Abstract

Let $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i(p)$ be a family of monic polynomial matrices smoothly dependent on a vector of real parameters $p = (p_1, \ldots, p_n)$. In this work we study behavior of a multiple eigenvalue of the monic polynomial family $P(\lambda)$.

Key Words: Polynomial matrix, Eigenvalues, Perturbation.

1. Introduction

Given a polynomial matrix $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$, where $A_i$ are square matrices over real or complex field, it is an important question from both the theoretical and the practical points of view to know how the eigenvalues and eigenvectors change when the elements of $P(\lambda)$ are subjected to small perturbations.

Eigenvalue problem for polynomial matrices $P(\lambda)v = 0$, appears among other applications modeling physical and engineering problems by means systems of $k$-order linear ordinary differential equations. The values of eigenvalues can correspond among others, to frequencies of vibration, critical values of stability parameters, or energy levels of atoms.

The eigenvalues of some matrices are sensitive to perturbations, it is well known that the eigenvalues of monic polynomial matrices are continuous functions of the entries of the matrix coefficients of the polynomial, but Small changes in the matrix elements can lead to large changes in the multiplicity of eigenvalues. For example a little perturbation of the matrix \( \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \) the double eigenvalue $\lambda = 0$ is perturbed to two different eigenvalues $\lambda = \pm \sqrt{\varepsilon}$ changing completely the structure of the polynomial matrix. Obviously if we consider the perturbation \( \begin{pmatrix} \lambda & 1+\varepsilon \\ 0 & \lambda \end{pmatrix} \) there are not changes in the structure.

Given a square complex matrix $A$, it is an important question from both the theoretical and the practical points of view to know how the eigenvalues and eigenvectors change when the elements of $A$ are subjected to small perturbations. The usual formulation of the problem introduces a perturbation parameter $\varepsilon$ belonging to some neighborhood of zero, and writes the perturbed matrix as $A + \varepsilon B$ for an arbitrary matrix $B$. In this situation, it is well known section II.1.2, that each eigenvalue or eigenvector of $A + \varepsilon B$ admits an expansion in fractional powers of $\varepsilon$, whose zero-th order term is an eigenvalue or eigenvector of the unperturbed matrix $A$.

In this paper, in section 1 we present an overview over polynomial matrices $P(\lambda)$ and the analysis of perturbation of simple eigenvalue $\lambda_0$ of $P(\lambda)$ such that $0$ is a simple eigenvalue of the linear map $P(\lambda_0)$. Finally, in section 3, we study the perturbation of a multiple eigenvalue with a simple eigenvector of a monic polynomial matrix smoothly depending on parameters.

The study of behavior of simple and multiple eigenvalues of a matrix depending smoothly of parameters has a great interest for its many applications. Perturbation theory for eigenvalues and eigenvectors of regular pencils is well established see [1],[10] for example and for vibrational systems in [9]. In this paper we extend some of these results to polynomial matrices.

2. Preliminaries

A square polynomial matrix of size $n$ and degree $k$ is a polynomial of the form

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i, \quad A_0, \ldots, A_k \in M_n(\mathbb{F})$$

where $\mathbb{F}$ is the field of real or complex numbers. Our focus is on monic polynomial matrices. A square polynomial matrix $P(\lambda)$ is said to be monic if $A_k = I_n$ is
The function of polynomial matrices

tions of the vector of parameters. We are going to re-
parameter family of polynomial matrices. Eigenvalues
depend on the vector of real parameters

eignevalue

\[ P(\lambda) \]

det

spect the variable

\[ \lambda \]

If

\[ \lambda \]

then there exists a vector

\[ v_0 \neq 0 \]

such that \( P(\lambda_0)(v_0) = 0 \), this vector is called an eigenvector.

We will call a Jordan chain of length \( k + 1 \) for \( P(\lambda) \) corresponding to complex number \( \lambda_0 \) to the sequence of \( n \)-dimensional vectors \( v_0, \ldots, v_k \) such that

\[
\sum_{\ell = 0}^{k} \frac{1}{\ell!} P^{(\ell)}(\lambda_0)v_{i-\ell} = 0, \quad i = 0, \ldots, k
\]

where \( P^{(\ell)} \) denotes the \( \ell \)-derivative of \( P(\lambda) \) with respect the variable \( \lambda \). If \( \lambda_0 \) is an eigenvalue there exists a Jordan chain of length at least 1 formed by the eigenvector.

