

# On Closed-Form Solutions to the Position Analysis of Baranov Trusses

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## Abstract

*The exact position analysis of a planar mechanism reduces to compute the roots of its characteristic polynomial. Obtaining this polynomial usually involves, as a first step, obtaining a system of equations derived from the independent kinematic loops of the mechanism. Although conceptually simple, the use of kinematic loops for deriving characteristic polynomials leads to complex variable eliminations and, in most cases, trigonometric substitutions. As an alternative, a method based on bilateration has recently been shown to permit obtaining the characteristic polynomials of the three-loop Baranov trusses without relying on variable eliminations or trigonometric substitutions. This paper shows how this technique can be applied to solve the position analysis of all catalogued Baranov trusses. The characteristic polynomials of them all have been derived and, as a result, the maximum number of their assembly modes has been obtained. A comprehensive literature survey is also included.*

*Keywords:* Position analysis, Baranov trusses, bilateration, characteristic polynomial.

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## 1. Introduction

A non-overconstrained linkage with zero-mobility from which an Assur group can be obtained by removing any of its links is defined as an Assur kinematic chain, basic truss [1, 2], or Baranov truss when no slider joints are considered [3]. Hence, a Baranov truss, named after the Russian kinematician G.G. Baranov who first presented the idea of this kind of truss in 1952 [4], corresponds to multiple Assur groups. The relevance of the Baranov trusses derive from the fact that, if the position analysis of a Baranov truss is solved, the same process can be applied to solve the position analysis of all its corresponding Assur groups. Baranov, in his seminal paper, presented 3 trusses of 7 links and 26 trusses of 9 links. In 1971, Manolescu and Erdelean identified two additional trusses of 9 links that were missing in the initial classification [5], thus completing the classification of Baranov trusses with up to 4 loops. In 1994, Yang and Yao found that the number of Baranov trusses with 11 links is 239 [6]. Unfortunately, their topologies

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were not made available and, to the best of our knowledge, they have not been published yet. Thus, strictly speaking, only the Baranov trusses with up to 9 links have been catalogued (see Table 1). This catalogue appears in Fig. 1 where each truss is identified using the nomenclature suggested by Manolescu [7].

| Links | Loops | Baranov<br>trusses | Available<br>topology | Coupling degree |    |   | Resulting<br>Assur groups |
|-------|-------|--------------------|-----------------------|-----------------|----|---|---------------------------|
|       |       |                    |                       | 1               | 2  | 3 |                           |
| 3     | 1     | 1                  | Yes                   | 1               | 0  | 0 | 1                         |
| 5     | 2     | 1                  | Yes                   | 1               | 0  | 0 | 2                         |
| 7     | 3     | 3                  | Yes                   | 3               | 0  | 0 | 10                        |
| 9     | 4     | 28                 | Yes                   | 24              | 4  | 0 | 173                       |
| 11    | 5     | 239                | No                    | 197             | 42 | 0 | 5442                      |
| 13    | 6     | ?                  | No                    | ?               | ?  | ? | 251638                    |

Table 1: Number of Baranov trusses as a function of the number of links (alternatively, number of loops), indication of whether the topologies are available in the literature, number of trusses with different coupling degrees, and number of different Assur groups resulting from eliminating one link from the trusses in each class.

The position analysis problem for a planar truss consists in, given the dimensions of all links, calculating all relative possible transformations between them all. This analysis is usually reduced to finding the roots of a polynomial in one variable, the *characteristic polynomial* of the truss. When this polynomial is obtained, it is said that the problem is solved in *closed form*. This approach is usually preferred to numerical approaches because the degree of the polynomial specifies the greatest possible number of assembly configurations of the linkage and modern software provides guaranteed and fast computation of all real roots of a polynomial equation and hence of all assembly configurations of the analyzed linkage.

The closed-form solution to the position analysis of the catalogued Baranov trusses is known for 22 of them. They have been solved on an ad hoc basis by several authors (see Tables 2-11 and the references therein) using mainly elimination techniques, as those based on Sylvester or Dixon resultants, applied to vector loop equations expressed in trigonometric or complex number terms. To the best of our knowledge, the closed-form position analysis of the trusses identified as  $9/B_2$ ,  $9/B_3$ ,  $9/B_4$ ,  $9/B_5$ ,  $9/B_6$ ,  $9/B_8$ ,  $9/B_9$ ,  $9/B_{13}$ ,  $9/B_{18}$ ,  $9/B_{19}$ , and  $9/B_{22}$  has not been reported in the literature. Nevertheless, the number of assembly modes of these trusses was studied in [8] using homotopy continuation. However, as it will be shown later, some of the reported results are erroneous. Beyond four loops, the closed-form position analysis of only two 11-link Baranov trusses and one 13-link Baranov truss have been reported. In [9], Lösch solved the five-loop version of the  $9/B_1$  Baranov truss with a procedure based on vector method and Gröbner basis. The same truss was analyzed by Wohlhart using Sylvester elimination [10]. Recently, Rojas and Thomas solved the five-loop and six-loop versions of the  $9/B_{10}$  Baranov truss [11].

An  $n$ -ary link in a Baranov truss introduces a set of distance constraints between the  $n$  involved joints. This translates into a set of quadratic equations from which an eliminant can be obtained to get a single equation in one variable. In this case, each equation is simple but the elimination process involve a large number of equations. A more compact elimination process is obtained when the set of  $2n + 1$  independent loop

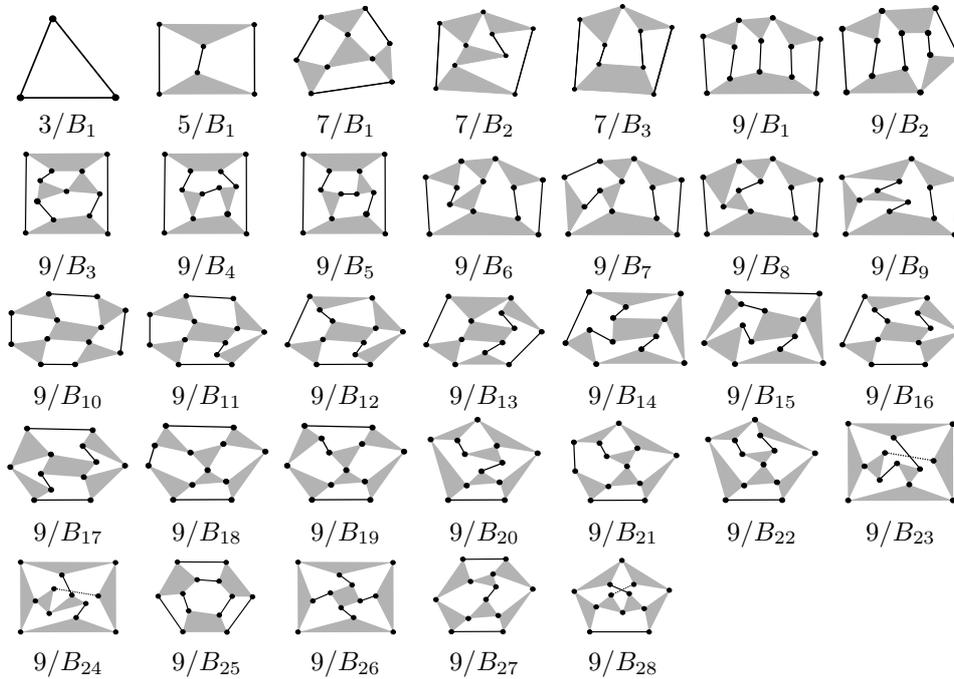


Figure 1: The catalogued Baranov trusses.

equations is derived. This has been the dominating technique which, in general, requires not only complex eliminations but also tangent-half-angle substitutions. An even more compact formulation is obtained by applying the following constructive process. Take one loop with a low number of joints and some of its joint variables as parameters which, when assigned to particular values, make the loop rigid. Then, the position of all links in the neighboring loops to this loop can also be obtained as a function of the chosen parameters taking, if needed, more joint variables as parameters. This process can be repeated till the locations of all links are expressed as functions of a set of parameters. Along the process, the locations of some links can be computed using different sets of parameters. This gives rise to constraints between the parameters which translate into equations. The number of these equations is called the *coupling degree* of the truss [6, 12]. For non-overconstrained trusses, as is the case of Baranov trusses, the number of resulting constraints always equals the number of joint variables taken as parameters. Since the coupling degree is always lower than the number of loops, the elimination process to get a characteristic polynomial is simplified. Actually, when the coupling degree is 1, eliminations are no longer required. Moreover, there are some trusses that form regular patterns whose coupling degree is independent from the number of its loops [11]. Although the idea is simple, its implementation using displacement transformations is not. This is probably why this approach has been belittled but, in this paper, we show how, by expressing the position analysis problem fully in terms of distances, this idea recovers interest because its implementation becomes straightforward. Moreover, as it will be presented in Section 4, for all the catalogued Baranov trusses the system of

kinematic equations is reduced to a single scalar equation in one variable, except for the Baranov trusses  $9/B_{25}$ ,  $9/B_{26}$ ,  $9/B_{27}$ , and  $9/B_{28}$  for which the system is formed by two scalar equations in two variables, a result in agreement with their coupling degrees found by Yang [6].

This paper is organized as follows. Section 2 introduces the basic tools needed for the application of the proposed technique. Section 3 directly presents how the position analysis of the  $9/B_{28}$  Baranov truss can be solved using the presented tools. This is the most difficult case as it is one of the only four cases that requires an elimination process and, in addition, it has a characteristic polynomial of degree 58, the highest degree among all the catalogued Baranov trusses. Section 4 contains a discussion on the results obtained on the application of the proposed technique to all other catalogued Baranov trusses, which are summarized in Tables 2-11, and gives prospects for further research.

## 2. Basic tools

### 2.1. Bilateralation

In what follows,  $P_i$  will denote a point,  $\overline{P_i P_j}$  the segment defined by  $P_i$  and  $P_j$ , and  $\triangle P_i P_j P_k$  the triangle defined by  $P_i$ ,  $P_j$ , and  $P_k$ . Moreover,  $\mathbf{p}_{i,j} = \overrightarrow{P_i P_j}$  and  $s_{i,j} = \|\mathbf{p}_{i,j}\|^2$ .

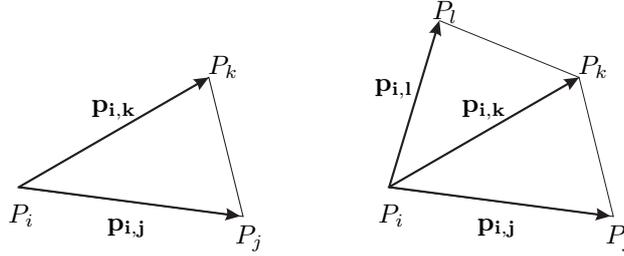


Figure 2: Left: The bilateralation problem. Right: Concatenation of two bilateralations.

The bilateralation problem consists in finding the feasible locations of a point, say  $P_k$ , given its distances to two other points, say  $P_i$  and  $P_j$ , whose locations are known. Then, according to Fig. 2(left), the result to this problem, in matrix form, can be expressed as:

$$\mathbf{p}_{i,k} = \mathbf{Z}_{i,j,k} \mathbf{p}_{i,j} \quad (1)$$

where

$$\mathbf{Z}_{i,j,k} = \frac{1}{2s_{i,j}} \begin{bmatrix} s_{i,j} + s_{i,k} - s_{j,k} & -4A_{i,j,k} \\ 4A_{i,j,k} & s_{i,j} + s_{i,k} - s_{j,k} \end{bmatrix}$$

is called a *bilateralation matrix*, and

$$A_{i,j,k} = \pm \frac{1}{4} \sqrt{(s_{i,j} + s_{i,k} + s_{j,k})^2 - 2(s_{i,j}^2 + s_{i,k}^2 + s_{j,k}^2)} \quad (2)$$

is the oriented area of  $\triangle P_i P_j P_k$  which is defined as positive if  $P_k$  is to the left of vector  $\mathbf{p}_{i,j}$ , and negative otherwise. The interested reader is addressed to [13] for a derivation of equation (1).

Given the triangle in Fig. 2(left), it is possible to compute six different bilaterations. By algebraically manipulating the obtained results, it is possible to prove that:

$$\mathbf{Z}_{i,j,k} = \mathbf{I} - \mathbf{Z}_{j,i,k}, \quad (3)$$

$$\mathbf{Z}_{i,j,k} = (\mathbf{Z}_{i,k,j})^{-1}, \quad (4)$$

$$\mathbf{Z}_{i,j,k} = -\mathbf{Z}_{k,j,i} \mathbf{Z}_{j,i,k}. \quad (5)$$

Moreover, it can be observed that the product of two bilateration matrices is commutative. Then, it is easy to prove that the set of bilateration matrices, that is, matrices of the form  $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ , constitute a commutative group under the product and addition operations.

Another important property, that will be useful later, comes from the fact that if  $\mathbf{v} = \mathbf{Z}\mathbf{w}$ , where  $\mathbf{Z}$  is a bilateration matrix, then  $\|\mathbf{v}\|^2 = \det(\mathbf{Z}) \|\mathbf{w}\|^2$ .

