Wigner Higher Order Moment Spectra: Definition, Properties, Computation and Application to Transient Signal Analysis

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Abstract—The Wigner higher order moment spectra (WHOS) are defined as extensions of the Wigner-Ville distribution (WD) to higher order moment spectra domains. A general class of time-frequency higher order moment spectra is also defined in terms of arbitrary higher order moments of the signal as generalizations of the Cohen's general class of time-frequency representations. The properties of the general class of time-frequency higher order moment spectra can be related to the properties of WHOS which are, in fact, extensions of the properties of the WD. Discrete time and frequency Wigner higher order moment spectra (DTF-WHOS) distributions are introduced for signal processing applications and are shown to be implemented with two FFT-based algorithms. One application is presented where the Wigner bispectrum (WB), which is a WHOS in the third-order moment domain, is utilized for the detection of transient signals embedded in noise. The WB is compared with the WD in terms of simulation examples and analysis of real sonar data. It is shown that better detection schemes can be derived, in low signal-to-noise ratio, when the WB is applied.

I. INTRODUCTION

Time-frequency distributions are transformations that attempt to describe how the spectral content of a signal is changing with time. They are known as distributions because somehow they describe the energy or intensity of a signal in time and in frequency simultaneously. Nevertheless, they are not distributions in a probabilistic sense since positivity can not usually be ensured. An infinite number of time-frequency distributions can be generated from Cohen's general class formulation [1]. Special cases of this general class include the spectrogram, Rihaczek, Page, Wigner-Ville and Choi-Williams distributions [2]. In particular, the Wigner-Ville distribution (WD), owing to its many desirable properties, has been of a special interest for the analysis of nonstationary signals, especially phase modulated signals common in radar and sonar [3]–[5]. In addition, the instantaneous energy and frequency, power spectral density, and group delay of the signal can be easily derived from the WD.

Cohen's class of distributions are bilinear expressions, i.e., they are based on the second-order moments of the signal. A definition of a general class of time-frequency distributions in terms of higher order moments could contribute to the understanding of time-varying higher order moment spectra (HOMS) [6] in the same way that Cohen's general class does for the time-varying second-order spectra. The definition of a general class requires, however, the formulation of the basic representation, i.e., the representation for which the kernel is equal to unity. In analogy with Cohen's general class, the Wigner higher order moment spectra (WHOS) are chosen as the basic representation. For every time instant t, the WHOS expresses the varying HOMS in the same way that the WD does for the instantaneous power spectrum. The third-order Wigner distribution was originally introduced by Gerr [7]. In this paper the continuous and discrete WHOS are defined and studied for an arbitrary order. This definition has been carefully conceived to preserve the properties of the WD. In particular, the properties related to the instantaneous power and spectral density function in the WD are now related to the instantaneous (k + 1)th-order moment and (k + 1)th-order HOMS. The properties of this higher order moment spectra derivation can differ substantially from a derivation based on higher order cumulant spectra (see Section II-E).

The organization of the paper is as follows. In Section II the WHOS is defined for an arbitrary order k. It is shown that this definition is consistent with the three fundamental properties of time frequency distributions [8], and a general class of time-frequency HOMS is defined. For practical signal processing applications, discrete time-frequency Wigner higher order moment spectra (DTF-WHOS) are derived in Section III. Their relation with the continuous WHOS and their nonaliasing constraints are also given in Section III. In Section IV, the Wigner bispectrum (WB), which is a WHOS in the third-order moment domain, is applied to the detection of transient signals in noise. A detection scheme is presented based on the WB and compared with simulations to a similar detection scheme based on the WD. Section V is devoted to concluding remarks.

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II. CONTINUOUS WIGNER HIGHER ORDER SPECTRA (WHOS)

The Wigner distribution was initially defined in the context of quantum mechanics [9] and later extended by Ville to the Wigner time-frequency distribution or Wigner-Ville distribution (WD) [10]. Ville's approach considered the characteristic function $M(\Omega, \tau)$, which is the two-dimensional Fourier transform of the distribution $P(t, f)$, as the expected value of $\exp (j2\pi\Omega t + j2\pi f\tau)$. That is,

$$M(\Omega, \tau) = E\{\exp (j2\pi\Omega t + j2\pi f\tau)\} = \int_{t_1}^{t_2} \int_{f_1}^{f_2} \exp (j2\pi\Omega t + j2\pi f\tau) P(t, f) \; dt \; df. \tag{2.1}$$

This expectation was calculated with the quantum mechanical method of associating operators with ordinary variables,

$$t \leftrightarrow j2\pi \frac{d}{df}$$

$$-j \frac{d}{dt} \leftrightarrow 2\pi f$$

depending on the domain, time or frequency, where the calculation is to be performed [2]. Nevertheless, this approach does not seem appropriate for the extension of the WD to higher order time-frequency representations, primarily because of the loss of symmetry which is present in the bilinear approach between time and frequency.

Another approach for deriving time-frequency distributions is by means of Fourier transform of local autocorrelation functions; i.e.,

$$P(t, f) = \int_{\tau} R_t(\tau) \exp (-j2\pi f\tau) \; d\tau. \tag{2.2}$$

If the local autocorrelation function is defined as

$$R_t(\tau) = x(t)x^*(t + \tau), \tag{2.3}$$

then the Rihaczek distribution is obtained from (2.2) [11]. On the other hand, the WD is obtained if

$$R_t(\tau) = x^*(t - \tau/2)x(t + \tau/2). \tag{2.4}$$

In general, $R_t(\tau)$ can be defined for an arbitrary delay $\alpha$ as follows:

$$R_t(\tau) = x^*(t - \alpha)x(t + \tau - \alpha) \tag{2.5}$$

and thus different distributions can be derived for each $\alpha$. Notwithstanding, as pointed out in [8], to give a particular distribution an interpretation as a distribution of the signal's energy in time and frequency, three twofold properties must be satisfied. First, shift in time (or frequency) of the signal should result in the corresponding shift of the distribution. Second, the integration of the distribution over all frequencies (or times) should be equal to the instantaneous power (or spectral density). Finally, the mean frequency (or time) of the distribution at each time (or frequency) should be equal to the instantaneous frequency (or group delay).

Notice that the first and second properties are satisfied by $P(t, f)$ in (2.2) for any value of $\alpha$. On the other hand, the third property requires the mean value of the arguments of $x(t)$ in $R_t(\tau)$ to be $t$; i.e., $R_t(\tau)$ be a centered local autocorrelation function. This constraint, known as the lag-centering condition, implies $\alpha = \tau/2$ and yields the WD. Note that a similar procedure will be followed to reach the definition of WHOS.

A. Definition

Let $x(t)$ be a complex deterministic signal. The WHOS of order $k$ of $x(t)$, $W_k(t, f_1, \cdots, f_k)$, is defined as

$$W_k(t, f_1, \cdots, f_k) = \int_{\tau_1}^{\tau_2} \cdots \int_{\tau_{k-1}}^{\tau_k} x^*(\tau_0) \prod_{i=1}^{k} \left( t - \frac{1}{k+1} \sum_{m=1}^{k} \tau_m \right)$$

$$\cdot \prod_{i=1}^{k} x(t + \frac{k}{k+1} \tau_i - \frac{1}{k+1} \sum_{j=i+1}^{k} \tau_j)$$

$$\cdot \exp (-j2\pi f_\tau, \tau_i) \; d\tau_i. \tag{2.6}$$

This definition was derived as follows. $W_k(t, f_1, \cdots, f_k)$ is a $k$-dimensional Fourier transform of a $k$-dimensional local moment function $R_k(\tau_1, \cdots, \tau_k)$; viz.,

$$W_k(t, f_1, \cdots, f_k) = \int_{\tau_1}^{\tau_2} \cdots \int_{\tau_{k-1}}^{\tau_k} R_k(\tau_1, \tau_2, \cdots, \tau_k)$$

$$\cdot \prod_{i=1}^{k} \exp (-j2\pi f_\tau, \tau_i) \; d\tau_i \tag{2.7}$$

where

$$R_k(\tau_1, \tau_2, \cdots, \tau_k) = x^*(t - \alpha) \prod_{i=1}^{k} x(t + \tau_i - \alpha) \tag{2.8}$$

is defined with an arbitrary time delay $\alpha$. Note that $R_k(\tau_1, \tau_2, \cdots, \tau_k)$ is defined such that one of the factors in the product is a delayed version of the conjugate of $x(t)$ and the rest are delayed versions of $x(t)$. It is shown in Appendix A that at least one conjugate term is necessary in the definition of $R_k(\tau_1, \tau_2, \cdots, \tau_k)$, if $x(t)$ is a deterministic analytic signal.

