is continuous and so is the calculation of the $\mathscr{T}$-transitive closure for finite universes. An algorithm to calculate the $\mathscr{T}$-transitive closure is provided.

Proposition 4.1 *If $\mathscr{T}$ is t-representable with T_1 and T_2 ($\mathscr{T} = [T_1, T_2]$) then an interval-valued relation $R : X^2 \to L$ is $\mathscr{T}$-transitive if and only if $\underline{R}$ is T_1-transitive and $\overline{R}$ is T_2-transitive.*

Definition 4.1 *Let $\mathscr{T}$ be a t-norm on L. Let $\mathscr{S}$ be a t-conorm on L and let $\bigtriangledown$ be the n-ary t-conorm on L defined by associativity. Let R_1 and R_2 be two interval-valued fuzzy relations on a finite set $X = \{c_1, \ldots, c_m\}$. The $\mathscr{S} - \mathscr{T}$-composition of R_1 and R_2 is defined as follows:*
$$(R_1 \diamond_{(\mathscr{S}\mathscr{T})} R_2)(c_q, c_r) =_L$$
$$\bigtriangledown_{c_k \in X} \mathscr{T}(R_1(c_q, c_k), R_2(c_k, c_r))$$
where:
$$\bigtriangledown_{c_k \in X} \mathscr{T}(R_1(c_q, c_k), R_2(c_k, c_r)) =$$
$$=_L \quad \mathscr{T}(R_1(c_q, c_1), R_2(c_1, c_r)) \bigtriangledown \ldots \bigtriangledown$$
$$\mathscr{T}(R_1(c_q, c_m), R_2(c_m, c_r))$$

Definition 4.2 *The local equality relation of a interval-valued fuzzy relation R on X is the fuzzy relation E_R is:*
$$E_R(a,b) =$$
$$\begin{cases} Sup_{L \, \forall c \in X} \max(R(a,c), R(c,a)) & a = b; \\ [0,0], & a \neq b. \end{cases}$$

Definition 4.3 *An interval-valued relation R is locally reflexive if $E_R \subseteq_L R$.*

It is easy to prove that if R is a reflexive interval-valued relation then R is a locally reflexive interval-valued relation.

Lemma 4.1 *R is $\mathscr{T}$-transitive if and only if $R \diamond_{Sup_L T} R \subseteq_L R$.*

Proof. The proof is a straightforward generalization of the proof for fuzzy relations.

- R is $\mathscr{T}$-transitive $\Rightarrow R \diamond_{Sup_L T} R \subseteq_L R$:
  $(R \diamond_{Sup_L T} R)(a,b) =_L$
  $Sup_{L \, c \in X}\{\mathscr{T}(R(a,c), R(c,b))\} \leqslant_L R(a,b)$
  due to $\mathscr{T}(R(a,c), R(c,b)) \leqslant_L R(a,b) \; \forall a,b,c \in X$

- $R \diamond_{Sup_L T} R \subseteq_L R \Rightarrow R$ is $\mathscr{T}$-transitive:
  $Sup_{L \, d \in X}\{\mathscr{T}(R(a,d), R(d,b))\} \leqslant_L R(a,b) \; \forall a,b \in X$
  Then $\forall c \in X \; \mathscr{T}(R(a,c), R(c,b)) \leqslant_L R(a,b)$

$\qquad\square$

Proposition 4.2 *Let X be an arbitrary universe. Let* $\{S_1,...,S_n\}$ *a set of interval-valued relations. If* $\mathscr{T}$ *is a sup-preserving generalized* $\mathscr{T}$*-norm, then it satisfies:*

$$R \diamond_{Sup_L \mathscr{T}} (Sup_{L \forall i=1..n}\{S_i\}) =_L$$
$$Sup_{L \forall i=1..n}\{R \diamond_{Sup_L \mathscr{T}} S_i\}$$

Proof. The proof is a straightforward generalization of the proof for fuzzy relations.