Now, let us consider the two triangles sharing one edge depicted in Fig. 2(right). Then,  $\mathbf{p}_{j,l}$  can be expressed in terms of  $\mathbf{p}_{i,j}$  by applying two consecutive bilaterations, as

$$\mathbf{p}_{i,l} = \mathbf{Z}_{i,k,l} \mathbf{p}_{i,k} = \mathbf{Z}_{i,k,l} \mathbf{Z}_{i,j,k} \mathbf{p}_{i,j}. \quad (6)$$

Actually, any vector involving  $P_i, P_j, P_k, P_l$  can be expressed in function of  $\mathbf{p}_{i,j}$  using bilateration matrices. For example,

$$\mathbf{p}_{j,l} = \mathbf{p}_{i,l} - \mathbf{p}_{i,j} = (\mathbf{Z}_{i,k,l} \mathbf{Z}_{i,j,k} - \mathbf{I}) \mathbf{p}_{i,j}. \quad (7)$$

As a consequence, the unknown squared distance between  $P_j$  and  $P_l$  can be obtained as:

$$s_{j,l} = \det(\mathbf{Z}_{i,k,l} \mathbf{Z}_{i,j,k} - \mathbf{I}) s_{i,j}. \quad (8)$$

If this result is compared to the one presented for example in [14, pp. 65-69], one starts to appreciate the ability of bilateration matrices to represent the solution of complex problems in a very compact form.

## 2.2. Squared distances in strips of triangles

The results presented in the previous subsection can be extended to strips of triangles —*i.e.*, series of connected triangles that share one edge with one neighbor and another with the next. As an example, let us suppose that we are interested in finding  $\mathbf{p}_{5,7}$  as a function of  $\mathbf{p}_{1,2}$  for the strip of five triangles appearing in Fig. 3. Then, to this end, we can perform the following sequence of bilaterations:

$$\begin{aligned} \mathbf{p}_{5,7} &= \mathbf{Z}_{5,6,7} \mathbf{p}_{5,6} \\ &= \mathbf{Z}_{5,6,7} \mathbf{Z}_{5,4,7} \mathbf{p}_{5,4} \\ &= \mathbf{Z}_{5,6,7} \mathbf{Z}_{5,4,7} (\mathbf{p}_{1,4} + \mathbf{p}_{5,1}) \\ &= \mathbf{Z}_{5,6,7} \mathbf{Z}_{5,4,7} (\mathbf{I} - \mathbf{Z}_{1,4,5}) \mathbf{p}_{1,4} \\ &= \mathbf{Z}_{5,6,7} \mathbf{Z}_{5,4,7} \mathbf{Z}_{4,1,5} \mathbf{p}_{1,4} \\ &= \mathbf{Z}_{5,6,7} \mathbf{Z}_{5,4,7} \mathbf{Z}_{4,1,5} \mathbf{Z}_{1,3,4} \mathbf{p}_{1,3} \\ &= \mathbf{Z}_{5,6,7} \mathbf{Z}_{5,4,7} \mathbf{Z}_{4,1,5} \mathbf{Z}_{1,3,4} \mathbf{Z}_{1,2,3} \mathbf{p}_{1,2}. \end{aligned}$$

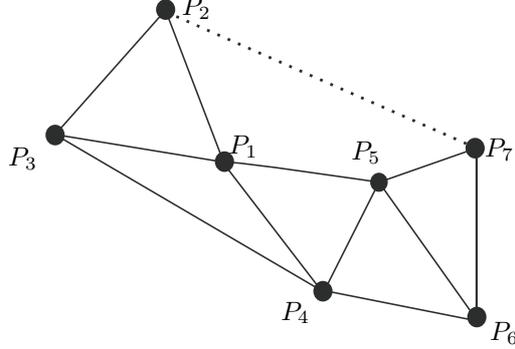


Figure 3: In a strip of triangles any unknown distance between any two of its vertices can be obtained by a set of bilaterations (see text for details).

Now, let us suppose that we want to compute  $\mathbf{p}_{2,7}$  as a function of  $\mathbf{p}_{1,2}$ . In this case  $\overline{P_2P_7}$  is not an edge of any triangle but clearly

$$\begin{aligned}
\mathbf{p}_{2,7} &= -\mathbf{p}_{1,2} + \mathbf{p}_{1,5} + \mathbf{p}_{5,7} \\
&= -\mathbf{p}_{1,2} + \mathbf{p}_{1,5} + \mathbf{Z}_{5,6,7} \mathbf{Z}_{5,4,6} \mathbf{p}_{5,4} \\
&= -\mathbf{p}_{1,2} + \mathbf{p}_{1,5} + \mathbf{Z}_{5,6,7} \mathbf{Z}_{5,4,6} (\mathbf{p}_{1,4} + \mathbf{p}_{5,1}) \\
&= -\mathbf{p}_{1,2} + \mathbf{p}_{1,5} + \mathbf{Z}_{5,6,7} \mathbf{Z}_{5,4,6} \mathbf{Z}_{4,1,5} \mathbf{p}_{1,4} \\
&= -\mathbf{p}_{1,2} + (\mathbf{Z}_{1,4,5} + \mathbf{Z}_{5,6,7} \mathbf{Z}_{5,4,6} \mathbf{Z}_{4,1,5}) \mathbf{p}_{1,4} \\
&= ((\mathbf{Z}_{1,4,5} + \mathbf{Z}_{5,6,7} \mathbf{Z}_{5,4,6} \mathbf{Z}_{4,1,5}) \mathbf{Z}_{1,3,4} \mathbf{Z}_{1,2,3} - \mathbf{I}) \mathbf{p}_{1,2}.
\end{aligned}$$

Therefore, the squared distance  $s_{2,7}$  can be expressed, as a function of all other known distance in the strip, as:

$$s_{2,7} = \det\left((\mathbf{Z}_{1,4,5} + \mathbf{Z}_{5,6,7} \mathbf{Z}_{5,4,6} \mathbf{Z}_{4,1,5}) \mathbf{Z}_{1,3,4} \mathbf{Z}_{1,2,3} - \mathbf{I}\right) s_{1,2}.$$

Observe that, if  $s_{2,7}$  is fixed to a given value, the above equation can be seen as a closure equation, a condition that is fulfilled if and only if the strip of triangles can be assembled so that the distance between  $P_2$  and  $P_7$  is the desired one. This way of obtaining closure conditions is extended in the next subsection to complex trusses.

### 2.3. Closure conditions using bilateration

Fig. 4(a) shows a strip of triangles equivalent to a planar truss consisting of 9 binary links that conforms 4 non-oriented triangles. In this truss, once points  $P_1$  and  $P_4$  have been located on the plane, points  $P_2$  and  $P_5$  can be positioned in 8 and 4 different locations, respectively. If  $\mathbf{p}_{1,6}$  is taken as reference, using a sequence of bilaterations, we have that

$$s_{2,5} = \det\left(\mathbf{I} - \mathbf{Z}_{1,3,2} \mathbf{Z}_{1,6,3} - \mathbf{Z}_{6,4,5} \mathbf{Z}_{6,1,4}\right) s_{1,6}. \quad (9)$$

Now, if the binary link  $P_1P_6$  is removed, the planar truss becomes a four-bar linkage [Fig. 4(b)]. But, if a binary link is then added between  $P_2$  and  $P_5$ , a truss is again

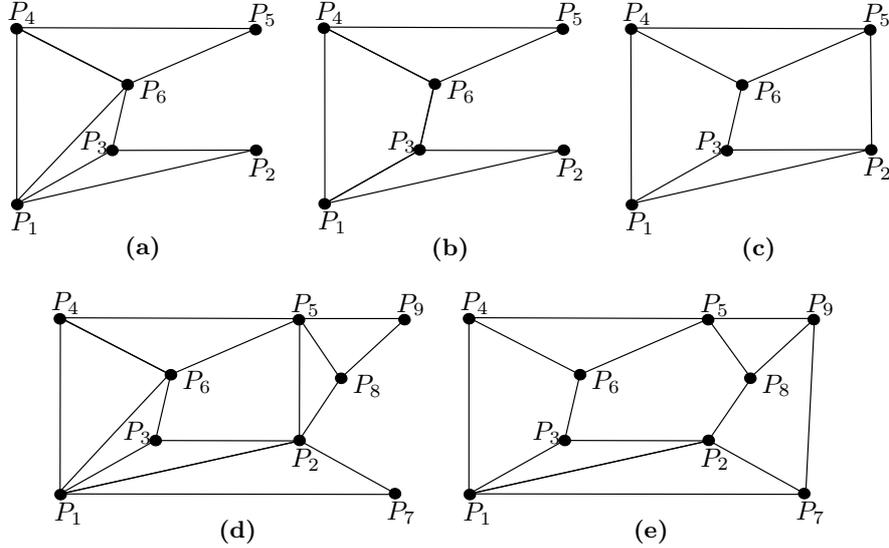


Figure 4: If the binary link  $P_1P_6$  is removed in the strip of triangles appearing in (a), a four-bar linkage is obtained (b). If a binary link is then added between  $P_2$  and  $P_5$ , a truss is again obtained (c). The strips of triangles  $\{\triangle P_1P_2P_7, \triangle P_1P_3P_2, \triangle P_1P_6P_3, \triangle P_1P_4P_6, \triangle P_4P_5P_6\}$  and  $\{\triangle P_2P_5P_8, \triangle P_5P_9P_8\}$  define a planar truss (d). If the binary links  $P_1P_6$  and  $P_2P_5$  are removed and a binary link is added between  $P_7$  and  $P_9$  a new truss is obtained whose closure condition can be expressed as the squared distance  $s_{7,9}$  as a function of  $s_{1,6}$  (e).

obtained [Fig. 4(c)]. What are the assembly modes of this new truss? Observe that the closure condition of this truss is given by equation (9). The solution of this equation gives the set of values of  $s_{1,6}$  compatible with the lengths of all the binary links of the new truss. Actually, when  $\triangle P_1P_3P_2$  and  $\triangle P_4P_5P_6$  are oriented —*i.e.*, when  $\triangle P_1P_3P_2$  and  $\triangle P_4P_5P_6$  are ternary links— this truss corresponds to the  $5/B_1$  Baranov truss.

The above process can be iterated further. Considering now that a new triangle is added to the strip of triangles of Fig. 4(a) to the edge  $\overline{P_1P_2}$ , and that points  $P_2$  and  $P_5$  also belong to another strip of triangles as shown in Fig. 4(d), the result defines a truss in which the squared distance  $s_{7,9}$  can be easily determined. Then, if the binary links  $\overline{P_1P_6}$  and  $\overline{P_2P_5}$  are removed and a binary link is added between  $P_7$  and  $P_9$  a new truss is obtained [Fig. 4(e)]. The assembly modes of this new truss can be computed using the expression obtained for  $s_{7,9}$ , a scalar equation in  $s_{1,6}$ . Actually, when  $\triangle P_1P_3P_2$ ,  $\triangle P_1P_2P_7$ ,  $\triangle P_4P_5P_6$ , and  $\triangle P_5P_9P_8$  are oriented, the resulting truss corresponds to the  $7/B_3$  Baranov truss.

#### 2.4. Closure conditions and symmetries

The symmetries of a truss are given by its automorphisms. An automorphism of a truss is a set of permutations of its joints that map the truss onto itself while preserving the connectivity between joints. The composition of two automorphisms is clearly another automorphism, and the set of automorphisms of a given truss, under the composition operation, forms a group, the automorphism group of the truss.

Finding the automorphism group and an irreducible set of its generators for a Baranov truss is an easy task using any of the available open-source software tools for computing

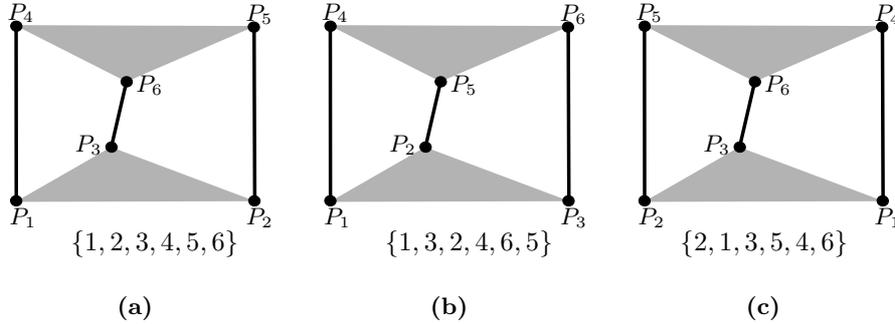


Figure 5: Three of the twelve automorphisms of the  $5/B_1$  Baranov truss.

graph automorphisms. For instance, the  $5/B_1$  Baranov truss, according to the notation of Fig. 4(c), has twelve automorphisms:  $\{1, 2, 3, 4, 5, 6\}$ ,  $\{1, 3, 2, 4, 6, 5\}$ ,  $\{2, 1, 3, 5, 4, 6\}$ ,  $\{2, 3, 1, 5, 6, 4\}$ ,  $\{3, 1, 2, 6, 4, 5\}$ ,  $\{3, 2, 1, 6, 5, 4\}$ ,  $\{4, 5, 6, 1, 2, 3\}$ ,  $\{4, 6, 5, 1, 3, 2\}$ ,  $\{5, 4, 6, 2, 1, 3\}$ ,  $\{5, 6, 4, 2, 3, 1\}$ ,  $\{6, 4, 5, 3, 1, 2\}$ , and  $\{6, 5, 4, 3, 2, 1\}$  (Fig. 5 depicts the first three). Observe that the closure condition of any of these symmetric trusses is obtained by the simple application of the corresponding permutation to equation (9). For example, the closure condition of the truss in Fig. 5(b) is obtained by applying the permutation

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 4 & 6 & 5 \end{bmatrix}.$$

That is,

$$s_{3,6} = \det(\mathbf{I} - \mathbf{Z}_{1,2,3}\mathbf{Z}_{1,5,2} - \mathbf{Z}_{5,4,6}\mathbf{Z}_{5,1,4})s_{1,5}.$$

### 3. Example: The $9/B_{28}$ Baranov truss

The  $9/B_{28}$  Baranov truss is one of the three catalogued Baranov trusses that cannot be represented as a planar graph (the other two are  $9/B_{23}$  and  $9/B_{24}$ ) [6], [15]. It was characterized for the first time by Baranov in 1952 [4] and Wang *et. al.* developed a procedure based on a complex number method and Dixon resultant to solve its position analysis in closed form [16]. They obtained a univariate polynomial of degree 64 but 6 extraneous roots were found, leading to the conclusion it can have up to 58 assembly modes—the largest number of assembly modes for a catalogued Baranov truss—, a result in agreement with that obtained by Hang *et. al.* using homotopy continuation [8].