To fulfill the three basic properties of time-frequency distributions, in a higher order moment spectrum domain, the value of $\alpha$ should be chosen properly. In particular, to attain the instantaneous frequency as the mean frequency in the multifrequency space at a given time, it will be shown that $R_k(\tau_1, \tau_2, \cdots, \tau_k)$ should be centered at time instant $t$, in such a way that

$$\frac{1}{k+1} \left( (t - \alpha) + \sum_{i=1}^{k} (t + \tau_i - \alpha) \right) = t. \tag{2.9}$$

All integrals are over $\Re$ where no limits are specifically marked.
Consequently,
\[ \alpha = \frac{1}{k+1} \sum_{i=1}^{k} \tau_i \]  
(2.10)
and
\[ R_{kr}(\tau_1, \tau_2, \cdots, \tau_k) = x^*\left( t - \frac{1}{k+1} \sum_{m=1}^{k} \tau_m \right) \]
\[ \cdot \prod_{i=1}^{k} x\left( t + \frac{k}{k+1} \tau_i - \frac{1}{k+1} \sum_{j=i+1}^{k} \tau_j \right). \]
(2.11)

This leads to definition (2.6). Special cases of WHOS include the Wigner bispectrum (WB) for \( k = 2 \),
\[ W_{2r}(t, f_1, f_2) = \int_{\tau_1} x^* (t - \frac{1}{2} \tau_1 - \frac{1}{2} \tau_2) \]
\[ \cdot x(t + \frac{1}{2} \tau_1 - \frac{1}{2} \tau_2) \exp (-j2\pi f_1 \tau_1) \]
\[ \cdot \exp (-j2πf_2τ_2) \quad \text{d}τ_1 \quad \text{d}τ_2, \]
(2.12)
and the Wigner trispectrum (WT) for \( k = 3 \),
\[ W_{3r}(t, f_1, f_2, f_3) = \int_{\tau_1} \int_{\tau_2} x^* (t - \frac{1}{3} \tau_1 - \frac{1}{3} \tau_2 - \frac{1}{3} \tau_3) \]
\[ \cdot x(t + \frac{1}{3} \tau_1 - \frac{1}{3} \tau_2 - \frac{1}{3} \tau_3) \]
\[ \cdot x(t + \frac{1}{3} \tau_2 - \frac{1}{3} \tau_3 - \frac{1}{3} \tau_1) \]
\[ \cdot \exp (-j2\pi f_1 \tau_1) \exp (-j2πf_2τ_2) \]
\[ \cdot \exp (-j2πf_3τ_3) \quad \text{d}τ_1 \quad \text{d}τ_2 \quad \text{d}τ_3. \]
(2.13)

Observe that for \( k = 1 \) the WD follows from (2.6):
\[ W_r(t, f) = \int_{\tau} x^* (t - \frac{1}{2} \tau) x(t + \frac{1}{2} \tau) \exp (-j2\pi f \tau) \quad \text{d}τ. \]
(2.14)

The definition of WB of (2.12) differs from the third-order Wigner distribution proposed by Gerr in [7] only in the dependence of the conjugate of the signal.

**B. Properties**

The definition of WHOS is consistent with most of the properties of the WD that have been extensively studied and reported, for example, in [12].

\[ \text{Re} \{ \{ f_m \} \} = \frac{\int_{f_1} \cdots \int_{f_k} \{ f_m - E \{ f_m \} \}^2 W_{kr}(t, f_1, \cdots, f_k) \prod_{i=1}^{k} df_i}{\int_{f_1} \cdots \int_{f_k} W_{kr}(t, f_1, \cdots, f_k) \prod_{i=1}^{k} df_i} = \frac{k}{k+1} \left[ \frac{(A(t))}{A(t)} - \frac{A^*(t)}{A(t)} \right] \]
(2.20)
for \( x(t) = A(t) \exp(j \phi(t)) \), \( A'(t) = (dA(t)/dt) \), \( A''(t) = (d^2A(t)/dt^2) \) and \( \forall m \).

5) Projection on Frequency: Projection on a \( k \)-dimensional multifrequency point (integration of the distribution over time) should be equal to the \((k+1)\)th-order moment spectrum. To prove that, the WHOS will be expressed as a function of the Fourier transform of the signal \( x(t) \), namely, \( X(f) \). In Appendix B it is shown that by taking inverse Fourier transform in (2.6), the following equality holds:

\[
W_{k}(t, f_1, \cdots, f_k) = \int_{0}^{1} X^* \left( \sum_{i=1}^{k} f_i + \frac{1}{k+1} \Omega \right) \prod_{j=1}^{k} X \left( f_j - \frac{1}{k+1} \Omega \right) \exp(-j2\pi \Omega t) d\Omega
\]  

(2.21)

Consequently,

\[
\int_{t}^{t} W_{k}(t, f_1, \cdots, f_k) dt = X^* \left( \sum_{i=1}^{k} f_i \right) \prod_{j=1}^{k} X(f_j)
\]  

(2.22)

6) Mean Time Calculation and Generalized Group Delay: From the point of view of classical time-frequency distributions, the group delay appears as the mean time at a given frequency. In WHOS the group delay becomes the mean time at a given multifrequency point. It is well known that the group delay, defined as the derivative of the phase of the Fourier transform of a signal, expresses the mean temporal position of the energy of this signal (or its second-order moment) as a function of frequency. That is,

\[
\int_{t}^{t} |x(t)|^2 dt = \int_{f} \phi'(f) |X(f)|^2 df
\]  

(2.23)

where \( x(t) \leftrightarrow X(f) = A(f) \exp(j \phi(f)) \). Similar expressions can be derived for the \((k+1)\)th-order moments. That is,

\[
\int_{t}^{t} t x^*(t) x^{(k)}(t) dt = \int_{f_1}^{f_k} \cdots \int_{f_k} \text{GD}_k(f_1, \cdots, f_k)
\]

\[
\times X^* \left( \sum_{i=1}^{k} f_i \right) \prod_{j=1}^{k} X(f_j) df_j
\]  

(2.24)

where the generalized group delay, \( \text{GD}_k(f_1, \cdots, f_k) \), can be calculated using WHOS as the mean time at a multifrequency point:

\[
E\{E\}_{f_1, \cdots, f_k} = \text{GD}_k(f_1, \cdots, f_k)
\]

\[
= \frac{\int_{t}^{t} W_{k}(t, f_1, \cdots, f_k) dt}{\int_{t}^{t} W_{k}(t, f_1, \cdots, f_k) dt}
\]  

(2.25)

In addition to the basic properties, the following properties are satisfied.

7) Time Support of \( x(t) \): In analogy with the WD, the WHOS do not have to be zero whenever the signal is zero. However,

\[
x(t) = 0 \ \forall \ t \not\in [T_1, T_2] \Rightarrow W_{k}(t, f_1, \cdots, f_k) = 0 \ \forall \ t \not\in [T_1, T_2].
\]  

(2.26)

This property is easily verified considering the lag-centering condition (2.9), i.e., that the mean value of the arguments in the right side of (2.8) should be \( t \). If (2.9) holds, then it is clear that some of the arguments of \( x(t) \) will be greater than \( t \) and some will be smaller. Consequently, if \( x(t) \) is zero except for \( t \in [T_1, T_2] \), so will be \( R(t_1, \tau_2, \cdots, \tau_k) \) for any value of \( \tau_1, \cdots, \tau_k \) and thus (2.26) follows.

8) Frequency Support of \( X(f) \): The region of support of WHOS in the frequency domain can be related to the support of \( X(f) \). For band-limited \( x(t) \), \( X(f) = 0 \ \forall \ f \not\in [F_1, F_2] \). Consequently, necessary conditions for the existence of \( W_{k}(t, f_1, \cdots, f_k) \) are that inequalities

\[
\sum_{i=1}^{k} f_i + \frac{1}{k+1} \Omega \geq F_1
\]  

(2.27.a)

\[
\sum_{i=1}^{k} f_i + \frac{1}{k+1} \Omega \leq F_2
\]  

(2.27.b)

\[
f_i - \frac{1}{k+1} \Omega \geq F_1
\]  

(2.27.c)

\[
f_i - \frac{1}{k+1} \Omega \leq F_2
\]  

(2.27.d)

are satisfied at least for one common value of \( \Omega (-\infty < \Omega < +\infty) \). Substitution of (2.27.a), (2.27.b) in (2.27.c), (2.27.d) leads to:

\[
2f + \sum_{i=1}^{k} f_i \geq 2F_1
\]  

(2.28.a)

\[
2f + \sum_{i=1}^{k} f_i \leq 2F_2.
\]  

(2.28.b)

And substitution of (2.27.c) into (2.27.d) results in

\[-B_1 \geq f_i - f_i \geq B_1 \quad \text{for} \quad B_2 = F_2 - F_1.
\]  

(2.28.c)

Notice that this region contains the domain of support of the higher order moment spectra of \( x(t) \). (Particular case of \( \Omega = 0 \) Fig. 1 illustrates this region of support for both the WB and the bispectrum of an analytic deterministic signal with bandwidth \( B_s \).

Considering (2.22) we see that integration of time of WHOS yields the higher order moment spectra of the signal, which is a desirable property for time frequency distributions. On the other hand, we do not observe any contradiction in the fact that the frequency support of WHOS at any particular time, which can be interpreted as the
instantaneous higher order moment spectra, exceeds the support of the HOS of the signal.

