

RIGIDITY OF HAMILTONIAN ACTIONS ON POISSON MANIFOLDS

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Dedicated to the memory of Hans Duistermaat

ABSTRACT. This paper is about the rigidity of compact group actions in the Poisson context. The main result is that Hamiltonian actions of compact semisimple type are rigid. We prove it via a Nash-Moser normal form theorem for closed subgroups of SCI-type. This Nash-Moser normal form has other applications to stability results that we will explore in a future paper. We also review some classical rigidity results for differentiable actions of compact Lie groups and export it to the case of symplectic actions of compact Lie groups on symplectic manifolds.

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1. INTRODUCTION

In this paper we prove rigidity results for Hamiltonian actions of compact Lie groups on Poisson manifolds.

One of the main concerns in the study of the geometry of group actions on a differentiable manifold is to understand their structural stability properties. When the group is compact, the classical technique of averaging allows to prove a “stability” property in a neighbourhood of a fixed point: Any group action is equivalent under conjugation to the linearized action. This result is due to Bochner [3]. As a consequence, any two “close” actions of a compact Lie group are equivalent. This phenomenon is known as local rigidity for compact group actions.

The question of global rigidity of compact group actions is a harder matter. Differentiable actions of compact Lie groups on compact manifolds are known to be rigid thanks to the work of Palais [27]. This result uses a Mostow-Palais embedding theorem.

When the manifold is endowed with additional geometrical structures one is interested in obtaining rigidity results for the group actions preserving those geometrical structures. This means that we also require that there exists a diffeomorphism that conjugates the two actions and that preserves the additional geometrical structure.

In the case the manifold is symplectic, the equivariant version of Darboux theorem [30] can be seen as a local rigidity for symplectic compact group actions. The main ingredients in proving this rigidity are Moser’s path method and averaging. These techniques allow to prove global rigidity for compact symplectic group actions on compact symplectic manifolds. For the sake of completeness, we include a proof of this fact in the first section of this paper.

When the manifold is Poisson, the equivariant version of Weinstein’s splitting theorem would entail local rigidity for compact group actions which preserve the Poisson structure. This equivariant version was obtained by Miranda and Zung in [23] under a mild additional condition of homological type on the Poisson structure which was called tameness of the Poisson structure. Roughly speaking, this tameness condition allows the path method to work in the Poisson context. There are other instances in the literature where the path method has been used in Poisson geometry (see for instance [1], [2], [14]) in all these examples the tameness condition is implicitly assumed.

In the global case of compact group actions on compact Poisson manifolds, it was Viktor Ginzburg who proved rigidity by deformations in [13]. However, Ginzburg's result does not imply rigidity because, a priori, Poisson manifolds do not form a tame Fréchet space ([13] and [17]) and hence we cannot use the “infinitesimal rigidity implies rigidity” argument.

In any event, it is still interesting to explore whether there is an “infinitesimal stability” result given by the vanishing of a cohomology group attached to the geometric problem. Any rigidity problem can be viewed as a problem about openness of orbits in an appropriate setting. For example, a group action can be viewed as a morphism from the Lie group to the group of diffeomorphisms of the manifold. Then two close actions are equivalent if they are conjugated by a diffeomorphism and, therefore, if they are on the same orbit of the group of diffeomorphisms acting on the set of actions. In the case the actions preserve an additional structure, we can require that this diffeomorphism and the actions in question also respect this given structure. This associated first cohomology group has coefficients in an infinite-dimensional space (usually the set of vector fields respecting the given additional structure) and it morally stands for the quotient of the “tangent space” to the variety of actions and the “tangent space” to the orbit. If the “set of actions” has good properties (either manifolds or tame Fréchet spaces), we can deduce stability from infinitesimal stability via an inverse function theorem (Nash-Moser inverse function in the case of tame Fréchet spaces). An inverse function theorem of Nash-Moser type was laid down by Richard Hamilton in his foundational paper [19]. Many examples and useful criteria to determine whether a set is a tame Fréchet manifold are given in [19]. However it is difficult to apply these criteria in order to prove that a certain given set is tame Fréchet. For instance, in the Poisson case, as observed by Ginzburg in [13], it is difficult to establish whether the set of Poisson vector fields constitute a Fréchet tame space. If this were the case, we could apply Nash-Moser inverse theorem straightaway to conclude structural stability or rigidity from infinitesimal stability. When the criteria given by Hamilton are hard or impossible to apply, we may still be able to apply Newton's iteration method used by Hamilton in [19] if the sets considered still satisfy some appropriate properties (SCI spaces and SCI actions). This infinitesimal stability result then leads to a stability result even if the “tameness” condition is hard to explore for the set of vector fields preserving the given structure.

We follow this philosophy to prove a rigidity result for Hamiltonian compact group actions on Poisson manifolds and the proof is based on the Nash-Moser method and cohomological considerations.

When M is a Poisson manifold, a Hamiltonian action of a Lie group G is given by a momentum map $\mu : M \rightarrow \mathfrak{g}^*$ where \mathfrak{g} is the Lie algebra of G and μ is a Poisson map with respect to the standard Poisson structure on \mathfrak{g}^* . When G is semisimple and compact, we call those actions, Hamiltonian actions of compact semisimple type. In order to prove the rigidity result for Hamiltonian actions, we first prove an infinitesimal stability result which lies on the vanishing of the first cohomology group of the Chevalley-Eilenberg complex associated to the representation of \mathfrak{g} on the set of smooth functions given by the momentum map. Our proof of infinitesimal stability is based on the techniques used by Conn [9] and [10] to prove linearization of Poisson structures whose linear part is semisimple of compact type. We can then

prove rigidity using an iteration process similar to that used by Conn in [10]; in turn, this iterative process is inspired by Newton's fast convergence method used by Hamilton to prove Moser's-Nash theorem [19]. Proving convergence of this iteration requires to have a close look at many estimates and carefully check out all the steps. Instead, we propose here a proof that smartly hides this iteration process via a Nash-Moser normal form theorem. This Normal form theorem is a refinement of a previous normal form theorem established by the two last authors of this paper in [24]. The condition on the Lie algebra to be semisimple of compact type is essential for the proof to work. Examples of non-linearizability (and in particular non-rigidity) for semisimple actions of non-compact type were already given by Guillemin and Sternberg [15]. Recently the Hamiltonian case has been considered by the first author of this paper in a short note [20].

The Nash-Moser normal form theorem used in the proof of the results of this paper seems to have many applications to structural stability problems concerning other geometrical structures (foliations, group actions, etc...). We plan to explore these applications in a future work.

Organization of this paper In Section 2 we recall some basic facts about rigidity of group actions and we give a proof for rigidity of compact symplectic group actions on a compact symplectic manifold which uses the path method. In the section 3 we review some known facts about rigidity in the Poisson case: infinitesimal rigidity, rigidity by deformations and local rigidity for tame Poisson structures. In section 4 we state the main results of this paper: local and global rigidity for Hamiltonian actions of semisimple actions of compact type. The semilocal result also holds when we replace a fixed point for the action by an invariant compact neighbourhood and we replace the norms by distance to the invariant manifold. In section 5 we prove an infinitesimal rigidity result for Poisson actions. In order to prove this result we prove the vanishing of a first Chevalley-Eilenberg cohomology group associated to a Hamiltonian action. In section 6 we give the proofs for local, semilocal and global rigidity for Hamiltonian actions of compact semisimple type. The proof uses a normal form Nash-Moser form for SCI-spaces and SCI-actions that we explain here. In the first appendix we prove a Nash-Moser normal form theorem for SCI spaces (theorem 7.7). In the second appendix we prove some technical lemmas needed in the proof of the main rigidity theorems in this paper (essentially to verify that our spaces fulfill the technical assumptions established in the Nash-Moser type theorem 7.7).

2. THE SYMPLECTIC CASE

Let G be a Lie group and let $\rho : G \times M \rightarrow M$ be a smooth action on a smooth manifold M . For each $g \in G$, we denote by $\rho(g)$ the diffeomorphism defined by $\rho(g)(x) := \rho(g, x), x \in M$.

Definition 2.1. Given two group actions, ρ_0 and ρ_1 , we say that they are C^k -equivalent if there exists a C^k -diffeomorphism conjugating the two actions, i.e, $\rho_0(g) \circ \phi = \phi \circ \rho_1(g)$.

In the case when we are given a local smooth action, ρ , and a fixed point p for the action we can define the linearized action, $\rho^{(1)}$ in a neighbourhood of p by the formula $\rho^{(1)}(g, x) = d_p(\rho(g))(x)$ for $g \in G$ and $x \in M$.

For compact Lie groups we have the following two equivalence results in the local and global settings:

Theorem 2.2 (Bochner [3]). *A local smooth action with a fixed point is locally equivalent to the linearized action.*

Theorem 2.3 (Palais [27]). *Two C^k -close group actions ($k \geq 1$) of a compact Lie group on a compact manifold are equivalent by a diffeomorphism of class C^k which belongs to the arc-connected component of the identity.*

We also have the following “global” linearization theorem due to Mostow-Palais for compact manifolds (which is also valid for neighbourhoods of compact invariant submanifolds).

Theorem 2.4 (Mostow-Palais, [26] [27]). *Let ρ be an action of a compact Lie group G act on a compact manifold M then there exists an equivariant embedding of M on a finite-dimensional vector space E such that via the embedding the action of ρ on M becomes part of a linear action ρ_0 on E .*

Now assume that we are given a symplectic manifold (M, ω) and a symplectic group action ρ . We can prove rigidity (local and global) imposing that the equivalence also preserves ω .

The first result in this direction is equivariant Darboux theorem ([29] and [4]). For a complete proof of equivariant Darboux theorem we refer either to the book by M. Chaperon [4] or to Appendix A in the book [11].

We can use the same ideas to prove the following global rigidity result. This proof was already included in the short note [22]. We include it here for the sake of completeness.

Theorem 2.5. *Let ρ_0 and ρ_1 be two C^2 -close symplectic actions of a compact Lie group G on a compact symplectic manifold (M, ω) . Then they can be made equivalent by conjugation via a symplectomorphism.*

Proof. Let φ be the diffeomorphism (given by Palais’ Theorem) that takes the action ρ_0 to ρ_1 . Define the symplectic structure $\omega_1 = \varphi^*(\omega_0)$. It remains to show that we can take ω_1 to ω_0 in an invariant way with respect to the ρ_0 -action.

Since φ belongs to the arc-connected component of the Identity, we can indeed construct an homotopy φ_t such that $\varphi_1 = \varphi$ and $\varphi_0 = id$.

We can use this homotopy to define a de Rham homotopy operator:

$$Q\omega = \int_0^1 \varphi_t^*(i_{v_t}\omega)dt$$

where v_t is the t-dependent vector field defined by the isotopy φ_t .

Via this formula, we can prove (see for instance pages 110 and 111 of the book by Guillemin-Sternberg [16]) that ω_1 belongs to the same cohomology class as ω_0 and we can write $\omega_1 = \omega_0 + d\alpha$ for a 1-form α (indeed $\alpha = Q\omega$ where Q is the de Rham homotopy operator).