In the  $9/B_{28}$  Baranov truss depicted in Fig. 6(a), the revolute pair centers of the six ternary links define the triangles  $\triangle P_1P_8P_3$ ,  $\triangle P_3P_6P_4$ ,  $\triangle P_4P_7P_5$ ,  $\triangle P_2P_5P_{11}$ ,  $\triangle P_8P_{12}P_9$ , and  $\triangle P_{10}P_{12}P_{11}$ . Next, it is shown how the kinematic equations of this truss can be reduced to compute  $s_{6,10}$  and  $s_{7,9}$  as a function of  $s_{1,4}$  and  $s_{2,4}$ . That is,  $s_{1,4}$  and  $s_{2,4}$  are used as parameters in terms of which the location of all other links of the truss can be expressed. Since two parameters are needed, the truss is said to have coupling number 2, as already observed by [6].

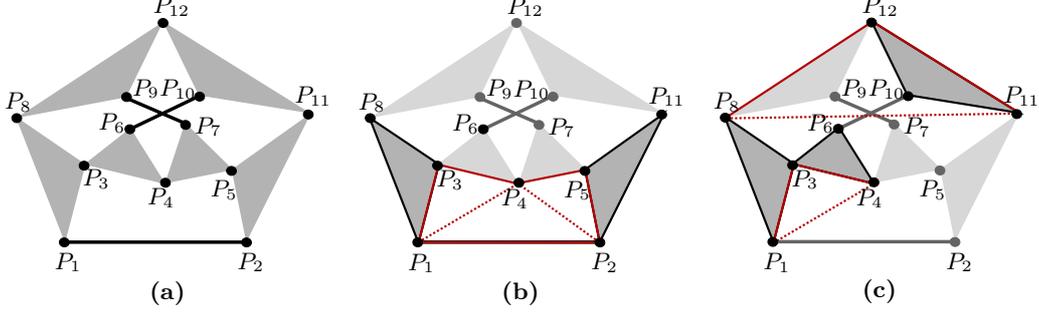


Figure 6: A  $9/B_{28}$  Baranov truss **(a)**. The strip of triangles  $\{\Delta P_1 P_8 P_3, \Delta P_1 P_3 P_4, \Delta P_1 P_4 P_2, \Delta P_4 P_2 P_5, \Delta P_2 P_5 P_{11}\}$  has been considered to compute  $s_{8,11}$  as a function of  $s_{1,4}$  and  $s_{2,4}$  **(b)**. The strips of triangles  $\{\Delta P_8 P_{12} P_{11}, \Delta P_{10} P_{12} P_{11}\}$ ,  $\{\Delta P_1 P_8 P_3, \Delta P_1 P_3 P_4\}$ , and  $\{\Delta P_3 P_4 P_6, \Delta P_1 P_3 P_4\}$  have also been considered to compute  $s_{6,10}$  as a function of  $s_{1,4}$  **(c)**.

### 3.1. Computing $s_{6,10}$ as a function of $s_{1,4}$ and $s_{2,4}$

On the one hand, for the strip of triangles in Fig. 6(b), we have

$$\begin{aligned}
\mathbf{p}_{8,11} &= -\mathbf{p}_{1,8} + \mathbf{p}_{1,4} - \mathbf{p}_{2,4} + \mathbf{p}_{2,11} \\
&= (-\mathbf{Z}_{1,3,8} \mathbf{Z}_{1,4,3} + \mathbf{I}) \mathbf{p}_{1,4} + (\mathbf{Z}_{2,5,11} \mathbf{Z}_{2,4,5} - \mathbf{I}) \mathbf{p}_{2,4} \\
&= (-\mathbf{Z}_{1,3,8} \mathbf{Z}_{1,4,3} + \mathbf{I} + (\mathbf{Z}_{2,5,11} \mathbf{Z}_{2,4,5} - \mathbf{I}) \mathbf{Z}_{4,1,2}) \mathbf{p}_{1,4}.
\end{aligned} \tag{10}$$

Therefore,

$$s_{8,11} = f(s_{1,4}, s_{2,4}) = \det\left(-\mathbf{Z}_{1,3,8} \mathbf{Z}_{1,4,3} + \mathbf{I} - \mathbf{Z}_{4,1,2} + \mathbf{Z}_{2,5,11} \mathbf{Z}_{2,4,5} \mathbf{Z}_{4,1,2}\right) s_{1,4}. \tag{11}$$

On the other hand, from the three strips of triangles in Fig. 6(c),

$$\begin{aligned}
\mathbf{p}_{6,10} &= -\mathbf{p}_{1,6} + \mathbf{p}_{1,8} + \mathbf{p}_{8,10} \\
&= (\mathbf{Z}_{4,3,6} \mathbf{Z}_{4,1,3} - \mathbf{I}) \mathbf{p}_{1,4} + \mathbf{Z}_{1,3,8} \mathbf{Z}_{1,4,3} \mathbf{p}_{1,4} + (\mathbf{I} - \mathbf{Z}_{11,12,10} \mathbf{Z}_{11,8,12}) \mathbf{p}_{8,11}.
\end{aligned} \tag{12}$$

Then, the substitution of equation (10) in equation (12) yields

$$\begin{aligned}
\mathbf{p}_{6,10} &= (-\mathbf{I} + \mathbf{Z}_{4,3,6} \mathbf{Z}_{4,1,3} + \mathbf{Z}_{1,3,8} \mathbf{Z}_{1,4,3} + (\mathbf{I} - \mathbf{Z}_{11,12,10} \mathbf{Z}_{11,8,12})) (-\mathbf{Z}_{1,3,8} \mathbf{Z}_{1,4,3} \\
&\quad + \mathbf{I} - \mathbf{Z}_{4,1,2} + \mathbf{Z}_{2,5,11} \mathbf{Z}_{2,4,5} \mathbf{Z}_{4,1,2}) \mathbf{p}_{1,4}.
\end{aligned} \tag{13}$$

Therefore,

$$\begin{aligned}
s_{6,10} &= f(s_{1,4}, s_{2,4}, s_{8,11}) \\
&= \det\left(-\mathbf{I} + \mathbf{Z}_{4,3,6} \mathbf{Z}_{4,1,3} + \mathbf{Z}_{1,3,8} \mathbf{Z}_{1,4,3} + (\mathbf{I} - \mathbf{Z}_{11,12,10} \mathbf{Z}_{11,8,12}) (-\mathbf{Z}_{1,3,8} \mathbf{Z}_{1,4,3} \right. \\
&\quad \left. + \mathbf{I} - \mathbf{Z}_{4,1,2} + \mathbf{Z}_{2,5,11} \mathbf{Z}_{2,4,5} \mathbf{Z}_{4,1,2})\right) s_{1,4}.
\end{aligned} \tag{14}$$

Then, the substitution of equation (11) in equation (14) yields a scalar equation in two variables:  $s_{1,4}$  and  $s_{2,4}$ .

### 3.2. Computing $s_{7,9}$ as a function of $s_{1,4}$ and $s_{2,4}$

The computation of  $s_{7,9}$  as a function of  $s_{2,4}$  and  $s_{2,4}$  can be simplified by considering the symmetries of the analyzed truss. Indeed, by applying the permutation

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 1 & 5 & 4 & 3 & 7 & 6 & 11 & 10 & 9 & 8 & 12 \end{bmatrix}$$

to equation (14), we conclude that

$$\begin{aligned} s_{7,9} &= f(s_{1,4}, s_{2,4}, s_{8,11}) \\ &= \det \left( -\mathbf{I} + \mathbf{Z}_{4,5,7} \mathbf{Z}_{4,2,5} + \mathbf{Z}_{2,5,11} \mathbf{Z}_{2,4,5} + (\mathbf{I} - \mathbf{Z}_{8,12,9} \mathbf{Z}_{8,11,12}) (-\mathbf{Z}_{2,5,11} \mathbf{Z}_{2,4,5} \right. \\ &\quad \left. + \mathbf{I} - \mathbf{Z}_{4,2,1} + \mathbf{Z}_{1,3,8} \mathbf{Z}_{1,4,3} \mathbf{Z}_{4,2,1}) \right) s_{2,4}. \end{aligned} \quad (15)$$

The expansion of the right hand side of equations (14) and (15), using equation (11) for substituting the unknown squared distance  $s_{1,8}$ , yields a system of two scalar equations in two variables:  $s_{1,4}$  and  $s_{2,4}$ . Next, it is shown how to algebraically manipulate this system to obtain the characteristic polynomial of the  $9/B_{28}$  Baranov truss.

### 3.3. Deriving the characteristic polynomial

The characteristic polynomial of the  $9/B_{28}$  Baranov truss can be obtained by: (i) converting the scalar radical equations (14) and (15) in polynomial equations in  $s_{1,4}$  and  $s_{2,4}$ , say  $\mathcal{P}_1(s_{1,4}, s_{2,4}) = 0$  and  $\mathcal{P}_2(s_{1,4}, s_{2,4}) = 0$ , respectively, by clearing all the involved square roots; and (ii) eliminating  $s_{2,4}$ , or  $s_{1,4}$ , from the resulting polynomial system to get a single polynomial in one variable.

By expanding equation (11), we get

$$\begin{aligned} s_{8,11} &= \frac{1}{s_{1,4} s_{2,4}} \left( \Lambda_1 + \Lambda_2 A_{1,4,3} + \Lambda_3 A_{1,4,2} + \Lambda_4 A_{2,4,5} + \Lambda_5 A_{1,4,3} A_{1,4,2} + \Lambda_6 A_{1,4,3} \right. \\ &\quad \left. A_{2,4,5} + \Lambda_7 A_{1,4,2} A_{2,4,5} + \Lambda_8 A_{1,4,3} A_{1,4,2} A_{2,4,5} \right), \end{aligned} \quad (16)$$

where  $A_{1,4,3}$ ,  $A_{1,4,2}$ , and  $A_{2,4,5}$  are the unknown oriented areas of  $\triangle P_1 P_4 P_3$ ,  $\triangle P_1 P_4 P_2$ , and  $\triangle P_2 P_4 P_5$ , respectively, and  $\Lambda_i$ ,  $i = 1, \dots, 8$  are polynomials in  $s_{1,4}$  and  $s_{2,4}$ .

Likewise, by expanding equation (14), we get

$$s_{6,10} = \frac{1}{s_{1,4} s_{2,4} s_{8,11}} \Psi, \quad (17)$$

where

$$\begin{aligned} \Psi &= \Psi_1 + \Psi_2 A_{1,4,3} + \Psi_3 A_{1,4,2} + \Psi_4 A_{2,4,5} + \Psi_5 A_{8,11,12} + \Psi_6 A_{1,4,3} A_{1,4,2} + \Psi_7 A_{1,4,3} \\ &\quad A_{2,4,5} + \Psi_8 A_{1,4,3} A_{8,11,12} + \Psi_9 A_{1,4,2} A_{2,4,5} + \Psi_{10} A_{1,4,2} A_{8,11,12} + \Psi_{11} A_{2,4,5} \\ &\quad A_{8,11,12} + \Psi_{12} A_{1,4,3} A_{1,4,2} A_{2,4,5} + \Psi_{13} A_{1,4,3} A_{1,4,2} A_{8,11,12} + \Psi_{14} A_{1,4,3} A_{2,4,5} \\ &\quad A_{8,11,12} + \Psi_{15} A_{1,4,2} A_{2,4,5} A_{8,11,12} + \Psi_{16} A_{1,4,3} A_{1,4,2} A_{2,4,5} A_{8,11,12}, \end{aligned}$$

with  $\Psi_i$ ,  $i = 1, \dots, 16$ , polynomials in  $s_{1,4}$ ,  $s_{2,4}$  and  $s_{8,11}$ .

