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A NOTE ON THE DYNAMICS AROUND THE LAGRANGE POINTS OF THE EARTH-MOON SYSTEM IN A COMPLETE SOLAR SYSTEM MODEL

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This paper studies the dynamics of a massless particle around the libration points of the Earth-Moon system in a full Solar System gravitational model. The study is based on the analysis of the quasi-periodic solutions around the equilibrium points. For the analysis and computation of the quasi-periodic orbits, a novel iterative algorithm is introduced which is a combination of the multiple shooting method and a refined Fourier analysis of the orbits computed with the multiple shooting. Using as initial seeds the libration point orbits of Circular Restricted Three Body Problem, determined by Lindstedt-Poincaré methods, the procedure is able to refine them in a complete Solar System model for large time-spans covering most of the relevant Sun-Earth-Moon periods. For the collinear points, the developed approach works well and reveals the strong relevance of the phase space around the equilibrium points in both models. For the triangular points, difficulties appear and an intermediate model (bicircular model, BCM) is introduced to aid the refinement.

I. INTRODUCTION

The purpose is to study of the phase space around the libration points in the Earth-Moon system when the remaining planets of the Solar System are taken into account. In the simplified model defined by the Circular Restricted Three Body Problem (CR3BP), the description of the phase space around L_1 and L_2 has been done in the past, either using semi-analytical techniques [1] or numerical ones [2]. When using more realistic models of motion, the refinement of different kinds of libration point orbits around both points has some problems for long time intervals. These problems are more evident for the Earth-Moon L_2 point due to a 1:2 resonance between the natural frequency (ω_h) of some halo orbits and the external frequency due to the perturbation of the Sun (ω_s, or ω_{B2} in the following sections); in fact $\omega_h \approx 2\omega_s$ for some orbits of the halo family. To analyze more closely this fact, Andreu [3] introduced an intermediate model between the CR3BP and the restricted n-body problem, which is the so-called Quasi-Bicircular Problem.

In this paper, using as gravitational model the restricted n-body problem and taking as primaries the Solar System bodies (SSRnBP), we compute a large set of libration orbits around the dynamical substitutes of libration points for this model. For this purpose, it has been necessary to develop a numerical procedure that, starting with orbits computed in the CR3BP, does their numerical refinements in the SSRnBP for large time intervals. In order to identify frequencies due to the perturbing Solar System bodies with a clear physical meaning, we use the following set of basic ones [4]:

- The mean longitude of the Moon, $\omega_{B1} = 1.0$,
- The mean elongation of the Moon from the Sun, $\omega_{B2} = 0.925195997455093$,
- The mean longitude of the lunar perigee, $\omega_{B3} = 8.45477852931292\times10^{-3}$,
- The longitude of the mean ascending node of the lunar orbit on the ecliptic, $\omega_{B4} = 4.01883841204748\times10^{-3}$,
- The Sun's mean longitude of perigee, $\omega_{B5} = 3.57408131981537\times10^{-6}$,

where the values are given in terms of cycles per lunar revolution.

In our case, the frequencies determined by the Fourier analysis of the quasi-periodic orbits computed, will be identified as linear combinations of the above set of fundamental basic frequencies and the ones intrinsic to the dynamics, which will be close to the ones of the CR3BP orbit. The accuracy of a linear combination is denoted by $|\varepsilon| = \left|\omega - \sum_{i=1}^{n} k_i\omega_i\right|$, where ω is the frequency obtained from the refined Fourier analysis, k_i and ω_i are respectively the integer coefficients and the basic frequencies.

II. THE NUMERICAL REFINEMENT PROCEDURE

The algorithm for the numerical computation of the quasi-periodic orbits has three main components [5].

- **CR3BP**. It is used to provide an initial seed for the shooting procedure. Various families of bounded orbits exist [1][2] in the CR3BP, which can be computed either using analytical or numerical methods. Their analytical expansions using Lindstedt-Poincaré methods up to high order give good approximations [6].
- **Parallel shooting** [7]. Taking as initial seed the nodes, equally or unequally spaced in time along a libration point orbit, it solves the matching equations using a modified Newton method. The output is a series of nodes along the refined trajectory. The matching errors are very small (less than 1 mm in position and 0.1 mm/day in velocity) so the orbit can be seen as a continuous one. For large time-spans, the convergence of the multiple shooting strongly depends on the accuracy of the initial guess as a "solution" of the SSRnBP.
- **Reined Fourier analysis** [8][9]. It is essentially a collocation method for approximating a quasi-periodic function with a trigonometric polynomial. For our problem, the Fourier analysis is done for $x(t)$, $y(t)$, $z(t)$ determined by the multiple shooting. The trigonometric approximations will be used to compute new guessed nodes.