$$R \diamond_{Sup_L \mathscr{T}} (Sup_{L \forall i=1..n}\{S_i\})(a,c) =_L$$
$$Sup_{L\, b \in X}\{\mathscr{T}(R(a,b), Sup_{L \forall i=1..n}\{S_i(b,c)\}\}$$
$$=_L Sup_{L \forall i=1..n} Sup_{L\, b \in X}\{\mathscr{T}(R(a,b), S_i(b,c))\}$$
$$=_L (Sup_{L \forall i=1..n}\{R \diamond_{Sup_L \mathscr{T}} S_i\})(a,c)$$

$\square$

Definition 4.4 *Given a t-norm on L* $\mathscr{T}$*, the* $\mathscr{T}$*-power* $R^{(n)\mathscr{T}}$ *of a fuzzy relation R on X is recursively defined as follows:*

1. $R^{(1)\mathscr{T}} \equiv R$

2. $R^{(n)\mathscr{T}} \equiv R^{(n-1)\mathscr{T}} \diamond_{Sup_L \mathscr{T}} R$

Lemma 4.2 *If* $A \subseteq_L B$ *then* $A^{(k)} \subseteq_L B^{(k)}$

Proof. Trivial due to monotony of $\mathscr{T}$ $\square$

Lemma 4.3 [8] *Let* $\mathscr{T}$ *be any t-norm on L, then:*
$$\mathscr{T}([x_1,x_2],[y_1,y_2]) \leqslant_L Inf_L([x_1,x_2],[y_1,y_2])$$
$$\forall [x_1,x_2],[y_1,y_2] \in L$$

Lemma 4.4 [8] *Let* $\mathscr{S}$ *be any t-conorm on L, then:*
$$\mathscr{S}([x_1,x_2],[y_1,y_2]) \geqslant_L Sup_L([x_1,x_2],[y_1,y_2])$$
$$\forall [x_1,x_2],[y_1,y_2] \in L$$

Lemma 4.5 *If* $\mathscr{T}$ *is t-representable with* T_1, T_2 *in* $([0,1],\leqslant)$ *then:*
$$R^{(n)\mathscr{T}} = [\underline{R}^{(n)T_1}, \overline{R}^{(n)T_2}]$$

Proof. $R^{(n)\mathscr{T}}(c_q,c_r) \equiv R_{q,r}^{(n)\mathscr{T}} =$

$$= Sup_{L\, k \in X} \mathscr{T}(R_{q,k}^{(n-1)\mathscr{T}}, R_{k,r})$$
$$= Sup_{L\, k \in X}[T_1(\underline{R}_{q,k}^{(n-1)\mathscr{T}}, \underline{R}_{k,r}), T_2(\overline{R}_{q,k}^{(n-1)\mathscr{T}}, \overline{R}_{k,r})]$$
$$= [\max T_1(\underline{R}_{q,k}^{(n-1)\mathscr{T}}, \underline{R}_{k,r}), \max T_2(\overline{R}_{q,k}^{(n-1)\mathscr{T}}, \overline{R}_{k,r})]$$
$$= [\underline{R}_{q,r}^{(n-1)\mathscr{T}} \circ_{T_1} \underline{R}_{q,r}, \overline{R}_{q,r}^{(n-1)\mathscr{T}} \circ_{T_2} \overline{R}_{q,r}]$$
$$= \quad [\max_{k \in X} T_1(\underline{R}_{q,k}^{(n-2)\mathscr{T}}, \underline{R}_{k,r}) \quad \circ_{T_1}$$
$$\underline{R}_{q,r}, \max_{k \in X} T_2(\overline{R}_{q,k}^{(n-2)\mathscr{T}}, \overline{R}_{k,r}) \circ_{T_2} \overline{R}_{q,r}]$$
$$= [(\underline{R}^{(n-2)\mathscr{T}} \circ_{T_1} \underline{R}^{(2)T_1})_{q,r}, (\overline{R}^{(n-2)\mathscr{T}} \circ_{T_2}$$

$$\overline{R}^{(2)T_2})_{q,r}]$$
$$= \underbrace{...}_{n-2\, times} = [\underline{R}^{(n)T_1}, \overline{R}^{(n)T_2}]$$