9) Convolution and Product Property: The corresponding properties of the WD apply now in relation with the product and convolution of signals. They become a frequency and time convolution in WHOS domains. Denoting by $\otimes$ and $\odot$, the frequency and time convolution, respectively,

$$x(t) = f(t)g(t) = W_{x}(t, f_{1}, \cdots, f_{k})$$
$$= W_{d}(t, f_{1}, \cdots, f_{k}) \otimes W_{k}(t, f_{1}, \cdots, f_{k})$$

(2.29.a)

$$x(t) = f(t)T g(t) = W_{x}(t, f_{1}, \cdots, f_{k})$$
$$= W_{d}(t, f_{1}, \cdots, f_{k}) \odot W_{k}(t, f_{1}, \cdots, f_{k})$$

(2.29.b)

10) Symmetry Property: WHOS are symmetric with respect to the hyperplanes $f_{i} = f_{j}$:

$$W_{x}(t, f_{1}, \cdots, f_{k}) = W_{x}(t, f_{1}, \cdots, f_{k}, f_{1}, \cdots, f_{k})$$

(2.30)

11) Derivation of the Signal $x(t)$, $X(f)$: The signal itself $x(t)$ and its Fourier transform $X(f)$ can be recovered from WHOS except for a constant term $x(t_{0})$ and $X(f_{0})$:

$$\int_{f_{0}} \cdots \int_{f_{0}} W_{x}(t, \cdots, f_{k}) \frac{1}{k+1} (t + kf_{0}), f_{1}, \cdots, f_{k} \prod_{i=1}^{k} \delta \left(f_{i} - \frac{2}{k+1} f_{0}\right).$$

(2.31.a)

$$\int_{f_{0}} \cdots \int_{f_{0}} W_{x}(t, f_{1}, \cdots, f_{k}) \frac{1}{k+1} (f + f_{0}), f_{1}, \cdots, f_{k} \prod_{i=1}^{k} \delta \left(f_{i} - \frac{2}{k+1} f_{0}\right)$$

(2.31.b)

C. Examples

1) Complex Exponential Signal: For $x(t) = A_{0} \exp (j2\pi f_{0}t)$, $X(f) = A_{0} \delta(f - f_{0})$, and thus substitution of $X(f)$ in (2.21) leads to

$$W_{x}(t, f_{1}, \cdots, f_{k}) = A_{0}^{(k+1)} \exp \left[j2\pi f_{0}(k - 1)\right]$$
$$\prod_{i=1}^{k} \delta \left(f_{i} - \frac{2}{k+1} f_{0}\right).$$

(2.32)

The impulse function of the Fourier transform of the signal is mapped into a multidimensional impulse function at frequencies $2f_{0}/(k + 1)$. Special cases include $k = 1$; i.e., the WD,

$$W_{x}(t, f) = A_{0}^{2} \delta(f - f_{0})$$

(2.33)

and $k = 2$ and $k = 3$, the WB and WT, respectively:

$$W_{x}(t, f_{1}, f_{2}) = A_{0}^{3} \exp \left(j2\pi f_{0}\right) \delta(f_{1} - \frac{1}{2} f_{0})$$
$$\delta(f_{2} - \frac{1}{2} f_{0})$$

(2.34)

$$W_{x}(t, f_{1}, f_{2}, f_{3}) = A_{0}^{4} \exp \left(j4\pi f_{0}\right) \delta(f_{1} - \frac{1}{2} f_{0})$$
$$\delta(f_{2} - \frac{1}{2} f_{0}) \delta(f_{3} - \frac{1}{2} f_{0}).$$

(2.35)

Observe that the WD is the only case where the impulse functions are centered at frequency $f_{0}$.

2) Impulse Function: For $x(t) = A_{0} \delta(t - t_{0})$, $X(f) = A_{0} \exp (-j2\pi f_{0})$, and thus substitution of $x(t)$ in (2.6) yields:

$$W_{x}(t, f_{1}, \cdots, f_{k}) = A_{0}^{(k+1)} \delta(t - t_{0}).$$

(2.36)

This is a somewhat expected result considering the projection on time (2.16) and the time support (2.26) properties.

D. General Class of Time-Frequency Higher Order Spectra

The unified approach given by Cohen to most time-frequency representations can be generalized to the case of higher order moment spectra. In [11], the general class of time-frequency representations is expressed as

$$P_{GEN}(t, f) = \int_{\Omega} \int_{\tau} \phi(\Omega, \tau) x^{*}(u - \tau/2)x(u + \tau/2)$$
$$\times \exp (-j2\pi \Omega \tau) \exp (-j2\pi \tau)$$
$$\times \exp (j2\pi \Omega \mu) du d\tau d\Omega,$$

(2.37)

where the properties of the distribution (e.g., time and frequency shifts, projection on time and frequency, and mean time and frequency) can be related to the properties of the kernel $\phi(\Omega, \tau)$. In particular, the WD corresponds to $\phi(\Omega, \tau) = 1$, the Rihaczek to $\phi(\Omega, \tau) = \exp (j\pi \Omega \tau)$ [11], and the Choi-Williams to $\phi(\Omega, \tau) = \exp (-2\pi \Omega \tau)^{2}/\sigma$ [8].

Expression (2.37) was obtained as the Fourier trans-
form of the general characteristic function $M_{\text{GEN}}(\Omega, \tau)$; viz:

$$P_{\text{GEN}}(t, f) = \int_{\Omega} \int_{\tau} M_{\text{GEN}}(\Omega, \tau) \exp(-j2\pi\Omega\tau) \cdot \exp(-j2\pi f\tau) \, d\Omega \, d\tau$$  \hspace{1cm} (2.38)

where $M_{\text{GEN}}(\Omega, \tau)$ is the product of the kernel $\phi(\Omega, \tau)$ with the inverse Fourier transform of the WD, i.e., the symmetrical ambiguity function,

$$M_{\text{GEN}}(\Omega, \tau) = \phi(\Omega, \tau) \int_{\tau} x^*(u - \tau/2) x(u + \tau/2) \, du$$  \hspace{1cm} (2.39)

Similarly, a characteristic function can be defined in the higher order moment spectra domains as the product of the multidimensional kernel $+^{HOS}(\Omega, \tau_1, \cdots, \tau_k)$ times the inverse Fourier transform of $W_{\text{HOS}}(t, f_1, \cdots, f_l)$; viz:

$$M_{\text{HOS}\text{GEN}}(\Omega, \tau_1, \cdots, \tau_k) = \phi_{\text{HOS}}(\Omega, \tau_1, \cdots, \tau_k) \int_{\tau_1} \cdots \int_{\tau_k} W_{\text{HOS}}(t, f_1, \cdots, f_l) \exp(j2\pi u\Omega)$$

$$\cdot \prod_{i=1}^{k} \exp(-j2\pi f_i\tau_i) \, df_i \, du.$$  \hspace{1cm} (2.40)

Combining (2.40) and (2.11), we obtain

$$M_{\text{HOS}\text{GEN}}(\Omega, \tau_1, \cdots, \tau_k) = \phi_{\text{HOS}}(\Omega, \tau_1, \cdots, \tau_k) \int_{\tau_1} \cdots \int_{\tau_k} R_{\text{HOS}}(\tau_1, \cdots, \tau_k)$$

$$\cdot \exp(j2\pi \Omega) \, du.$$  \hspace{1cm} (2.41)

Hence, a general class of higher-order time-frequency distributions, $\text{PHOS}_{\text{GEN}}(t, f_1, \cdots, f_l)$, can be generated from the Fourier transform of the generalized characteristic function $M_{\text{HOS}\text{GEN}}(\Omega, \tau_1, \cdots, \tau_k)$ as

$$\text{PHOS}_{\text{GEN}}(t, f_1, \cdots, f_l) = \int_{\tau_1} \cdots \int_{\tau_k} \text{MHOS}_{\text{GEN}}(\Omega, \tau_1, \cdots, \tau_k)$$

$$\cdot \exp(j2\pi \Omega) \prod_{i=1}^{k} \exp(-j2\pi f_i\tau_i) \, d\tau_i \, d\Omega.$$  \hspace{1cm} (2.42)

or, in terms of the local higher order moment function,

$$\text{PHOS}_{\text{GEN}}(t, f_1, \cdots, f_l) = \int_{\Omega} \int_{\tau_1} \cdots \int_{\tau_k} \phi_{\text{HOS}}(\Omega, \tau_1, \cdots, \tau_k)$$

$$\cdot R_{\text{HOS}}(\tau_1, \cdots, \tau_k) \exp(j2\pi \Omega)$$

$$\cdot \exp(-j2\pi \Omega) \prod_{i=1}^{k} \exp(-j2\pi f_i\tau_i) \, d\tau_i \, d\Omega \, du.$$  \hspace{1cm} (2.43)

The properties of any particular member of the general class can be obtained via the properties of the WHOS and its particular kernel $\phi_{\text{HOS}}(\Omega, \tau_1, \cdots, \tau_k)$. One of the main difficulties in signal analyses using the WD is the presence of interference terms between different time-frequency components of signals. The Choi–Williams distribution [8], or in general, RID [14] have been conceived to reduce the spurious terms without sacrificing highly desirable properties of time-frequency representations. The properties and applications of a generalized exponential kernel,

$$\phi(\Omega, \tau_1, \cdots, \tau_k) = \exp(-\Omega^2(\tau_1^2 + \tau_2^2 + \cdots + \tau_k^2)/\sigma)$$  \hspace{1cm} (2.44)

are currently under investigation.