Let \( \lambda_0 \) be an eigenvalue of \( P(\lambda) \), then \( P^t(\lambda_0) = \det P(\lambda_0) = 0 \), so \( \lambda_0 \) is an eigenvalue of \( P^t(\lambda) \). For this eigenvalue there exists an eigenvector \( u_0 \), that is \( P^t(\lambda_0)(u_0) = 0 \), equivalently \( u_0^t P(\lambda_0) = 0 \). The vector \( u_0 \) is called left eigenvector corresponding to the eigenvalue \( \lambda_0 \) of \( P(\lambda) \).

For more information see [4], or [7] for example.

Let \( P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i \) be now, a polynomial matrix and we assume that the matrices \( A_i \) smoothly depend on the vector of real parameters \( p = (p_1, \ldots, p_r) \). The function \( P(\lambda;p) = \sum_{i=0}^{k} \lambda^i A_i(p) \) is called a multi-parameter family of polynomial matrices. Eigenvalues of the polynomial matrix function are continuous functions of the vector of parameters. We are going to review the behavior of a simple eigenvalue of the family of polynomial matrices \( P(\lambda;p) \).

Let \( \lambda(p) \) be a simple eigenvalue of the polynomial matrix \( P(\lambda;p) \). Since \( \lambda(p) \) is a simple root of the scalar polynomial \( \det P(\lambda;p) \), we have

\[
\frac{\partial}{\partial \lambda} \det P(\lambda;p) \neq 0.
\]

The expression (4) permit us to make use the implicit function theorem to the equation \( \det P(\lambda;p) = 0 \), and we observe that the eigenvalue \( \lambda(p) \) of the family

\[
\frac{\partial \lambda(p)}{\partial p_i} = - \frac{u_0^t \frac{\partial P(\lambda;p)}{\partial p_i}}{u_0^t \frac{\partial P^t(\lambda;p)}{\partial p_i} u_0(p)}
\]

\[
= - T_0^{-1} \left( \frac{\partial \lambda}{\partial p_i} \left( P'(\lambda;p) + \frac{\partial P'(\lambda;p)}{\partial p_i} \right) \right) v_0(p).
\]

where \( T_0 = \left( P(\lambda_0;p_0) \right)^{-1} \)

\[
\frac{\partial^2 \lambda}{\partial p_i \partial p_j} = - \frac{a}{b},
\]

with

\[
a = \left( u_0 \left( \frac{\partial \lambda}{\partial p_i} \frac{\partial \lambda}{\partial p_j} P'(\lambda;p) + \frac{\partial \lambda}{\partial p_i} \frac{\partial P'(\lambda;p)}{\partial p_i} \right) \frac{\partial \lambda}{\partial p_i} \frac{\partial P'(\lambda;p)}{\partial p_j} \right) v_0(p)
\]

\[
+ u_0 \left( P'(\lambda;p) \frac{\partial \lambda}{\partial p_i} + \frac{\partial P'(\lambda;p)}{\partial p_i} \frac{\partial \lambda}{\partial p_i} \right) \frac{\partial v_0}{\partial p_i}
\]

and

\[
b = u_0^t P'(\lambda_0;p_0) v_0(p_0).
\]
\[
\frac{\partial^2 v_0(p)}{\partial p_i \partial p_j} T_0^{-1} (\frac{\partial}{\partial \lambda} P' (\lambda; p) v_0(p) + 
\left( \frac{\partial \lambda}{\partial p_i} \frac{\partial}{\partial p_j} P' (\lambda; p) + \frac{\partial \lambda}{\partial p_i} \frac{\partial}{\partial p_j} P' (\lambda; p) \right) v_0(p) 
+ \left( \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_j} P' (\lambda; p) \right) v_0(p) \bigg|_{(\lambda, p_0)}.
\]

The proof is analogous to that given in [9] for matrix pencils and for vibrational systems.