$\square$

Theorem 4.1 *Let* $\mathscr{T}$ *be a t-representable t-norm* $(\mathscr{T} = [T_1, T_2])$ *and let* $R = [\underline{R}, \overline{R}]$ *be a interval-valued relation. Then, the* $\mathscr{T}$*-transitive closure interval-valued of R,* $R^{\mathscr{T}}$*, satisfies:*

$$R^{\mathscr{T}} = [\underline{R}^{T_1}, \overline{R}^{T_2}]$$

Proof. Trivial due to lemma 4.5 $\square$

Lemma 4.6 *If* $\mathscr{T}$ *is pseudo-t-representable with T, then:*
$$R^{(n)\mathscr{T}} = [\underline{R}^{(n)T}, \max_{k:1..n}\{\underline{R}^{(k-1)T} \circ_T \overline{R} \circ_T \underline{R}^{(n-k)T}\}]$$

Proof. Note that the composition of fuzzy relations is associative and distributes over "inf" and "sup" fuzzy relations. Note that $R^{(1)T}$ is the relation identity. It is verified to $n = 2$. Let's suppose that

$$R^{(n)\mathscr{T}} = [\underline{R}^{(n)T}, \max_{k:1..n}\{\underline{R}^{(k-1)T} \circ_T \overline{R} \circ_T \underline{R}^{(n-k)T}\}]$$

Then:
$R^{(n+1)\mathscr{T}} =$

$$= [\underline{R}^{(n+1)T}, \max\{\overline{R}^{(n)\mathscr{T}} \circ_T \underline{R}, \underline{R}^{(n)\mathscr{T}} \circ_T \overline{R}\}]$$
$$= [\underline{R}^{(n+1)T}, \max\{\max_{k:1..n}\{\underline{R}^{(k-1)T} \circ_T \overline{R} \circ_T \underline{R}^{(n-k)T}\} \circ_T \underline{R}, \underline{R}^{(n)T} \circ_T \overline{R}\}]$$
$$= [\underline{R}^{(n+1)T}, \max_{k:1..n+1}\{\underline{R}^{(k-1)T} \circ_T \overline{R} \circ_T \underline{R}^{(n-k)T}\}]$$

$\square$

Theorem 4.2 *Let X be an arbitrary universe and* $\mathscr{T}$ *a sup-preserving t-norm. The transitive closure of R is:*

$$R^{\mathscr{T}} =_L Sup_{L \forall k \in \mathbb{N}}\{R^{(k)\mathscr{T}}\}$$

Proof. Let $S = Sup_{L \forall k \in \mathbb{N}}\{R^{(k)\mathscr{T}}\}$ be. By proposition 4.2:

$$S^{(2)\mathscr{T}} =_L (Sup_{L \forall k \in \mathbb{N}}\{R^{(k)\mathscr{T}}\}) \diamond_{Sup_L \mathscr{T}}$$
$$(Sup_{L \forall l \in \mathbb{N}}\{R^{(l)\mathscr{T}}\}) =_L Sup_{L \forall k,l \in \mathbb{N}}\{R^{(k+l)\mathscr{T}}\} =_L$$
$$Sup_{L \forall m \in \mathbb{N}\setminus\{0,1\}}\{R^{(m)\mathscr{T}}\} \subseteq S$$

By lemma 4.1, it follows that S is T-transitive.

Moreover, if $R \subseteq S'$ and S' is T-transitive, then again by lemma 4.1 and 4.2 it holds that:

$R^{(2)\mathscr{T}} \subseteq_L S'^{(2)\mathscr{T}} \subseteq_L S', ..., R^{(k)\mathscr{T}} \subseteq_L S'^{(k)\mathscr{T}} \subseteq_L S'$
hence $S \subseteq_L S'$ and $R^{\mathscr{T}} =_L S$ ☐

Corollary 4.1 *If $\mathscr{T}$ is t-representable with T_1, T_2 in $([0,1],\leqslant)$, the transitive closure of R is:*
$$R^{\mathscr{T}} =_L Sup_{L \, \forall k \in \mathbb{N}}\{[\underline{R}^{(k)_{T_1}}, \overline{R}^{(k)_{T_2}}]\}$$