E. Higher Order Moment Versus Cumulant Spectra

Most time-frequency representations, including Cohen's general class, are normally analyzed with nonrandom signals only. In fact, deterministic signals buried in random noise are often the desired signal in many of the applications of time-frequency distributions. For these signals, such as finite energy signals and periodic signals, the higher order moment spectra are defined as the multidimensional Fourier transform of the higher order moment sequence [6], i.e.,

$$r_{k_1}(\tau_1, \tau_2, \cdots, \tau_k) = r_{k_1}(\tau) = \int_{\tau} x^*(t) \prod_{i=1}^{k} x(t + \tau_i) \, dt$$  \hspace{1cm} (2.45)

$$\tau = [\tau_1, \cdots, \tau_k]^T$$  \hspace{1cm} (2.46)

$$M_{k_1}(f) = M_{k_1}(f) = \int_{\tau} r_{k_1}(\tau)$$

$$\cdot \exp(-j2\pi f^T \tau) \, dt = X^*$$

$$\cdot \left( \prod_{i=1}^{k} f_i \right) \prod_{j=1}^{k} X(f_j)$$  \hspace{1cm} (2.47)

$$f = [f_1, \cdots, f_k]^T; \quad dt = d\tau_1, \cdots, d\tau_k.$$  \hspace{1cm} (2.48)

The definitions of WHOS, only for deterministic signals, is consistent with this formulation, and so is the projection on frequency property. Let us consider, however, the random signal case and compare a moment-based versus a cumulant-based definition of WHOS. For a complex random process $\{x(t)\}$, nonstationary in general, we define

$$x = [x^*(t - \alpha), x(t + \tau_1 - \alpha), \cdots, x(t + \tau_k - \alpha)]^T; \quad \alpha = \frac{1}{k+1} \sum_{i=1}^{k} \tau_i.$$  \hspace{1cm} (2.49)
Higher order cumulants and moments are denoted by
\[ \text{cum} \{ x \} = \text{cum} (t, \tau_1, \cdots, \tau_e) \]
\[ = \text{cum} (t, \tau); \]
\[ m \{ x \} = m(t, \tau_1, \cdots, \tau_e) \]
\[ = m(t, \tau). \]  
(2.50)
The cumulant-based WHOS, recently proposed in [15], is defined as
\[ C_{\text{WHOS}}(t, f) = \int \text{cum} (t, \tau) \exp (-j2\pi f \tau) \, d\tau \]  
(2.51)
whereas the moment-based WHOS or expectation of the WHOS defined in this paper as
\[ M_{\text{WHOS}}(t, f) = E \{ \text{WHOS} (t, f) \} \]
\[ = \int m(t, \tau) \exp (-j2\pi f \tau) \, d\tau. \]  
(2.52)
The properties of \( C_{\text{WHOS}}(t, f) \) versus \( M_{\text{WHOS}}(t, f) \) are the following:
\[ P1: \] If \( \{ x(t) \} \) and \( \{ y(t) \} \) are independent random processes and \( \{ z(t) \} \) is defined by
\[ z(t) = x(t) + y(t) \]  
(2.53)
then,
\[ C_{\text{WHOS}}(t, f) = C_{\text{WHOS}}(t, f) + C_{\text{WHOS}}(t, f) \]  
(2.54.a)
\[ M_{\text{WHOS}}(t, f) = M_{\text{WHOS}}(t, f) + M_{\text{WHOS}}(t, f) \]  
(2.54.b)
due to the separation property of cumulants of independent random variables that does not hold for moments.
\[ P2: \] If \( \{ x(t) \} \) is complex jointly Gaussian with zero mean and with independent and equally correlated real and imaginary parts (stationary or nonstationary), then (see Appendix C),
\[ C_{\text{WHOS}}(t, f) = 0; \]
\[ M_{\text{WHOS}}(t, f) = \text{WHOS}(t, f) \]  
(2.55)
Of course, \( C_{\text{WHOS}}(t, f) \) is also zero for any Gaussian signal.
\[ P3: \] If \( \{ x(t) \} \) is deterministic then
\[ C_{\text{WHOS}}(t, f) = 0; \]
\[ M_{\text{WHOS}}(t, f) = \text{WHOS}(t, f) \]  
(2.56)
Deterministic signals can be regarded as Gaussian signals with zero variance and thus are suppressed by the cumulant-based definition. Although additive Gaussian signals are suppressed using the cumulant-based definition, deterministic signals suffer the same cancellation. Using the definition based on moments, one important class of Gaussian noise, stationary or not, is also suppressed in the absence of any other signal. Furthermore, the presence of deterministic signals can be detected using \( M_{\text{WHOS}} \), and it is exactly this property that we exploit later for the detection of transients in noise.

III. DISCRETE TIME AND FREQUENCY WIGNER HIGHER ORDER SPECTRA

A. Definition

When it comes to practical signal processing applications of WHOS a discretization in time and frequency is necessary. A discrete version of the WD was initially defined by Claasen and Mecklenbrauker [16]. Later, Peyrin and Prost [17] proposed a natural and unified definition for the discrete-time, discrete-frequency, and discrete-time/frequency Wigner distribution. The nonaliasing constraints, which are due to discretization in both time and frequency domains, became apparent; and one interpolation formula allowing the reconstruction of the continuous WD was derived.

We followed the second aforementioned approach to define the discrete time and frequency WHOS. Initially, the continuous definition is applied to a signal discretized in time to derive the discrete time WHOS (DT-WHOS). Then, the definition of WHOS in the continuous frequency domain is used with a periodic signal, discrete in frequency, to derive the discrete frequency WHOS (DF-WHOS). Finally, a discretization in time and frequency is considered simultaneously, by applying the original continuous definition to a discrete-time and periodic signal resulting in the DTF-WHOS. Numerical expressions are derived and the periodicity studied. The relation between the discretized expressions and the original WHOS of the signal before sampling is discussed at the end of the section.

1) Discrete Time WHOS (DT-WHOS): Let \( x_c(t) \) be a continuous, complex in general, signal and \( x_c(t) \) the signal derived after uniform sampling of \( x_c(t) \) by a train of ideal impulse functions of period \( T \). The resultant discrete sequence is denoted by \( x(n) = x_c(nT) \), i.e.,
\[ x_c(t) = \sum_n x_c(nT) \delta(t - nT) = \sum_n x(n) \delta(t - nT). \]  
(3.1)
Applying expression (2.6) to \( x_c(t) \), we obtain
\[ W_{\text{cx}}(t, f_1, \cdots, f_k) = \sum_n W_{\text{cx}}(n, f_1, \cdots, f_k) \delta\left(t - n \frac{T}{k + 1}\right) \]  
(3.2)
\[ ^3\text{Summations are over } \mathbb{Z} \text{ where no limits are specifically marked.} \]
where

\[ W_{k}(n, f_{1}, \ldots, f_{k}) = \sum_{m_{1}} \cdots \sum_{m_{k}} x^{*}_{m_{1}} \left( n - \sum_{q=1}^{k} m_{q} \right) \prod_{j=1}^{k} x(m_{j}) \cdot \exp \left[ -j 2\pi \left( 2m_{j} + \sum_{i=1, i \neq j}^{k} m_{i} - n \right) \right] . \]

(3.3)

It is clear from (3.2) that the WHOS of the discretized signal is sampled in time at rate \((k + 1)/T\), which is \((k + 1)\) times faster than the sampling rate of the original signal. In analogy with the definition of the discrete sequence \(x(n)\), it is natural to define the DT-WHOS as the weight of the impulse functions, \(W_{k}(n, f_{1}, \ldots, f_{k})\), discrete in time but still a continuous function in the multifrequency space. The DT-WHOS is periodic in each frequency, with the same periodicity \(1/T\) as the Fourier transform of the sampled signal.

2) Discrete Frequency WHOS (DF-WHOS): Consider a continuous temporal signal, nonperiodic in general. Its Fourier transform will be a continuous function in frequency. Let \(X_{a}(f)\) be generated after uniform sampling of \(X_{a}(f)\) by a train of ideal impulse functions of period \(\Delta\). Then, \(X_{p}(f)\) becomes the Fourier transform of a periodic repetition of the original nonperiodic signal with period \(1/\Delta\). The resulting discrete frequency sequence is defined by

\[ X_{p}(f) = \sum_{n} X_{p}(n\Delta) \delta(f - n\Delta) = \sum_{n} X(n) \delta(f - n\Delta). \]

(3.4)

Substitution of \(X_{p}(f)\) in the continuous frequency definition of WHOS given by (2.21) leads to

\[ W_{k}(t, f_{1}, \ldots, f_{k}) = \sum_{p_{1}} \cdots \sum_{p_{k}} W_{k}(t, p_{1}, \ldots, p_{k}) \cdot \prod_{i=1}^{k} \delta \left[ f_{i} - \left( kp_{i} - \sum_{j=1, j \neq i}^{k} p_{j} \right) \frac{\Delta}{k + 1} \right] . \]

(3.5)

where

\[ W_{k}(t, p_{1}, \ldots, p_{k}) = \sum_{m} X^{*}(m) \prod_{j=1}^{k} X(p_{j} - m) \cdot \exp \left( -j 2\pi \left[ (k + 1) m - \sum_{i=1}^{k} p_{i} \right] \Delta t \right) . \]

(3.6)

Notice that \(W_{k}(t, p_{1}, \ldots, p_{k})\) is the weight of an impulse function situated in the multifrequency space of WHOS at position:

\[ f_{i} = \left( kp_{i} - \sum_{j=1, j \neq i}^{k} p_{j} \right) \frac{\Delta}{k + 1} , \]

for \(i = 1 \ldots k \)

(3.7.a)

\[ l_{i} = kp_{i} - \sum_{j=1, j \neq i}^{k} p_{j} \]

(3.7.b)

where the index \(l_{i}\) is used to represent the position of \(W_{k}(t, p_{1}, \ldots, p_{k})\) in the frequency component \(f_{i}\) as a function of all the indexes \(p_{1}, \ldots, p_{k}\). Although all integer values are valid for \(p_{i}\), the only possible values of \(l_{i}\) are the integers satisfying

\[ \frac{1}{k + 1} \left( 2l_{i} + \sum_{j=1, j \neq i}^{k} p_{j} \right) = p_{i} = \text{integer value}, \]

for \(i = 1 \ldots k \).