3. Perturbation of eigenvalue of arbitrary multiplicity with single eigenvector

Let \( P(\lambda; p) = \lambda^2 I_2 + A(p) \) with \( A(p) = \begin{pmatrix} -1 & p \\ p & 0 \end{pmatrix} \) be a one parameter family of polynomial matrices. The eigenvalues are

\[
\lambda_i = \pm \sqrt{1 + \frac{1 + 4p^2}{2}},
\]

that they are branches of one quadruple-valued analytic function \( \lambda(p) = \sqrt{1 + \frac{1 + 4p^2}{2}} \)

- \( p = \frac{1}{2} i \) and the eigenvalues are \( \pm \frac{\sqrt{2}}{2} \) both being double.
- \( p = -\frac{1}{2} i \) and the eigenvalues are \( \pm \frac{\sqrt{2}}{2} \) both being double.
- \( p = 0 \) and the eigenvalues are \( +1, -1 \) both being simple and zero being double.

We observe that for \( p = 0 \), the polynomial matrix \( P(\lambda; p) \) has a single eigenvector up to a non-zero scaling factor for the double eigenvalue \( \lambda = 0 \).

We next consider the behavior of the eigenvalues in the neighborhood of one of the exceptional points. Concretely we take \( p = 0 \). In this case the eigenvalues are not differentiable functions of the parameter at \( p = 0 \), just where the double eigenvalue appears. Therefore the analysis of perturbations of multiple eigenvalues with single eigenvector, must be treated in a different manner.

Let \( P(\lambda; p) \) be a monic polynomial matrix family and \( \lambda_0 \) an eigenvalue of arbitrary multiplicity \( \ell \) with single eigenvector up to a non-zero scaling factor at the point \( p = p_0 \), then, there exists a Jordan chain \( v_0, \ldots, v_{\ell-1} \) such that

\[
P(\lambda, p_0) v_0 = 0, \quad P'(\lambda, p_0) v_0 + P(\lambda, p_0) v_1 = 0, \quad \frac{1}{(\ell - 1)!} P^{\ell-1}(\lambda, p_0) v_0 + \ldots + P(\lambda, p_0) v_{\ell-1} = 0,
\]

and, there exists a left Jordan chain \( u_0, \ldots, u_{\ell-1} \) such that

\[
u_0^T P(\lambda, p_0) = 0, \quad u_0^T P'(\lambda, p_0) + u_1^T P(\lambda, p_0) = 0, \quad \frac{1}{(\ell - 1)!} u_0^T P^\ell(\lambda, p_0) + \ldots + u_{\ell-1}^T P(\lambda, p_0) = 0.
\]

Remark 1. a) \( u_0^T P'(\lambda, p_0) v_0 = 0 \),

b) \( u_0^T P'(\lambda, p_0) v_0 = 0 \Leftrightarrow u_1^T P(\lambda, p_0) v_1 = 0 \Leftrightarrow u_0^T P'(\lambda, p_0) v_1 = 0 \),

c) \( u_0^T P'(\lambda; p_0) v_1 = u_1^T P'(\lambda; p_0) v_0 \).

In order to analyze the behavior of two eigenvalues \( \lambda(p) \) that merge to \( \lambda_0 \) at \( p_0 \), we consider a perturbation of the parameter along a smooth curve \( p = p(\varepsilon) \), where \( \varepsilon \geq 0 \) is a small real perturbation parameter and \( p(0) = p_0 \).

Along the curve \( p(\varepsilon) = (p_1(\varepsilon), \ldots, p_r(\varepsilon)) \) we have a one parameter matrix family \( P(\lambda, p(\varepsilon)) \), which can be represented in the form of Taylor expansion

\[
P(\lambda, p(\varepsilon)) = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \ldots,
\]

with \( P_0 = P(\lambda, p_0), P_1 = \sum_{i=1}^{r} \frac{\partial P(\lambda, p_0)}{\partial p_i} dp_i \frac{dp_i}{d\varepsilon} \),

\[
P_2 = \frac{1}{2} \left( \sum_{i=1}^{r} \frac{\partial^2 P(\lambda, p_0)}{\partial p_i^2} (\frac{dp_i}{d\varepsilon})^2 + \sum_{i \neq j=1}^{r} \frac{\partial^2 P(\lambda, p_0)}{\partial p_i \partial p_j} (\frac{dp_i}{d\varepsilon}) (\frac{dp_j}{d\varepsilon}) \right),
\]

where the derivatives are evaluated at \( p_0 \).