We first consider the linear path of symplectic structures

$$\omega_t = t\omega_1 + (1 - t)\omega_0, \quad t \in [0, 1].$$

It is a path of symplectic structures since ω_0 and ω_1 are close. Let X_t be the vector field,

$$i_{X_t}\omega_t = -\alpha.$$

Now we consider the averaged vector field with respect to a Haar measure on G :

$$X_t^G = \int_G \rho(g)_*(X_t) d\mu.$$

Since the diffeomorphism ϕ conjugates the actions ρ_0 and ρ_1 which both preserve the initial symplectic form ω_0 , then the path of symplectic forms ω_t is ρ_0 -invariant. This vector field satisfies the equation,

$$i_{X_t^G}\omega_t = - \int_G \rho(g)^*(\alpha) d\mu.$$

The new invariant 1-form $\alpha_G = \int_G \rho(g)^*(\alpha) d\mu$ fulfills the cohomological equation $\omega_1 = \omega_0 + d\alpha_G$ due to ρ_0 -invariance of the forms ω_t .

Let ϕ_t^G be defined by the equation,

$$(2.1) \quad X_t^G(\phi_t^G(q)) = \frac{\partial \phi_t^G}{\partial t}(q).$$

Observe that there is a loss of one degree of differentiability with respect to the degree of differentiability of φ . Therefore, in order to be able to guarantee the existence of ϕ_t , we need degree of differentiability at least 1 and therefore we need φ to be of degree at least 2. Palais theorem guarantees that if the initial to action are C^2 -close the conjugating diffeomorphism φ is of class C^2 .

The compactness of M guarantees the existence of ϕ_t^G , $\forall t \in [0, 1]$. Then ϕ_t^G commutes with the action of G given by ρ_0 and satisfies $\phi_t^{G*}(\omega_t) = \omega_0$.

Therefore the time-1-map ϕ_1 of X_t^G takes ω_1 to ω_0 in an equivariant way. □

Remark 2.6. The path method also works very well for contact structures [12]. Indeed in the local case a linearization result for compact contact group actions was already established by Marc Chaperon [4]. In the global case, we can use the techniques of Gray [12] and reproduce the same ideas of the proof of the symplectic case.

Remark 2.7. This proof uses the path method to conclude and therefore requires C^2 -closedness of the initial two actions ρ_0 and ρ_1 . However, we do not know any example of close C^1 -symplectic actions which can be shown that are not equivalent. The theorem might still hold for C^1 -closed actions but this method of proof is not powerful enough to guarantee this (current methods in symplectic topology could be useful for this end).

3. RIGIDITY BY DEFORMATIONS AND LINEARIZATION

Let (P, Π) stand for a Poisson manifold and let ρ stand for a Poisson action of a compact Lie group G .

Ginzburg proved in [13] that Poisson actions are rigid by deformations.

Theorem 3.1 (Ginzburg). *Let ρ_t be a family of Π -preserving actions smoothly parametrized by $t \in [0, 1]$. Then there exists a family of Poisson diffeomorphisms $\phi_t : M \rightarrow M$ which sends ρ_0 to ρ_t such that $\rho_t(g)x = \phi_t(\rho_0(g)\phi_t^{-1}(x))$ for all $x \in M$, $g \in G$ and $\phi_0 = Id$.*

Remark 3.2. In [13] it was first proved that they are infinitesimally rigid (here meaning vanishing of certain cohomology group associated to a group representation) and then used the homotopy method to prove rigidity by deformations.

In the Poisson case the phenomenon “infinitesimal rigidity implies rigidity by deformations” is observed. However, if we would like to prove “infinitesimal rigidity implies rigidity” we would need to check that the space of G -Poisson actions is “tame Fréchet” (as observed by Ginzburg on [13]) but this seems out of reach.

In the case when we are not given a path of actions connecting the two actions, the first attempt is to try to use Moser’s path method as we did for symplectic actions.

Unlike the symplectic case, the path method does not seem to work so well for Poisson structures. We need to impose additional hypothesis.

We recall the definition of tameness given in [23]:

Definition 3.3. Let (P^n, Π) be a smooth Poisson manifold and p a point in P . We will say that Π is *tame* at p if for any pair X_t, Y_t of germs of smooth Poisson vector fields near p which are tangent to the symplectic foliation of (P^n, Π) and which may depend smoothly on a (multi-dimensional) parameter t , then the function $\Pi^{-1}(X_t, Y_t)$ is smooth and depends smoothly on t .

In [23] this tameness condition is studied several examples of tameness and non-tameness are given. In particular, all 2 and 3-dimensional Lie algebras are tame Poisson structures and all semisimple Lie algebras of compact type are tame.

For these Poisson structures, we can prove the result contained in [23].

Theorem 3.4 (Miranda, Zung). *Let (P^n, Π) be a smooth Poisson manifold, p a point of P , $2k = \text{rank } \Pi(p)$, and G a compact Lie group which acts on P in such a way that the action preserves Π and fixes the point p . Assume that the Poisson structure Π is tame at p . Then there is a smooth canonical local coordinate system $(x_1, y_1, \dots, x_k, y_k, z_1, \dots, z_{n-2k})$ near p , in which the Poisson structure Π can be written as*

$$(3.1) \quad \Pi = \sum_{i=1}^k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{i,j} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j},$$

and in which the action of G is linear and preserves the subspaces $\{x_1 = y_1 = \dots x_k = y_k = 0\}$ and $\{z_1 = \dots = z_{n-2k} = 0\}$.

This result implies local rigidity for compact Poisson group actions.

By using Conn’s linearization theorem [10] for semisimple Lie algebra’s of compact type, we can prove an equivariant linearization theorem also contained in [23].

Theorem 3.5 (Miranda, Zung). *Let (P^n, Π) be a smooth Poisson manifold, p a point of P , $2r = \text{rank } \Pi(p)$, and G a compact Lie group which acts on P in*

such a way that the action preserves Π and fixes the point p . Assume that the linear part of transverse Poisson structure of Π at p corresponds to a semisimple compact Lie algebra \mathfrak{k} . Then there is a smooth canonical local coordinate system $(x_1, y_1, \dots, x_r, y_r, z_1, \dots, z_{n-2r})$ near p , in which the Poisson structure Π can be written as

$$(3.2) \quad \Pi = \sum_{i=1}^r \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \frac{1}{2} \sum_{i,j,k} c_{ij}^k z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j},$$

where c_{ij}^k are structural constants of \mathfrak{k} , and in which the action of G is linear and preserves the subspaces $\{x_1 = y_1 = \dots x_r = y_r = 0\}$ and $\{z_1 = \dots = z_{n-2r} = 0\}$.

In the paper [23], we also asserted that in the case of Hamiltonian actions, we could use Nash-Moser techniques to prove the existence of an equivariant splitting Weinstein theorem regardless of the ‘‘tameness condition’’ on the Poisson structure. This result is a corollary of our local rigidity theorem for the semisimple actions of compact type.

4. RIGIDITY OF HAMILTONIAN GROUP ACTIONS ON POISSON MANIFOLDS

In this paper, we will show that if two Hamiltonian group actions of compact semisimple type on a Poisson manifold are close then they are equivalent. We do it in the following settings:

- Local: For two close Hamiltonian actions of a compact Lie group neighbourhood of a common fixed point for the two actions.
- Semilocal: For two close Hamiltonian actions in a neighbourhood of an invariant compact manifold.
- Global: For two close Hamiltonian actions in a compact manifold.

4.1. Norms in the space of actions. In this section, we clarify what we mean in this paper by ‘‘close’’.

An action $\rho : G \times M \longrightarrow M$ of a Lie group G on a manifold M is a morphism from G to the group of diffeomorphisms $Diff(M)$. In particular, we can view this action as an element in $Map(G \times M, M)$ and use the C^k -topology there.

In this paper we consider Hamiltonian actions so we can define the topology by using the associated momentum maps: If two momentum maps $\mu_1 : M \longrightarrow \mathfrak{g}^*$ and $\mu_2 : M \longrightarrow \mathfrak{g}^*$ then the two Hamiltonian actions are close.

We will need the following norms depending on the case considered: in the local case we will consider the C_ρ^k norms and in the global case the C^k norms. In the semilocal case we just consider the distance to the invariant manifold.

In the local case, the manifold is $M = (\mathbb{R}^n, 0)$, for each positive number $r > 0$, denote by B_r the closed ball of radius r in \mathbb{R}^n centered at 0. In order to make estimates, we will use the following norms on the vector space of smooth functions on B_r :

$$(4.1) \quad \|F\|_{k,r} := \sup_{|\alpha| \leq k} \sup_{z \in B_r} |D^\alpha F(z)|$$

for each smooth function F on B_r , where the sup runs over all partial derivatives of degree $|\alpha|$ at most k . More generally, if $F = (F_1, \dots, F_m)$ is a smooth mapping from B_r to \mathbb{R}^m we can define

$$(4.2) \quad \|F\|_{k,r} := \sup_i \sup_{|\alpha| \leq k} \sup_{z \in B_r} |D^\alpha F_i(z)|.$$

Similarly, for a vector field $X = \sum_{i=1}^n X_i \partial / \partial x_i$ on B_r we put

$$(4.3) \quad \|X\|_{k,r} := \sup_i \sup_{|\alpha| \leq k} \sup_{z \in B_r} |D^\alpha X_i(z)|.$$

Finally, let us fix a basis $\{\xi_1, \dots, \xi_m\}$ of \mathfrak{g} . If $\mu : U \rightarrow \mathfrak{g}^*$ is a momentum map (with respect to the Poisson structure above) then the map $\xi_i \circ \mu$ is a smooth function on U for each i . We then define the C^k -norms of μ on $B_r \subset U$ as

$$(4.4) \quad \|\mu\|_{k,r} := \max_i \{\|\xi_i \circ \mu\|_{k,r}\}.$$

The above norms are called C_r^k -norms in the local case (strictly speaking, they are semi-norms).

In the global case, we use a norm which gives the standard C^k -topology in the space of mappings (the C^k norms are equivalent).

In the semilocal case: Let N be the compact invariant submanifold of M , we can consider a tubular neighbourhood of the submanifold N in the manifold M , $U_\rho(N) = \{x \in M, d(x, N) < \rho\}$, where ρ is the distance between a given point to the manifold with respect to some fixed Riemannian metric on M . To define a topology in the set of Hamiltonian actions that have N as invariant manifold, it suffices to replace the ball by a ρ -tubular neighbourhood,

That is to say:

For each positive number $r > 0$, we consider the following norms:

$$(4.5) \quad \|F\|_{k,r} := \sup_{|\alpha| \leq k} \sup_{z \in U_\rho(N)} |D^\alpha F(z)|.$$

These norms are also called C_ρ^k -norms.

We will also need Sobolev norms (to be defined below) which turn our spaces into (pre)Hilbert-spaces.

4.2. The results. In this section we state the main theorems of this paper. These theorems are rigidity results for Hamiltonian actions in the local, semilocal and global setting respectively.