The whole procedure is an iterative process whose flow chart is given in Fig. 1, where T_0 is the initial epoch, ΔT the time-span covered by nodes, ΔT^* the desired final time-span, N the number of nodes, N_{min} the minimum number of nodes required and γ_P an enlarging factor. Typical values are: T_0 = J2000.0, ΔT_0 = 400 TU, ΔT^* = 60 years, N_{min} = 20000 and γ_P = 1.3.

III. DYNAMICAL SUBSTITUTES OF THE LAGRANGE POINTS

The SSRnBP equations of motion are non-autonomous, and there are no relative equilibrium points in this model. The dynamical substitutes of the equilibrium points are defined as those solutions of the equations of motion that have as basic frequencies only those of the perturbing bodies. These substitutes are not unique since they depend, for instance, on the initial epoch at which they are computed.

The initial seeds are the CR3BP equilibrium points along a certain time interval. The orbits have been refined to achieve ΔT^* = 200 years. The intersections of the refinements with z = 0 is shown in Fig. 2. Table 1 shows that the frequencies determined by the Fourier analysis can be written as linear combinations of the perturbing basic frequencies, thus the refinements are the dynamical substitutes, which are in agreement with those obtained by other methods [10][11].

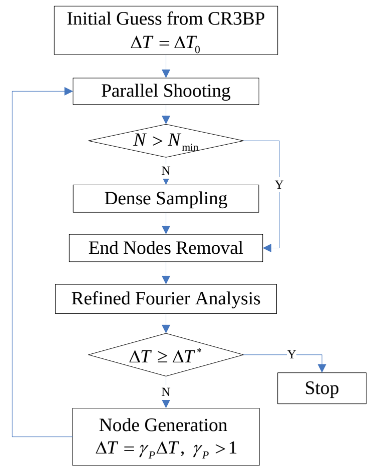

Fig. 1: The basic procedure of the proposed algorithm to refine orbits for large time-spans in the SSRnBP.

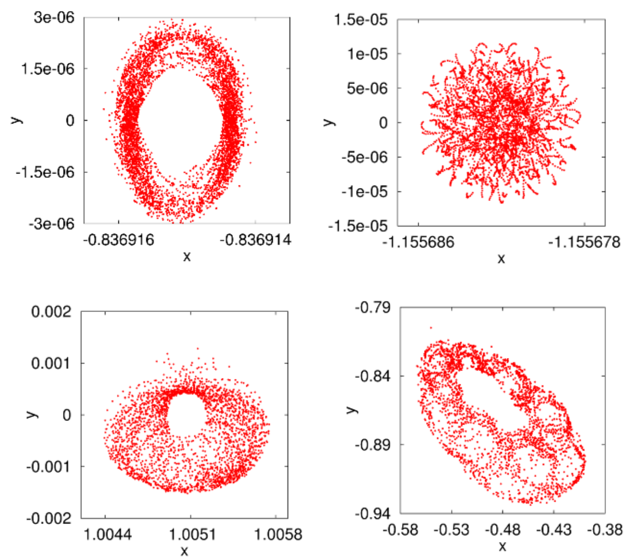

Fig. 2: Poincaré map representation of the **dynamical substitutes** of the L_1 (top left), L_2 (top right), L_3 (bottom left) and L_4 (bottom right) points in the SSRnBP (200 years).

ω_x	Amp.	k_1	k_2	k_3	k_4	$\lvert\varepsilon\rvert$
2.77558756	8.5E−7	0	3	0	0	4.4E−7
0.92519548	1.1E−7	0	1	0	0	5.2E−7
3.76713251	9.6E−8	1	3	−1	0	7.1E−7
2.70078718	8.8E−8	−1	4	0	0	3.2E−6

Table 1: Fourier analysis results of the x function of the L_2 dynamical substitute (200 years).