Now, by expressing equation (17) as a linear equation in  $A_{8,11,12}$  (*i.e.*,  $a + b A_{8,11,12} = 0$ ), squaring it (*i.e.*,  $a^2 - b^2 A_{8,11,12}^2 = 0$ ), and replacing equation (16) in the result, a equation in  $A_{1,4,3}$ ,  $A_{1,4,2}$ , and  $A_{2,4,5}$  is obtained. Repeating this process for  $A_{2,4,5}$ , we get

$$\Phi_1 + \Phi_2 A_{1,4,3} + \Phi_3 A_{1,4,2} + \Phi_4 A_{1,4,3} A_{1,4,2} = 0, \quad (18)$$

where  $\Phi_1$ ,  $\Phi_2$ ,  $\Phi_3$ , and  $\Phi_4$  are polynomials in  $s_{1,4}$  of degree 16, 15, 15, and 14, respectively, and in  $s_{2,4}$  of degree 8 in all cases. Then, to clear radicals, the terms in equation (18) can be rearranged to obtain

$$\Phi_2 A_{1,4,3} + \Phi_3 A_{1,4,2} = -(\Phi_1 + \Phi_4 A_{1,4,3} A_{1,4,2}).$$

Now, by squaring both sides of the above equation, expanding the result, and rearranging terms, we get

$$\Phi_2^2 A_{1,4,3}^2 + \Phi_3^2 A_{1,4,2}^2 - \Phi_4^2 A_{1,4,3}^2 A_{1,4,2}^2 - \Phi_1^2 = 2(\Phi_1 \Phi_4 - \Phi_2 \Phi_3) A_{1,4,3} A_{1,4,2}.$$

Finally, if both sides of the above equation are again squared and expanded, the following equation is obtained:

$$\begin{aligned} & -\Phi_4^4 A_{1,4,3}^4 A_{1,4,2}^4 + 2\Phi_4^2 \Phi_2^2 A_{1,4,3}^2 A_{1,4,2}^2 + 2\Phi_4^2 \Phi_3^2 A_{1,4,3}^2 A_{1,4,2}^2 - \Phi_2^4 A_{1,4,3}^4 - \Phi_3^4 A_{1,4,2}^4 \\ & - \Phi_1^4 + (2\Phi_2^2 \Phi_3^2 - 8\Phi_2 \Phi_3 \Phi_4 \Phi_1 + 2\Phi_4^2 \Phi_1^2) A_{1,4,3}^2 A_{1,4,2}^2 + 2\Phi_1^2 \Phi_2^2 A_{1,4,3}^2 + 2\Phi_1^2 \Phi_3^2 A_{1,4,2}^2 = 0. \end{aligned} \quad (19)$$

If the above procedure is applied to equation (16), we get a polynomial in  $s_{1,4}$ ,  $s_{2,4}$ , and  $s_{8,11}$ , say  $\mathcal{F}(s_{1,4}, s_{2,4}, s_{8,11})$ . Finally, the full expansion of equation (19) factorizes as:

$$s_{1,4}^{16} s_{2,4}^{16} \mathcal{F}(s_{1,4}, s_{2,4}, 0) \mathcal{P}_1 = 0 \quad (20)$$

where  $\mathcal{P}_1$  is a non-homogeneous bivariate polynomial in  $s_{1,4}$  and  $s_{2,4}$  with leading term  $s_{1,4}^{32} s_{2,4}^{14}$ . The roots of the term  $s_{1,4}^{16} s_{2,4}^{16} \mathcal{F}(s_{1,4}, s_{2,4}, 0)$  were introduced when clearing denominators to obtain equation (17), so they can be dropped. To obtain  $\mathcal{P}_2$ , we can proceed in the same way as for the derivation of  $\mathcal{P}_1$ .

Finally, to obtain the characteristic polynomial, we have to eliminate either  $s_{2,4}$  or  $s_{1,4}$  from the polynomial system  $\{\mathcal{P}_1(s_{1,4}, s_{2,4}) = 0, \mathcal{P}_2(s_{1,4}, s_{2,4}) = 0\}$ . This can be implemented using, for example, Sylvester or Bézout resultants. If we eliminate either  $s_{2,4}$  or  $s_{1,4}$ , the associated Sylvester and Bézout matrices have dimensions  $68 \times 68$  and  $36 \times 36$ , respectively. In any case —*i.e.*, eliminating either  $s_{2,4}$  or  $s_{1,4}$  and using either Sylvester or the Bézout resultants— the result is a polynomial equation of degree 1826. When  $s_{2,4}$  is the eliminated variable, this polynomial factors into 15 polynomials

$$\mathcal{T}(s_{1,4}) \prod_{i=1}^{14} \mathcal{D}_i(s_{1,4}) = 0 \quad (21)$$

where the roots of polynomials  $\mathcal{D}_1, \dots, \mathcal{D}_{14}$  are not solutions of the original system of radical equations formed by equations (14) and (15) and  $\mathcal{T}$ , the characteristic polynomial

of the  $9/B_{28}$  Baranov truss, is of degree 58 in  $s_{1,4}$ . As expected, the same result is obtained when  $s_{1,4}$  is the eliminated variable.

For each of the real roots of  $\mathcal{T}(s_{1,4}) = 0$ , we can determine the Cartesian position of the revolute pair centers given by  $P_2, P_4, P_5, P_6, P_7, P_9, P_{10}, P_{11}$ , and  $P_{12}$ , with respect to the ternary link given by  $\triangle P_1 P_8 P_3$ , computing  $s_{2,4}$  from the system  $\{\mathcal{P}_1(s_{1,4}, s_{2,4}) = 0, \mathcal{P}_2(s_{1,4}, s_{2,4}) = 0\}$  and using the set of equations

$$\begin{aligned} \mathbf{p}_{1,4} &= \mathbf{Z}_{1,3,4} \mathbf{p}_{1,3} \\ \mathbf{p}_{3,6} &= \mathbf{Z}_{3,4,6} \mathbf{p}_{3,4} \\ \mathbf{p}_{1,2} &= \mathbf{Z}_{1,4,2} \mathbf{p}_{1,4} \\ \mathbf{p}_{2,5} &= \mathbf{Z}_{2,4,5} \mathbf{p}_{2,4} \\ \mathbf{p}_{5,7} &= \mathbf{Z}_{4,5,7} \mathbf{p}_{4,5} \\ \mathbf{p}_{2,11} &= \mathbf{Z}_{2,5,11} \mathbf{p}_{2,5} \\ \mathbf{p}_{8,12} &= \mathbf{Z}_{8,11,12} \mathbf{p}_{8,11} \\ \mathbf{p}_{8,9} &= \mathbf{Z}_{8,12,9} \mathbf{p}_{8,12} \\ \mathbf{p}_{11,10} &= \mathbf{Z}_{11,12,10} \mathbf{p}_{11,12}. \end{aligned}$$

This process leads up to 16 combinations of locations for the couples  $P_6, P_{10}$  and  $P_7, P_9$ , and at least one of them must satisfy simultaneously the distances imposed by the binary links connecting them.

### 3.4. Numerical example

According to the notation used in Fig. 6, let us suppose that  $s_{1,2} = 3185$ ,  $s_{1,3} = \frac{6610}{9}$ ,  $s_{1,8} = \frac{16525}{9}$ ,  $s_{2,5} = \frac{820}{9}$ ,  $s_{2,11} = \frac{15826}{9}$ ,  $s_{3,4} = 225$ ,  $s_{3,6} = 180$ ,  $s_{3,8} = 661$ ,  $s_{4,5} = 400$ ,  $s_{4,6} = 225$ ,  $s_{4,7} = 452$ ,  $s_{5,7} = 676$ ,  $s_{5,11} = 1202$ ,  $s_{6,10} = 625$ ,  $s_{7,9} = 625$ ,  $s_{8,9} = 305$ ,  $s_{8,12} = 1600$ ,  $s_{9,12} = 625$ ,  $s_{10,11} = 676$ ,  $s_{10,12} = 484$ , and  $s_{11,12} = 1600$ . This corresponds to the example used by Wang *et. al.* in [16]. Then, proceeding as explained above, we obtain the characteristic polynomial

$$\begin{aligned} &1.8481 10^{239} s_{1,4}^{58} - 1.5725 10^{244} s_{1,4}^{57} + 6.5857 10^{248} s_{1,4}^{56} - 1.8099 10^{253} s_{1,4}^{55} \\ &+ 3.6712 10^{257} s_{1,4}^{54} - 5.8610 10^{261} s_{1,4}^{53} + 7.6692 10^{265} s_{1,4}^{52} - 8.4571 10^{269} s_{1,4}^{51} \\ &+ 8.0199 10^{273} s_{1,4}^{50} - 6.6413 10^{277} s_{1,4}^{49} + 4.8604 10^{281} s_{1,4}^{48} - 3.1737 10^{285} s_{1,4}^{47} \\ &+ 1.8635 10^{289} s_{1,4}^{46} - 9.9019 10^{292} s_{1,4}^{45} + 4.7871 10^{296} s_{1,4}^{44} - 2.1152 10^{300} s_{1,4}^{43} \\ &+ 8.5737 10^{303} s_{1,4}^{42} - 3.1985 10^{307} s_{1,4}^{41} + 1.1012 10^{311} s_{1,4}^{40} - 3.5073 10^{314} s_{1,4}^{39} \\ &+ 1.0354 10^{318} s_{1,4}^{38} - 2.8379 10^{321} s_{1,4}^{37} + 7.2324 10^{324} s_{1,4}^{36} - 1.7158 10^{328} s_{1,4}^{35} \\ &+ 3.7930 10^{331} s_{1,4}^{34} - 7.8194 10^{334} s_{1,4}^{33} + 1.5042 10^{338} s_{1,4}^{32} - 2.7012 10^{341} s_{1,4}^{31} \\ &+ 4.5296 10^{344} s_{1,4}^{30} - 7.0937 10^{347} s_{1,4}^{29} + 1.0375 10^{351} s_{1,4}^{28} - 1.4168 10^{354} s_{1,4}^{27} \\ &+ 1.8060 10^{357} s_{1,4}^{26} - 2.1479 10^{360} s_{1,4}^{25} + 2.3817 10^{363} s_{1,4}^{24} - 2.4603 10^{366} s_{1,4}^{23} \\ &+ 2.3655 10^{369} s_{1,4}^{22} - 2.1141 10^{372} s_{1,4}^{21} + 1.7540 10^{375} s_{1,4}^{20} - 1.3487 10^{378} s_{1,4}^{19} \\ &+ 9.5930 10^{380} s_{1,4}^{18} - 6.2979 10^{383} s_{1,4}^{17} + 3.8066 10^{386} s_{1,4}^{16} - 2.1121 10^{389} s_{1,4}^{15} \end{aligned}$$

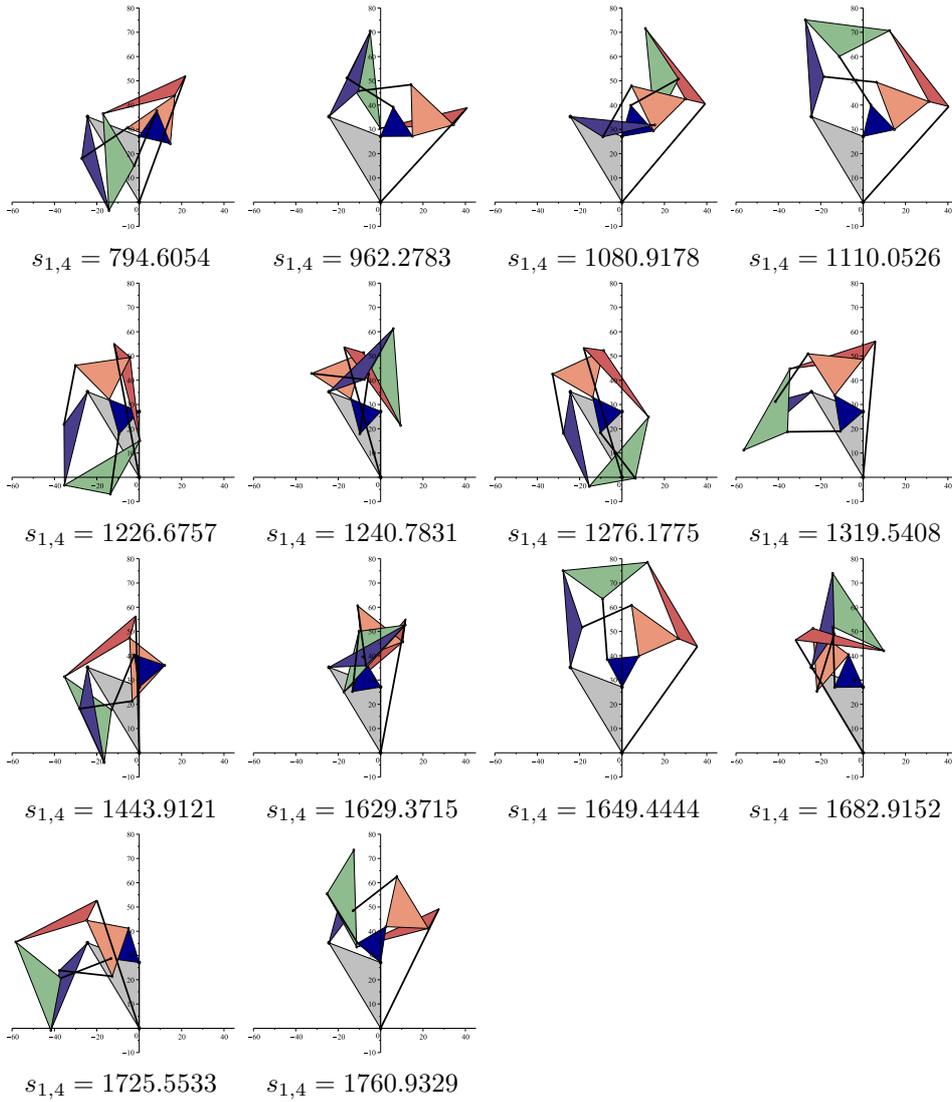


Figure 7: The assembly modes of the analyzed  $9/B_{28}$  Baranov truss.

$$\begin{aligned}
& + 1.0722 \cdot 10^{392} s_{1,4}^{14} - 4.9598 \cdot 10^{394} s_{1,4}^{13} + 2.0812 \cdot 10^{397} s_{1,4}^{12} - 7.8783 \cdot 10^{399} s_{1,4}^{11} \\
& + 2.6731 \cdot 10^{402} s_{1,4}^{10} - 8.0665 \cdot 10^{404} s_{1,4}^9 + 2.1442 \cdot 10^{407} s_{1,4}^8 - 4.9610 \cdot 10^{409} s_{1,4}^7 \\
& + 9.8395 \cdot 10^{411} s_{1,4}^6 - 1.6396 \cdot 10^{414} s_{1,4}^5 + 2.2327 \cdot 10^{416} s_{1,4}^4 - 2.3865 \cdot 10^{418} s_{1,4}^3 \\
& + 1.8780 \cdot 10^{420} s_{1,4}^2 - 9.6768 \cdot 10^{421} s_{1,4} + 2.4499 \cdot 10^{423}.
\end{aligned}$$

This polynomial has 14 real roots, a result in agreement with the solution reported in [16]. The values of these roots, as well as the corresponding assembly modes, for the case in which  $P_1 = (0, 0)^T$ ,  $P_3 = (0, \frac{1}{3} \sqrt{6610})^T$ , and  $P_8 = (-\frac{3}{10} \sqrt{6610}, \frac{13}{30} \sqrt{6610})^T$ , appear in Fig. 7. The coefficients of the above polynomial have to be computed in exact rational arithmetic. Otherwise, numerical problems make impracticable the correct computation of its roots. Although these coefficients are given here in floating point arithmetic for space limitation reasons, they could be of interest for comparison with other possible methods. To our knowledge, this is the first time that the characteristic polynomial of a  $9/B_{28}$  Baranov truss is explicitly obtained.