4.2.1. Local Rigidity. When we consider two close Hamiltonian actions in a neighbourhood of a fixed point, we can prove the following:

Theorem 4.1. *Consider a Poisson structure $\{, \}$ defined on a neighborhood U of 0 in \mathbb{R}^n containing a closed ball B_R of radius $R > 0$ and a semisimple Lie algebra \mathfrak{g} of compact type. Let $\lambda : U \rightarrow \mathfrak{g}^*$ be a momentum map with respect to the Poisson structure.*

There exist a positive integer l and two positive real numbers α and β such that, if μ is another momentum map on U with respect to the same Poisson structure

and Lie algebra, satisfying

$$(4.6) \quad \|\lambda - \mu\|_{l,R} \leq \alpha \quad \text{and} \quad \|\lambda - \mu\|_{2l-1,R} \leq \beta$$

then, there exists a smooth diffeomorphism Φ_∞ of class $C^{(k, \frac{R}{2})}$ (with $k \geq l$) on $B_{R/2}$ such that $\mu \circ \Phi_\infty = \lambda$.

Remark 4.2. The integer l can be determined from some inequalities in the proof of the iteration process in [24] (page 365).

Remark 4.3. Indeed as a consequence of the techniques used for the proof we do not need λ to be of C^∞ -type. It is enough to require λ to be of class at least $2l - 1$ where l depends on the iteration process spelled out in [24]

As a corollary we obtain an equivariant Weinstein's splitting theorem for Hamiltonian actions of compact semisimple type which doesn't need the tameness conditions but which requires the Poisson action to be Hamiltonian (compare with the splitting theorem in [23]).

Theorem 4.4. *Let (P^n, Π) be a smooth Poisson manifold, p a point of P , $2k = \text{rank } \Pi(p)$, and G a semisimple compact Lie group which acts on P in a Hamiltonian way fixing the point p . Then there is a smooth canonical local coordinate system $(x_1, y_1, \dots, x_k, y_k, z_1, \dots, z_{n-2k})$ near p , in which the Poisson structure Π can be written as*

$$(4.7) \quad \Pi = \sum_{i=1}^k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{ij} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j},$$

and in which the action of G is linear and preserves the subspaces $\{x_1 = y_1 = \dots x_k = y_k = 0\}$ and $\{z_1 = \dots = z_{n-2k} = 0\}$.

Proof. Because of Weinstein's splitting theorem [30], we can assume that there exists a canonical local coordinate system $(x_1, y_1, \dots, x_k, y_k, z_1, \dots, z_{n-2k})$ in a neighbourhood of the fixed point for the action p , in which the Poisson structure Π can be written as

$$(4.8) \quad \Pi = \sum_{i=1}^k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{ij} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}.$$

Let ρ be the Hamiltonian action given by a momentum map $\mu : (P, \Pi) \rightarrow \mathfrak{g}^*$ since $\rho^{(1)}$ is the linearization of ρ at the fixed point p then it is close to ρ . Recall that, $\rho^{(1)}$ is defined with the formula $\rho^{(1)}(g, x) = d_p(\rho(g))(x)$ for $g \in G$ and $x \in M$ and, in particular, is also a Poisson action. Not only that, but it is also Hamiltonian as we are going to check now.

Since $\mu : (P, \Pi) \rightarrow \mathfrak{g}^*$ is the moment map for ρ we know that it is a Poisson map. Now consider the differential, $d\mu$ at the fixed point p defined as $\langle d\mu(x), v \rangle = d_p(\langle \mu(x), v \rangle)$ for $p \in M$ and $v \in \mathfrak{g}$. Using differentiation we obtain that $d\mu : (P, \Pi) \rightarrow \mathfrak{g}^*$ defines a Poisson map. In particular, the action $\rho^{(1)}$ is Hamiltonian.

Now let α and β such that,

$$(4.9) \quad \|\mu - d\mu\|_{l,R} \leq \alpha \quad \text{and} \quad \|\mu - d\mu\|_{2l-1,R} \leq \beta,$$

Because of theorem 4.1, there exists a smooth diffeomorphism Φ_∞ of class $C^{(k, \frac{R}{2})}$ (with $k \geq l$) on $B_{R/2}$ such that $\mu \circ \Phi_\infty = d\mu$. Since we started from a splitted Poisson structure, this proves the existence of a linear-equivariant Weinstein's splitting theorem for Hamiltonian actions of semisimple compact type. This concludes the proof of the corollary. \square

4.2.2. *Semilocal Rigidity: Hamiltonian actions on Poisson manifolds in the neighbourhood of a compact invariant manifold.* We prove the following theorem in the semilocal case:

Theorem 4.5. *Consider a Poisson structure $\{, \}$ defined on a neighborhood N_ε of a compact submanifold N of M . Let ρ be a Hamiltonian action defined on the given G -invariant neighbourhood given by a momentum map $\lambda : U_p(N) \longrightarrow \mathfrak{g}^*$ with \mathfrak{g} a semisimple Lie algebra of compact type.*

There exist a positive integer l and two positive real numbers α and β such that, if μ is another momentum map on $U_p(N)$ with respect to the same Poisson structure and Lie algebra which also has N as an invariant set, satisfying

$$(4.10) \quad \|\lambda - \mu\|_{l,R} \leq \alpha \quad \text{and} \quad \|\lambda - \mu\|_{2l-1,R} \leq \beta$$

then, there exists a smooth diffeomorphism Φ_∞ of class $C^{(k, \frac{R}{2})}$ (with $k \geq l$) on $B_{R/2}$ such that $\mu \circ \Phi_\infty = \lambda$.

4.2.3. *Global Rigidity: Hamiltonian actions on Poisson manifolds of compact semisimple type.* We prove the following

Theorem 4.6. *Consider a Poisson manifold (M, Π) and a semisimple Lie algebra \mathfrak{g} of compact type. Let $\lambda : U \longrightarrow \mathfrak{g}^*$ be a momentum map.*

There exist a positive integer l and two positive real numbers α and β such that, if μ is another momentum map on M with respect to the same Poisson structure and Lie algebra, satisfying

$$(4.11) \quad \|\lambda - \mu\|_l \leq \alpha \quad \text{and} \quad \|\lambda - \mu\|_{2l-1} \leq \beta$$

then, there exists a smooth diffeomorphism Φ_∞ of class C^k (with $k \geq l$) on M such that $\mu \circ \Phi_\infty = \lambda$.

Remark 4.7. For regular Poisson manifolds (even if they are not compact) with a symplectic foliation with compact leaves, we can prove this result directly via Moser's theorem under some homotopical constraints on the symplectic foliation. We would first prove leafwise equivalence by carefully applying the path method leafwise as we did in the symplectic case. There are some conditions on the symplectic foliation specified in [6] (pages 126-127) under which we can easily extend this leaf-wise equivalence to the Poisson manifold since the leaves are symplectically embedded (the conditions on the foliation entail that the manifold is Lie-Dirac).

In the case that our Poisson manifold is not compact but it is a Poisson manifold of compact type (see [8]) (that is, its symplectic groupoid is compact), we may try to work directly on the symplectic groupoid. However, it seems that the only examples known so far are regular [8] and we could try to apply the strategy explained above.

5. INFINITESIMAL RIGIDITY OF HAMILTONIAN ACTIONS

The introduction of this section follows the spirit of Guillemin, Ginzburg and Karshon in [17] which views the step of infinitesimal rigidity implies rigidity as an application of the inverse function in the appropriate setting following the guidelines of Kuranishi for deformation of complex structures.

Following [17], an action of a compact Lie group $\rho : G \times M \rightarrow M$ can be considered like a group morphism to the group of diffeomorphisms $\alpha : G \rightarrow \text{Diff}(M)$.

In the case of Poisson actions, it can be considered as a morphism in the group of Poisson diffeomorphisms, $\text{Diff}(M, \Pi)$.

Two actions are equivalent if there exist an element ϕ in the corresponding diffeomorphism group, say \mathcal{G} , such that ϕ conjugates both actions. Let \mathcal{M} stand for $\mathcal{M} = \text{Hom}(G, \mathcal{G})$, we may as well consider the action,

$$(5.1) \quad \begin{aligned} \beta : \mathcal{G} \times \mathcal{M} &\longrightarrow \mathcal{M} \\ (\phi, \alpha) &\longmapsto \phi \circ \alpha \circ \phi^{-1} \end{aligned}$$

Then two actions ρ_1 and ρ_2 are equivalent if they are on the same orbit by the new action β . Then the rigidity result that we want to prove in the corresponding category boils down to proving that the orbits of the action β are open. If the a priori, infinite dimensional set \mathcal{M} was a manifold and the tangent space to the orbits equals the tangent space to the whole space, then a straightforward application of the inverse function theorem would entail that the orbit is an open set inside the manifold. In other words, the orbits would be open in \mathcal{M} and therefore the actions would be rigid.

As it is clearly explained in [17], in the Palais case, the measure of the tangent to the orbits to fail to be equal the tangent to the whole space is given by the first cohomology group of the “group cohomology” associated to the action with coefficients in the space of smooth vector fields.

In the case that the actions we are comparing are Hamiltonian on a given Poisson manifold, we have a natural representation of \mathfrak{g} on the set of smooth functions which naturally leads to a Chevalley-Eilenberg complex. This is the Lie algebra cohomology that we will consider (instead of the group cohomology considered in [17]).

The substitute for the inverse function theorem for infinite-dimensional sets is the Nash-Moser theorem. The sets considered have to fulfill some conditions explained in the foundational paper by Hamilton [19]. The sets are required to be tame Fréchet manifolds and the considered mappings to be tame. In the Poisson case we would need to check that the set of Poisson vector fields satisfy the conditions explained by Hamilton. This seems a difficult endeavour. Our strategy would be based rather on applying the method of proof in [19] which is inspired by Newton’s iterative method.

This is why we compute the first cohomology group of the Lie algebra cohomology associated to a Hamiltonian action in this section.

Remark 5.1. Throughout this section (M, Π) stands for a Poisson manifold that can either be a ball, an ε -neighbourhood of a compact submanifold inside a Poisson

manifold or a compact Poisson manifold. Attention has to be paid when considering the norms and estimates for these manifolds because they might differ according to the context.

5.1. Chevalley-Eilenberg complex associated to a Hamiltonian action. In this subsection we define a Chevalley-Eilenberg complex associated to the Hamiltonian action.

Let $F : (M, \Pi) \rightarrow \mathfrak{g}^*$ be a momentum map with component functions $F = (f_1, \dots, f_n)$. Pick an orthonormal basis ξ_1, \dots, ξ_n of \mathfrak{g} .

The Lie algebra \mathfrak{g} defines a representation ρ of \mathfrak{g} on $\mathcal{C}^\infty(M)$ defined on the base as $\rho_{\xi_i}(h) := \{f_i, h\}$, $\forall i$.

The set $\mathcal{C}^\infty(M)$ can be then viewed as a \mathfrak{g} -module and we can introduce the corresponding Chevalley-Eilenberg complex [5].