Taking into account that \( P(\lambda, p(\varepsilon)) = \sum_{i=0}^{k} \lambda^i A_i(p(\varepsilon)) \) \( A_i(p(\varepsilon)) = I_n \), we have that

\[
P(\lambda, p(\varepsilon)) = \sum_{i=0}^{k} \lambda^i (A_{i_0} + \varepsilon A_{i_1} + \varepsilon^2 A_{i_2} + \ldots) \]

where \( A_{i_0} + \varepsilon A_{i_1} + \varepsilon^2 A_{i_2} + \ldots = I_n \), \( A_{i_0} = A_0(p_0), A_{i_1} = \sum_{i=1}^{r} \frac{\partial A_i(p(\varepsilon))}{\partial p_i} dp_i \),

\[
A_{i_2} = \frac{1}{2} \left( \sum_{i=1}^{r} \frac{\partial^2 A_i(p(\varepsilon))}{\partial p_i^2} (\frac{dp_i}{d\varepsilon})^2 + \sum_{i \neq j=1}^{r} \frac{\partial^2 A_i(p(\varepsilon))}{\partial p_i \partial p_j} (\frac{dp_i}{d\varepsilon}) (\frac{dp_j}{d\varepsilon}) \right),
\]

and the derivatives are evaluated at \( p_0 \).

If \( \lambda_0 \) is a \( \ell \)-multiplicity eigenvalue of \( P(\lambda; p_0) \) having a unique eigenvector \( v_0 \) up to a non-zero scaling factor
the perturbation theory (see [8], for example) tell us that the $\ell$-fold eigenvalue $\lambda_0$ generally splits into $\ell$ of simple eigenvalues $\lambda$ under perturbation of the polynomial matrix $P(\lambda; p_0)$. These eigenvalues $\lambda$ and the corresponding eigenvectors $v$ can be represented in the form of the Puiseux series:

$$\lambda = \lambda_0 + \varepsilon^{1/\ell}u_1 + \varepsilon^{2/\ell}u_2 + \varepsilon^{3/\ell}u_3 + \varepsilon^{4/\ell}u_4 + \ldots$$

$$v = v_0 + \varepsilon^{1/\ell}w_1 + \varepsilon^{2/\ell}w_2 + \varepsilon^{3/\ell}w_3 + \varepsilon^{4/\ell}w_4 + \ldots$$

(12)

**Lemma 1.** Let $p_0$ be a point such that $\lambda(p_0) = \lambda_0$ is a $\ell$-multiplicity eigenvalue with single eigenvector $v_0(p_0)$ and $u_0$ a corresponding left eigenvector. Then, $[u_0]^\dagger = \text{Im} P(\lambda_0, p_0)$.

**Proof.** Let $z \in \text{Im} P(\lambda_0, p_0)$, then there exists a vector $x$ such that $P(\lambda_0, p_0)x = z$. So $u_0^\dagger u_0^\dagger P(\lambda_0, p_0)x = 0 \implies x = 0$, consequently $\text{Im} P(\lambda_0; p_0) \subset [u_0]^\dagger$. And taking into account that $\text{rank} P(\lambda_0, p_0) = \text{dim} \text{Im} P(\lambda_0, p_0) = n - 1 = \text{dim}[u_0]^\dagger$, we conclude the result.

**Corollary 1.** With the same conditions as the previous lemma, we have.

$$\frac{1}{\ell!}u_0^\dagger P(\lambda_0; p_0)v_0 + \frac{1}{(\ell - 1)!}u_0^\dagger P(\lambda_0; p_0)v_0 + \ldots + u_0^\dagger P(\lambda_0; p_0)v_{\ell - 1} \neq 0.$$ 

**Proof.** Suppose $\frac{1}{\ell!}u_0^\dagger P(\lambda_0; p_0)v_0 + \frac{1}{(\ell - 1)!}u_0^\dagger P(\lambda_0; p_0)v_0 + \ldots + P(\lambda_0; p_0)v_{\ell - 1} = \text{Im} P(\lambda_0, p_0)$, and $\frac{1}{\ell!}P(\lambda^0; p_0)v_0 + \frac{1}{(\ell - 1)!}P(\lambda^0; p_0)v_0 + \ldots + P(\lambda^0; p_0)v_{\ell - 1} = P(\lambda_0; p_0)x$. Equivalently:

$$\frac{1}{\ell!}P(\lambda^0; p_0)v_0 + \frac{1}{(\ell - 1)!}P(\lambda^0; p_0)v_0 + \ldots + P(\lambda^0; p_0)v_{\ell - 1} + P(\lambda_0; p_0)(-x) = 0,$$

but the Jordan chains of the $P(\lambda; p_0)$ for $\lambda = \lambda_0$ are length $\ell$, so there is no vector $x$ verifying (13).

