

# Pattern and velocity selection of fronts propagating in modulated media

J. Armero<sup>1</sup>, A.M. Lacasta<sup>2</sup>, L. Ramírez-Piscina<sup>2</sup>,  
J. Casademunt<sup>1</sup>, J.M. Sancho<sup>1,4</sup> and F. Sagués<sup>3</sup>

<sup>1</sup> *Departament d'Estructura i Constituents de la Matèria, Universitat de Barcelona,  
Av. Diagonal 647, E-08028 Barcelona, Spain.*

<sup>2</sup> *Departament de Física Aplicada, Universitat Politècnica de Catalunya,  
Av. Dr. Gregorio Marañón 50, E-08028 Barcelona, Spain.*

<sup>3</sup> *Departament de Química Física, Universitat de Barcelona,  
Av. Diagonal 647, E-08028 Barcelona, Spain.*

<sup>4</sup> *Institute for Nonlinear Science, University of California, San Diego,  
9500 Gilman Drive, Dept 0407, La Jolla, CA 92093-0407, USA*

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We study the problem of pattern and velocity selection of morphologically stable two-dimensional fronts propagating in a spatially modulated medium. The generic system is governed by a local equation and evolves towards a non-trivial steady state with a spatial structure which arises from non-local competition effects and does not necessarily mimic the local structure externally fixed by the modulation. The dynamical process leading to this steady state is studied both analytically and numerically.

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Velocity and shape selection in pattern forming interfaces has been an issue of broad interest in recent years, in a variety of contexts including viscous fingering, dendritic growth, directional solidification, flame propagation, etc [1]. In many of these systems the selection of a steady state velocity is closely related to the selection of a specific spatial pattern. The degeneracy of the steady state velocity and shape, however, may only be apparent, due to the neglect of surface tension, which acts as a singular perturbation. It has been shown that when surface tension effects are properly taken into account, the spatial scale and velocity become uniquely fixed. A more genuine problem of velocity selection arises when the front of a stable state invades an unstable state. In this case a continuous degeneracy of solutions may exist and the selection of the steady state solution becomes a dynamical problem [2–6]. This situation has received a great deal of attention in recent years and has been explained satisfactorily [4–6]. In the context of these references, however, a planar front in a two-dimensional configuration is morphologically stable and the velocity selection is thus not related to the emergence of any spatial structure.

In this letter we address an intermediate situation in which an otherwise stable front, develops a spatial structure as a consequence of the spatial modulation of an external control parameter. Our study has been motivated by some experimental studies of chemical waves propagating into modulated excitable media [7,8]. In these

references stationary patterns were generated by spatial modulations of the chemical excitability, either from variations in the illumination conditions of a photosensitive reaction [7], or from a smooth concentration profile radially maintained in an annular reactor [8]. Here we will focus on two aspects of the generic case. First, we address the determination of the stationary shape and velocity of the two-dimensional front for a given generic spatial modulation of an external control parameter. Second, we study the dynamic process of competition of local structures leading to the final configuration. The problem exhibits some interesting similarities but also some qualitative differences from the case of pattern selection of morphologically unstable interfaces. For instance, curvature effects in the present case act also as a singular perturbation and are crucial in determining the steady state shape but are rather unimportant to set the actual steady state velocity. An important difference of the present case of front propagation with respect to the above examples of unstable interface dynamics is the fact that the front dynamics is basically local. Surface tension effects however are also crucial in the slow process of competition dynamics leading to the non-trivial steady state solution.

In order to set the problem let us consider the evolution of a linear interface  $y(x, t)$  that is moving in the  $y$ -direction. In a broad class of systems the normal velocity of the interface is given locally by the planar interface velocity plus a correction coming from the curvature [9,10]. This result can be extended to the case in which a sufficiently smooth modulation of an external parameter is considered. Then, in terms of the local velocity  $u(x)$  corresponding to the planar front solution, we have

$$v_n(x, t) = u(x) + \kappa(x, t) \quad (1)$$

where  $v_n(x, t)$  is the local normal velocity and  $\kappa(x, t)$  is the curvature.