(3.8)

Consequently, the WHOS of a frequency sampled signal results in a nonuniformly sampled function in all frequency axes at some, but not all, integer multiplies of \(\Delta/(k + 1)\). Following the same arguments given for the DT-WHOS, it is natural to define the DF-WHOS as the discrete function in each frequency component, but still continuous in time. Observe that, in analogy with the inverse Fourier transform of \(X_{p}(f)\), \(W_{k}(t, p_{1}, \ldots, p_{k})\) is a periodic function in time with periodicity \(1/\Delta\).

3) Discrete Time and Frequency WHOS (DTF-WHOS): The previous definitions for the DT-WHOS and DF-WHOS permit a formulation of the DTF-WHOS. Consider a continuous signal that is uniformly sampled in both time and frequency domains. The sampling period in time and frequency is \(T\) and \(1/NT\), respectively.

\[ x_{p}(t) = \sum_{n} x(n) \delta(t - nT) \otimes \sum_{m} \delta(t - mNT). \]

(3.9)

It follows from definitions in the time domain (2.6) or in the frequency domain (2.21) and the convolution and product property (2.29) that

\[ W_{k}(t, f_{1}, \ldots, f_{k}) = \sum_{p_{1}} \cdots \sum_{p_{k}} W_{k}(n, p_{1}, \ldots, p_{k}) \cdot \prod_{i=1}^{k} \delta \left[ t - n \Delta t - \frac{1}{k + 1} \right] . \]

(3.10)
main expression would yield

\[ W_{kr}(n, p_1, \ldots, p_k) = \frac{1}{N} \exp \left[ j \frac{2\pi}{N(k+1)} \left( \frac{k}{N} \sum_{i=1}^{k} p_i \right) \right] \]

\[ \cdot \prod_{m=1}^{N} X^*(m) \prod_{j=1}^{k} X(p_j - m) \exp \left( -j \frac{2\pi}{N} mn \right). \]  

(3.11)

The equivalent expression in the frequency domain is obtained using (2.22):

\[ W_{kr}(n, p_1, \ldots, p_k) = \frac{1}{N} \exp \left[ j \frac{2\pi}{N(k+1)} \left( \frac{k}{N} \sum_{i=1}^{k} p_i \right) \right] \]

\[ \cdot \prod_{m=1}^{N} X^*(m) \prod_{j=1}^{k} X(p_j - m) \exp \left( -j \frac{2\pi}{N} mn \right). \]

(3.12)

The WHOS of \( x_w(t) \) becomes a sampled function at positions \( nT/(k+1) \) and \( f_j/(k+1)NT \) in time and every frequency component, respectively. The resultant temporal sampling is uniform. However, the WHOS is not uniformly sampled in the frequency domain because not all integers are permitted for \( f_j \) (but only the ones defined in (3.7) and (3.8)). The DTF-WHOS is then defined as the weight of the impulse functions in the WHOS of the discrete and periodic signal \( W_{kr}(n, p_1, \ldots, p_k) \).

This is actually a periodic function with respect to the indexes \( n \) and \( p_i \) with period \( (k+1)N \). Nonetheless a careful analysis reveals that only knowledge of \( W_{kr}(n, p_1, \ldots, p_k) \) for \( 0 \leq n \leq N - 1, 0 \leq p_i \leq N - 1 \) is necessary to reconstruct \( W_{kr}(n, p_1, \ldots, p_k) \) elsewhere since

\[ W_{kr}(n+qN, p_1, \ldots, p_k) = \exp \left[ j \frac{2\pi q}{(k+1)} \left( \frac{k}{N} \sum_{i=1}^{k} p_i \right) \right] W_{kr}(n, p_1, \ldots, p_k) \]

(3.13.a)

\[ W_{kr}(n, p_1, \ldots, p_k + qN) = \exp \left[ j \frac{2\pi q}{(k+1)} n \right] W_{kr}(n, p_1, \ldots, p_k) \]

(3.13.b)

for any integer \( q \). Fig. 2 illustrates the nonredundant multifrequency domain of the DTF-WHOS definition for \( k = 2 \).

4) Computation of DTF-WHOS: Expressions (3.11) and (3.12) suggest a fast algorithm for the calculation of

the DTF-WHOS. Relation (3.11) can be expressed as

\[ W_{kr}(n, p_1, \ldots, p_k) = \frac{1}{N} \exp \left[ j \frac{2\pi}{N(k+1)} \left( \frac{k}{N} \sum_{i=1}^{k} p_i \right) \right] \]

\[ \cdot \text{DFT}^k \{ f_0(m_1, \ldots, m_k) \}_{p_1, \ldots, p_k} \]

(3.14.a)

where

\[ f_0(m_1, \ldots, m_k) = X^* \left( n - \sum_{j=1}^{k} m_j \right) \prod_{j=1}^{k} X(m_j). \]

(3.14.b)

For every temporal sample \( n \), the function \( f_0(m_1, \ldots, m_k) \) can be defined and a \( k \)-dimensional FFT be utilized for the calculation of the DTF-WHOS, except for a phase factor. Thus \( W_{kr}(n, p_1, \ldots, p_k) \) can be computed in the nonredundant region of support, \( 0 \leq n \leq N - 1, 0 \leq p_i \leq N - 1 \), with \( N^k \) one-dimensional FFT's of order \( N \).

If (3.12) is considered for the DTF-WHOS,

\[ W_{kr}(n, p_1, \ldots, p_k) = \frac{1}{N} \exp \left[ j \frac{2\pi}{N(k+1)} \left( \frac{k}{N} \sum_{i=1}^{k} p_i \right) \right] \]

\[ \cdot \text{DFT} \{ f_{p_1, \ldots, p_k}(m) \}_{n} \]

(3.15.a)

where

\[ f_{p_1, \ldots, p_k}(m) = X^*(m) \prod_{j=1}^{k} X(p_j - m). \]

(3.15.b)

For every multifrequency point \( p_1, \ldots, p_k \), the function \( f_{p_1, \ldots, p_k}(m) \) can be Fourier transformed to derive the DTF-WHOS at that point and for any temporal index \( 0 \leq n \leq N - 1 \). Consequently, \( N^k \) one-dimensional FFT's of order \( N \) are necessary to compute \( W_{kr}(n, p_1, \ldots, p_k) \) in its entire nonredundant domain of support.

Assuming that the \( k \)-dimensional FFT are implemented by means of one-dimensional FFT's, both possible implementations require \( O(N^{k+1} \log_2 N) \) operations.
B. Relation Between the WHOS and the DTF-WHOS After Sampling

Continuous signals are usually sampled to obtain discrete sequences in real data analyses applications. Thus, it becomes useful to relate the WHOS to the DTF-WHOS after sampling. In order to derive this relation, we will calculate initially the WHOS of a periodic train of impulses.

1) WHOS of a Periodic Train of Impulses: The impulse train function of period \( L \) and its Fourier transform are denoted by

\[
d_{L}(t) = \sum_{m} \delta(t - mL) \tag{3.16}
\]

\[
D_{L}(f) = \frac{1}{L} \sum_{n} \delta(f - n \frac{1}{L}). \tag{3.17}
\]

The procedure used to define the expression of DT-WHOS can be used to calculate the WHOS of \( d_{L}(t) \) substituting \( x(n) = 1, T = L \) in (3.2) and (3.3), since the impulse train can be viewed as a sampled signal with unity valued samples. That is,

\[
W_{k_{L}}(t_{L}, f_{1}, \ldots, f_{k}) = \sum_{n} \sum_{p_{1}} \cdots \sum_{p_{k}} \prod_{j=1}^{k}_{j \neq i} \delta_{n}
\]

\[
\cdot \exp \left[ -j2\pi f_{i} \left( 2p_{i} + \sum_{i=1}^{k} p_{i} - n \right) L \right]
\]

\[
\cdot \delta \left( t - n \frac{L}{k + 1} \right). \tag{3.18}
\]

Another equivalent expression can be obtained considering the definition of DF-WHOS, \( \Delta = 1/L \) in (3.5) (3.6), since a train of impulses in the time domain is represented by another impulse train in the frequency domain:

\[
W_{k_{L}}(t_{L}, f_{1}, \ldots, f_{k}) = \frac{1}{L^{(k-1)}} \sum_{p_{1}} \cdots \sum_{p_{k}} \sum_{m}
\]

\[
\cdot \exp \left[ -j2\pi f \left( k + 1 \right) m - \sum_{i=1}^{k} p_{i} \right] \frac{1}{L}
\]

\[
\cdot \prod_{j=1}^{k} \delta_{j}
\]

\[
\cdot \frac{1}{L^{(k-1)}} \left( k + 1 \right) \delta \left( t_{L} - \left( kp_{i} - \frac{1}{j-1} \sum_{j \neq i} p_{j} \right) \frac{1}{L} \right). \tag{3.19}
\]

These expression can be used to calculate the effect of sampling a continuous signal in relation with its WHOS.