IV. NUMERICAL REFINEMENT OF ORBITS AROUND COLLINEAR POINTS

IV.I Refined Orbits and Their Fourier Analysis

IV.I.I The Planar and Vertical Lyapunov Families

According to Lyapunov theorem [12], two Lyapunov families of periodic orbits emanate at the collinear libration points. The planar Lyapunov orbit is located on the x–y plane, while the vertical Lyapunov orbit is 3D. For either of the two Lyapunov families, orbits of different amplitudes are associated to different energy levels. In Fig. 3, we show three refinements of the planar and vertical Lyapunov orbits around L_2, whose $C_J = 3.162$, 3.174, and 3.184, respectively.

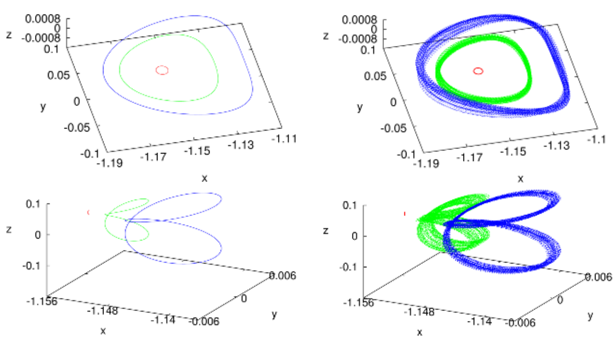

Fig. 3: CR3BP **Lyapunov** orbits and their 60-year refinements around L_2 (top, planar; bottom, vertical).

Taking for instance a L_1 vertical Lyapunov orbit, Table 2 shows the integer linear combinations of the obtained frequencies of the x coordinate in terms of the intrinsic frequency ω_v and those of the perturbation basic set. The frequencies of the y coordinate coincide with those of x.

ω_x	Amp.	k_v	k_1	k_2	k_3	k_4	$\lvert\varepsilon\rvert$
4.46019019	1.8E−3	2	0	0	0	0	2E−8
5.45173525	3.6E−4	2	1	0	−1	0	2E−7
3.46864514	3.2E−4	2	−1	0	1	0	2E−7
0.99154505	8.3E−5	0	1	0	−1	0	2E−7

Table 2: Fourier analysis results of the x function of the L_1 vertical Lyapunov orbit ($C_J = 3.186$, 60 years).

IV.I.II The Halo Families

Halo orbits in the CR3BP are spatial periodic solutions that bifurcate from the planar Lyapunov orbits at a given amplitude. They can be characterized by an amplitude parameter β. Some of these orbits ($\beta = 0.12$, 0.24, 0.36, 0.48) are displayed in Fig. 4 for L_2.

Figure 5 shows the evolution of the halo frequency ω_h for different time-spans. On the left are the results for L_1, where ω_h is very close to the frequency in the CR3BP. For the L_2 case on the right, for $\beta \in (0.2, 0.3)$, the refined orbits seem to be attracted by the resonance, and ω_h remains almost constant to $2\omega_{B2}$. Similar to Lyapunov orbits, Fourier analysis of the halo refinements show that all the identified frequencies can be written as simple linear combinations of ω_h and the basic frequency set.

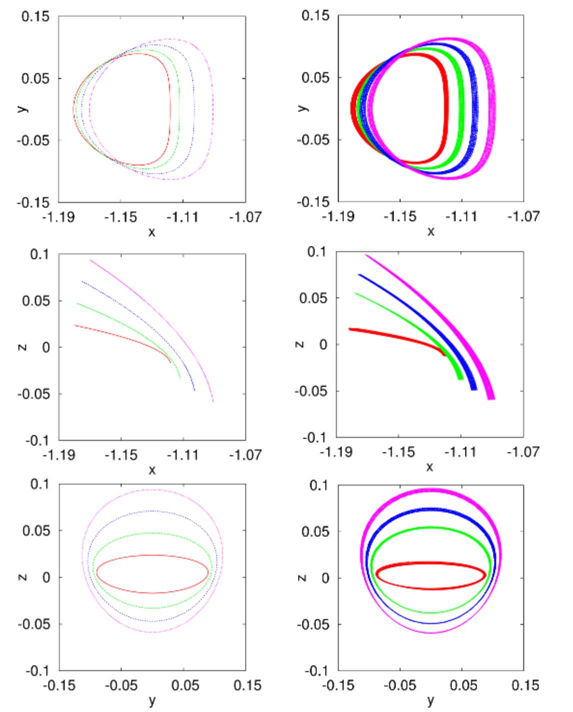

Fig. 4: CR3BP **halo** orbits and their 60-year refinements around L_2 (left, CR3BP; right, SSRnBP).