The generic situation can be summarized as follows. If the modulating function  $u(x)$  has  $n$  local maxima, the front will adapt, after a short transient, to a shape with

the same amount of local maxima (which we will call fingers by analogy with similar pattern forming interfaces). In Fig.1 it is shown the temporal evolution of the interface for a  $u(x)$  with  $n = 8$  local maxima. Starting with a planar front, a configuration with  $n = 8$  fingers is formed. At early times, the shape and velocity of each of these fingers are strongly dominated by the neighborhood of the corresponding local maxima of  $u(x)$ . However, since the fingers advance, in general, with different velocities, a slow process of competition among them sets in. As a result, some slow fingers (three) will be eliminated and some of them (five) will survive. One of the questions we address in this paper is the determination of the actual number  $m \leq n$  of surviving fingers in the steady state. The nature of this competition process is more clearly shown in Fig.2, where the local velocity in the  $y$ -direction ( $\frac{\partial y}{\partial t}$ ) is plotted for a particular case of  $n = 2$  and  $m = 1$ .

Our analysis starts from Eq. (1). Without loss of generality we assume that the modulation of the system is periodic with period  $L$ , which we will take as the system size. We assume that this modulation is sufficiently smooth, so we can take  $\frac{1}{L}$  as a small parameter. In order to construct a perturbative scheme, it is useful to write Eq. (1) in terms of the angle of the front ( $\tan \theta = \frac{\partial y}{\partial x}$ , see Fig.2.a), and to rescale variables according to  $z = x/L$  and  $\tau = t/L$  which gives

$$\frac{\partial \tan \theta}{\partial \tau} = \frac{\partial}{\partial z} \left( \frac{u(z)}{\cos \theta} + \frac{1}{L} \frac{\partial \theta}{\partial z} \right) \quad (2)$$

For the fingers emerging from a planar interface, the value of the expression in parenthesis in the right hand side of Eq. (2) adopts a different constant value, which from Eq. (1) is equal to the velocity of the finger. In this way the time derivative in Eq. (2) is very small in each finger and its shape is practically unchanged. From this situation all the interesting dynamics occurs in the contact points between fingers, where the value of the spatial derivative in Eq. (2) is nonzero (see Fig.2).

The spirit of the scaling of variables in Eq. (2) is to extract explicitly the part of the solution that scales with the system size  $L$ . This scaling information is basically contained in the lowest order solution of Eq. (2). In this equation the small parameter appears multiplying the highest order derivative, which makes the perturbative scheme singular. That means that the solution obtained from this equation as an expansion in  $1/L$  (the so-called outer solution) will necessarily break down in regions where the highest order derivative take large values. In these regions, called boundary layers, a different expansion has to be done (defining the so-called inner solution). In our problem, the derivatives of  $\theta$  are related with the curvature of the interface, so boundary layers correspond to regions where the shape of the interface does not scale with the system size. Inner and outer solutions have to

be asymptotically matched order by order to get explicit approximations. However, here we will not be concerned with the actual construction of explicit perturbative solutions, but rather we will extract some simple rules for the competition dynamics and for the selection of fingers from the lowest orders of this scheme.

After the initial transient, for each local maximum  $i$  of  $u(x)$  there is a roughly stationary finger propagating with a velocity  $v^i$  given by the value of the parenthesis in Eq.(2). Substituting the expansions  $v^i = v_0^i + L^{-1}v_1^i + \dots$  and  $\theta^i = \theta_0^i + L^{-1}\theta_1^i + \dots$  in Eq.(2), we get at lowest order

$$\cos \theta_0^i = \frac{u}{v_0^i} \quad (3)$$

and for the velocities

$$v_0^i = u_m^i \quad ; \quad v_1^i = -\sqrt{\frac{|u_m^{ii}|}{u_m^i}} \quad (4)$$

where  $u_m^i$  is the maximum of  $u(z)$  in the finger  $i$ , and  $u_m^{ii}$  its second derivative at the same maximum. We see that the curvature correction for the velocity of the finger is given by a length, which is nothing but the length scale of the spatial variation of the modulating function  $u$  at its local maximum.