Corollary 4.2 *If $\mathscr{T}$ is pseudo-t-representable with T, the transitive closure of R is:*
$$R^{\mathscr{T}} =_L Sup_{L \, \forall k \in \mathbb{N}}\{[\underline{R}^{(n)_T}, \max_{q:1..k}\{\underline{R}^{(q-1)_T} \circ_T \overline{R} \circ_T$$
$$\underline{R}^{(k-q)_T}\}]\}$$

Due associativity of $\mathscr{T}$:
$$\mathscr{T}([x_1,x_2], \mathscr{T}([y_1,y_2],[z_1,z_2])) =_L$$
$$\mathscr{T}(\mathscr{T}([x_1,x_2],[y_1,y_2]),[z_1,z_2]) =_L [x_1,x_2] \triangle$$
$$[y_1,y_2] \triangle [z_1,z_2]$$
$$Inf_L([x_1,x_2], Inf_L([y_1,y_2],[z_1,z_2])) =_L$$
$$Inf_L(Inf_L([x_1,x_2],[y_1,y_2]),[z_1,z_2]) =_L$$
$$[x_1,x_2] \triangle_{L-inf} [y_1,y_2] \triangle_{L-inf} [z_1,z_2]$$

Lemma 4.7 *Let $\mathscr{T}$ be an arbitrary t-norm on L. Let P be a path with a cycle:*
$$P \equiv R(a,a_1) \triangle R(a_1,a_2) \triangle ... \triangle R(a_{t-1},a_t) \triangle$$
$$R(a_t,c_1) \triangle R(c_1,c_2) \triangle ... \triangle R(c_q,a_t) \triangle$$
$$R(a_t,a_{t+1}) \triangle ... \triangle R(a_k,b)$$
then
$$P \equiv R(a,a_1) \triangle R(a_1,a_2) \triangle ... \triangle R(a_{t-1},a_t) \triangle$$
$$R(a_t,c_1) \triangle R(c_1,c_2) \triangle ... \triangle R(c_q,a_t) \triangle R(a_t,a_{t+1}) \triangle$$
$$... \triangle R(a_k,b) \leqslant_L R(a,a_1) \triangle R(a_1,a_2) \triangle ... \triangle$$
$$R(a_{t-1},a_t) \triangle R(a_t,a_{t+1}) \triangle ... \triangle R(a_k,b)$$

Proof. Trivial due to $\mathscr{T}([x_1,x_2],[y_1,y_2]) \leqslant_L [y_1,y_2]$ for all $[x_1,x_2],[y_1,y_2]$ in L and associativity of $\mathscr{T}$ ☐

Theorem 4.3 *Let X be a finite universe with cardinality n. The transitive closure of R, $R^{\mathscr{T}}$, is:*
$$R^{\mathscr{T}} =_L Sup_{L \, k=1..n}\{R^{(k)\mathscr{T}}\}$$

Proof.
$$R^{(k)_T}(a,b) =_L$$

$$=_L Sup_{L \, a_1,a_2,...,a_{k-1}} R(a,a_1) \triangle R(a_1,a_2) \triangle$$
$$... \triangle R(a_{k-1},b)$$
$$\leqslant_L \quad Sup_{L \, a_1,a_2,...,a_{k-1}} R(a,a_1) \quad \triangle_{Inf_L}$$
$$R(a_1,a_2) \triangle_{Inf_L} ... \triangle_{Inf_L} R(a_{k-1},b) \text{ according to lemma 4.3.}$$

Now suppose $k > n$. Then $R^{(k)} \subseteq_L R^{(n)}$ because it exists a cycle (lemma 4.7). Therefore, the cases $k > n$ need not be considered ☐

Corollary 4.3 *If $\mathscr{T}$ is t-representable with T_1, T_2 in $([0,1],\leqslant)$, the transitive closure of R is:*
$$R^{\mathscr{T}} =_L Sup_{L \, k=1..n}\{[\underline{R}^{(k)_{T_1}}, \overline{R}^{(k)_{T_2}}]\}$$