2) Nonaliasing Constraints for DT-WHOS: WHOS of Sampled Signals: A sampled signal can be represented as a continuous signal multiplied by an impulse train function,

\[
x_{s}(t) = x(t) \sum_{n} \delta(t - nT). \tag{3.20}
\]

Taking into account the product property of WHOS in (2.29.a), the DT-WHOS may be viewed as a result of a convolution in the multifrequency domain between the WHOS of the continuous signal and the WHOS of the impulse train of period \( L = T \). Taking for \( W_{k_{L}}(t_{L}, f_{1}, \ldots, f_{k}) \) the expression (3.19) and applying the product property (2.29.a) the following relation between the WHOS of any signal prior and after sampling is obtained:

\[
\sum_{n} W_{k_{L}}(n, f_{1}, \ldots, f_{k}) \delta \left( t_{L} - n \frac{T}{k + 1} \right)
\]

\[
= \frac{1}{T^{(k-1)}} \sum_{p_{1}} \cdots \sum_{p_{k}} \sum_{m}
\]

\[
\cdot \exp \left[ -j2\pi f \left( k + 1 \right) m - \sum_{i=1}^{k} p_{i} \right] \frac{1}{Tt}
\]

\[
\cdot W_{k_{L}}(t_{L}, f_{1}, \ldots, f_{k} - l, \ldots, f_{k}) \frac{1}{T(k + 1)} \cdots
\]

for

\[
l_{i} = \left( kp_{i} - \frac{1}{j-1} \sum_{j \neq i} p_{j} \right). \tag{3.21}
\]

We observe that the DT-WHOS becomes a superposition of frequency delayed versions of the original WHOS before sampling. Again, we have defined the index \( l_{i} \) as a function of all \( p_{j} \), for \( j = 1, \ldots, k \). Every integer is valid for \( p_{j} \), although only certain combinations of integers, the ones satisfying (3.8), are possible for \( l_{i} \). This nonuniformity in the frequency aliasing due to sampling is of paramount importance since it determines the surprising nonaliasing constraints.

To avoid aliasing in the multifrequency domain, the regions of support of WHOS of the continuous signal should not overlap when displaced in frequency by \( l_{i} / (T(k + 1)) \). In Appendix D, it is shown that aliasing is avoided if

\[
\frac{1}{T(k + 1)} \geq \frac{2B_{s}}{(k + 1)} = B_{s} \leq \frac{1}{2T} \tag{3.22}
\]

i.e., the sampling frequency is at least two times the bandwidth of the continuous signal. This result is illustrated in Fig. 3 for \( k = 2 \).

3) Nonaliasing Constraints for DF-WHOS: WHOS of Periodic Signals: It is well known that a time-limited signal may be represented uniquely by its Fourier coefficients, i.e., samples of its Fourier transform. This representation is equivalent to considering a periodic repetition of the signal in the time domain. Using the DF-WHOS with the Fourier coefficients of the signal

\[
X_{p}(f) = X_{sp}(f) \sum_{n} \delta(f - n\Delta) \tag{3.23}
\]
C. Example

Discrete Exponential and Chirp-Line Signals: The Wigner bispectrum (e.g., DTF-WHOS fork $k = 2$) of order $N = 256$ was calculated for the generalized discrete chirp signal considering for $W_{k,n}(t, f_1, \cdots, f_k)$ equation (3.18), and applying the convolution property (2.29 b), we obtain

$$\sum_{p_1} \cdots \sum_{p_k} W_{k,n}(t, p_1, \cdots, p_k) \cdot \prod_{i=1}^{k} \delta(f_i - l_i) \cdot \Delta = \sum_{n} \sum_{p_1} \cdots \sum_{p_k} \prod_{j=1}^{k} \left[ \exp \left[-j2\pi f_j \left( \frac{1}{\Delta} + \sum_{i=1}^{k} p_i - n \right) \right] \cdot W_{k,n} \left( t - n \frac{1}{(k+1)\Delta}, f_1, \cdots, f_k \right) \right]$$

(3.25)

where $l_i$ is given by (3.7) and (3.8). The effect of a time limited observation is that the DF-WHOS turns out to be a superposition of time-delayed versions of the original WHOS, before frequency sampling of $\Delta$. Consequently, temporal aliasing is avoided when the temporal duration of WHOS of the nonperiodic signal is shorter than $1/(k + 1)\Delta$.

4) Nonaliasing Constraints for DTF-WHOS: WHOS of Periodic and Sampled Signals: If the nonaliasing constraints for DT-WHOS and DF-WHOS are combined, they define the constraints for DTF-WHOS. In this case, $\Delta = 1/NT$ and both frequency and time aliasing have to be taken into account. As mentioned for the DT-WHOS (due to the nonuniform aliasing in the multifrequency domain), the sampling frequency prior to the computation of the DTF-WHOS should be at least twice the Nyquist frequency. If the signal cannot be filtered, then interpolation by a factor of two becomes necessary. To avoid time aliasing, the signal must be limited to one interval of length $NT/(k+1)$ in time. Considering property (2.26), the time support of WHOS is the same with the duration of the signal and, therefore, the signal itself should be limited to $NT/(k+1)$. As such, zero padding of $kN/(k+1)$ samples is necessary.

IV. DETECTION OF TRANSIENT SIGNALS: AN IMPORTANT APPLICATION OF DTF-WHOS

A. Introduction

The detection of unknown deterministic transient signals in additive stationary noise is an important problem in underwater acoustics and radar. Recently, some higher order spectra-based methods have been reported for the detection and parameter estimation of transient signals [18], [19]. Some additional methods have also been proposed for the detection of transients based on time-frequency representations [20]–[22]. In [20], [21] the Gabor coefficients are employed for the detection of signals of short duration as compared to the observation interval. In [22] a methodology is developed for the detection and classification of cylinder firings of marine engines. The autocorrelation function is used for the detection and the Wigner–Ville distribution (WD) for the classification.
and

\[ W_{1s}(n, p_1, p_2) = \frac{1}{N} \exp \left( j \pi N^{-1} \sum_{m=0}^{N-1} x^*(n - m) x(m) \exp \left( -j \frac{2\pi}{N} mp \right) \right) \]

In addition, the output of a matched filter is calculated for comparison purposes. Assuming that the noise is white in the bandwidth of the transient signal \( s(n) \), the matched filter is designed with impulse response equal to

\[ h(n) = s(T - n) \]

where \( T \) is the duration, in samples, of the deterministic transient \( s(n) \). If \( x(n) \) is the received signal (noise only or signal plus noise), the output of the matched filter is given by

\[ y_{mf}(n) = x(n) \otimes h(n). \]

In the analysis of long data sequences, windowing becomes necessary. When the WD or WB are applied, the window length is determined by the order \( N \) used in the computation of (4.1) or (4.2). Time aliasing is avoided if \( x(n) \) is limited from \( n = 0 \) to \( N/2 - 1 \) and \( N/3 - 1 \) in (4.1) and (4.2), respectively. Furthermore, to avoid frequency aliasing, an interpolation of order 2 of the data is necessary in both expressions.

**B. Detection Statistics Derived from Time-Frequency Distributions**

Let us assume that the signal to be analyzed is passed through an antialiasing low-pass filter of cutoff frequency \( f_s/4 \). The output of the filter is sampled at rate \( f_s \), to obtain \( x(n) \). The following hypotheses are considered for the received signal:

\[ H_0: x(n) = w(n) \]

where \( w(n) \) is a zero mean stationary noise process and

\[ H_1: x(n) = s(n) + w(n) \]

where \( s(n) \) is a deterministic transient signal and, consequently, \( x(n) \) is nonstationary.

For \( H_0 \), the expected value of both the WD and WB are independent of \( n \); they correspond to the power spectrum and bispectrum of the noise, respectively. For \( H_1 \), on the other hand, the nonstationarity of \( x(n) \) will be reflected in the WD and in the WB. In other words, both the WD and WB functions will be changing with the temporal index \( n \).

---

**Fig. 5.** Illustration of the Wigner bispectrum of order \( N = 256 \) of a complex chirp signal of mean frequency \( \beta = 40 \) and \( \alpha = 60 \). (a) at the temporal sample \( n = N/4 \), (b) at \( n = N/2 \), and (c) at \( n = 3N/4 \).
A decision variable of a function $f(n)$, $DS_y\{f(n)\}$, may be defined as

$$DS_y\{f(n)\} = \left( \prod_{i=0}^{M-1} |f(n-i)|^{\gamma} \right)^{1/\Gamma}$$

for

$$\Gamma = \sum_{i=0}^{M-1} \gamma^i|f(n-i)|$$

where $M$ is the number of available data points of $f(n)$ and $0 < \gamma \leq 1$. $DS_y\{f(n)\}$ is the geometric overs the arithmetic mean of the sequence $|f(n)|$. It can be shown that

i) $0 \leq DS_y\{f(n)\} \leq 1$

ii) $DS_y\{f(n)\} = 1 \Leftrightarrow |f(i)| = |f(j)| \quad \forall i, j.$

(4.8)

The forgetting factor $\gamma$ emphasizes recent samples by means of one-sided exponential windowing and can be set to 1 if the rectangular window is assumed. The decision variable (4.6) can be used to detect the transient signal as follows.