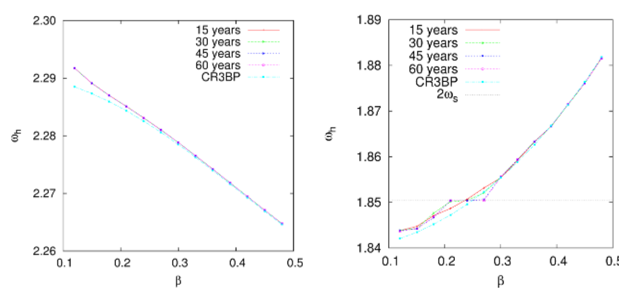

Fig. 5: Evolution with β of the frequencies for $\Delta T^* = 15$, 30, 45, 60 years for the refined **halo** orbits around L_1 (left) and L_2 (right).

IV.I.III The Quasi-halo Families

Quasi-halo orbits are quasi-periodic trajectories librating about halo orbits. They have two intrinsic frequencies, ω_h being that of the baseline halo orbit, and ω_q the frequency of the "transversal" motion. There are two associated amplitudes: β is the z-amplitude of the halo and γ measures separation from it [7].

Figure 6 shows the results of the L_2 quasi-halo refinements for 30 years. The γ–ω_h lines are with β between 0.12 and 0.48 (0.06 step size). There are no large differences for L_1 orbits computed in both models.

For the L_2 quasi-halo orbits, ω_h tends to the 1:2 resonant value 1.850392 in $\beta \in (0.18, 0.30)$. The curves of $\beta - \omega_q$ have a slightly perturbed region, which lies around β = 0.36. Fourier analysis of the quasi-halo refinements show that all the identified frequencies can be written as simple linear combinations of ω_h, ω_q and the basic frequency set.

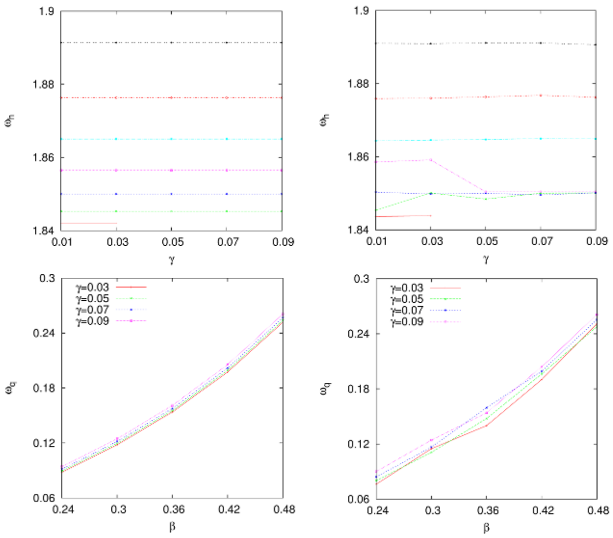

Fig. 6: Behavior of ω_h and ω_q versus β and γ of quasi-halo orbits around L_2 (left, CR3BP; right, SSRnBP).

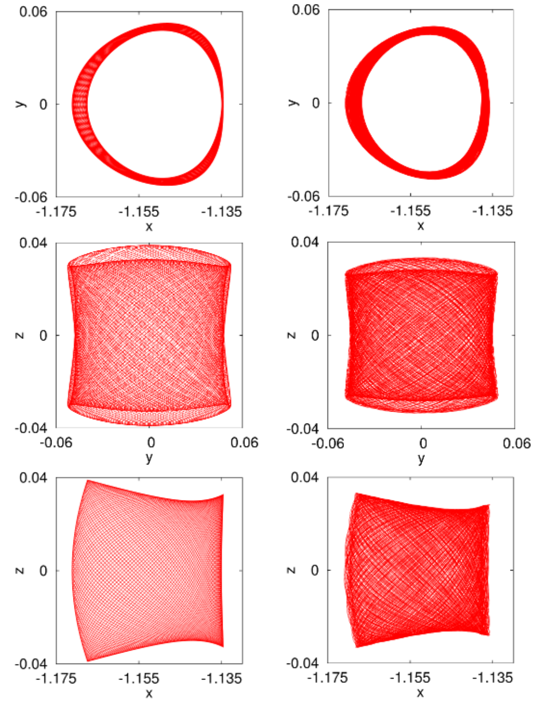

Fig. 7: CR3BP **Lissajous** orbits and their 60-year refinements around L_2 (left, CR3BP; right, SSRnBP).