Corollary 4.4 *If $\mathscr{T}$ is pseudo-t-representable with T, the transitive closure of R is:*
$$R^{\mathscr{T}} =_L Sup_{L \, k=1..n}\{[\underline{R}^{(n)_T}, \max_{q:1..k}\{\underline{R}^{(q-1)_T} \circ_T \overline{R} \circ_T$$
$$\underline{R}^{(k-q)_T}\}]\}$$

Theorem 4.4 *Let X be a finite universe with cardinality n. If R is a locally reflexive relation, the transitive closure of R is:*
$$R^{\mathscr{T}} =_L Sup_{L \, k=1..n-1}\{R^{(k)\mathscr{T}}\}$$

The proof is a straightforward generalization of the proof for fuzzy relations.

Proof. If R is a locally reflexive relation then $R(a,b) \leqslant_L R(a,a)$. Consider a cycle of length n, then:
$$R^{(n)\mathscr{T}}(a,a) =_L$$

$$=_L Sup_{L \, a_1,a_2,...,a_{k-1}} R(a,a_1) \triangle R(a_1,a_2) \triangle$$
$$...,\triangle R(a_{k-1},a) \leqslant_L$$
$$Sup_{L \, a_1,a_2,...,a_{k-1}} R(a,a_1) \triangle_{Inf_L} R(a_1,a_2) \triangle_{Inf_L}$$
$$... \triangle_{Inf_L} R(a_{k-1},a) \leqslant_L R(a,a)$$

☐

Theorem 4.5 *Let X be a finite universe with cardinality n. If R is a reflexive fuzzy relation on X then the transitive closure of R is:*
$$R^{\mathscr{T}} =_L R^{(n-1)\mathscr{T}}$$

Proof. *The proof is a straightforward generalization of the proof for fuzzy relations.*
$$R(a,b)^{(k)\mathscr{T}} =_L$$

$$=_L \mathscr{T}(R(a,a), R^{(k)\mathscr{T}}(a,b))$$
$$\leqslant_L Sup_{L \, c \in X}\{T(R(a,c), R^{(k)\mathscr{T}}(c,b))\}$$
$$=_L R^{(k+1)\mathscr{T}}(a,b)$$

Therefore $R(a,b)^{(k)\mathscr{T}} \leqslant_L R^{(k+1)\mathscr{T}}(a,b) \Leftrightarrow R^{(k)\mathscr{T}} \subseteq R^{(k+1)\mathscr{T}}$ ☐

Corollary 4.5 *If $\mathscr{T}$ is t-representable with T_1, T_2 in $([0,1],\leqslant)$, the transitive closure of R is:*
$$R^{\mathscr{T}} =_L \{[\underline{R}^{(n-1)_{T_1}}, \overline{R}^{(n-1)_{T_2}}]\}$$

Corollary 4.6 *If $\mathscr{T}$ is pseudo-t-representable with T, the transitive closure of R is:*
$$R^{\mathscr{T}} =_L$$
$$\{[\underline{R}^{(n-1)_T}, \max_{q:1..n-1}\{\underline{R}^{(q-1)_T} \circ_T \overline{R} \circ_T \underline{R}^{(n-1-q)_T}\}]\}$$

Definition 4.5 *A distance d of the supremum on L is defined for all $x_1, x_2, y_1, y_2 \in [0,1]$ by $d([x_1,x_2],[y_1,y_2]) = \max(|x_1 - y_1|, |x_2 - y_2|)$.*

Definition 4.6 *Let R,S be two interval-valued fuzzy relations on a set X. The distance d between R and S is defined by:*
$$d(R,S) = sup_{x,y \in X} d(R(x,y), S(x,y))$$

Lemma 4.8 *d is a distance on the set R_X of interval-valued fuzzy relations on X.*

Proof.

The supremum of distances is a distance. $\quad\square$

Theorem 4.6 *Let T be a generalized continuous t-norm and R_X the set of interval-valued fuzzy relations on X. R_X with the sup-$\mathcal{T}$ product is an ordered topological semigroup.*

Proof.