**4-1. Perturbation of double eigenvalue with single eigenvector**

Firstly and for a more understanding, we analyze the case where $\ell = 2$

Substituting (12) into (11) we obtain

$$P(\lambda; p(\varepsilon)) = (\lambda_0^2 - 1)A_{k-10} + \ldots + \lambda_0A_{k1} + A_{k0} + \varepsilon^{1/2}(k\lambda_0^2 - 1)A_{k-1} + \ldots + \lambda_0A_{k1} + \varepsilon((k\lambda_0^k + \frac{1}{2}(k - 1)k\lambda_0^2)A_{k-10} + ((k - 1)k\lambda_0^2 - 2)A_{k-1} + \varepsilon((k\lambda_0^k - 2)A_{k-1} + \lambda_0A_{k1} + \lambda_0A_{k1} + \ldots + A_{k1}) + \ldots$$

If $v$ is an eigenvector for the eigenvalue $\lambda$, we have that

$$P(\lambda; p(\varepsilon))v = P(\lambda; p(\varepsilon))(v_0 + \varepsilon^{1/2}w_1 + \varepsilon w_2 + \ldots) = 0.$$ 

Then, we find the chain of equations for the unknowns $\lambda_1, \lambda_2, \ldots$ and $w_1, w_2, \ldots$

$$P(\lambda_0; p_0)v_0 = 0,$$ 

$$\lambda_1 P(\lambda_0; p_0)v_0 + P(\lambda_0, p_0)w_1 = 0,$$ 

$$P(\lambda_0; p_0)w_2 + \lambda_1 P(\lambda_0; p_0)w_1 + \frac{1}{2}\lambda_1^2 P(\lambda_0; p_0)v_0 + \lambda_2 P(\lambda_0; p_0)v_0 + P_1(\lambda_0; p_0)w_2 = 0,$$ 

$$P(\lambda_0; p_0)w_3 + \lambda_1 P(\lambda_0; p_0)w_2 + \frac{1}{2}\lambda_1^2 P(\lambda_0; p_0)v_0 + \lambda_2 P(\lambda_0; p_0)v_0 + P_1(\lambda_0; p_0)w_3 + \lambda_1 P_1(\lambda_0; p_0)v_0 = 0,$$ 

$$P(\lambda_0; p_0)w_4 + \lambda_1 P(\lambda_0; p_0)w_3 + \frac{1}{2}\lambda_1^2 P(\lambda_0; p_0)v_0 + \lambda_2 P(\lambda_0; p_0)v_0 + P_1(\lambda_0; p_0)w_4 + \lambda_1 P_1(\lambda_0; p_0)v_0 = 0.$$ 

Equation (14) is satisfied because $v_0$ is an eigenvector corresponding to the eigenvalue $\lambda_0$. Comparing equation (15) with (3) for $i = 1$ we observe that $w_1 = \lambda_1 v_1 + \beta v_0$ for all $\beta$ is a solution, we take $w_1 = \lambda_1 v_1$.

To find the value of $\lambda_1$ we premultiply equation (16) by $u_0^\dagger$, using the given value for $w_1$ and taking into account $u_0^\dagger P(\lambda_0; p_0) = 0$ and $u_0^\dagger P(\lambda_0; p_0)v_0 = 0$ we obtain

$$\lambda_1^2(u_0^\dagger P(\lambda_0; p_0)v_1 + \frac{1}{2}u_0^\dagger P(\lambda_0; p_0)v_0 + u_0^\dagger P_1(\lambda_0; p_0)v_0 = 0.$$ 

Taking into account corollary 1 we can find

$$\lambda_1 = \pm \sqrt{-\frac{u_0^\dagger P(\lambda_0; p_0)v_1 + \frac{1}{2}u_0^\dagger P(\lambda_0; p_0)v_0 + u_0^\dagger P_1(\lambda_0; p_0)v_0}{u_0^\dagger P(\lambda_0; p_0)v_0 + \frac{1}{2}u_0^\dagger P(\lambda_0; p_0)v_0}}.$$ 

If $u_0^\dagger P_1(\lambda_0; p_0)v_0 \neq 0$ we have two values of $\lambda_1$ that determine leading terms in expansions for two different eigenvalues $\lambda$ that bifurcate from the double eigenvalue $\lambda_0$. 