IV.I.IV The Lissajous Families

Lissajous orbits are quasi-periodic trajectories around the libration points parameterized by two amplitudes, the in-plane amplitude (α) and the out-of-plane amplitude (β), and two phases. Orbits with α = 0 correspond to the vertical Lyapunov family, and β = 0 the planar Lyapunov one. For each pair (α, β), there are two intrinsic frequencies associated, one of which is the in-plane frequency (ω_p) and the other the out-of-plane frequency (ω_v).

Figure 7 shows a CR3BP L_2 Lissajous orbit and its refinement in the SSRnBP for 60 years (10 years displayed). All the identified frequencies can be written as simple linear combinations of ω_p and ω_v and the basic frequency set.

IV.II A Poincaré Map Representation of the Phase Space around $L_{1,2}$

Figure 8 shows the Poincaré maps defined by $\{ z = 0, \dot{z} > 0 \}$ associated to the quasi-halo orbits around $L_{1,2}$ points. For each point, β is constant (β = 0.36 for L_1 and β = 0.42 for L_2) and γ = 0.01, 0.03, 0.05, 0.07, 0.09. Since the value of C_J is different for each orbit, some overlapping is observed. As can be seen in Fig. 8, the refined quasi-halo orbits are enlarged in size and the tori on which they lie has some "thickness".

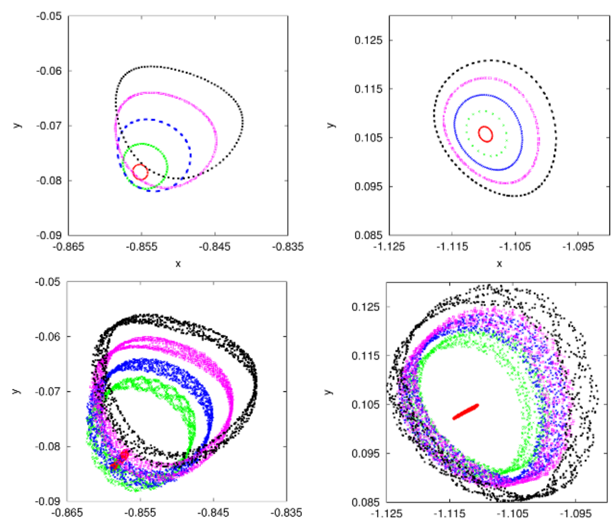

Fig. 8: Poincaré map representation with $\{z = 0, \dot{z} > 0\}$ of the **quasi-halo** orbits around L_1 (left) and L_2 (right) of the Earth-Moon system.

The results associated to the Lissajous orbits are shown in Fig. 9 (L_1) and Fig. 10 (L_2). The CR3BP intersections in the same plot share the same C_J, 3.188, 3.194, 3.200 for L_1, and 3.166, 3.174, 3.184 for L_2. The outermost curve corresponds to the entire planar Lyapunov orbit, and the fixed point in the middle represents a vertical Lyapunov orbit. In the SSRnBP, the intersections do not change much, which means that the geometry of CR3BP orbits sustains in a much more realistic dynamical model. Nevertheless, orbits around L_1 preserve their geometries quite well, while orbits around L_2 seem to have been perturbed in size.