#### 4. Discussion and conclusions

The simplest Baranov truss is the  $3/B_1$  truss or triad. This truss has three links and up to two assembly modes. Its position analysis is equivalent to solve the bilateration problem. Other analytical approaches can be found, for instance, in [17], [18]. When the technique described in this paper is applied to the position analysis of the other catalogued Baranov trusses, one observes that the problem is reduced to solve a single scalar radical equation in one variable for all cases, except for trusses  $9/B_{25}$ ,  $9/B_{26}$ ,  $9/B_{27}$ , and  $9/B_{28}$ , for which the resulting system is formed by two scalar radical equations in two variables. This is in agreement with the coupling degree of the catalogued Baranov trusses presented in [6].

Since for the Baranov trusses with coupling degree 1 the obtained system of kinematic equations is a single scalar radical equation, their characteristic polynomials are obtained by simply clearing square roots, as it was already explained for trusses  $7/B_1$ ,  $7/B_2$ , and  $7/B_3$  in [13]. Obtaining the characteristic polynomials of trusses  $9/B_{25}$ ,  $9/B_{26}$ ,  $9/B_{27}$ , and  $9/B_{28}$  requires converting the resulting scalar radical equations in bivariate polynomials, by clearing square roots as well, and using classical elimination techniques, as it has been explained in this paper. Then, it is important to realize that 197 trusses, out of the total of 239 Baranov trusses with 5 loops, could also be solved in an elementary way and, for the remaining 42 trusses, the problem could be reduced to the solution of a system of two equations in two variables. Thus, compared to the approaches based on vector loop equations and elimination techniques, the application of the proposed technique to the position analysis of Baranov trusses seems clearly superior.

Tables 2 to 11 present, for all the catalogued Baranov trusses, the system of kinematic equations derived using the presented technique, the number of resulting assembly modes, and references to previous reported solutions using not only analytical approaches but also numerical ones. Using the kinematic equations presented in these tables, the closed-form position analysis of trusses  $9/B_2$ ,  $9/B_3$ ,  $9/B_4$ ,  $9/B_5$ ,  $9/B_6$ ,  $9/B_8$ ,  $9/B_9$ ,  $9/B_{13}$ ,  $9/B_{18}$ ,  $9/B_{19}$ , and  $9/B_{22}$ , can be straightforwardly solved for the first time for the best of our knowledge. It can also be concluded from these tables that the number of assembly

| Baranov truss  | AM | Reported solutions   |
|--|----|--|
| <p>5/B<sub>1</sub></p>   | 6  | <p><b>Analytical:</b> Peisach [21], Pennock and Kassner [22], Gosselin <i>et. al.</i> [23], [24], Wohlhart [25], Husty [26], Kong and Gosselin [27], Collins [28] (Basic elimination theory), Han <i>et. al.</i> [29], Liu <i>et. al.</i> [30], Luo [31] (Wu method), Rojas and Thomas [32], [33] (Bilateration method). <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation), Luo <i>et. al.</i> [34] (Newton iterative method), Chandra and Rolland [35] (Hybrid metaheuristics)</p>   |
| $s_{2,5} = f(s_{1,6}) = \det(\mathbf{I} - \mathbf{z}_{1,3,2} \mathbf{z}_{1,6,3} - \mathbf{z}_{6,4,5} \mathbf{z}_{6,1,4}) s_{1,6}$  |    |  |
| <p>7/B<sub>1</sub></p>   | 14 | <p><b>Analytical:</b> Kong and Yang [36], Innocenti [37] (Vector method with Sylvester resultant), Dhingra <i>et. al.</i> [38] (Vector method with Sylvester resultant), Rojas and Thomas [13] (Bilateration method). <b>Numerical:</b> Liu and Yang [39] (Homotopy continuation), Hang <i>et. al.</i> [8] (Vector method with homotopy continuation)</p>  |
| $s_{5,8} = f(s_{1,4}, s_{6,7}) = \det(-\mathbf{z}_{1,3,5} \mathbf{z}_{1,4,3} + \mathbf{I} - \mathbf{z}_{4,3,6} \mathbf{z}_{4,1,3} + \mathbf{z}_{6,9,8} \mathbf{z}_{6,7,9} \Omega_1) s_{1,4}$ |    |  |
| $s_{6,7} = f(s_{1,4}) = \det(\Omega_1) = \det(\mathbf{z}_{4,3,6} \mathbf{z}_{4,1,3} - \mathbf{z}_{4,2,7} \mathbf{z}_{4,1,2}) s_{1,4}$  |    |  |
| <p>7/B<sub>2</sub></p>   | 16 | <p><b>Analytical:</b> Innocenti [40] (Vector method with an ad-hoc elimination procedure), Almadi <i>et. al.</i> [41] (Vector method with Sylvester resultant), Dhingra <i>et. al.</i> [38] (Vector method with Sylvester resultant), Dhingra <i>et. al.</i> [42] (Vector method with Gröbner basis and Sylvester resultant), Wampler [43] (Complex number method with Dixon resultant), Rojas and Thomas [13] (Bilateration method). <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation), Shen <i>et. al.</i> [44] (Single-opened-chains iterative method)</p> |
| $s_{1,6} = f(s_{4,7}, s_{3,8}) = \det(-\mathbf{z}_{3,2,1} \mathbf{z}_{3,8,2} \Omega_1 - \mathbf{z}_{4,5,3} \mathbf{z}_{4,7,5} + \mathbf{z}_{4,9,6} \mathbf{z}_{4,7,9}) s_{4,7}$              |    |  |
| $s_{3,8} = f(s_{4,7}) = \det(\Omega_1) = \det(-\mathbf{z}_{4,5,3} \mathbf{z}_{4,7,5} + \mathbf{I} - \mathbf{z}_{7,9,8} \mathbf{z}_{7,4,9}) s_{4,7}$  |    |  |

Table 2: Position analysis of all the catalogued Baranov trusses (part 1).

modes previously reported in the literature for trusses 9/B<sub>4</sub>, 9/B<sub>8</sub>, 9/B<sub>18</sub>, and 9/B<sub>19</sub> do not seem accurate.

Finally, it is worth to mention that the concept of Baranov truss has been extended to trusses with joints involving more than two links. It has been shown that there are such 125 Baranov trusses with up to four loops [19]. The closed-form solution to the position analysis of at least one of these trusses, the Dixon-Wunderlich linkage, has already been reported [20]. The application of the proposed technique to all other members of this family of trusses is a point that deserves further attention.

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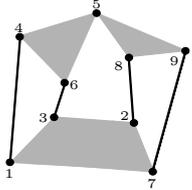
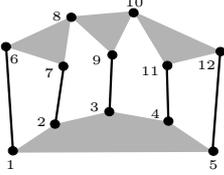
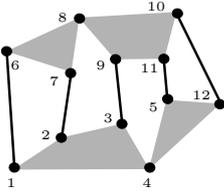
|         | Baranov truss   | AM | Reported solutions   |
|---------|---|----|--|
| $7/B_3$ |    | 18 | <p><b>Analytical:</b> Innocenti [45], [46] (Vector method with Sylvester resultant), Han <i>et. al.</i> [47] (Complex number method with Sylvester resultant), Dhingra <i>et. al.</i> [38] (Vector method with Sylvester resultant), Wang <i>et. al.</i> [48] (Complex number method with Wu method), Ni <i>et. al.</i> [49] (Conformal geometric algebra with Dixon resultant), Rojas and Thomas [13] (Bilateration method).<br/> <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation)</p>  |
|         |   |    | $s_{7,9} = f(s_{1,6}, s_{2,5}) = \det((\mathbf{z}_{1,3,2} - \mathbf{z}_{1,3,7})\mathbf{z}_{1,6,3} + (\mathbf{I} - \mathbf{z}_{5,8,9} \mathbf{z}_{5,2,8}) \Omega_1) s_{1,6}$<br>$s_{2,5} = f(s_{1,6}) = \det(\Omega_1) = \det(-\mathbf{z}_{1,3,2} \mathbf{z}_{1,6,3} + \mathbf{I} - \mathbf{z}_{6,4,5} \mathbf{z}_{6,1,4}) s_{1,6}$   |
| $9/B_1$ |   | 54 | <p><b>Analytical:</b> Lösch [9] (Vector method with Gröbner basis), Dhingra <i>et. al.</i> [50] (Vector method with Gröbner basis), Wei <i>et. al.</i> [51] (Complex number method with Sylvester resultant), Wohlhart [52] (Vector method with Sylvester resultant).<br/> <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation)</p>  |
|         |   |    | $s_{5,12} = f(s_{1,7}, s_{3,8}, s_{4,10}) = \det(\mathbf{z}_{4,1,5} \mathbf{z}_{1,3,4} \mathbf{z}_{1,2,3} \mathbf{z}_{1,7,2} + (\mathbf{I} - \mathbf{z}_{10,11,12} \mathbf{z}_{10,4,11}) \Omega_2) s_{1,7}$<br>$s_{4,10} = f(s_{1,7}, s_{3,8}) = \det(\Omega_2) = \det(-\mathbf{z}_{1,3,4} \mathbf{z}_{1,2,3} \mathbf{z}_{1,7,2} + \mathbf{z}_{1,2,3} \mathbf{z}_{1,7,2} + (\mathbf{I} - \mathbf{z}_{8,9,10} \mathbf{z}_{8,3,9}) \Omega_1) s_{1,7}$<br>$s_{3,8} = f(s_{1,7}) = \det(\Omega_1) = \det(-\mathbf{z}_{1,2,3} \mathbf{z}_{1,7,2} + \mathbf{I} - \mathbf{z}_{7,6,8} \mathbf{z}_{7,1,6}) s_{1,7}$   |
| $9/B_2$ |  | 54 | <p><b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation)</p>  |
|         |   |    | $s_{10,12} = f(s_{1,7}, s_{3,8}, s_{4,11}) = \det(-\mathbf{z}_{1,2,3} \mathbf{z}_{1,7,2} - (\mathbf{I} - \mathbf{z}_{8,9,10} \mathbf{z}_{8,3,9}) \Omega_1 + \mathbf{z}_{1,3,4} \mathbf{z}_{1,2,3} \mathbf{z}_{1,7,2} + \mathbf{z}_{4,5,12} \mathbf{z}_{4,11,5} \Omega_2) s_{1,7}$<br>$s_{4,11} = f(s_{1,7}, s_{3,8}) = \det(\Omega_2) = \det(-\mathbf{z}_{1,3,4} \mathbf{z}_{1,2,3} \mathbf{z}_{1,7,2} + \mathbf{z}_{1,2,3} \mathbf{z}_{1,7,2} + (\mathbf{I} - \mathbf{z}_{8,9,11} \mathbf{z}_{8,3,9}) \Omega_1) s_{1,7}$<br>$s_{3,8} = f(s_{1,7}) = \det(\Omega_1) = \det(-\mathbf{z}_{1,2,3} \mathbf{z}_{1,7,2} + \mathbf{I} - \mathbf{z}_{7,6,8} \mathbf{z}_{7,1,6}) s_{1,7}$ |

Table 3: Position analysis of all the catalogued Baranov trusses (part 2).