The space of cochains is defined as follows:

For $q \in \mathbb{N}$, $C^q(\mathfrak{g}, \mathcal{C}^\infty(M)) = \text{Hom}(\wedge^q \mathfrak{g}, \mathcal{C}^\infty(M))$ is the space of alternating q -linear maps from \mathfrak{g} to $\mathcal{C}^\infty(M)$, with the convention $C^0(\mathfrak{g}, \mathcal{C}^\infty(M)) = \mathcal{C}^\infty(M)$. The associated differential is denoted by δ_i .

Let us give an explicit expression for δ_0 , δ_1 and δ_2 since they will show up in the proof of the main theorem of this paper.

$$\mathcal{C}^\infty(M) \xrightarrow{\delta} C^1(\mathfrak{g}, \mathcal{C}^\infty(M)) \xrightarrow{\delta} C^2(\mathfrak{g}, \mathcal{C}^\infty(M))$$

$$\delta_0(f)(\xi_1) = \rho(\xi_1)(f), \quad f \in \mathcal{C}^\infty(M)$$

$$\delta_1(\alpha)(\xi_1 \wedge \xi_2) = \rho(\xi_1)(\alpha(\xi_2)) - \rho(\xi_2)(\alpha(\xi_1)) - \alpha([\xi_1, \xi_2]), \quad \alpha \in C^1(\mathfrak{g}, \mathcal{C}^\infty(M))$$

$$\delta_2(\beta)(\xi_1 \wedge \xi_2 \wedge \xi_3) = \sum_{\sigma \in A_3} (\rho(\xi_{\sigma(1)}))(\beta(\xi_{\sigma(2)} \wedge \xi_{\sigma(3)}) + \beta(\xi_{\sigma(1)} \wedge [\xi_{\sigma(2)}, \xi_{\sigma(3)}]), \quad \beta \in C^2(\mathfrak{g}, \mathcal{C}^\infty(M))$$

where $\xi_1, \xi_2, \xi_3 \in \mathfrak{g}$.

As proved by Chevalley-Eilenberg the differential satisfies $\delta_i \circ \delta_{i-1} = 0$. Therefore we can define the quotients

$$H^i(\mathfrak{g}, \mathcal{C}^\infty(M)) = \ker(\delta_i) / \text{Im}(\delta_{i-1}) \quad \forall i \in \mathbb{N}.$$

This complex was used in the abelian case for instance in [21] to construct a deformation complex for integrable systems on symplectic manifolds. A similar complex was used by Conn in [10] and [9]. This complex was associated to the infinitesimal version of the adjoint representation of the Lie algebra associated to the linear part of the Poisson structure.

Geometrical interpretation of $H^1(\mathfrak{g}, \mathcal{C}^\infty(M))$

The first cohomology group can be interpreted as infinitesimal deformations of the Hamiltonian action modulo trivial deformations.

In the compact semisimple case, it is known that the first and second cohomology groups vanish.

Namely,

Lemma 5.2. *Let \mathfrak{g} be semisimple of compact type and let V be a Fréchet space then $H^1(\mathfrak{g}, V) = H^2(\mathfrak{g}, V) = 0$*

This lemma can be seen as an infinite-dimensional version of Whitehead's lemma (for a proof see [13]). Note that the set $V = \mathcal{C}^\infty(M)$ is a Fréchet space.

Therefore we know that there exist homotopy operators h_i satisfying,

$$\delta_i \circ h_i + h_{i+1} \circ \delta_{i+1} = id_{\mathcal{C}^{i+1}(\mathfrak{g}, \mathcal{C}^\infty(M))}$$

for $i = 0, 1$.

$$\mathcal{C}^\infty(M) \begin{array}{c} \xrightarrow{\delta} \\ \xleftarrow{h_0} \end{array} \mathcal{C}^1(\mathfrak{g}, \mathcal{C}^\infty(M)) \begin{array}{c} \xrightarrow{\delta} \\ \xleftarrow{h_1} \end{array} \mathcal{C}^2(\mathfrak{g}, \mathcal{C}^\infty(M))$$

We will use those homotopy operators in the proof of the main theorem and we will need to control and bound the norms and we will need to control their regularity properties. We spell this in the next subsection.

5.2. Estimates for homotopy operators in the compact semisimple case.

Regularity properties of the homotopy operators are well-known for Sobolev spaces. That is why, following the spirit of Conn's proof in [10], we first need to consider the extended Chevalley-Eilenberg complex which comes from considering the induced representation on the set of complex-valued functions on closed balls (which will be denoted in the sequel by $C_C^\infty(B_r)$) and consider the Sobolev norm there. However, a main point in the proof of Conn is that those Sobolev norms are invariant by the action of the group which is linear (in his case the coadjoint representation of G in \mathfrak{g}). Since the action is linear we can really use projections to decompose the Hilbert space into invariant spaces.

In our case, we have a Hamiltonian action which is semisimple of compact type. Since the action is of semisimple type and the support is compact there exists compact Lie group action integrating the Lie algebra action. Thanks to Bochner's theorem [3] in the local case and to Mostow-Palais theorem ([26], [27]) in the global case and semilocal ¹ case, we can assume that this action is linear (by using an appropriate G -equivariant embedding). By virtue of these result we can even assume $G \subset O(N)$ where N is the dimension of the ambient space.

By using an orthonormal basis in the vector space E for this action we can define the corresponding Sobolev norms in the ambient spaces given by Mostow-Palais.

¹the semilocal case for compact invariant N can be easily inferred from the proof given by Mostow in his detailed paper [26]. Indeed, the theorem of Mostow-Palais is valid in full generality for separable metric spaces and his proof starts by constructing equivariant embeddings of neighbourhood of orbits. Our invariant compact manifold N is just a union of orbits. So by compactness of N we can find a finite covering from a given covering of this "basic" Mostow-orbit neighbourhoods to find the equivariant embedding.

$$(5.2) \quad \langle F_1, F_2 \rangle_{k,r}^H := \int_{B_r} \sum_{|\alpha| \leq k} \left(\frac{|\alpha|!}{\alpha!} \right) \left(\frac{\partial^{|\alpha|} F_1}{\partial z^\alpha}(z) \right) \left(\frac{\partial^{|\alpha|} F_2}{\partial z^\alpha}(z) \right) d\mu(z),$$

Remark 5.3. In the global case we consider the corresponding Sobolev norms by integrating on the manifold:

$$(5.3) \quad \langle F_1, F_2 \rangle_{k,r}^H := \int_{B_r} \sum_{|\alpha| \leq k} \left(\frac{|\alpha|!}{\alpha!} \right) \left(\frac{\partial^{|\alpha|} F_1}{\partial z^\alpha}(z) \right) \left(\frac{\partial^{|\alpha|} F_2}{\partial z^\alpha}(z) \right) d\mu(z),$$

Remark 5.4. In the semilocal case we consider the corresponding Sobolev norms by integrating on closed tubular neighbourhoods:

$$(5.4) \quad \langle F_1, F_2 \rangle_{k,r}^H := \int_{N_r} \sum_{|\alpha| \leq k} \left(\frac{|\alpha|!}{\alpha!} \right) \left(\frac{\partial^{|\alpha|} F_1}{\partial z^\alpha}(z) \right) \left(\frac{\partial^{|\alpha|} F_2}{\partial z^\alpha}(z) \right) d\mu(z),$$

From now on, we just check all the assertions for the local norms for the sake of simplicity. The semilocal and global case can be treated in the same way.

Since these Sobolev norms are expressed in an orthonormal basis for the linear action. The norm $H_{1,r} = \langle F_1, F_2 \rangle_{k,r}^H := \int_{B_r} f(z)g(z)d\mu(z)$ is invariant by G (because of standard change of variable $\bar{z} = \rho_g(z)$ and the fact that the modulus of the determinant of this change is 1).

Now the chain rule yields invariance of the other norms by the linear group action.

For the sake of simplicity, we just include here explicit propositions and theorems only in the local case. Of course, all the estimates hold in the global and semilocal the same results hold by replacing the $C^{k,R}$ norms by the C^k norms in the global case and considering the ρ -neighbourhood topology in the tubular neighbourhood of N as we spelled out in the preceding section.

We now proceed to study the regularity properties of the homotopy operators with respect to these Sobolev norms and then deduce regularity properties of the initial norms by looking at the real part.

From now on we will closely follow notation and results contained in [10].

The set $\mathcal{C}_C^\infty(B_r)$ can be considered naturally as a \mathfrak{g} -module after considering the representation defined on an orthonormal basis ξ_i of \mathfrak{g} by $\rho_{\xi_i}(h) := \{f_i, h\}, \forall i$. To this representation we can associate a Chevalley-Eilenberg complex and define the differential operator δ as we did before.

We will need the following result contained in [10]. This result was used with respect to a representation ρ associated to the linearization problem considered by Conn.

Conn proves in [10] a similar result when the action considered is given by $\rho(\xi_i)(f) = \sum c_{ij}^k x_k \frac{\partial f}{\partial x_j}$.

In our case, the representation is associated to a momentum map and is linear after applying the Mostow-Palais embedding. Therefore Conn's Proposition 2.1 in page 576 in [10] still holds true in our case:

Lemma 5.5. *In the Chevalley-Eilenberg complex associated to ρ :*

$$\mathcal{C}_C^\infty(M) \xrightarrow{\delta} C^1(\mathfrak{g}, \mathcal{C}_C^\infty(M)) \xrightarrow{\delta} C^2(\mathfrak{g}, \mathcal{C}_C^\infty(M))$$

there exists a chain of homotopy operators

$$\mathcal{C}_C^\infty(M) \xleftarrow{h_0} C^1(\mathfrak{g}, \mathcal{C}_C^\infty(M)) \xleftarrow{h_1} C^2(\mathfrak{g}, \mathcal{C}_C^\infty(M))$$

such that

$$\delta_0 \circ h_0 + h_1 \circ \delta_1 = id_{C^1(\mathfrak{g}, \mathcal{C}_C^\infty(M))}$$

and

$$\delta_1 \circ h_1 + h_2 \circ \delta_2 = id_{C^1(\mathfrak{g}, \mathcal{C}_C^\infty(M))}$$

Moreover, there exists a real constant $C > 0$ which is independent of the radius R of M such that

$$\|h_j(S)\|_{H_s(M)} \leq C\|S\|_{H_s(M)}, \quad j = 0, 1, 2$$

for all $S \in C^{j+1}(\mathfrak{g}, \mathcal{C}_C^\infty(M))$ and $s \geq 0$. These mappings h_j are real operators.

Since the operators h_j are real operators we can use this bound to obtain bounds for the initial representation on $\mathcal{C}_C^\infty(B_r)$ instead of $C^\infty(B_r)$. In [10] the bounds in the lemma before are reinterpreted for the $\|\cdot\|_{s,k}$ using the Sobolev lemma to obtain bounds which control the loss of differentiability in the norms $\|\cdot\|_{k,s}$.

We recall the arguments contained in [10]; Denote by μ_R the homothety of scale R .