- Associativity is a straightforward exercise.
- The interval-valued relation $E(x,y) = \begin{cases} [1,1], & x = y; \\ [0,0], & \text{otherwise.} \end{cases}$ is the identity element of R_X.
- Continuity: Since T is defined on a compact set, it is uniformly continuous. Therefore:
$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that}$$
$$\forall m,n,m',n',a,b,a',b' \in [0,1]$$
$$\max(|m-a|,|m'-a'|,|n-b|,|n'-b'|) \Rightarrow$$
$$|\mathcal{T}([m,m'],[n,n']) - \mathcal{T}([a,a'],[b,b']) < \varepsilon(*)$$
We want to prove that given two interval-valued fuzzy relations $A, B \in R_X$:
$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall M,N \in R_X$$
$$\max(d(M,A),d(N,B)) < \delta \Rightarrow d(M \circ N, A \circ B) < \varepsilon$$
Given ε, we take $\delta > 0$ satisfying (*). Then:
$$d(M \circ N, A \circ B) =$$
$$sup_{x,y \in X}|sup_{z \in X}\mathcal{T}(M(x,z),N(z,y)) -$$
$$sup_{z \in X}\mathcal{T}(A(x,z),B(z,y))| \leqslant$$
$$sup_{x,y \in X}sup_{z \in X}|\mathcal{T}(M(x,z),N(z,y)) -$$
$$\mathcal{T}(A(x,z),B(z,y))| \leqslant \varepsilon$$
- Monotonicity is an immediate consequence of the monotonicity of $\mathcal{T}$

$$\square$$

Corollary 4.7 *For any positive integer n the map assigning to an interval-valued fuzzy relation M on*

a finite set its nth power is non-decreasing and continuous if the corresponding t-norm on L $\mathcal{T}$ is continuous.

Corollary 4.8 *If a t-norm on L $\mathcal{T}$ is continuous, the map that assigns the $\mathcal{T}$-transitive closure to the interval-valued fuzzy relations on a finite set is non-decreasing and continuous.*

It is given a method to calculate the transitive closure of a interval-valued fuzzy relation. Nevertheless, there are more effective methods and algorithms. For example, it is possible to use an extension of the Floyd-Warshall algorithm [17]

5. Algorithm to compute the $\mathcal{T}$-transitive closure of an interval-valued fuzzy relation

Let R be an interval-valued relation on a finite universe X with cardinality n and let $\mathcal{T}$ be a t-norm on L. The $\mathcal{T}$-transitive closure of R, $R^{\mathcal{T}}$, can be computed using the following algorithm:

```
FOR{k=1}{n}
 FOR{i=1}{n}
  FOR{j=1}{n}
   R(i,j)=Sup_L(R(i,j),T(R(i,k),R(k,j)))
  ENDFOR
 ENDFOR
ENDFOR
```

Example 5.1 *Let $\mathcal{T}$ be a t-norm on L*

$$Inf_L(\{[x_1,x_2],[y_1,y_2]\}) =_L [\min(x_1,y_1),\min(x_2,y_2)]$$

and let $R : X \times X \to L$ be the following interval-valued relation:

$$R = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0,0] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

The computation of the $\mathcal{T}$-transitive closure of R is the following:

$$R = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0,0] \\ & [1,1] & [0.6,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

$$R = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0.6,0.8] \\ & [1,1] & [0.6,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

$$R = \begin{pmatrix} [1,1] & [0.6,0.9] & [0.6,0.9] & [0.6,0.9] \\ & [1,1] & [0.6,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

$$R^{\mathcal{T}} = \begin{pmatrix} [1,1] & [0.6,0.9] & [0.6,0.9] & [0.6,0.9] \\ & [1,1] & [0.6,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

Note that $R^{\mathcal{T}} = [\underline{R}^{\min}, \overline{R}^{\min}]$ which we already by theorem 4.1.