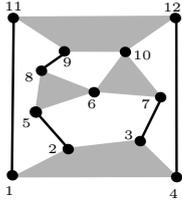
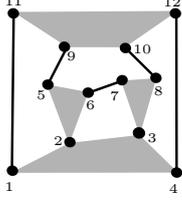
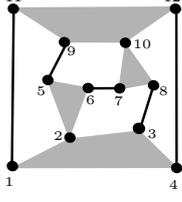
| Baranov truss  | AM | Reported solutions  |
|--|----|---|
| <p>9/B<sub>3</sub></p>    | 48 | <i>Numerical:</i> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation)  |
| $s_{8,9} = f(s_{1,12}, s_{3,10}, s_{2,6}) = \det \left( -\mathbf{z}_{1,4,3} \mathbf{z}_{1,12,4} - (\mathbf{I} - \mathbf{z}_{10,7,6} \mathbf{z}_{10,3,7}) \boldsymbol{\Omega}_1 + \mathbf{z}_{6,5,8} \mathbf{z}_{6,2,5} \boldsymbol{\Omega}_2 + \mathbf{I} - \mathbf{z}_{12,11,9} \mathbf{z}_{12,1,11} \right) s_{1,12}$ $s_{2,6} = f(s_{1,12}, s_{3,10}) = \det(\boldsymbol{\Omega}_2) = \det \left( -\mathbf{z}_{1,4,2} \mathbf{z}_{1,12,4} + \mathbf{z}_{1,4,3} \mathbf{z}_{1,12,4} + (\mathbf{I} - \mathbf{z}_{10,7,6} \mathbf{z}_{10,3,7}) \boldsymbol{\Omega}_1 \right) s_{1,12}$ $s_{3,10} = f(s_{1,12}) = \det(\boldsymbol{\Omega}_1) = \det \left( -\mathbf{z}_{1,4,3} \mathbf{z}_{1,12,4} + \mathbf{I} - \mathbf{z}_{12,11,10} \mathbf{z}_{12,1,11} \right) s_{1,12}$ |    |   |
| <p>9/B<sub>4</sub></p>   | 42 | <i>Numerical:</i> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation, the reported number of AM is incorrect (38)) |
| $s_{6,7} = f(s_{1,12}, s_{3,10}, s_{2,9}) = \det \left( -\mathbf{z}_{1,4,2} \mathbf{z}_{1,12,4} - \mathbf{z}_{2,5,6} \mathbf{z}_{2,9,5} \boldsymbol{\Omega}_2 + \mathbf{z}_{1,4,3} \mathbf{z}_{1,12,4} + \mathbf{z}_{3,8,7} \mathbf{z}_{3,10,8} \boldsymbol{\Omega}_1 \right) s_{1,12}$ $s_{2,9} = f(s_{1,12}) = \det(\boldsymbol{\Omega}_2) = \det \left( -\mathbf{z}_{1,4,2} \mathbf{z}_{1,12,4} + \mathbf{I} - \mathbf{z}_{12,11,9} \mathbf{z}_{12,1,11} \right) s_{1,12}$ $s_{3,10} = f(s_{1,12}) = \det(\boldsymbol{\Omega}_1) = \det \left( -\mathbf{z}_{1,4,3} \mathbf{z}_{1,12,4} + \mathbf{I} - \mathbf{z}_{12,11,10} \mathbf{z}_{12,1,11} \right) s_{1,12}$  |    |   |
| <p>9/B<sub>5</sub></p>    | 48 | <i>Numerical:</i> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation)  |
| $s_{6,7} = f(s_{1,12}, s_{3,10}, s_{2,9}) = \det \left( -\mathbf{z}_{1,4,2} \mathbf{z}_{1,12,4} - \mathbf{z}_{2,5,6} \mathbf{z}_{2,9,5} \boldsymbol{\Omega}_2 + \mathbf{z}_{1,4,3} \mathbf{z}_{1,12,4} + (\mathbf{I} - \mathbf{z}_{10,8,7} \mathbf{z}_{10,3,8}) \boldsymbol{\Omega}_1 \right) s_{1,12}$ $s_{2,9} = f(s_{1,12}) = \det(\boldsymbol{\Omega}_2) = \det \left( -\mathbf{z}_{1,4,2} \mathbf{z}_{1,12,4} + \mathbf{I} - \mathbf{z}_{12,11,9} \mathbf{z}_{12,1,11} \right) s_{1,12}$ $s_{3,10} = f(s_{1,12}) = \det(\boldsymbol{\Omega}_1) = \det \left( -\mathbf{z}_{1,4,3} \mathbf{z}_{1,12,4} + \mathbf{I} - \mathbf{z}_{12,11,10} \mathbf{z}_{12,1,11} \right) s_{1,12}$  |    |   |

Table 4: Position analysis of all the catalogued Baranov trusses (part 3).

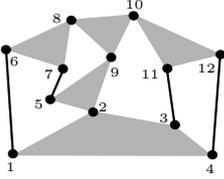
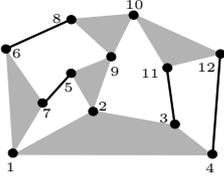
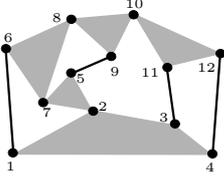
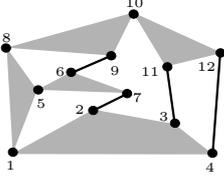
|                  | Baranov truss   | AM | Reported solutions   |
|------------------|---|----|--|
| 9/B <sub>6</sub> |    | 48 | <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation)   |
|                  |   |    | $s_{4,12} = f(s_{7,9}, s_{2,6}, s_{3,10}) = \det \left( (-\mathbf{z}_{2,3,4} + \mathbf{I}) \mathbf{z}_{2,1,3} \mathbf{z}_{2,6,1} \Omega_1 + (\mathbf{I} - \mathbf{z}_{10,11,12} \mathbf{z}_{10,3,11}) \Omega_2 \right) s_{7,9}$<br>$s_{3,10} = f(s_{7,9}, s_{2,6}) = \det(\Omega_2) = \det(\mathbf{z}_{9,5,2} \mathbf{z}_{9,7,5} - \mathbf{z}_{2,1,3} \mathbf{z}_{2,6,1} \Omega_1 - \mathbf{z}_{9,8,10} \mathbf{z}_{9,7,8}) s_{7,9}$<br>$s_{2,6} = f(s_{7,9}) = \det(\Omega_1) = \det(-\mathbf{I} + \mathbf{z}_{9,5,2} \mathbf{z}_{9,7,5} + \mathbf{z}_{7,8,6} \mathbf{z}_{7,9,8}) s_{7,9}$  |
| 9/B <sub>7</sub> |    | 48 | <b>Analytical:</b> Han <i>et. al.</i> [53] (Complex number method with Sylvester resultant). <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation)  |
|                  |   |    | $s_{4,12} = f(s_{1,5}, s_{6,9}, s_{3,10}) = \det \left( -\mathbf{z}_{1,3,4} \mathbf{z}_{1,2,3} \mathbf{z}_{1,5,2} + \mathbf{z}_{1,2,3} \mathbf{z}_{1,5,2} + (\mathbf{I} - \mathbf{z}_{10,11,12} \mathbf{z}_{10,3,11}) \Omega_2 \right) s_{1,5}$<br>$s_{3,10} = f(s_{1,5}, s_{6,9}) = \det(\Omega_2) = \det \left( -\mathbf{z}_{1,2,3} \mathbf{z}_{1,5,2} + \mathbf{z}_{1,7,6} \mathbf{z}_{1,5,7} + (\mathbf{I} - \mathbf{z}_{9,8,10} \mathbf{z}_{9,6,8}) \Omega_1 \right) s_{1,5}$<br>$s_{6,9} = f(s_{1,5}) = \det(\Omega_1) = \det \left( -\mathbf{z}_{1,7,6} \mathbf{z}_{1,5,7} + \mathbf{I} - \mathbf{z}_{5,2,9} \mathbf{z}_{5,1,2} \right) s_{1,5}$              |
| 9/B <sub>8</sub> |  | 48 | <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation, the reported number of AM is incorrect (44))  |
|                  |   |    | $s_{4,12} = f(s_{1,7}, s_{5,8}, s_{3,10}) = \det \left( -\mathbf{z}_{1,3,4} \mathbf{z}_{1,2,3} \mathbf{z}_{1,7,2} + \mathbf{z}_{1,2,3} \mathbf{z}_{1,7,2} + (\mathbf{I} - \mathbf{z}_{10,11,12} \mathbf{z}_{10,3,11}) \Omega_2 \right) s_{1,7}$<br>$s_{3,10} = f(s_{1,7}, s_{5,8}) = \det(\Omega_2) = \det \left( -\mathbf{z}_{1,2,3} \mathbf{z}_{1,7,2} + \mathbf{I} - \mathbf{z}_{7,2,5} \mathbf{z}_{7,1,2} + (\mathbf{I} - \mathbf{z}_{8,9,10} \mathbf{z}_{8,5,9}) \Omega_1 \right) s_{1,7}$<br>$s_{5,8} = f(s_{1,7}) = \det(\Omega_1) = \det \left( \mathbf{z}_{7,2,5} \mathbf{z}_{7,1,2} - \mathbf{z}_{7,6,8} \mathbf{z}_{7,1,6} \right) s_{1,7}$               |
| 9/B <sub>9</sub> |  | 42 | <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation)   |
|                  |   |    | $s_{4,12} = f(s_{1,7}, s_{6,8}, s_{3,10}) = \det \left( -\mathbf{z}_{1,3,4} \mathbf{z}_{1,2,3} \mathbf{z}_{1,7,2} + \mathbf{z}_{1,2,3} \mathbf{z}_{1,7,2} + (\mathbf{I} - \mathbf{z}_{10,11,12} \mathbf{z}_{10,3,11}) \Omega_2 \right) s_{1,7}$<br>$s_{3,10} = f(s_{1,7}, s_{6,8}) = \det(\Omega_2) = \det \left( -\mathbf{z}_{1,2,3} \mathbf{z}_{1,7,2} + \mathbf{I} - \mathbf{z}_{7,5,6} \mathbf{z}_{7,1,5} + (\mathbf{I} - \mathbf{z}_{8,9,10} \mathbf{z}_{8,6,9}) \Omega_1 \right) s_{1,7}$<br>$s_{6,8} = f(s_{1,7}) = \det(\Omega_1) = \det \left( -\mathbf{I} + \mathbf{z}_{7,5,6} \mathbf{z}_{7,1,5} + \mathbf{z}_{1,5,8} \mathbf{z}_{1,7,5} \right) s_{1,7}$ |

Table 5: Position analysis of all the catalogued Baranov trusses (part 4).

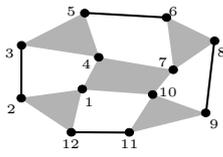
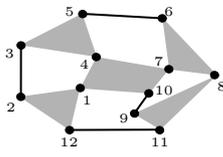
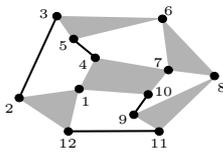
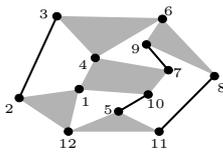
| Baranov truss   | AM | Reported solutions   |
|---|----|--|
|  <p>9/B<sub>10</sub></p>   | 30 | <p><b>Analytical:</b> Han <i>et. al.</i> [54] (Complex number method with Sylvester resultant), Dhingra <i>et. al.</i> [55] (Vector method with Sylvester resultant), Borràs and Di Gregorio [56] (Vector method with Sylvester dialytic elimination method). <b>Numerical:</b> Liu and Yang [39] (Homotopy continuation), Hang <i>et. al.</i> [8] (Vector method with homotopy continuation), Cai <i>et. al.</i> [57]</p> |
| $s_{11,12} = f(s_{1,3}, s_{5,7}, s_{8,10}) = \det(-\mathbf{z}_{1,7,10}\mathbf{z}_{1,4,7}\mathbf{z}_{1,3,4} + \mathbf{z}_{10,9,11}\mathbf{z}_{10,8,9}\Omega_2 + \mathbf{z}_{1,2,12}\mathbf{z}_{1,3,2}) s_{1,3}$  |    |  |
| $s_{8,10} = f(s_{1,3}, s_{5,7}) = \det(\Omega_2) = \det(-\mathbf{z}_{1,4,7}\mathbf{z}_{1,3,4} + \mathbf{z}_{7,6,8}\mathbf{z}_{7,5,6}\Omega_1 + \mathbf{z}_{1,7,10}\mathbf{z}_{1,4,7}\mathbf{z}_{1,3,4}) s_{1,3}$  |    |  |
| $s_{5,7} = f(s_{1,3}) = \det(\Omega_1) = \det(\mathbf{z}_{1,4,7}\mathbf{z}_{1,3,4} - \mathbf{I} + \mathbf{z}_{3,4,5}\mathbf{z}_{3,1,4}) s_{1,3}$  |    |  |
|  <p>9/B<sub>11</sub></p>   | 36 | <p><b>Analytical:</b> Wei <i>et. al.</i> [58] (Complex number method with Sylvester resultant). <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation, the reported number of AM is incorrect (34))</p>  |
| $s_{11,12} = f(s_{1,3}, s_{5,7}, s_{8,10}) = \det(-\mathbf{z}_{1,7,10}\mathbf{z}_{1,4,7}\mathbf{z}_{1,3,4} + (\mathbf{I} - \mathbf{z}_{8,9,11}\mathbf{z}_{8,10,9})\Omega_2 + \mathbf{z}_{1,2,12}\mathbf{z}_{1,3,2}) s_{1,3}$                              |    |  |
| $s_{8,10} = f(s_{1,3}, s_{5,7}) = \det(\Omega_2) = \det(-\mathbf{z}_{1,4,7}\mathbf{z}_{1,3,4} + \mathbf{z}_{7,6,8}\mathbf{z}_{7,5,6}\Omega_1 + \mathbf{z}_{1,7,10}\mathbf{z}_{1,4,7}\mathbf{z}_{1,3,4}) s_{1,3}$  |    |  |
| $s_{5,7} = f(s_{1,3}) = \det(\Omega_1) = \det(\mathbf{z}_{1,4,7}\mathbf{z}_{1,3,4} - \mathbf{I} + \mathbf{z}_{3,4,5}\mathbf{z}_{3,1,4}) s_{1,3}$  |    |  |
|  <p>9/B<sub>12</sub></p>   | 40 | <p><b>Analytical:</b> Dhingra <i>et. al.</i> [55] (Vector method with Sylvester resultant). <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation, the reported number of AM is incorrect, (54))</p>   |
| $s_{2,3} = f(s_{4,6}, s_{8,10}, s_{1,11}) = \det(-\mathbf{z}_{4,7,1}\mathbf{z}_{4,6,7} - \mathbf{z}_{1,12,2}\mathbf{z}_{1,11,12}\Omega_2 + \mathbf{I} - \mathbf{z}_{6,5,3}\mathbf{z}_{6,4,5}) s_{4,6}$  |    |  |
| $s_{1,11} = f(s_{4,6}, s_{8,10}) = \det(\Omega_2) = \det(-\mathbf{z}_{4,7,1}\mathbf{z}_{4,6,7} + \mathbf{I} - \mathbf{z}_{6,7,8}\mathbf{z}_{6,4,7} + \mathbf{z}_{8,9,11}\mathbf{z}_{8,10,9}\Omega_1) s_{4,6}$   |    |  |
| $s_{8,10} = f(s_{4,6}) = \det(\Omega_1) = \det(-\mathbf{I} + \mathbf{z}_{6,7,8}\mathbf{z}_{6,4,7} + \mathbf{z}_{4,7,10}\mathbf{z}_{4,6,7}) s_{4,6}$   |    |  |
|  <p>9/B<sub>13</sub></p>   | 40 | <p><b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation)</p>  |
| $s_{8,11} = f(s_{2,4}, s_{6,7}, s_{10,12}) = \det(\mathbf{z}_{4,3,6}\mathbf{z}_{4,2,3} - \mathbf{z}_{6,9,8}\mathbf{z}_{6,7,9}\Omega_1 - \mathbf{z}_{4,1,10}\mathbf{z}_{4,2,1} + (\mathbf{I} - \mathbf{z}_{12,5,11}\mathbf{z}_{12,10,5})\Omega_2) s_{2,4}$ |    |  |
| $s_{10,12} = f(s_{2,4}, s_{6,7}) = \det(\Omega_2) = \det(-\mathbf{I} + \mathbf{z}_{4,1,10}\mathbf{z}_{4,2,1} + \mathbf{z}_{2,1,12}\mathbf{z}_{2,4,1}) s_{2,4}$  |    |  |
| $s_{6,7} = f(s_{2,4}) = \det(\Omega_1) = \det(\mathbf{z}_{4,3,6}\mathbf{z}_{4,2,3} - \mathbf{z}_{4,1,7}\mathbf{z}_{4,2,1}) s_{2,4}$   |    |  |