ISBN: 978-1-61804-023-7 103
Suppose then, that \( u_0^t P_1(\lambda_0; p_0) v_0 \neq 0 \). Premultiplying (17) by \( u_0^t \),

\[
\begin{align*}
\lambda_1 u_0^t P^r(\lambda_0; p_0) w_2 + & \frac{1}{2} \lambda_1^2 u_0^t P^{rr}(\lambda_0; p_0) v_1 \\
\lambda_1 \lambda_2 u_0^t P^r(\lambda_0; p_0) v_1 + & \lambda_1 u_0^t P_1(\lambda_0; p_0) v_0 \\
\lambda_1 \lambda_2^2 u_0^t P^{rr}(\lambda_0; p_0) v_0 + & \frac{1}{3} \lambda_1 \lambda_2^3 u_0^t P^{rrr}(\lambda_0; p_0) v_0 \\
+ \lambda_1 \lambda_2^3 u_0^t P_1(\lambda_0; p_0) v_0 & = 0.
\end{align*}
\]

Premultiplying (16) by \( u_1^t \) and according to 1, we have:

\[
\begin{align*}
u_0^t P^r(\lambda_0; p_0) w_2 = & \lambda_1 u_1^t P^r(\lambda_0; p_0) v_1 + \frac{1}{2} \lambda_1^2 u_1^t P^{rr}(\lambda_0; p_0) v_0 \\
+ \lambda_2^2 u_1^t P^r(\lambda_0; p_0) v_0 + & \lambda_1^2 u_1^t P_1(\lambda_0; p_0) v_0.
\end{align*}
\]

So, taking into account (18)

\[
\lambda_1 \lambda_2^2 (2 u_0^t P^r(\lambda_0; p_0) v_1 + \lambda_1 u_0^t P^{rr}(\lambda_0; p_0) v_0)
\]

\[
= -\left( \lambda_1 \left( u_0^t P(\lambda_0; p_0) v_1 + \frac{1}{2} u_0^t P^r(\lambda_0; p_0) v_0 + \frac{1}{2} u_0^t P^{rr}(\lambda_0; p_0) v_1 \\
+ \frac{1}{3} u_0^t P^{rrr}(\lambda_0; p_0) v_0 + \lambda_1 u_0^t P_1(\lambda_0; p_0) v_0 + \\
+ u_0^t P_1(\lambda_0; p_0) v_1 + u_0^t P_1(\lambda_0; p_0) v_0) \right) \right)
\]

Since \( \lambda_1 (u_0^t P^r(\lambda_0; p_0) v_1 + \frac{1}{2} u_0^t P^{rr}(\lambda_0; p_0) v_0) \neq 0 \) we obtain

\[
\lambda_2 = -\frac{\lambda_1^2 (u_0^t P^{rr}(\lambda_0; p_0) v_1 + \frac{1}{2} u_0^t P^{rrr}(\lambda_0; p_0) v_0) - 2(u_0^t P^r(\lambda_0; p_0) v_1 + \frac{1}{2} u_0^t P^{rr}(\lambda_0; p_0) v_0)}{2(u_0^t P^r(\lambda_0; p_0) v_1 + \frac{1}{2} u_0^t P^{rr}(\lambda_0; p_0) v_0)}
\]

(19)

Now, we can compute \( w_2 \). We have

\[
P(\lambda_0; p_0) w_2 = -\lambda_1 P^r(\lambda_0; p_0) w_1 - \frac{1}{2} \lambda_1^2 P^{rr}(\lambda_0; p_0) v_0 - \\
\lambda_2 P^r(\lambda_0; p_0) v_0 + P_1(\lambda_0; p_0) v_0
\]

(20)

**Lemma 2.** Following condition \( u_0^t P_1(\lambda_0; p_0) v_0 \neq 0 \) we have that \( P(\lambda_0; p_0) + u_0 u_0^t P_1(\lambda_0; p_0) v_0 v_0^t \) is an invertible matrix.