Under hypothesis $H_0$, $x(n)$ is a stationary process and thus

$$DS_y\{E\{W_{2s}(n, p_1, p_2)\}\} = 1$$

$$DS_y\{E\{W_{1s}(n, p)\}\} = 1$$

$$DS_y\{E\{y_{op}(n)\}\} = 1.$$  

(4.9)

Under $H_1$, however, $x(n)$ is nonstationary and

$$DS_y\{E\{W_{2s}(n, p_1, p_2)\}\} < 1$$

$$DS_y\{E\{W_{1s}(n, p_1, p_2)\}\} < 1$$

$$DS_y\{E\{y_{op}(n)\}\} < 1.$$  

(4.10)

The ability of each transformation to detect the nonstationarity will be reflected in the discriminating power of $DS_y$ between $H_0$ and $H_1$.

C. Simulation Results

Simulation results are given for the transient signal $s(n)$ illustrated in Fig. 6(a). White Gaussian noise was added to $s(n)$ to generate $x(n)$. The signal-to-noise ratio, SNR, is defined as

$$\text{SNR} = 10 \log_{10} \left( \frac{S^2_{\text{max}}}{\sigma^2} \right).$$

(4.11)

Fig. 6(b) illustrates $x(n)$ for SNR = 0 dB.

At every $L = 20$ samples, the following averaged functions are calculated:

$$\text{AWB}(m, p_1, p_2) = \sum_{i=0}^{L/2-1} W_{2s}(Lm - i, p_1, p_2)$$

(4.12.a)

$$\text{AWD}(m, p) = \sum_{i=0}^{L/2-1} W_{1s}(Lm - i, p)$$

(4.12.b)

$$\text{AOD}(m) = \sum_{i=0}^{L/2-1} y_{op}(Lm - i)$$

(4.12.c)

and their corresponding detection statistics follows by combining (4.12) with (4.6); viz.,

1) $\text{WB Detector}$:

$$DS-WB(m) = \sum_{p_1=0}^{N-1} \sum_{p_2=0}^{N-1} DF_y\{\text{AWB}(m, p_1, p_2)\}$$

(4.13.a)

2) $\text{WD Detector}$:

$$DS-WD(m) = \sum_{p=0}^{N-1} DF_y\{\text{AWD}(m, p)\}$$

(4.13.b)

3) $\text{Matched Filter Detector}$:

$$DS-MF(m) = DF_y\{\text{AOD}(m)\}.$$  

(4.13.c)
Fig. 7. Detection statistic for SNR = 10, 5, and 0 dB: (a) DS-WB(m), (b) DS-WD(m), and (c) DS-MF(m).

Figs. 7(a)-(c) show DS-WB(m), DS-WD(m) and DS-MF(m) as functions of m, respectively, for SNR = 10, 5, and 0 dB.

From these figures, it is apparent that detection statistics DS-WB(m), DS-WD(m), and DS-MF(m) attain initially a constant value at 0.7, 0.65, and 0.85, respectively and then decrease in value around a point in the data record that corresponds to the beginning of the transient signal. Note that a constant value in detection statistics over a time interval corresponds to stationary data (i.e., noise only) whereas dramatic change in their value corresponds to presence of nonstationarity, i.e., transient signal in stationary noise. Comparing Figs. 7(a) and (b), we see that for SNR > 0 the Wigner bispectrum (WB) detection statistic performs better than the detection statistic based on the WD. The function DS-MF(m), shown in Fig. 7(c), illustrates the detection performance obtained with the optimum method, which requires exact knowledge of the shape of the transient signal. Note that the detection statistics based on WB and WD do not require a priori knowledge of the shape of the transient signal. Similar performance for DS-WB(m), DS-WD(m), and DS-MF(m) has been observed for other types of chirp-like transient signals. Nevertheless, it is worth mention-
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Fig. 9. Detection statistics in the detection of signal C, (a) DS-WB(m) for quiet ocean noise and SNR = 0 dB, (b) DS-WB(m) for noisy ocean noise and SNR = -18 dB. (Continued on next page.)

Fig. 9. Detection statistics in the detection of signal C, (a) DS-WB(m) for quiet ocean noise and SNR = 0 dB, (b) DS-WB(m) for noisy ocean noise and SNR = -18 dB. (Continued on next page.)

D. Analysis of Real Sonar Signals

Both DS-WB and DS-WD detection methods have been applied to real underwater sonar signals of the U.S. Navy Standard Transient Data Set. Several transient signals have been analyzed under different real ocean conditions. Under these real data conditions, and for different types of transient signal and noise scenarios, the Wigner-bispectrum-based detection method was found to outperform the detection method based on the Wigner distribution. Fig. 8 illustrates the transient signals that were embedded in real "quiet" and "noisy" ocean noise for different SNR. In particular, Fig. 8(a) shows the transient signal C and Fig. 8(b) the transient signal D.

Fig. 9 illustrates the detection statistics, DS-WB(m) and DS-WD(m), generated using time series that contain transient signal C in different types of ocean noises with different SNR: 9(a) DS-WB(m) for "quiet" ocean noise and SNR = 0 dB; 9(b) DS-WB(m) for "noisy" ocean noise and SNR = -18 dB; 9(c) DS-WD(m) for "quiet" ocean noise and SNR = 0 dB; and 9(d) DS-WD(m) for "noisy"
V. SUMMARY

In this paper, we have derived the extension of the Wigner–Ville distribution (WD) to higher order moment spectral domains. Initially, we defined the continuous Wigner higher order moment spectra (WHOS) and studied their properties. After that, we defined a general class of time-frequency higher order moment spectra representations, in analogy with Cohen’s general class of distributions. Numerical expressions for the discrete time and frequency WHOS (DTF-WHOS) were derived and their relation with the continuous WHOS established. The non-aliasing constraints have also been established and discussed. The detection of transient signals was addressed as a problem where WHOS can be applied and do better than the Wigner distribution. The Wigner bispectrum (WB) was compared to the WD in an analogous detection scheme using synthetic data and real sonar signals. Simulation results were given and the advantages using the WB illustrated.
Fig. 10. Detection statistics in the detection of signal $D$. (a) DS-WB (m) for quiet ocean noise and SNR = 0 dB. (b) DS-WB (m) for noisy ocean noise and SNR = −18 dB. (c) DS-WD (m) for quiet ocean noise and SNR = 0 dB. (Continued on next page.)
APPENDIX A

We can define the higher order moment spectra of a deterministic signal \( x(t) \) as

\[
HOS(f_1, \cdots, f_k) = \sum_{\tau_1} \cdots \int_{\tau_k} R_{\eta}(\tau_1, \tau_2, \cdots, \tau_k) \cdot \prod_{i=1}^{k} \exp(-j2\pi f_i \tau_i) \, d\tau_i \, dt \tag{A.1}
\]

where \( R_{\eta}(\tau_1, \tau_2, \cdots, \tau_k) \) can be any valid local higher order moment function. If \( R_{\eta}(\tau_1, \tau_2, \cdots, \tau_k) \) is defined as in (2.11) then

\[
HOS(f_1, \cdots, f_k) = X^* \left( \sum_{i=1}^{k} f_i \right) \prod_{j=1}^{k} X(f_j). \tag{A.2}
\]

However, considering the alternative definition

\[
R_{\eta}^{*}(\tau_1, \tau_2, \cdots, \tau_k) = x(t - \alpha) \prod_{i=1}^{k} x(t + \tau_i - \alpha) \tag{A.3}
\]

or

\[
R_{\eta}^*(\tau_1, \tau_2, \cdots, \tau_k) = x^*(t - \alpha) \prod_{i=1}^{k} x^*(t + \tau_i - \alpha) \tag{A.4}
\]

we obtain

\[
HOS^*(f_1, \cdots, f_k) = X \left( - \sum_{i=1}^{k} f_i \right) \prod_{j=1}^{k} X(f_j) \tag{A.5}
\]

\[
HOS^*(f_1, \cdots, f_k) = X^* \left( \sum_{i=1}^{k} f_i \right) \prod_{j=1}^{k} X(-f_j). \tag{A.6}
\]

In applications of the WD, \( x(t) \) is very often an analytic signal and \( X(f) \) is only defined for \( 0 \leq f \leq B_x \). As such, \( HOS^*(f_1, \cdots, f_k) = \text{HOS}^*(f_1, \cdots, f_k) = 0 \). Consequently, at least one dependence with \( x^*(t) \) and one with \( x(t) \) becomes necessary in the definition of the local higher order moment function. Definition (2.11) is just one arbitrary definition satisfying the aforementioned requirement.