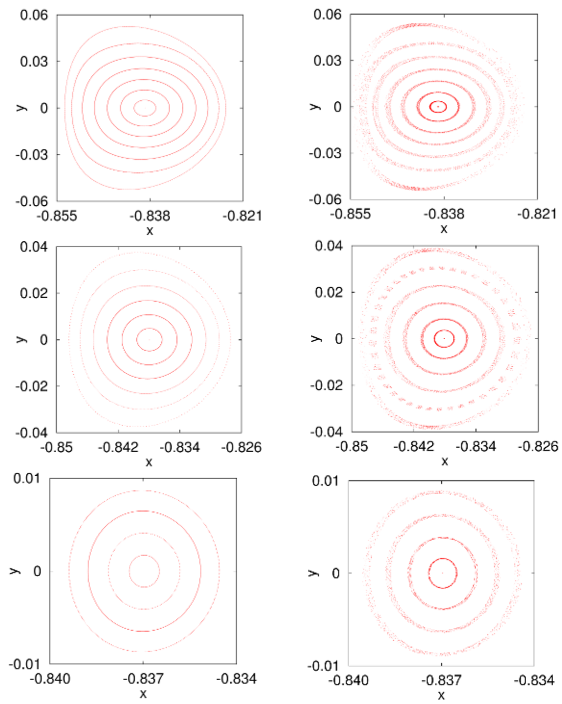

Fig. 9: Poincaré map representation with $\{z = 0, \dot{z} > 0\}$ of the **Lissajous and Lyapunov** periodic orbits around the L_1 point of the Earth-Moon system.

V. NUMERICAL REFINEMENT OF ORBITS AROUND TRIANGULAR POINTS

Three intrinsic frequencies exist for the triangular points, therefore, there are three types of Lyapunov orbits, which are the long-term family, the short-term family, and the vertical family. Besides, the three frequencies also give birth to quasi-periodic orbits. Lindstedt-Poincaré expansions can also be obtained for these orbits defined by three amplitudes (A_1, A_2, A_3) or, correspondingly, three frequencies (ω_1, ω_2, ω_3).

V.I Orbits in the Earth-Moon System

A CR3BP vertical Lyapunov orbit (A_1, A_2, A_3) = (0, 0, 0.03) around L_4 is selected, whose 10-year refinement is given in Fig. 11. It can be seen that the refined orbit is very different from its CR3BP seed. Moreover, it is difficult to obtain very long lasting orbits using the proposed algorithm. Therefore, an intermediate model, i.e., the bicircular model (BCM), is introduced. The CR3BP orbit is firstly refined in the BCM, and then used as initial guess for the ephemeris model. This approach works, but the resultant refinements are still quite far away from its CR3BP counterpart.

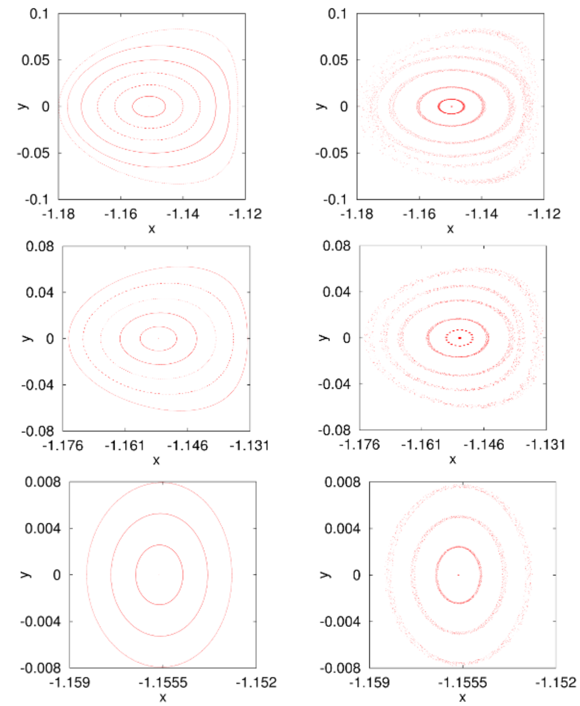

Fig. 10: Poincaré map representation with $\{z = 0, \dot{z} > 0\}$ of the **Lissajous and Lyapunov** periodic orbits around the L_2 point of the Earth-Moon system.