**Proof.** Let \( x = \alpha v_0 + w \) with \( w \in [v_0]^\perp \), be a vector in the null space, then \( (P(\lambda_0; p_0) + u_0 u_0^t P_1(\lambda_0; p_0) v_0 v_0^t) x = 0 \).

Premultiplying by \( u_0^t \) we have

\[
u_0^t (P(\lambda_0; p_0) + u_0 u_0^t P_1(\lambda_0; p_0) v_0 v_0^t) x = 0,
\]

\[
0 = u_0 u_0^t P_1(\lambda_0; p_0) w_1 (\alpha v_0 + w) = \alpha u_0^t P_1(\lambda_0; p_0) v_0 \alpha v_0 + u_0^t P_1(\lambda_0; p_0) v_0 w.
\]

Then \( \alpha = 0 \).

Consequently, \( x = w \in [v_0]^\perp \) and \( x \in \text{Ker} u_0 u_0^t P_1(\lambda_0; p_0) v_0 v_0^t \), so \( x \in \text{Ker} P(\lambda_0; p_0) \) and \( x = \beta v_0 \), but \( x \in [v_0]^\perp \), then \( \beta = 0 \).

Now we consider the normalization condition \( u_0^t w_2 = 0 \), and adding \( u_0 u_0^t P_1(\lambda_0; p_0) v_0 v_0^t \) from the left to equation (20) and using lemma 2, we find vector \( w_2 \).

Using these calculations we have the following theorem.

**Theorem 2.** Let \( \lambda_0 \) be a double eigenvalue of the polynomial matrix \( P(\lambda; p_0) \), with a single eigenvector up to a non-zero scaling factor, and let \( v_0, v_1 \) be a Jordan chain and \( u_0, u_1 \) a left Jordan chain. We consider a perturbation of the parameter vector along the curve \( \rho(\varepsilon) \) starting at \( p_0 \) satisfying the condition \( \lambda_1 \neq 0 \).

Then, the double eigenvalue \( \lambda_0 \) bifurcates into two simple eigenvalues given by the relation

\[
\lambda = \lambda_0 + \varepsilon^{1/2} \lambda_1 + \varepsilon \lambda_2 + o(\varepsilon),
\]

with \( \lambda_1 \) and \( \lambda_2 \) as (18) and (19) respectively.

### 4.2. Perturbation of a \( \pm \)-multiplicity eigenvalue with single eigenvector

Now, we analyze the general case.

Analogously, substituting (12) into (11) we obtain

\[
P(\lambda; p(\varepsilon)) = (\lambda_0 + \varepsilon^{1/2} \lambda_1 + \varepsilon^{2/2} \lambda_2 + \ldots + \varepsilon \lambda_k + \ldots)^k I_n + \ldots + (\lambda_0 + \varepsilon^{1/2} \lambda_1 + \varepsilon^{2/2} \lambda_2 + \ldots)(A_{1} + \varepsilon A_{2} + \ldots + A_{m}) + \ldots + A_{n} = \varepsilon_{A_{1} + \ldots + A_{m}} + \ldots + \varepsilon^{1/2} \lambda_1 A_{k-1} + \ldots + \varepsilon^{2/2} \lambda_2 + \ldots + \varepsilon^{n/2} \lambda_{n} + \ldots
\]

(21)

If \( v \) is an eigenvector for the eigenvalue \( \lambda \) we have that

\[
(\lambda; p(\varepsilon)) v = P(\lambda; p(\varepsilon))(v_0 + \varepsilon^{1/2} w_1 + \varepsilon^{2/2} w_2 + \ldots) = 0
\]

Then, we find the chain of equations for the unknowns \( \lambda_1, \lambda_2, \ldots \) and \( w_1, w_2, \ldots \)

\[
P(\lambda_0; p_0) w_0 = 0,
\]

(22)

\[
P(\lambda_0; p_0) w_2 + \lambda_1 P^r(\lambda_0; p_0) v_0 + \frac{1}{2} \lambda_1^2 P^{rr}(\lambda_0; p_0) v_0 + \lambda_2 P^r(\lambda_0; p_0) v_0 + P_1(\lambda_0; p_0) v_0 = 0
\]