Example 5.2 *Let $\mathcal{T}$ be a t-norm on L $T_w([x_1,x_2],[y_1,y_2]) =_L [\max(0, x_1 + y_1 - 1), \max(0, x_2 + y_2 - 1)]$ and let $R : X \times X \to L$ be the following interval-valued relation:*

$$R = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0,0] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

The computation of the $\mathcal{T}$-transitive closure of R is the following:

$$R = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0,0] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

$$R = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0.2,0.7] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

$$R = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0.2,0.8] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

$$R^{\mathcal{T}} = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0.2,0.8] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

Note that $R^{\mathcal{T}} = [\underline{R}^W, \overline{R}^W]$ also known by theorem 4.1

Example 5.3 *Let $\mathcal{T}$ be a t-norm on L $T([x_1,x_2],[y_1,y_2]) =_L [W(x_1,y_1), \min(x_2,y_2)]$ and let $R : X \times X \to L$ be the following interval-valued relation:*

$$R = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0,0] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

The computation of the $\mathcal{T}$-transitive closure of R is the following:

$$R = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0,0] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

$$R = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0.2,0.8] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

$$R = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0.2,0.9] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

$$R^{\mathcal{T}} = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0.2,0.9] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

Note that $R^{\mathcal{T}} = [\underline{R}^W, \overline{R}^{\min}]$ also known by theorem 4.1.

Example 5.4 *Let $\mathcal{T}$ be a t-norm on L $T_W([x_1,x_2],[y_1,y_2]) =_L [\max(0, x_1 + y_1 - 1), \max(0, x_1 + y_2 - 1, x_2 + y_1 - 1)]$ and let $R : X \times X \to L$ be the following interval-valued relation:*

$$R = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0,0] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

The computation of the $\mathcal{T}$-transitive closure of R is the following:

$$R = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0,0] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

$$R = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0.2,0.5] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

$$R = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0.2,0.5] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

$$R^{\mathcal{T}} = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0.2,0.5] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

Note that the generalized $\mathcal{T}$-norm in this example is not t-representable.

6. An Application

It is possible to model preference relations using Interval-valued Fuzzy Sets. For example, it is possible to model the relation "be better or equal than". Let R be a relation which describes the degree of preference between pairs of objects. This relation is reflexive and transitive. In this context, the $\mathcal{T}$-transitive closure can be used to impose coherence to a set of dates.

Example 6.1 *Let R be an interval-valued relation which models the degree of the relation "be better or equal than" between four objects $\{a,b,c,d\}$. It is possible to model the transitivity of this relation using the t-norm on $L *_L([x_1,x_2],[y_1,y_2]) =_L [x_1*y_1, x_2*y_2]$:*

$$R = \begin{pmatrix} [1,1] & [0.6,0.9] & [0.6,0.7] & [0,0] \\ & [1,1] & [0,0] & [0.2,0.8] \\ & & [1,1] & [0,0] \\ & & & [1,1] \end{pmatrix}$$

However, R is not a $*_L$-transitive relation so some membership degrees must be wrong.

$$R^{\mathcal{T}} = \begin{pmatrix} [1,1] & [0.6,0.9] & [0.6,0.7] & [0.16,0.72] \\ & [1,1] & [0.48,0.63] & [0.2,0.8] \\ & & [1,1] & [0.096,0.5] \\ & & & [1,1] \end{pmatrix}$$

It is possible to impose transitivity replacing R by $R^{\mathcal{T}}$.

7. Conclusions

In this paper the main properties of interval-valued fuzzy relations are studied: reflexivity, symmetry, composition, local reflexivity and T-transitivity. The existence and uniqueness of the T-transitive closure for $\mathcal{L}$-fuzzy relations has been proved when $\mathcal{L}$ is a complete lattice. Several expressions to compute the T-transitive closure of $\mathscr{IVFR}$ are proposed for any t-norm on L, any generalized t-representable t-norm and any t-pseudo representable t-norm. Finally, an algorithm based on the Floyd-Warshall method to compute the T-transitive closure of $\mathscr{IVFR}$ for any t-norm on L T is showed. Some examples are provided.

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