Because of the definition of Sobolev norms, the following inequality holds ($n = \dim M$):

$$R^{n/2}\|f \circ \mu_R\|_{H_s(M)} \leq \|f\|_{H_s(M)} \leq R^{n/2}B^{1/2}(n+1)^s\|f\|_{s,R}, \quad \forall s \geq 0.$$

Furthermore, A weak version of Sobolev lemma implies that there exists a constant $S > 0$ such that $\forall l \geq 0$ and $s \geq l + [n/2] + 1$ and $0 < R \leq 1$ the following inequality holds:

$$\|f\|_{s,R} \leq R^{-n/2}S\|f\|_{H_s(M)}.$$

Combining those two inequalities with the lemma above we obtain.

Lemma 5.6. *In the Chevalley-Eilenberg complex associated to ρ :*

$$C^\infty(M) \xrightarrow{\delta} C^1(\mathfrak{g}, C^\infty(M)) \xrightarrow{\delta} C^2(\mathfrak{g}, C^\infty(M))$$

there exists a chain of homotopy operators

$$C^\infty(M) \xleftarrow{h_0} C^1(\mathfrak{g}, C^\infty(M)) \xleftarrow{h_1} C^2(\mathfrak{g}, C^\infty(M))$$

such that

$$\delta_0 \circ h_0 + h_1 \circ \delta_1 = id_{C^1(\mathfrak{g}, C^\infty(M))}$$

and

$$\delta_1 \circ h_1 + h_2 \circ \delta_2 = id_{C^1(\mathfrak{g}, C^\infty(B_r))}$$

Moreover, for each k , there exists a real constant $C_k > 0$ which is independent of the radius R of M such that

$$\|h_j(S)\|_{k,R} \leq C_k \|S\|_{k+s,R}, \quad j = 0, 1, 2$$

for all $S \in C^{j+1}(\mathfrak{g}, C^\infty(B_r))$

Remark 5.7. This lemma gives bounds on the norms of the homotopy operators. However, the combination of estimates coming from the definition and a weak version of Sobolev lemma do not guarantee regularity. That is why in each step of the iterative process in the proof of the main theorem of this paper we have a loss of differentiability. In order to overcome this loss of differentiability one needs to use smoothing operators as defined by Hamilton in [19]. These smoothing operators were used by Conn in his proof of normal forms of Poisson structures and we will also use them in the proof of theorem 7.7 that is needed to conclude the rigidity result.

Remark 5.8. Instead of using the cohomology associated of the Lie algebra representation, we could try to use the group cohomology (since the semisimple Lie algebra of compact type integrates to a group action) and the operators associated to it. One might guess that via the Chevalley-Eilenberg complex associated to the group representation, rather than the Lie algebra representation, we could guarantee a regularity result and therefore skip the hard techniques required from geometrical analysis.

In fact, we can use Hodge decomposition to try to write explicit formulas for the homotopy operators. Then the problem amounts to finding explicit formulas for the Green operators associated to the harmonic forms. However, we have not been able to obtain those explicit formulas that guarantee the necessary regularity.

6. PROOF OF THE MAIN THEOREMS

In this section we prove Theorems 4.1, 4.5 and 4.6, modulo a Nash-Moser type normal form theorem which will be proved in Appendix 7.

As we mentioned in section 5, infinitesimal rigidity suggests that the “tangent space” to the orbit of the action defined in 5.1 and the tangent space to the space of G -actions on the group of Poisson diffeomorphisms coincide.

If those manifolds were smooth or tame Fréchet we would be able to apply the inverse function theorem or Nash-Moser theorem to find the conjugating diffeomorphism. However, a priori, our sets are not known to be tame by any of the criteria proposed by Hamilton in [19]. The plan B would be to attack the problem

by reproducing an iterative fast converging method used by Hamilton in his proof of the Nash-Moser theorem.

6.1. Idea of the proof. As we will see later, the proof of the results consists of applying a general and abstract normal form theorem given in Appendix 1.

In fact, the proof of this normal form theorem, and then the proof of our results, is just an iterative process inspired by Newton's fast convergence method.

Let λ and μ be two close momentum maps. The idea is to construct a sequence of momentum maps $(\mu_d)_{d \geq 0}$ defined on closed balls B_{r_d} ($(r_d)_{d \geq 0}$ is an appropriate decreasing sequence of positive numbers which has a strictly positive limit) which are equivalent, with $\mu_0 = \mu$ and such that μ_d tends to λ when d tends to $+\infty$.

Let us explain how to construct μ_{d+1} from μ_d :

- We define the following 1-cochain $f_d : \mathfrak{g} \rightarrow C^\infty(B_{r_d})$ for the Chevalley-Eilenberg complex associated to the action of \mathfrak{g} on $C^\infty(B_{r_d})$ defined in the previous section :

$$(6.1) \quad f_d(\xi) = \xi \circ (\mu_d - \lambda) = \xi \circ \mu_d - \xi \circ \lambda \quad \text{for all } \xi \in \mathfrak{g}.$$

- Even if f_d is not a 1-cocycle (i.e. $\delta f_d \neq 0$), we will apply to it the homotopy operator h introduced in Lemma 5.6. We then define the Hamiltonian vector field $X_d = X_{S_{t_d}(h(f_d))}$ associated to the smooth function $S_{t_d}(h(f_d))$ with respect to our Poisson structure, where S_{t_d} is a smoothing operator (with a well chosen t_d). Denote by

$$(6.2) \quad \varphi_d = \varphi_{X_d}^1 = Id + \chi_d$$

the time-1 flow of X_d .

- Finally, μ_{d+1} is defined as

$$(6.3) \quad \mu_{d+1} = \mu_d \circ \varphi_d.$$

We then can check that we have, grosso modo,

$$(6.4) \quad \|\mu_{d+1} - \lambda\|_{k, r_{d+1}} \leq \|\mu_d - \lambda\|_{k, r_d}^2 \quad \text{for all } k \in \mathbb{N}.$$

The reason why we use the smoothing operators is that the estimate of the homotopy operator h in Lemma 5.6 introduces a loss of differentiability.

Of course, one has to check the convergence (with respect to each C^k -norms) of the sequence of Poisson diffeomorphisms Φ_d defined by $\Phi_d = \varphi_0 \circ \dots \circ \varphi_d$, which is the hard part of this work. ²

However this is not the proof we are going to implement here even if this is the leit-motif for it (this is why this section is called *idea* of the proof). Instead of checking convergence of this sequence of Poisson diffeomorphisms we are going to evoke a more general theorem that works in other settings too. This is a Nash-moser normal form theorem that we state and prove in full detail in Appendix 1. (Appendix 2 is then devoted to technical lemmas that let us conclude that our

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We gave here the idea of the guideline of proof in the local case. The same construction still works for the semilocal and global cases. In the semilocal case, we may replace the sets $C^\infty(B_{r_d})$ by the set of functions of type $C^\infty(\overline{\mathcal{U}_{r_d}(N)})$, where we recall that $\mathcal{U}_{r_d}(N)$ stands of an r_d -closed neighbourhood of the invariant manifold N . In the global case, we work with the set of smooth functions $C^\infty(M)$ on a compact manifold M .

problem satisfies the hypothesis of this Nash-Moser normal form theorem.) Of course, sooner or later, we will need to struggle to find hard estimates to prove convergence but this will happen in Appendix 1.

Notations used in the proof below are the ones used in the statement of this Nash-Moser theorem. The reader of this proof is highly encouraged to read the statement of theorem 7.7 and notations at the beginning of Appendix 1 before facing this proof.

6.2. Proof of Theorem 4.1. The proof of this theorem is just an application of the general normal form theorem which we will state and prove in Section 7 (see Theorem 7.7). We explain here how we can adapt the formalism of section 7 to our situation. In fact, we are going to apply the affine version of the general normal form theorem 7.7, see Remark 7.12.

Recall that we have a Poisson structure $\{, \}$ in a neighbourhood U of 0 in \mathbb{R}^n , a semisimple real Lie algebra of compact type \mathfrak{g} and a momentum map $\lambda : U \rightarrow \mathfrak{g}^*$. For each positive real number, we denote by B_r the closed ball of radius r and center 0.

We first define the SCI-space \mathcal{T} by the spaces $\mathcal{T}_{k,r}$ of C^k -differentiable maps from the balls B_r to \mathfrak{g}^* equipped with the norms $\| \cdot \|_{k,r}$ defined above. The SCI-space \mathcal{S} is given by the momentum maps with respect to the Poisson structure. Of course, we choose λ as the origin (f_0 in the formulation of the theorem) of the affine space and $\mathcal{F} = \mathcal{N} = \{0\}$.

The SCI-group \mathcal{G} consists of the local C^k -diffeomorphisms on the balls B_r and the action is the classical right action : $\phi \cdot \mu := \mu \circ \phi$ with $\phi \in \mathcal{G}$ and $\mu \in \mathcal{T}$. One can check the axioms of SCI-action looking at [10], [24] and also in Appendix 2. The closed subset \mathcal{G}^0 of \mathcal{G} is given by the Poisson diffeomorphisms (i.e. diffeomorphisms preserving the Poisson structure). It is clear that the elements of \mathcal{G}^0 preserves \mathcal{S} .

We define the SCI-space \mathcal{H} by the spaces $\mathcal{H}_{k,r}$ of C^k -differentiable functions on the balls B_r . The application $H : \mathcal{S} \rightarrow \mathcal{H}$ is defined as follows. A C^k -differentiable map $\mu : B_r \rightarrow \mathfrak{g}^*$ can be obviously viewed as a 1-cochain in the Chevalley-Eilenberg complex defined above. The image of μ by H is then just $h_0(\mu - \lambda)$ where h_0 is the homotopy operator given in Lemma 5.6. The relation (7.16) is then obvious. Finally, for every element g of \mathcal{H} , we denote by X_g the Hamiltonian vector field associated to g with respect to the Poisson structure (i.e. $X_g = \{g, \}$). We then define $\Phi(g) = \phi_{X_g}^1$ the time-1 flow of the vector field X_g . Of course, by definition, the diffeomorphisms $\Phi(g)$ preserves the Poisson structure and the SCI-set of momentum maps \mathcal{S} . Now, we just have to check that the estimates (7.17), (7.18) and (7.19) are satisfied. These three estimates are direct consequences of the lemmas 8.4, 8.4 and 8.5 given in the Appendix 2. The affine version of Theorem 7.7 then gives the result.

6.3. Proof of Theorems 4.5 and 4.6. In the semilocal case (neighbourhood of invariant compact submanifold N), we can use exactly the same procedure replacing balls of radius ρ by a ρ -neighbourhood of the compact invariant submanifold N in the definition of the norms implied. All the technical lemmas 8.4, 8.4 and 8.5 given in the Appendix 2 work. A quick way to see that is using normal coordinates via the exponential map and arguing in the same way as in the local case (by considering

balls in the normal fibers to the submanifold N). In particular this proves that the estimates in (7.17), (7.18) and (7.19) are satisfied for these norms.

Now we can apply exactly the same scheme of proof that we did for the local case to apply Theorem 7.7 in Section 7.

For the global compact case: Indeed this case is easier because we could approach the initial proof using the iteration. The loss of differentiability is easy to control in this case since the radius of the ball does not shrink in each step of the iteration.