(23)
\[ \lambda_3 P'(\lambda_0; p_0)v_0 + \frac{1}{3!}\lambda_3^3 P'''(\lambda_0; p_0)v_0 + \]
\[ \frac{1}{2} \lambda_1 \lambda_2 P''(\lambda_0; p_0)v_0 + \lambda_2 P'(\lambda_0; p_0)v_1 + \]
\[ \frac{1}{2} \lambda_1^2 P''(\lambda_0; p_0)v_1 + \lambda_1 P'(\lambda_0; p_0)v_2 + P(\lambda_0; p_0)v_3 = 0, \]
\[ \ldots \]
\[ P(\lambda_0; p_0)v_0 + \lambda_1 P'(\lambda_0; p_0)v_{\ell-1} + \]
\[ \frac{1}{2} \lambda_1^2 P''(\lambda_0; p_0)v_{\ell-2} + \lambda_2 P'(\lambda_0; p_0)v_1 + \ldots + \]
\[ \lambda_{\ell-1} P'(\lambda_0; p_0)v_1 + P_1(\lambda_0; p_0)v_0 = 0, \]

where \( P_1(\lambda_0; p_0) = \lambda_0^{k-1} A_{k-1} + \lambda_0 A_{k-2} + \ldots + \lambda_0 A_1 + A_0. \)

Equation (21) is satisfied because \( v_0 \) is an eigenvector corresponding to the eigenvalue \( \lambda_0. \) Comparing equation (22) with (3) for \( i = 1 \) we observe that \( w_1 = \lambda_1 v_1 + \beta_0 v_0 \) is a solution, comparing equation (23) with (3) for \( i = 2 \) we have \( \lambda_2^2 v_1 + \lambda_2 v_2 = 0 \) is a solution, following in this sense \( w_2 = \lambda_1^2 v_3 + \lambda_1 v_2 + \lambda_3 v_1 \).

**Theorem 3.** Let \( \lambda_0 \) be a \( \ell \)-multiplicity eigenvalue of the polynomial matrix \( P(\lambda; p_0) \), with a single eigenvector up to a non-zero scaling factor, and let \( v_0, \ldots, v_{\ell-1} \) be a Jordan chain and \( u_0, \ldots, u_{\ell-1} \) a left Jordan chain. We consider a perturbation of the parameter vector along the curve \( \rho(\varepsilon) \) starting at \( p_0 \). Suppose \( u_0^T P_1(\lambda_0; p_0) v_0 \neq 0 \), then, the eigenvalue \( \lambda_0 \) bifurcates into \( \ell \) simple eigenvalues given by the relation

\[ \lambda = \lambda_0 + \varepsilon^{1/\ell} \lambda_1 + o(\varepsilon), \]

with

\[ \lambda_1 = \sqrt{\frac{-u_0^T P_1(\lambda_0; p_0) v_0}{\frac{1}{\ell!} u_0^T P'(\lambda_0; p_0) v_0 + \ldots + u_0^T P'(\lambda_0; p_0) v_{\ell-1}}}}. \]

**Remark 2.** Condition \( u_0^T P_1(\lambda_0; p_0) v_0 \neq 0 \) holds for all perturbations.

**Proof.** To find the value of \( \lambda_1 \) using \( w_1 = \lambda_1 v_1 + \beta_0 v_0 \) in equation (16) and premultiply it by \( u_0^T \) and taking into account remark 1 and normalization condition \( u_0^T P(\lambda_0; p_0) v_0 = 0 \), we obtain

\[ \lambda_1^\ell \left( \frac{1}{\ell!} u_0^T P'(\lambda_0; p_0) v_0 + \frac{1}{(\ell - 1)!} u_0^T P^{\ell-1}(\lambda_0; p_0) v_1 + \ldots + u_0^T P(\lambda_0; p_0) v_{\ell-1} \right) + u_0^T P_1(\lambda_0; p_0) v_0 = 0. \]

Now, corollary 1 ensures the result. \( \square \)

5. Conclusion

In this paper the perturbation of a multiple eigenvalue with a simple eigenvector of a monic polynomial matrix smoothly depending on parameters is analyzed.

**References**