Table 6: Position analysis of all the catalogued Baranov trusses (part 5).

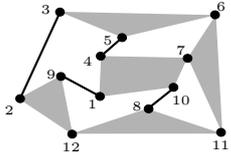
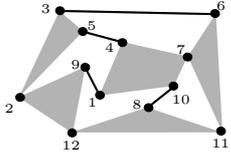
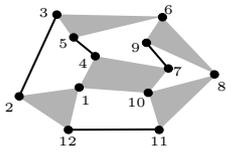
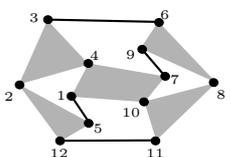
| Baranov truss   | AM | Reported solutions  |
|---|----|---|
|  <p>9/B<sub>14</sub></p>   | 42 | <p><b>Analytical:</b> Han <i>et. al.</i> [59] (Complex number method with Sylvester resultant). <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation, the reported number of AM is incorrect (52))</p> |
| $s_{2,3} = f(s_{4,6}, s_{10,11}, s_{1,12}) = \det(-\mathbf{z}_{4,7,1}\mathbf{z}_{4,6,7} - (\mathbf{I} - \mathbf{z}_{12,9,2}\mathbf{z}_{12,1,9})\Omega_2 + \mathbf{I} - \mathbf{z}_{6,5,3}\mathbf{z}_{6,4,5})^{s_{4,6}}$ $s_{1,12} = f(s_{4,6}, s_{10,11}) = \det(\Omega_2) = \det(-\mathbf{z}_{4,7,1}\mathbf{z}_{4,6,7} + \mathbf{z}_{4,7,10}\mathbf{z}_{4,6,7} + (\mathbf{I} - \mathbf{z}_{11,8,12}\mathbf{z}_{11,10,8})\Omega_1)^{s_{4,6}}$ $s_{10,11} = f(s_{4,6}) = \det(\Omega_1) = \det(-\mathbf{z}_{4,7,10}\mathbf{z}_{4,6,7} + \mathbf{I} - \mathbf{z}_{6,7,11}\mathbf{z}_{6,4,7})^{s_{4,6}}$   |    |   |
|  <p>9/B<sub>15</sub></p>   | 52 | <p><b>Analytical:</b> Dhingra <i>et. al.</i> [55] (Vector method with Sylvester resultant), <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation, the reported number of AM is incorrect (26))</p>     |
| $s_{3,6} = f(s_{10,11}, s_{1,12}, s_{2,4}) = \det(-\mathbf{z}_{10,7,1}\mathbf{z}_{10,11,7} - (\mathbf{I} - \mathbf{z}_{12,9,2}\mathbf{z}_{12,1,9})\Omega_1 - \mathbf{z}_{2,5,3}\mathbf{z}_{2,4,5}\Omega_2 + \mathbf{I} - \mathbf{z}_{11,7,6}\mathbf{z}_{11,10,7})^{s_{10,11}}$ $s_{2,4} = f(s_{10,11}, s_{1,12}) = \det(\Omega_2) = \det(-\mathbf{z}_{10,7,1}\mathbf{z}_{10,11,7} - (\mathbf{I} - \mathbf{z}_{12,9,2}\mathbf{z}_{12,1,9})\Omega_1 + \mathbf{z}_{10,7,4}\mathbf{z}_{10,11,7})^{s_{10,11}}$ $s_{1,12} = f(s_{10,11}) = \det(\Omega_1) = \det(-\mathbf{z}_{10,7,1}\mathbf{z}_{10,11,7} + \mathbf{I} - \mathbf{z}_{11,8,12}\mathbf{z}_{11,10,8})^{s_{10,11}}$ |    |   |
|  <p>9/B<sub>16</sub></p>   | 44 | <p><b>Analytical:</b> Han <i>et. al.</i> [60] (Complex number method with Sylvester resultant). <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation)</p>  |
| $s_{2,3} = f(s_{7,8}, s_{4,6}, s_{1,11}) = \det(-\mathbf{z}_{7,10,1}\mathbf{z}_{7,8,10} - \mathbf{z}_{1,12,2}\mathbf{z}_{1,11,12}\Omega_2 + \mathbf{z}_{7,10,4}\mathbf{z}_{7,8,10} + (\mathbf{I} - \mathbf{z}_{6,5,3}\mathbf{z}_{6,4,5})\Omega_1)^{s_{7,8}}$ $s_{1,11} = f(s_{7,8}, s_{4,6}) = \det(\Omega_2) = \det(-\mathbf{z}_{7,10,1}\mathbf{z}_{7,8,10} + \mathbf{I} - \mathbf{z}_{8,10,11}\mathbf{z}_{8,7,10})^{s_{7,8}}$ $s_{4,6} = f(s_{7,8}) = \det(\Omega_1) = \det(-\mathbf{z}_{7,10,4}\mathbf{z}_{7,8,10} + \mathbf{I} - \mathbf{z}_{8,9,6}\mathbf{z}_{8,7,9})^{s_{7,8}}$   |    |   |
|  <p>9/B<sub>17</sub></p>   | 44 | <p><b>Analytical:</b> Han <i>et. al.</i> [61] (Complex number method with Sylvester resultant). <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation, the reported number of AM is incorrect (66))</p> |
| $s_{3,6} = f(s_{1,2}, s_{10,12}, s_{7,8}) = \det(-\mathbf{I} + \mathbf{z}_{2,4,3}\mathbf{z}_{2,1,4} + \mathbf{z}_{1,4,7}\mathbf{z}_{1,2,4} + (\mathbf{I} - \mathbf{z}_{8,9,6}\mathbf{z}_{8,7,9})\Omega_2)^{s_{1,2}}$ $s_{7,8} = f(s_{1,2}, s_{10,12}) = \det(\Omega_2) = \det(-\mathbf{z}_{1,4,7}\mathbf{z}_{1,2,4} + \mathbf{z}_{1,4,10}\mathbf{z}_{1,2,4} + \mathbf{z}_{10,11,8}\mathbf{z}_{10,12,11}\Omega_1)^{s_{1,2}}$ $s_{10,12} = f(s_{1,2}) = \det(\Omega_1) = \det(-\mathbf{z}_{1,4,10}\mathbf{z}_{1,2,4} + \mathbf{I} - \mathbf{z}_{2,5,12}\mathbf{z}_{2,1,5})^{s_{1,2}}$   |    |   |

Table 7: Position analysis of all the catalogued Baranov trusses (part 6).

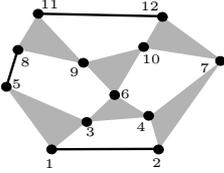
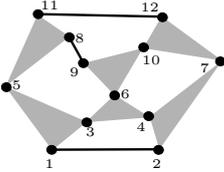
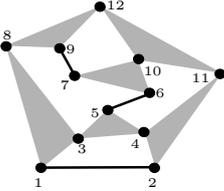
| Baranov truss  | AM | Reported solutions   |
|--|----|--|
|  <p>9/B<sub>18</sub></p>  | 38 | <p><b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation, the reported number of AM is incorrect (34))</p>   |
| $s_{11,12} = f(s_{1,4}, s_{6,7}, s_{5,9}) = \det(-\mathbf{z}_{1,3,5}\mathbf{z}_{1,4,3} - (\mathbf{I} - \mathbf{z}_{9,8,11}\mathbf{z}_{9,5,8})\Omega_2 + \mathbf{I} - \mathbf{z}_{4,3,6}\mathbf{z}_{4,1,3} + (\mathbf{I} - \mathbf{z}_{7,10,12}\mathbf{z}_{7,6,10})\Omega_1)^{s_{1,4}}$ |    |  |
| $s_{5,9} = f(s_{1,4}, s_{6,7}) = \det(\Omega_2) = \det(-\mathbf{z}_{1,3,5}\mathbf{z}_{1,4,3} + \mathbf{I} - \mathbf{z}_{4,3,6}\mathbf{z}_{4,1,3} + \mathbf{z}_{6,10,9}\mathbf{z}_{6,7,10}\Omega_1)^{s_{1,4}}$  |    |  |
| $s_{6,7} = f(s_{1,4}) = \det(\Omega_1) = \det(\mathbf{z}_{4,3,6}\mathbf{z}_{4,1,3} - \mathbf{z}_{4,2,7}\mathbf{z}_{4,1,2})^{s_{1,4}}$  |    |  |
|  <p>9/B<sub>19</sub></p>   | 46 | <p><b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation, the reported number of AM is incorrect (62))</p>   |
| $s_{11,12} = f(s_{1,4}, s_{6,7}, s_{5,9}) = \det(-\mathbf{z}_{1,3,5}\mathbf{z}_{1,4,3} - \mathbf{z}_{5,8,11}\mathbf{z}_{5,9,8}\Omega_2 + \mathbf{I} - \mathbf{z}_{4,3,6}\mathbf{z}_{4,1,3} + (\mathbf{I} - \mathbf{z}_{7,10,12}\mathbf{z}_{7,6,10})\Omega_1)^{s_{1,4}}$                |    |  |
| $s_{5,9} = f(s_{1,4}, s_{6,7}) = \det(\Omega_2) = \det(-\mathbf{z}_{1,3,5}\mathbf{z}_{1,4,3} + \mathbf{I} - \mathbf{z}_{4,3,6}\mathbf{z}_{4,1,3} + \mathbf{z}_{6,10,9}\mathbf{z}_{6,7,10}\Omega_1)^{s_{1,4}}$  |    |  |
| $s_{6,7} = f(s_{1,4}) = \det(\Omega_1) = \det(\mathbf{z}_{4,3,6}\mathbf{z}_{4,1,3} - \mathbf{z}_{4,2,7}\mathbf{z}_{4,1,2})^{s_{1,4}}$  |    |  |
|  <p>9/B<sub>20</sub></p>  | 46 | <p><b>Analytical:</b> Wang <i>et. al.</i> [62] (Complex number method with Dixon resultant and Sylvester resultant), Wang <i>et. al.</i> [63] (Complex number method with Sylvester resultant). <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation, the reported number of AM is incorrect (64)), Luo and Liu [64] (Complex number method with chaos least square method)</p> |
| $s_{5,6} = f(s_{1,4}, s_{8,11}, s_{9,10}) = \det(-\mathbf{I} + \mathbf{z}_{4,3,5}\mathbf{z}_{4,1,3} + \mathbf{z}_{1,3,8}\mathbf{z}_{1,4,3} + \mathbf{z}_{8,12,9}\mathbf{z}_{8,11,12}\Omega_1 + (\mathbf{I} - \mathbf{z}_{10,7,6}\mathbf{z}_{10,9,7})\Omega_2)^{s_{1,4}}$               |    |  |
| $s_{9,10} = f(s_{1,4}, s_{8,11}) = \det(\Omega_2) = \det((-\mathbf{z}_{8,12,9}\mathbf{z}_{8,11,12} + \mathbf{I} - \mathbf{z}_{11,12,10}\mathbf{z}_{11,8,12})\Omega_1)^{s_{1,4}}$   |    |  |
| $s_{8,11} = f(s_{1,4}) = \det(\Omega_1) = \det(-\mathbf{z}_{1,3,8}\mathbf{z}_{1,4,3} + \mathbf{I} - \mathbf{z}_{4,2,11}\mathbf{z}_{4,1,2})^{s_{1,4}}$  |    |  |