Alternatively we could also use the SCI theorem for these we need to check that estimates are also satisfied with these norms. Out of compassion for the reader, we just give the general idea of how to do this: the norms we use are defined on the compact manifold M but can be easily related to the norms $\|\cdot\|_{k,s}$ on each ball via an adequate partition of unity (U_i, ϕ_i) of the manifold M . Therefore, since we have the estimates for the norms $\|\cdot\|_{k,s}$ we also have the estimates in (7.17), (7.18) and (7.19) for the norms defined on M . We now reproduce exactly the same proof as we did in the local case and we apply Theorem 7.7.

7. APPENDIX 1 : A NORMAL FORM THEOREM

7.1. The setting. Grosso modo, the situation is as follows: we have a group \mathcal{G} (say of diffeomorphisms) which acts on a set \mathcal{S} (of structures). Inside \mathcal{S} there is a subset \mathcal{N} (of structures in normal form). We want to show that, under some appropriate conditions, each structure can be put into normal form, i.e. for each element $f \in \mathcal{S}$ there is an element $\phi \in \mathcal{G}$ such that $\phi.f \in \mathcal{N}$. We will assume that \mathcal{S} is a subset of a linear space \mathcal{T} (a space of tensors) on which \mathcal{G} acts, and \mathcal{N} is the intersection of \mathcal{S} with a linear subspace \mathcal{F} of \mathcal{T} . To formalize the situation involving smooth *local* structures (defined in a neighborhood of something), let us introduce the following notions of *SCI-spaces* and *SCI-groups*. Here SCI stands for *scaled C^∞ type*. Our aim here is not to create a very general setting, but just a setting which works and which can hopefully be adjusted to various situations. So our definitions below (especially the inequalities appearing in them) are probably not “optimal”, and can be improved, relaxed, etc.

SCI-spaces. An *SCI-space* \mathcal{H} is a collection of Banach spaces $(\mathcal{H}_{k,\rho}, \|\cdot\|_{k,\rho})$ with $0 < \rho \leq 1$ and $k \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ (ρ is called the *radius* parameter, k is called the *smoothness parameter*; we say that $f \in \mathcal{H}$ if $f \in \mathcal{H}_{k,\rho}$ for some k and ρ , and in that case we say that f is k -smooth and defined in radius ρ) which satisfies the following properties:

- If $k < k'$, then for any $0 < \rho \leq 1$, $\mathcal{H}_{k',\rho}$ is a linear subspace of $\mathcal{H}_{k,\rho}$: $\mathcal{H}_{k',\rho} \subset \mathcal{H}_{k,\rho}$.
- If $0 < \rho' < \rho \leq 1$, then for each $k \in \mathbb{Z}_+$, there is a given linear map, called the *projection map*, or *radius restriction map*,

$$\pi_{\rho,\rho'} : \mathcal{H}_{k,\rho} \rightarrow \mathcal{H}_{k,\rho'}.$$

These projections don't depend on k and satisfy the natural commutativity condition $\pi_{\rho,\rho''} = \pi_{\rho,\rho'} \circ \pi_{\rho',\rho''}$. If $f \in \mathcal{H}_{k,\rho}$ and $\rho' < \rho$, then by abuse of language we will still denote by f its projection to $\mathcal{H}_{k,\rho'}$ (when this notation does not lead to confusions).

- For any f in \mathcal{H} we have

$$(7.1) \quad \|f\|_{k,\rho} \geq \|f\|_{k',\rho'} \quad \forall k \geq k', \rho \geq \rho'.$$

In the above inequality, if f is not in $\mathcal{H}_{k,\rho}$ then we put $\|f\|_{k,\rho} = +\infty$, and if f is in $\mathcal{H}_{k,\rho}$ then the right hand side means the norm of the projection of f to $\mathcal{H}_{k',\rho'}$, of course.

- There is a smoothing operator for each ρ , which depends continuously on ρ . More precisely, for each $0 < \rho \leq 1$ and each $t > 1$ there is a linear map, called the *smoothing operator*,

$$(7.2) \quad S_\rho(t) : \mathcal{H}_{0,\rho} \longrightarrow \mathcal{H}_{\infty,\rho} = \bigcap_{k=0}^{\infty} \mathcal{H}_{k,\rho}$$

which satisfies the following inequalities: for any $p, q \in \mathbb{Z}_+$, $p \geq q$ we have

$$(7.3) \quad \|S_\rho(t)f\|_{p,\rho} \leq C_{\rho,p,q} t^{p-q} \|f\|_{q,\rho}$$

$$(7.4) \quad \|f - S_\rho(t)f\|_{q,\rho} \leq C_{\rho,p,q} t^{q-p} \|f\|_{p,\rho}$$

where $C_{\rho,p,q}$ is a positive constant (which does not depend on f nor on t) and which is continuous with respect to ρ .

In the same way as for the Fréchet spaces (see for instance [28]), the two properties (7.3) and (7.4) of the smoothing operator imply the following inequality called *interpolation inequality*: for any positive integers p, q and r with $p \geq q \geq r$ we have

$$(7.5) \quad (\|f\|_{q,\rho})^{p-r} \leq C_{p,q,r} (\|f\|_{r,\rho})^{p-q} (\|f\|_{p,\rho})^{q-r},$$

where $C_{p,q,r}$ is a positive constant which is continuous with respect to ρ and does not depend on f .

Example 7.1. The main example that we have in mind is the space of functions in a neighbourhood of 0 in the Euclidean space \mathbb{R}^n : here ρ is the radius and k is the smoothness class, i.e. $\mathcal{H}_{k,\rho}$ is the space of C^k -functions on the closed ball of radius ρ and centered at 0 in \mathbb{R}^n , together with the maximal norm (of each function and its partial derivatives up to order k); the projections are restrictions of functions to balls of smaller radii. In the same way, others basic examples are given by the differential forms or multivectors defined in a neighbourhood of the origin in \mathbb{R}^n .

By an *SCI-subspace* of an SCI-space \mathcal{H} , we mean a collection \mathcal{V} of subspaces $\mathcal{V}_{k,\rho}$ of $\mathcal{H}_{k,\rho}$, which themselves form an SCI-space (under the induced norms, induced smoothing operators, induced inclusion and radius restriction operators from \mathcal{H} - it is understood that these structural operators preserve \mathcal{V}).

By a *subset* of an SCI-space \mathcal{H} , we mean a collection \mathcal{F} of subsets $\mathcal{F}_{k,\rho}$ of $\mathcal{H}_{k,\rho}$, which are invariant under the inclusion and radius restriction maps of \mathcal{H} .

Remark 7.2. Of course, if \mathcal{H} is an SCI-space then each $\mathcal{H}_{\infty,\rho}$ is a tame Fréchet space.

The above notion of SCI-spaces generalizes at the same time the notion of tame Fréchet spaces and the notion of scales of Banach spaces [32]. Evidently, the scale parameter is introduced to treat local problems. When things are globally defined (say on a compact manifold), then the scale parameter is not needed, i.e. $\mathcal{H}_{k,\rho}$ does

not depend on ρ and we get back to the situation of tame Fréchet spaces, as studied by Sergeraert [28] and Hamilton [18, 19].

SCI-groups. An *SCI-group* \mathcal{G} consists of elements ϕ which are written as a (formal) sum

$$(7.6) \quad \phi = Id + \chi,$$

where χ belongs to an SCI-space \mathcal{W} , together with *scaled group laws* to be made more precise below. We will say that \mathcal{G} is modelled on \mathcal{W} , if $\chi \in \mathcal{W}_{k,\rho}$ then we say that $\phi = Id + \chi \in \mathcal{G}_{k,\rho}$ and $\chi = \phi - Id$ (so as a space, \mathcal{G} is the same as \mathcal{W} , but shifted by Id), $Id = Id + 0$ is the neutral element of \mathcal{G} .

Scaled composition (product) law. There is a positive constant c (which does not depend on ρ or k) such that if $0 < \rho' < \rho \leq 1$, $k \geq 1$, and $\phi = Id + \chi \in \mathcal{G}_{k,\rho}$ and $\psi = Id + \xi \in \mathcal{G}_{k,\rho}$ such that

$$(7.7) \quad \rho'/\rho \leq 1 - c\|\xi\|_{1,\rho}$$

then we can compose ϕ and ψ to get an element $\phi \circ \psi$ with $\|\phi \circ \psi - Id\|_{k,\rho'} < \infty$, i.e. $\phi \circ \psi$ can be considered as an element of $\mathcal{G}_{k,\rho'}$ (if $\rho'' < \rho'$ then of course $\phi \circ \psi$ can also be considered as an element of $\mathcal{G}_{k,\rho''}$, by the restriction of radius from ρ' to ρ''). Of course, we require the composition to be *associative* (after appropriate restrictions of radii).

Scaled inversion law. There is a positive constant c (for simplicity, take it to be the same constant as in Inequality (7.7)) such that if $\phi \in \mathcal{G}_{k,\rho}$ such that

$$(7.8) \quad \|\phi - Id\|_{1,\rho} < 1/c$$

then we can define an element, denoted by ϕ^{-1} and called the inversion of ϕ , in $\mathcal{G}_{k,\rho'}$, where $\rho' = (1 - \frac{1}{2}c\|\phi - Id\|_{1,\rho})\rho$, which satisfies the following condition: the compositions $\phi \circ \phi^{-1}$ and $\phi^{-1} \circ \phi$ are well-defined in radius $\rho'' = (1 - c\|\phi - Id\|_{1,\rho})\rho$ and coincide with the neutral element Id there.

Continuity conditions. We require that the above scaled group laws satisfy the following continuity conditions i), ii) and iii) in order for \mathcal{G} to be called an SCI-group.

i) For each $k \geq 1$ there is a polynomial $P = P_k$ (of one variable), such that for any $\chi \in \mathcal{W}_{2k-1,\rho}$ with $\|\chi\|_{1,\rho} < 1/c$ we have

$$(7.9) \quad \|(Id + \chi)^{-1} - Id\|_{k,\rho'} \leq \|\chi\|_{k,\rho} P(\|\chi\|_{k,\rho}),$$

where $\rho' = (1 - c\|\chi\|_{1,\rho})\rho$.

ii) If $(\phi_m)_{m \geq 0}$ is a sequence in $\mathcal{G}_{k,\rho}$ which converges (with respect to $\|\cdot\|_{k,\rho}$) to ϕ , then the sequence $(\phi_m^{-1})_{m \geq 0}$ also converges to ϕ^{-1} in $\mathcal{G}_{k,\rho'}$, where $\rho' = (1 - c\|\phi - Id\|_{1,\rho})\rho$.

iii) For each $k \geq 1$ there are polynomials P , Q , R and T (of one variable) such that if $\phi = Id + \chi$ and $\psi = Id + \xi$ are in $\mathcal{G}_{k,\rho}$ and if ρ' and ρ satisfy Relation (7.7), then we have the two inequalities

$$(7.10) \quad \|\phi \circ \psi - \phi\|_{k,\rho'} \leq \|\xi\|_{k,\rho} P(\|\xi\|_{k,\rho}) + \|\chi\|_{k+1,\rho} \|\xi\|_{k,\rho} Q(\|\xi\|_{k,\rho}).$$

and

$$(7.11) \quad \|\phi \circ \psi - Id\|_{k,\rho'} \leq \|\xi\|_{k,\rho} R(\|\xi\|_{k,\rho}) + \|\chi\|_{k,\rho} (1 + \|\xi\|_{k,\rho} T(\|\xi\|_{k,\rho})).$$

Example 7.3. The main example of a SCI-group is given by the local differentiable diffeomorphisms in a neighbourhood of 0 in \mathbb{R}^n . If we have in mind this example, the relation (7.10) is just a consequence of the mean value theorem. These estimates are proved in this case for instance in [10] or [24].