The dynamics of finger competition can be found at the lowest order from the solution inside the boundary layers between them. The corresponding equation is obtained from Eq. (1) by making  $u$  constant. In terms of the original variables  $x, t$  it reads

$$\frac{\partial \theta}{\partial t} = \cos^2 \theta \frac{\partial^2 \theta}{\partial x^2} + u \sin \theta \frac{\partial \theta}{\partial x} \quad (5)$$

which has to be solved with the appropriate boundary conditions to match the outer solution. For a boundary layer placed between two fingers moving at velocities  $v_-$  and  $v_+$ , the boundary conditions at the lowest order will be

$$\lim_{x \rightarrow \pm \infty} \theta_0^{in}(x) = \theta_{\pm} \quad ; \quad \cos \theta_{\pm} = \frac{u}{v_{\pm}} \quad (6)$$

Eq. (5) has a stationary solution that can be seen as a one-dimensional front moving laterally at velocity  $c$ , which represents the invasion of the slower finger by the faster one. By taking  $\theta(x, t) = \theta(x - ct)$  into Eq. (5) and imposing the boundary conditions Eq. (6), the velocity of the lateral front turns out to be

$$c = \frac{v_+ - v_-}{\tan \theta_- - \tan \theta_+} \quad (7)$$

In view of the solutions found for the shape and velocity of each finger, it becomes clear that up to the lowest orders in the inverse system size, the behavior of each finger during the competition process is dominated by the local properties of the medium in which it moves, and different fingers are roughly independent from each other.

The invasion of a slow finger by a faster one is basically a kinematic process, in which the boundary moves as dictated by the velocities of each finger. A similar competition process appears in the deterministic KPZ equation but evolving to a different steady state [11].

The next point to address is the final stationary state. From the lowest order approximation, it can be shown that for large enough  $L$ , only the fastest finger given by the absolute maximum  $u_M$  of the modulating function will survive. In that case the velocity of the whole front is given by

$$v = u_M - \sqrt{\frac{|u_M''|}{u_M}} + \dots \quad (8)$$

where the derivative refers to the original  $x$  variable. The shape of the whole front is now given by the same solution as Eq. (3) but applied to the values  $u_M$  and  $u_M''$  of the absolute maximum of  $u(x)$ . A rather counterintuitive feature of the solution is that only one tip (maximum of the front) remains, even if there exist several local maxima of the modulating function. This statement, true for large  $L$ , is valid as long as the velocity selected by the front and given by Eq. (8), remains greater than the value of the other local maxima of the  $u$  function. If it is not the case, Eq. (1) would give negative curvatures for the secondary maxima and then they will be also local maxima for the shape of the front.

In that way, the solution of the front shape for a not very large  $L$  may differ qualitatively from the scaling solution, with the existence of additional fingers, and the perturbative scheme as described above may fail. However, the number of surviving fingers (i.e. number of maxima of the stationary front shape) are given by a simple comparison between the actual selected velocity  $v$  and the local maxima of  $u(x)$ . The front shape will develop fingers at those maxima of  $u(x)$  larger than  $v$ . A good estimate of the selected velocity  $v$  can be obtained in turn from the perturbative analysis and corresponds to the largest of the different values taken by  $v^i$  when evaluated at the different local maxima of  $u(z)$ . In any case, for a given modulating function  $u(z)$ , there always exists a system size  $L$  above which the scaling solution, always with a single finger, gives the correct shape, and the whole perturbative analysis works. Fig.3 shows the steady state reached by systems with different values of  $L$  but the same two-maxima modulating function. It can be seen that for small  $L$ , two distinct fingers survive, but for larger values of  $L$  the solution approaches the scaling solution with a single finger.

In summary we have seen that the steady state front velocity and the number of surviving fingers follow very simple rules found from a local analysis at the different maxima of the modulation  $u(x)$ .

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FIG. 1. Temporal evolution of an initially flat interface. At the bottom it is shown the modulating function  $u(x)$ .

FIG. 2. Competition process between two fingers. (a) Different stages in time of the front shape. The fronts are plotted in a frame moving at the steady velocity  $v$ . (b) Vertical component of the local velocity of the front shown for the same evolution. The effective transversal velocity  $c$  (see Eq. (7) in the text) is also indicated.

FIG. 3. Scaled stationary front shapes reached by systems of different sizes  $L$  (cf. figure legend). The inset shows the two-maxima modulating function  $u(x/L)$  common to all systems.