Table 8: Position analysis of all the catalogued Baranov trusses (part 7).

| Baranov truss  | AM | Reported solutions   |
|--|----|--|
| <p>9/B<sub>21</sub></p>  | 40 | <p><b>Analytical:</b> Wang <i>et. al.</i> [65] (Complex number method with Dixon resultant and Sylvester resultant). <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation, the reported number of AM is incorrect (30))</p>                                 |
| $s_{5,8} = f(s_{1,4}, s_{6,11}, s_{7,12}) = \det(-\mathbf{z}_{1,3,5}\mathbf{z}_{1,4,3} + \mathbf{I} - \mathbf{z}_{4,3,6}\mathbf{z}_{4,1,3} + \mathbf{z}_{6,10,7}\mathbf{z}_{6,11,10}\Omega_1 + (\mathbf{I} - \mathbf{z}_{12,9,8}\mathbf{z}_{12,7,9})\Omega_2)^{s_{1,4}}$ $s_{7,12} = f(s_{1,4}, s_{6,11}) = \det(\Omega_2) = \det((-z_{6,10,7}z_{6,11,10} + \mathbf{I} - z_{11,10,12}z_{11,6,10})\Omega_1)^{s_{1,4}}$ $s_{6,11} = f(s_{1,4}) = \det(\Omega_1) = \det(\mathbf{z}_{4,3,6}\mathbf{z}_{4,1,3} - \mathbf{z}_{4,2,11}\mathbf{z}_{4,1,2})^{s_{1,4}}$  |    |  |
| <p>9/B<sub>22</sub></p>  | 50 | <p><b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation)</p>  |
| $s_{7,10} = f(s_{1,4}, s_{8,11}, s_{5,9}) = \det(-\mathbf{I} + \mathbf{z}_{4,3,5}\mathbf{z}_{4,1,3} - \mathbf{z}_{5,6,7}\mathbf{z}_{5,9,6}\Omega_2 + \mathbf{z}_{1,3,8}\mathbf{z}_{1,4,3} + (\mathbf{I} - \mathbf{z}_{11,12,10}\mathbf{z}_{11,8,12})\Omega_1)^{s_{1,4}}$ $s_{5,9} = f(s_{1,4}, s_{8,11}) = \det(\Omega_2) = \det(-\mathbf{I} + \mathbf{z}_{4,3,5}\mathbf{z}_{4,1,3} + \mathbf{z}_{1,3,8}\mathbf{z}_{1,4,3} + \mathbf{z}_{8,12,9}\mathbf{z}_{8,11,12}\Omega_1)^{s_{1,4}}$ $s_{8,11} = f(s_{1,4}) = \det(\Omega_1) = \det(-\mathbf{z}_{1,3,8}\mathbf{z}_{1,4,3} + \mathbf{I} - \mathbf{z}_{4,2,11}\mathbf{z}_{4,1,2})^{s_{1,4}}$ |    |  |
| <p>9/B<sub>23</sub></p>  | 46 | <p><b>Analytical:</b> Zhuang <i>et. al.</i> [66] (Complex number method with Sylvester resultant), Wang <i>et. al.</i> [67] (Complex number method with Dixon resultant and Sylvester resultant). <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation)</p> |
| $s_{10,11} = f(s_{1,4}, s_{6,8}, s_{5,7}) = \det(-\mathbf{z}_{6,12,10}\mathbf{z}_{6,8,12}\Omega_1 - \mathbf{z}_{1,3,6}\mathbf{z}_{1,4,3} + \mathbf{z}_{1,2,5}\mathbf{z}_{1,4,2} + \mathbf{z}_{5,9,11}\mathbf{z}_{5,7,9}\Omega_2)^{s_{1,4}}$ $s_{5,7} = f(s_{1,4}, s_{6,8}) = \det(\Omega_2) = \det(-\mathbf{z}_{1,2,5}\mathbf{z}_{1,4,2} + \mathbf{I} - \mathbf{z}_{4,3,7}\mathbf{z}_{4,1,3})^{s_{1,4}}$ $s_{6,8} = f(s_{1,4}) = \det(\Omega_1) = \det(-\mathbf{z}_{1,3,6}\mathbf{z}_{1,4,3} + \mathbf{I} - \mathbf{z}_{4,2,8}\mathbf{z}_{4,1,2})^{s_{1,4}}$   |    |  |

Table 9: Position analysis of all the catalogued Baranov trusses (part 8).

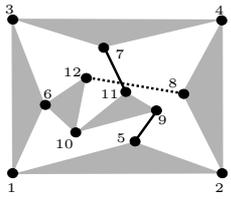
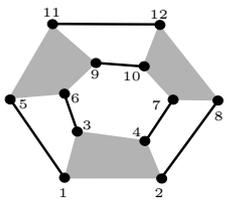
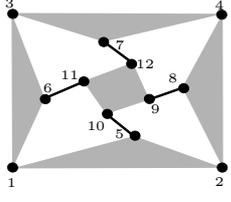
| Baranov truss   | AM | Reported solutions  |
|---|----|---|
| <p>9/B<sub>24</sub></p>    | 50 | <p><b>Analytical:</b> Wang <i>et. al.</i> [68] (Complex number method with Dixon resultant and Sylvester resultant).<br/> <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation)</p>  |
| $s_{7,11} = f(s_{1,4}, s_{6,8}, s_{5,10}) = \det(-\mathbf{I} + \mathbf{Z}_{4,3,7}\mathbf{Z}_{4,1,3} + \mathbf{Z}_{1,2,5}\mathbf{Z}_{1,4,2} + (i - \mathbf{Z}_{10,9,11}\mathbf{Z}_{10,5,9})\mathbf{\Omega}_2) s_{1,4}$ $s_{5,10} = f(s_{1,4}, s_{6,8}) = \det(\mathbf{\Omega}_2) = \det(-\mathbf{Z}_{1,2,5}\mathbf{Z}_{1,4,2} + \mathbf{Z}_{1,3,6}\mathbf{Z}_{1,4,3} + \mathbf{Z}_{6,12,10}\mathbf{Z}_{6,8,12}\mathbf{\Omega}_1) s_{1,4}$ $s_{6,8} = f(s_{1,4}) = \det(\mathbf{\Omega}_1) = \det(-\mathbf{Z}_{1,3,6}\mathbf{Z}_{1,4,3} + \mathbf{I} - \mathbf{Z}_{4,2,8}\mathbf{Z}_{4,1,2}) s_{1,4}$   |    |   |
| <p>9/B<sub>25</sub></p>   | 52 | <p><b>Analytical:</b> Wang <i>et. al.</i> [69] (Complex number method with Dixon resultant and Sylvester resultant).<br/> <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation)</p>  |
| $s_{9,10} = f(s_{1,6}, s_{2,7}) = \det(-\mathbf{I} + \mathbf{Z}_{6,5,9}\mathbf{Z}_{6,1,5} + \mathbf{Z}_{1,3,2}\mathbf{Z}_{1,6,3} + (\mathbf{I} - \mathbf{Z}_{7,8,10}\mathbf{Z}_{7,2,8})\mathbf{Z}_{2,4,7}(-\mathbf{Z}_{1,3,2}\mathbf{Z}_{1,6,3} + \mathbf{Z}_{1,3,4}\mathbf{Z}_{1,6,3})) s_{1,6}$ <p><math>s_{11,12} = f(s_{1,6}, s_{2,7})</math> is obtained applying the permutation</p> $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 11 & 12 & 9 & 10 \end{bmatrix}$ to the above equation   |    |   |
| <p>9/B<sub>26</sub></p>    | 56 | <p><b>Analytical:</b> Wohlhart [70], [71] (Vector method with Sylvester resultant), Lösch [9] (Vector method with Gröbner basis), Wampler [43] (Complex number method with Dixon resultant), Wang <i>et. al.</i> [72] (Complex number method with Dixon resultant), Wang <i>et. al.</i> [73] (Complex number method with Dixon resultant and Sylvester resultant).<br/> <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation, the reported number of AM is incorrect (38)), He <i>et. al.</i> [74] (Complex number method with hyper-chaotic Newton downhill method)</p> |
| $s_{5,10} = f(s_{3,11}, s_{3,12}, s_{1,4}) = \det(-\mathbf{Z}_{11,12,10}\mathbf{Z}_{11,3,12} + \mathbf{I} - \mathbf{Z}_{3,6,1}\mathbf{Z}_{3,11,6} + \mathbf{Z}_{1,2,5}\mathbf{Z}_{1,4,2}\mathbf{\Omega}_1) s_{3,11}$ $s_{1,4} = f(s_{3,11}, s_{3,12}) = \det(\mathbf{\Omega}_1) = \det(\mathbf{Z}_{3,6,1}\mathbf{Z}_{3,11,6} - \mathbf{Z}_{3,7,4}\mathbf{Z}_{3,12,7}\mathbf{Z}_{3,11,12}) s_{3,11}$ <p><math>s_{8,9} = f(s_{3,12}, s_{3,11}, s_{1,4})</math> and <math>s_{1,4} = f(s_{3,12}, s_{3,11})</math> are obtained applying the permutation</p> $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 4 & 2 & 3 & 1 & 8 & 7 & 6 & 5 & 10 & 9 & 12 & 11 \end{bmatrix}$ to the above equations |    |   |

Table 10: Position analysis of all the catalogued Baranov trusses (part 9).

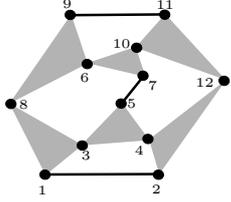
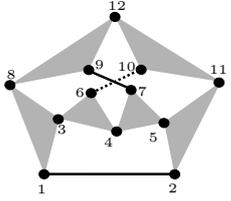
| Baranov truss   | AM | Reported solutions  |
|---|----|---|
| <p>9/B<sub>27</sub></p>    | 50 | <p><b>Analytical:</b> Wang <i>et. al.</i> [75] (Complex number method with Dixon resultant and Sylvester resultant). <b>Numerical:</b> Hang <i>et. al.</i> [8] (Vector method with homotopy continuation, the reported number of AM is incorrect (36))</p>  |
| $s_{9,11} = f(s_{5,12}, s_{7,12}, s_{3,6}) = \det(-\mathbf{Z}_{7,10,6}\mathbf{Z}_{7,12,10} + \mathbf{Z}_{6,8,9}\mathbf{Z}_{6,3,8}\Omega_1 + \mathbf{I} - \mathbf{Z}_{12,10,11}\mathbf{Z}_{12,7,10}) s_{7,12}$ $s_{3,6} = f(s_{5,12}, s_{7,12}) = \det(\Omega_1) = \det(-\mathbf{Z}_{5,4,3}\mathbf{Z}_{5,12,4}(-\mathbf{Z}_{7,12,5} + \mathbf{I}) - \mathbf{Z}_{7,12,5} + \mathbf{Z}_{7,10,6}\mathbf{Z}_{7,12,10}) s_{7,12}$ $s_{1,2} = f(s_{7,12}, s_{5,12}, s_{3,6}) \text{ and } s_{3,6} = f(s_{7,12}, s_{5,12}) \text{ are obtained applying the permutation}$ $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 9 & 11 & 6 & 10 & 7 & 3 & 5 & 8 & 1 & 4 & 2 & 12 \end{bmatrix} \text{ to the above equations}$ |    |   |
| <p>9/B<sub>28</sub></p>    | 58 | <p><b>Analytical:</b> Wang <i>et. al.</i> [16] (Complex number method with Dixon resultant). <b>Numerical:</b> Almadi <i>et. al.</i> [76] (Homotopy continuation), Hang <i>et. al.</i> [8] (Vector method with homotopy continuation), Luo and Liu [77] (Complex number method with hyper-chaotic Newton downhill method)</p> |
| $s_{6,10} = f(s_{1,4}, s_{2,4}, s_{8,11}) = \det(-\mathbf{I} + \mathbf{Z}_{4,3,6}\mathbf{Z}_{4,1,3} + \mathbf{Z}_{1,3,8}\mathbf{Z}_{1,4,3} + (\mathbf{I} - \mathbf{Z}_{11,12,10}\mathbf{Z}_{11,8,12})\Omega_1) s_{1,4}$ $s_{8,11} = f(s_{1,4}, s_{2,4}) = \det(\Omega_1) = \det(-\mathbf{Z}_{1,3,8}\mathbf{Z}_{1,4,3} + \mathbf{I} - \mathbf{Z}_{4,1,2} + \mathbf{Z}_{2,5,11}\mathbf{Z}_{2,4,5}\mathbf{Z}_{4,1,2}) s_{1,4}$ $s_{7,9} = f(s_{2,4}, s_{1,4}, s_{8,11}) \text{ and } s_{8,11} = f(s_{2,4}, s_{1,4}) \text{ are obtained applying the permutation}$ $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 1 & 5 & 4 & 3 & 7 & 6 & 11 & 10 & 9 & 8 & 12 \end{bmatrix} \text{ to the above equations}$   |    |   |

Table 11: Position analysis of all the catalogued Baranov trusses (part 10).

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