SCI-actions. We will say that there is a *linear left SCI-action* of an SCI-group \mathcal{G} on an SCI-space \mathcal{H} if there is a positive integer γ (and a positive constant c) such that, for each $\phi = Id + \chi \in \mathcal{G}_{k,\rho}$ and $f \in \mathcal{H}_{k,\rho'}$ with $\rho' = (1 - c\|\chi\|_{1,\rho})\rho$, the element $\phi \cdot f$ (the image of the action of ϕ on f) is well-defined in $\mathcal{H}_{k,\rho'}$, the usual axioms of a left group action modulo appropriate restrictions of radii (so we have *scaled action laws*) are satisfied, and the following inequalities expressing some continuity conditions are also satisfied:

i) For each k there are polynomials Q and R (which depend on k) such that

$$(7.12) \quad \|(Id + \chi) \cdot f\|_{2k-1,\rho'} \leq \|f\|_{2k-1,\rho} (1 + \|\chi\|_{k+\gamma,\rho} Q(\|\chi\|_{k+\gamma,\rho})) \\ + \|\chi\|_{2k-1+\gamma,\rho} \|f\|_{k,\rho} R(\|\chi\|_{k+\gamma,\rho})$$

ii) There is a polynomial function T of 2 variables such that

$$(7.13) \quad \|(\phi + \chi) \cdot f - \phi \cdot f\|_{k,\rho'} \leq \|\chi\|_{k+\gamma,\rho} \|f\|_{k+\gamma,\rho} T(\|\phi - Id\|_{k+\gamma,\rho}, \|\chi\|_{k+\gamma,\rho})$$

In the above inequalities, ρ' is related to ρ by a formula of the type $\rho' = (1 - c(\|\chi\|_{1,\rho} + \|\phi - Id\|_{1,\rho}))\rho$. ($\phi = Id$ in the first two inequalities).

Note that a consequence of the property i) is the following inequality, where P is a polynomial function depending on k :

$$(7.14) \quad \|(Id + \chi) \cdot f\|_{k,\rho'} \leq \|f\|_{k,\rho} (1 + \|\chi\|_{k+\gamma,\rho} P(\|\chi\|_{k+\gamma,\rho})) .$$

Remark 7.4. Of course, we can define in the same way the notion of *linear right SCI-action*.

Example 7.5. The main examples of a SCI-action that we have in mind is the action of the SCI-group of local diffeomorphisms of $(\mathbb{R}^n, 0)$ on the SCI-space of local tensors of a given type on $(\mathbb{R}^n, 0)$.

If the tensors are for instance k -vectors fields, like in [10] and [24], we have a left SCI-action by push-forward. If the tensors are for instance smooth maps, like in this paper, or differential forms, we get a right SCI-action.

7.2. Normal form theorem. Roughly speaking, the following theorem says that whenever we have a “fast normalizing algorithm” in an SCI setting then it will lead to the existence of a smooth normalization. “Fast” means that, putting loss of differentiability aside, one can “quadraturize” the error term at each step (going from “ ε -small” error to “ ε^2 -small” error).

Remark 7.6. In this technical part, we are going to adopt the following notations in order to simplify the writing of equations.

- The notation $Poly(\|f\|_{k,r})$ denotes a polynomial term in $\|f\|_{k,r}$ where the polynomial has positive coefficients and does not depend on f (it may depend on k and on r continuously).
- The notation $Poly_{(p)}(\|f\|_{k,r})$, where p is a strictly positive integer, denotes a polynomial term in $\|f\|_{k,r}$ where the polynomial has positive coefficients

and does not depend on f (it may depend on k and on r continuously) and which contains terms of degree greater or equal to p .

Theorem 7.7. *Let \mathcal{T} be a SCI-space, \mathcal{F} a SCI-subspace of \mathcal{T} , and \mathcal{S} a subset of \mathcal{T} , $\mathcal{F} \ni 0$. Denote $\mathcal{N} = \mathcal{F} \cap \mathcal{S}$. Assume that there is a projection $\pi : \mathcal{T} \rightarrow \mathcal{F}$ (compatible with restriction and inclusion maps) such that for every f in $\mathcal{T}_{k,\rho}$, the element $\zeta(f) = f - \pi(f)$ satisfies*

$$(7.15) \quad \|\zeta(f)\|_{k,\rho} \leq \|f\|_{k,\rho} \text{Poly}(\|f\|_{[(k+1)/2],\rho})$$

for all $k \in \mathbb{N}$ (or at least for all k sufficiently large), where $[\]$ is the integer part.

Let \mathcal{G} be an SCI-group acting on \mathcal{T} by a linear left SCI-action and let \mathcal{G}^0 be a closed subset of \mathcal{G} formed by elements preserving \mathcal{S} .

Let \mathcal{H} be a SCI-space and assume that there exist maps $\mathbf{H} : \mathcal{S} \rightarrow \mathcal{H}$ and $\Phi : \mathcal{H} \rightarrow \mathcal{G}^0$ and an integer $s \in \mathbb{N}$ such that for every $0 < \rho \leq 1$, every f in \mathcal{S} and g in \mathcal{H} , and for all k in \mathbb{N} (or at least for all k sufficiently large) we have the three properties :

$$(7.16) \quad \begin{aligned} \|\mathbf{H}(f)\|_{k,\rho} &\leq \|\zeta(f)\|_{k+s,\rho} \text{Poly}(\|f\|_{[(k+1)/2]+s,\rho}) \\ &\quad + \|f\|_{k+s,\rho} \|\zeta(f)\|_{[(k+1)/2]+s,\rho} \text{Poly}(\|f\|_{[(k+1)/2]+s,\rho}), \end{aligned}$$

$$(7.17) \quad \|\Phi(g) - Id\|_{k,\rho'} \leq \|g\|_{k+s,\rho} \text{Poly}(\|g\|_{[(k+1)/2]+s,\rho})$$

and

$$(7.18) \quad \begin{aligned} \|\Phi(g_1) \cdot f - \Phi(g_2) \cdot f\|_{k,\rho'} &\leq \|g_1 - g_2\|_{k+s,\rho} \|f\|_{k+s,\rho} \text{Poly}(\|g_1\|_{k+s,\rho}, \|g_2\|_{k+s,\rho}) \\ &\quad + \|f\|_{k+s,\rho} \text{Poly}_{(2)}(\|g_1\|_{k+s,\rho}, \|g_2\|_{k+s,\rho}) \end{aligned}$$

if $\rho' \leq \rho(1 - c\|g\|_{2,\rho})$ in (7.17) and $\rho' \leq \rho(1 - c\|g_1\|_{2,\rho})$ and $\rho' \leq \rho(1 - c\|g_2\|_{2,\rho})$ in (7.18).

Finally, for every f in \mathcal{S} denote $\phi_f = Id + \chi_f = \Phi(\mathbf{H}(f)) \in \mathcal{G}^0$ and assume that there is a positive real number δ such that we have the inequality

$$(7.19) \quad \|\zeta(\phi_f \cdot f)\|_{k,\rho'} \leq \|\zeta(f)\|_{k+s,\rho}^{1+\delta} Q(\|f\|_{k+s,\rho}, \|\chi_f\|_{k+s,\rho}, \|\zeta(f)\|_{k+s,\rho}, \|f\|_{k,\rho})$$

(if $\rho' \leq \rho(1 - c\|\chi_f\|_{1,\rho})$) where Q is a polynomial of four variables and whose degree in the first variable does not depend on k and with positive coefficients.

Then there exist $l \in \mathbb{N}$ and two positive constants α and β with the following property: for all $p \in \mathbb{N} \cup \{\infty\}$, $p \geq l$, and for all $f \in \mathcal{S}_{2p-1,R}$ with $\|f\|_{2l-1,R} < \alpha$ and $\|\zeta(f)\|_{l,R} < \beta$, there exists $\psi \in \mathcal{G}_{p,R/2}^0$ such that $\psi \cdot f \in \mathcal{N}_{p,R/2}$.

Remark 7.8. Of course, this theorem still works if we have a linear *right* SCI-action and in facts, as we will see later, the proof in this case is a bit easier.

Remark 7.9. It is necessary to try to explain the role that play all the SCI-spaces of this theorem.

- \mathcal{T} is the space of "tensors" (for instance 2-vectors, smooth maps, differential forms, etc.).

- \mathcal{F} is the subspace of normal forms in \mathcal{T} (for instance *linear* 2-vectors, etc.).
- \mathcal{S} is the set of structures (like Poisson structures, momentum maps, etc.).
- \mathcal{N} is the set of normal forms in \mathcal{S} (like linear Poisson structures, etc.).
- \mathcal{G} represents the group of local diffeomorphisms acting on \mathcal{T} .

Actually, \mathcal{G}^0 does not need to be a subgroup of \mathcal{G} but it is important that it is closed.

Finally, the *raison d'être* of the SCI-space \mathcal{H} is purely technical. Indeed, the estimates given in the definition of the SCI-actions and in the hypothesis of the theorem make appear a loss of differentiability. A classical idea to compensate this loss of differentiability is to use the smoothing operators. But it is not always possible to apply these smoothing operators directly in the SCI-group ; that is why we apply them in the intermediary SCI-space \mathcal{H} .

Remark 7.10. We can illustrate this theorem with the basic example of linearization of smooth Poisson structures proved by J. Conn in [10]. In this case, \mathcal{T} is the SCI-space of bivectors fields, \mathcal{F} the subspace of linear bivectors, \mathcal{S} the subset of Poisson structures, \mathcal{N} the subset of linear Poisson structures. The group \mathcal{G} is the group of local diffeomorphisms (and $\mathcal{G}^0 = \mathcal{G}$) ; the action is given by the push-forward. The SCI-space \mathcal{H} is given by the smooth vector fields and the map $H : \mathcal{S} \rightarrow \mathcal{H}$ is defined by $H(\pi) = h(\pi - \pi^{(1)})$ where h is the homotopy operator defined by Conn and $\pi^{(1)}$ is the linear part of π . Finally, the map $\Phi : \mathcal{H} \rightarrow \mathcal{G}$ is defined by $\Phi(X) = Id + X$.

Remark 7.11. The estimates of the theorem come from trying generalizing some concrete examples of normal forms (linearization of Poisson structures in [10], Levi decomposition of Poisson structures in [24] and the rigidity of momentum maps in this paper), that is why they look artificially complicated. There must be a clever way to present this theorem, we have looked for it but we didn't find.