APPENDIX B

Substitution of \( x(t) \) in definition (2.6), as the inverse Fourier transform of \( X(\Omega) \)

\[
x(t) = \int_{\Omega} X(\Omega) \exp(j2\pi\Omega t) \, d\Omega \tag{B.1}
\]

leads to

\[
W_{\omega}(t_1, f_1, \cdots, f_k) = \int_{\xi_0} \cdots \int_{\xi_k} X^*(-\Omega_0) \exp(-j2\pi\Omega_0 \tau) \, d\Omega_0 \prod_{i=1}^{k} X(\Omega_i) \exp(-j2\pi\Omega_i \tau) \, d\Omega_i
\]

\[
\cdot \int_{\nu_1} \cdots \int_{\nu_k} \exp \left[ -j2\pi\Omega \left( \frac{1}{k+1} \sum_{m=1}^{k} \tau_m \right) \right]
\]

\[
\cdot \prod_{i=1}^{k} \exp \left[ -j2\pi\Omega_i \left( \frac{k}{k+1} \tau_i - \frac{1}{k+1} \sum_{j=1, j \neq i}^{k} \tau_j \right) \right] \exp(-j2\pi f_i \tau_i) \, d\tau_i. \tag{B.2}
\]
Equivalently,

\[ W_{k}(t, f_{1}, \ldots, f_{k}) = \int_{0}^{1} \cdots \int_{0}^{1} X^{*}(-\Omega_{0}) \exp(-j2\pi\Omega_{0}t) d\Omega_{0} \prod_{i=1}^{k} X(\Omega_{i}) \exp(-j2\pi\Omega_{i}) d\Omega_{i} \]

\[ \cdot \int_{r_{1}}^{1} \cdots \int_{r_{k+1}}^{1} \exp \left[ -j2\pi r_{i} \left( f_{i} + \frac{1}{k+1} \Omega_{0} - \frac{k}{k+1} \Omega_{i} + \frac{1}{k+1} \sum_{j=1, j \neq i}^{k} \Omega_{j} \right) \right] dr_{i} \]

\[ = \int_{0}^{1} \cdots \int_{0}^{1} X^{*}(-\Omega_{0}) \exp(-j2\pi\Omega_{0}t) d\Omega_{0} \prod_{i=1}^{k} X(\Omega_{i}) \exp(-j2\pi\Omega_{i}) d\Omega_{i} \]

\[ \cdot \prod_{i=1}^{k} \delta \left( f_{i} + \frac{1}{k+1} \Omega_{0} - \frac{k}{k+1} \Omega_{i} + \frac{1}{k+1} \sum_{j=1, j \neq i}^{k} \Omega_{j} \right). \]

The impulse product defines the following system of equations:

\[ \frac{k}{k+1} \Omega_{1} - \frac{1}{k+1} \sum_{j=1, j \neq 1}^{k} \Omega_{j} = f_{1} + \frac{1}{k+1} \Omega_{0} \]

\[ \frac{k}{k+1} \Omega_{2} - \frac{1}{k+1} \sum_{j=1, j \neq 2}^{k} \Omega_{j} = f_{2} + \frac{1}{k+1} \Omega_{0} \]

\[ \cdots \]

\[ \frac{k}{k+1} \Omega_{k} - \frac{1}{k+1} \sum_{j=1, j \neq k}^{k} \Omega_{j} = f_{k} + \frac{1}{k+1} \Omega_{0}, \]

which can be solved as

\[ \Omega_{i} = \Omega_{0} + 2f_{i} + \sum_{j=1, j \neq i}^{k} f_{j}, \quad \text{for } i = 1, \ldots, k. \]

Substitution in (B.3) and calculation of the impulse integrals gives

\[ W_{k}(t, f_{1}, \ldots, f_{k}) = (k + 1) \int_{0}^{1} \cdots \int_{0}^{1} X^{*}(-\Omega_{0}) \exp(-j2\pi\Omega_{0}t) d\Omega_{0} \prod_{i=1}^{k} X(\Omega_{i}) \exp(-j2\pi\Omega_{i}) d\Omega_{i} \]

\[ \cdot \prod_{i=1}^{k} \delta \left[ \Omega_{i} - \left( \Omega_{0} + 2f_{i} + \sum_{j=1, j \neq i}^{k} f_{j} \right) \right] \]

\[ = (k + 1) \int_{0}^{1} \cdots \int_{0}^{1} X^{*}(-\Omega_{0}) \prod_{i=1}^{k} X\left( \Omega_{0} + 2f_{i} + \sum_{j=1, j \neq i}^{k} f_{j} \right) \]

\[ \cdot \exp \left[ -j2\pi(k + 1) \left( \Omega_{0} + \sum_{j=1}^{k} f_{j} \right) \right] d\Omega_{0} \]

where \((k + 1)\) appears as the normalization factor equal to the inverse of the determinant of the system of equations \((1/k + 1)\). For

\[ \Omega = - (k + 1) \left( \Omega_{0} + \sum_{j=1}^{k} f_{j} \right) \]

definition (2.21) is obtained

\[ W_{k}(t, f_{1}, \ldots, f_{k}) = \int_{0}^{1} \left( \int_{0}^{1} X^{*} \left( \sum_{i=1}^{k} f_{i} + \frac{1}{k+1} \Omega \right) \right) \]

\[ \cdot \prod_{j=1}^{k} X \left( f_{j} - \frac{1}{k+1} \Omega \right) \]

\[ \cdot \exp (-j2\pi\Omega t) d\Omega. \]

**APPENDIX C**

Let us define

\[ x = [x^{*}(t - \alpha), x(t + \tau_{1} - \alpha), \cdots, x(t + \tau_{k} - \alpha)]^{T} \]

\[ = [x^{*}, x_{1}, \cdots, x_{k}]^{T} \]

\[ \alpha = \frac{1}{k+1} \sum_{i=1}^{k} \tau_{i}. \]

The following properties hold:

**P1**: If \(x(t)\) is Gaussian, by the fundamental property of Gaussian processes

\[ \text{cum} \{x\} = \text{cum} (t, \tau_{1}, \cdots, \tau_{k}) = 0. \]
If \( x(t) \) is complex, jointly Gaussian with zero mean and equally correlated real and imaginary parts, then

\[
P3: \quad x(t): \text{Gaussian (C.3.a)}
\]

\[
E\{x(t)\} = E\{a(t) + jb(t)\}
\]

\[
= m_x(t) + jm_y(t) = 0 \quad \text{for } k > 1 \quad (C.8)
\]

and thus,

\[
C_{WHOS_x}(t, f) = M_{WHOS_x}(t, f) = 0, \quad \text{for } k > 1. \quad (C.9)
\]

**Appendix D**

The region of support in the multifrequency domain of WHOS was given in (2.28):

\[
P3: \quad \text{If } x(t) \text{ is Gaussian and } \text{Cum}_{k}\{x\} = m_k\{x\} = 0, \quad \text{for } k > 1, \quad \text{then (3.21):}
\]

\[
\sum \limits_{n} W_{K_0}(n, f_1, \ldots, f_k) \delta(t_n - n \frac{T}{k + 1}) = \frac{1}{T^{d+1}} \sum \sum \sum \sum \exp \left\{ -j 2\pi \left[ (k + 1) m - \sum \limits_{i=1}^{k} p_i \right] \frac{1}{T} \right\} \cdot W_{K_0}(t, \ldots, f_i - \frac{l_i}{T(k + 1)} \ldots) \quad (D.2)
\]
for
\[ l_i = \left( k p_i - \sum_{j=1}^{k} p_j \right) \]

denoting the following change of variables (homeomorphism in \( \mathbb{R}^k \)),
\[ g_i = \frac{1}{k + 1} \left( 2 f_i + \sum_{j \neq i}^{k} f_j \right), \quad \text{for } i = 1 \cdots k \]

(D.3.a)
invertible by
\[ f_i = \left( k g_i - \sum_{j \neq i}^{k} g_j \right), \quad \text{for } i = 1 \cdots k \]

(D.4)

then expressions (D.1) result in the new variables:
\[ g_i \geq \frac{2 F_1}{k + 1} \]

(D.5.a)
\[ g_i \leq \frac{2 F_2}{k + 1} \]

(D.5.b)
\[ -\frac{B_s}{k + 1} \geq g_i - g_j \geq \frac{B_s}{k + 1} \]

for \( B_s = F_2 - F_1 \) \hspace{1cm} (D.5.c)

for the region of support in the multifrequency domain of the continuous signal. The expression in \( g_i \) of the WHOS of the sampled signal is
\[
\sum_n W_{k_i}(n, g_1, \cdots, g_k) \delta \left( t - n \frac{T}{k + 1} \right) = \frac{1}{T(k + 1)} \sum_{p_1} \cdots \sum_{p_m} \sum_{n} \exp \left[ -j2\pi \left( k + 1 \right) m - \sum_{i=1}^{k} p_i \right] \frac{1}{T} \]
\[ \cdot W_{k_i}(f_1, \cdots, f_k, g_1, \cdots, g_k, -p_i, \frac{1}{T(k + 1)}, \cdots) \]

for any integer value of \( p_i \) and for \( i = 1, \cdots k \).

In this domain it is apparent that aliasing is avoided for
\[ \frac{1}{T(k + 1)} \geq \frac{2 B_s}{k + 1} \Rightarrow B_s \leq \frac{1}{2 T} \]

for all values of \( k \), i.e., the sampling rate must be twice the one suggested by Nyquist theorem for the signal independently of the value of \( k \).

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