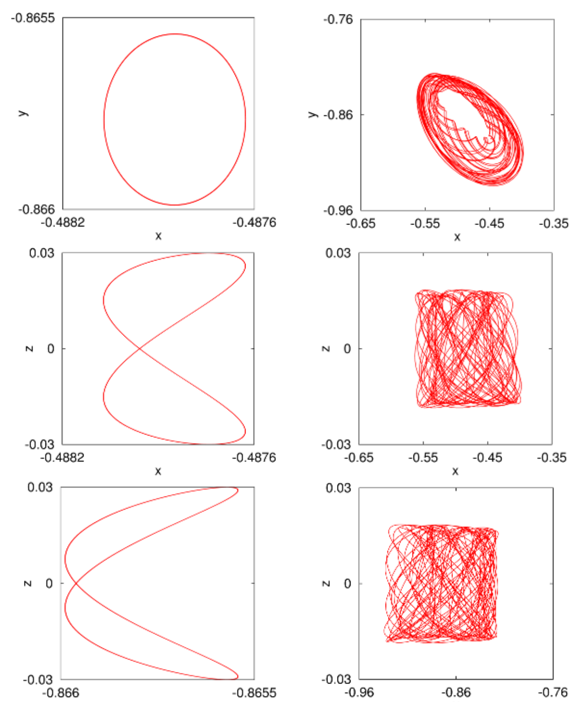

Fig. 11: CR3BP initial seed (left) and SSRnBP refinement (right) of **(0, 0, 0.03)** orbit around the Earth-Moon L_4 point (10 years).

The accuracy of Fourier analysis on the position components is worse, being on the order of 1×10^{-2}, and only a few frequencies can be detected. This could be the reason why the orbit loses its original geometry during the refining process.

V.II Orbits in the Sun-Earth/Moon System

In order to gain some insight into the problem, we turn to check some orbits in the Sun-Earth/Moon system. First we use Lindstedt-Poincaré expansions to generate a CR3BP initial seed covering 200 years with (A_1, A_2, A_3) = (0.05, 0.03, 0.03). The refinement can be done in the SSRnBP model directly using this initial guess, which is shown in Fig. 12. As can been seen, the SSRnBP refinement is in good accordance with the CR3BP initial seed in shape and size, which means that CR3BP is a good approximation of the real dynamics for the Sun-Earth/Moon system. However, the associated Fourier analysis results also show poor accuracy with very few frequencies detected.

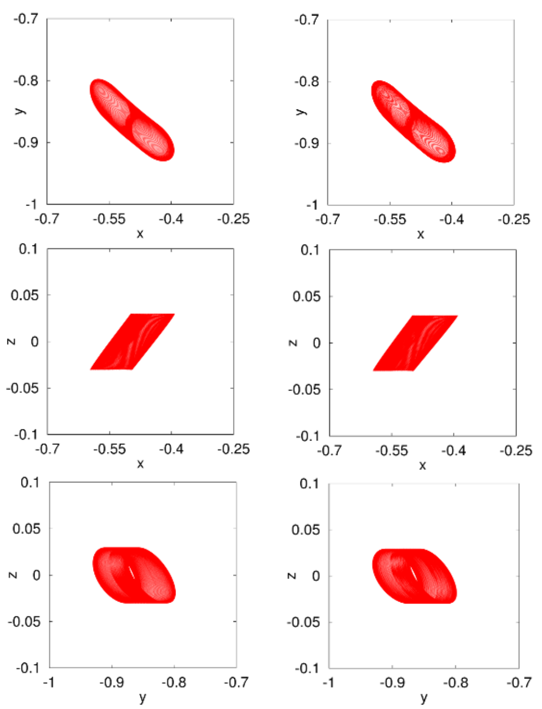

Fig. 12: CR3BP initial seed (left) and SSRnBP refinement (right) of **(0.05, 0.03, 0.03)** orbit around the Sun-Earth/Moon L_4 point (200 years).

VI. CONCLUSIONS

This paper presents a novel algorithm capable of refining large time-span orbits in a Solar System model using real ephemeris. The algorithm is iterative and combines parallel shooting with refined Fourier analysis, starting with an initial guess produced in the CR3BP. Based on this algorithm, dynamical substitutes of the Lagrange points and the orbits around L_1 and L_2, including periodic ones and quasi-periodic ones, are obtained and analyzed for 60 years in the Earth-Moon system. All the results obtained shows that the phase space around the collinear points in the CR3BP is in good accordance with that around dynamical substitutes in the ephemeris model. For the triangular point L_4, the so far obtained results are not able to show a good similarity between the two models. The proposed algorithm loses its effectiveness in refining some of the triangular orbits in the Earth-Moon system, but works well in the Sun-Earth/Moon system.

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