The small shift $+s$ in the estimates of the theorem is needed to compensate the loss of differentiability that appears initially in the paper of J. Conn [10] when he constructs the homotopy operator. In the same way, the shift $+\gamma$ in the axiom of SCI-actions may appear when writing explicitly the estimates of particular SCI-actions (see [24]).

In some estimates of this theorem and also in the axiom (7.12) we can notice the presence at the same time of the smoothness degrees k and $[(k+1)/2]$ (or k and $2k-1$). We will see later that to show the $\|\cdot\|_k$ -convergence of the sequence of "diffeomorphisms" (Ψ_d) , we need to control their $(2k-1)$ -differentiability too.

Remark 7.12. Affine version of Theorem 7.7.

Stated in that way, Theorem 7.7 cannot be applied directly to our situation of rigidity of momentum maps. In fact, we can state a kind of *affine* version of this theorem that we will be able to apply. The formulation is very close to the original one. The notations are the same but we pick here an element f_0 in \mathcal{S} ($\subset \mathcal{T}$) that will be considered as the *origin* in \mathcal{T} . In particular, we define now the affine norms in \mathcal{T} by :

$$(7.20) \quad \|f\|_{k,\rho}^{Aff} := \|f - f_0\|_{k,\rho} \quad \text{if } f \in \mathcal{T}.$$

Now, the formulation of the affine version of Theorem 7.7 is exactly the same. We just have to replace in (7.15), (7.16) and (7.19) the norm $\|\cdot\|_{k,\rho}$ by the affine norm $\|\cdot\|_{k,\rho}^{Aff}$ for all the elements in \mathcal{T} (but not for the elements of \mathcal{G} or \mathcal{H} !).

The conclusion is then : *There exist $l \in \mathbb{N}$ and two positive constants α and β with the following property: for all $p \in \mathbb{N} \cup \{\infty\}$, $p \geq l$, and for all $f \in \mathcal{S}_{2p-1,R}$ with $\|f\|_{2l-1,R}^{Aff} < \alpha$ and $\|\zeta(f)\|_{l,R}^{Aff} < \beta$, there exists $\psi \in \mathcal{G}_{p,R/2}^0$ such that $\psi \cdot f \in \mathcal{N}_{p,R/2}$.*

7.3. Proof of Theorem 7.7. We construct, by induction, a sequence $(\psi_d)_{d \geq 1}$ in \mathcal{G}^0 , and then a sequence $f^d := \psi_d \cdot f$ in \mathcal{S} , which converges to $\psi_\infty \in \mathcal{G}_{p,R/2}^0$ (since \mathcal{G}^0 is closed) and such that $f^\infty := \psi_\infty \cdot f \in \mathcal{N}_{p,R/2}$ (i.e. $\zeta(f^\infty) = 0$).

In order to simplify, we can assume that the constant s of the theorem is the same as the integer γ defined by the SCI-action of \mathcal{G} on \mathcal{H} (see (7.12), (7.13) and (7.14)). We first fix some parameters. Let $A = 8s + 5$ (actually, A just has to be strictly larger than $8s + 4$). Recall that τ and δ are introduced in the statement of Theorem 7.7. We consider a positive real number $\varepsilon < 1$ such that

$$(7.21) \quad -(1 - \varepsilon) + A\varepsilon < -\frac{4}{5}.$$

and

$$(7.22) \quad -\delta(1 - \varepsilon) < -\frac{7}{10}.$$

Finally, we fix a positive integer $l > 6s + 1$ which satisfies

$$(7.23) \quad \frac{3s + 3}{l - 1}(1 + \delta + \tau) < \varepsilon.$$

and

$$(7.24) \quad -\frac{8}{5} + A\frac{s}{l - 1} < -\frac{3}{2}.$$

The definition of the parameters A , ε , l and s by the inequalities above has a purely technical origin and will be used in the proofs of the two technical lemmas given later.

The construction of the sequences is the following : Let $t_0 > 1$ be a real constant ; this constant is still not really fixed and will be chosen according to Lemma 7.13. We then define the sequence $(t_d)_{d \geq 0}$ by $t_{d+1} := t_d^{3/2}$. We also define the sequence $r_d := (1 + \frac{1}{d+1})R/2$. This is a decreasing sequence such that $R/2 \leq r_d \leq R$ for all d . Note that we have $r_{d+1} = r_d(1 - \frac{1}{(d+2)^2})$. We will see later that, technically, in order to use the relations (7.17) and (7.18) we have to define an intermediate sequence of radii : $\rho_d := r_d(1 - \frac{1}{2(d+2)^2})$. Of course, we have $r_{d+1} \leq \rho_d \leq r_d$ for all d .

Let $p \geq l$ and f in $\mathcal{S}_{2p-1,R}$. We start with $f_0 := f \in \mathcal{S}_{2p-1,R}$. Now, assume that we have constructed $f^d \in \mathcal{S}_{2p-1,r_d}$ for $d \geq 0$. We put $\phi_{d+1} := \Phi(\mathbf{H}(f^d)) = Id + \chi^{d+1}$ and $\hat{\phi}_{d+1} := \Phi(S(t_d)\mathbf{H}(f^d)) = Id + \hat{\chi}^{d+1}$. Then, f^{d+1} is defined by

$$(7.25) \quad f^{d+1} = \hat{\phi}_{d+1} \cdot f^d.$$

Roughly speaking, the idea is that the sequence $(f^d)_{d \geq 0}$ will satisfy, grosso modo :

$$(7.26) \quad \|\zeta(f^{d+1})\|_{p,r_{d+1}} \leq \|\zeta(f^d)\|_{p,r_d}^{1+\delta}.$$

For every $d \geq 1$, we put $\psi_d = \hat{\phi}_d \circ \dots \circ \hat{\phi}_1$. We then have to show that we can choose two positive constants α and β such that if $\|f\|_{2l-1,R} \leq \alpha$ and $\|\zeta(f)\|_{l,R} \leq \beta$ then, the sequence $(\psi_d)_{d \geq 1}$ converges with respect to $\|\cdot\|_{p,R/2}$. It will follow from these two technical lemmas that we will prove later :

Lemma 7.13. *There exists a real number $t_0 > 1$ such that for any $f \in \mathcal{F}_{2p-1,r_0}$ satisfying the conditions $\|f^0\|_{2l-1,r_0} < t_0^A$, $\|\zeta(f^0)\|_{2l-1,r_0} < t_0^A$ and $\|\zeta(f^0)\|_{l,r_0} < t_0^{-1}$ then, with the construction above, we have for all $d \geq 0$,*

- (1_d) $\|\hat{\chi}^{d+1}\|_{l+s,\rho_d} < t_d^{-1/2}$
- (2_d) $\|f^d\|_{l,r_d} < C \frac{d+1}{d+2}$ where C is a positive constant
- (3_d) $\|f^d\|_{2l-1,r_d} < t_d^A$
- (4_d) $\|\zeta(f^d)\|_{2l-1,r_d} < t_d^A$
- (5_d) $\|\zeta(f^d)\|_{l,r_d} < t_d^{-1}$

Lemma 7.14. *Suppose that for an integer $k \geq l$, there exists a constant C_k and an integer $d_k \geq 0$ such that for any $d \geq d_k$ we have $\|f^d\|_{2k-1,r_d} < C_k t_d^A$, $\|\zeta(f^d)\|_{2k-1,r_d} < C_k t_d^A$, $\|f^d\|_{k,r_d} < C_k \frac{d+1}{d+2}$ and $\|\zeta(f^d)\|_{k,r_d} < C_k t_d^{-1}$. Then, there exists a positive constant C_{k+1} and an integer $d_{k+1} > d_k$ such that for any $d \geq d_{k+1}$ we have*

- (i) $\|\hat{\chi}^{d+1}\|_{k+1+s,\rho_d} < C_{k+1} t_d^{-1/2}$
- (ii) $\|f^d\|_{k+1,r_d} < C_{k+1} \frac{d+1}{d+2}$
- (iii) $\|f^d\|_{2k+1,r_d} < C_{k+1} t_d^A$
- (iv) $\|\zeta(f^d)\|_{2k+1,r_d} < C_{k+1} t_d^A$
- (v) $\|\zeta(f^d)\|_{k+1,r_d} < C_{k+1} t_d^{-1}$

End of the proof of Theorem 7.7 : We choose t_0 as in Lemma 7.13. Then we fix two positive constants α and β such that $t_0^A \geq \alpha$ and $t_0^{-1} \geq \beta$. Now, if $f \in \mathcal{F}_{2p-1,R}$ satisfies $\|f\|_{2l-1,R} \leq \alpha$ and $\|\zeta(f)\|_{l,R} \leq \beta$ then, using Lemma 7.13 and applying Lemma 7.14 repetitively, we get that for each $k \geq l$ there exists a positive integer d_k such that for all $d \geq d_k$,

$$(7.27) \quad \|\hat{\chi}^{d+1}\|_{k+s,\rho_d} < C_k t_d^{-1/2}.$$

Actually it is more convenient to prove the convergence of the sequence $(\psi_d^{-1})_{d \geq 1}$. The point ii) of the continuity conditions in the definition of SCI-group will then give the convergence of $(\psi_d)_{d \geq 1}$. Choose, for every d , a radius ρ'_d such that $\rho_{d+1} \leq \rho'_d \leq \rho_d(1 - c\|\hat{\chi}^{d+1}\|_{1,\rho_d})$. For all positive integer d , we have $\psi_d^{-1} = \hat{\phi}_1^{-1} \circ \dots \circ \hat{\phi}_d^{-1}$ and if we denote $\hat{\phi}_d^{-1} = Id + \hat{\xi}^d$, the axiom (7.9) implies

$$(7.28) \quad \|\hat{\xi}^{d+1}\|_{k+s,\rho'_d} < M_k t_d^{-1/2},$$

for all $d \geq d_p$, where M_k is a positive constant independent of d . Now, since $\sum t_d^{-1/2}$ is convergent, the inequality (7.11) gives that for each $p \geq l$ the sequence $(\|\psi_d^{-1} - Id\|_{p+s+1,\rho'_d})_{d \geq 1}$ is bounded and (7.10) gives then the $\|\cdot\|_{p+s,R/2}$ -convergence of $(\psi_d^{-1})_{d \geq 1}$ in \mathcal{G}^0 that is closed. Theorem 7.7 is then proved. \square

Remark 7.15. Note that in the case of a *right* SCI-action, it is easier because we can prove directly the convergence of the sequence $(\psi_d)_{d \geq 1}$ without working with $(\psi_d^{-1})_{d \geq 1}$.

Proof of Lemma 7.13 : We prove this lemma by induction. Note that in this proof, the letter M denotes a positive constant and P denotes a polynomial with positive real coefficients, which do not depend on d and which vary from inequality to inequality.

At the step $d = 0$ the only thing we have to verify is the point (1₀) (for the point (2₀) we just choose the constant C such that $C > 2\|f^0\|_{l,r_0}$).

We have, by definition, $\hat{\chi}^1 = \Phi(S(t_0)\mathbf{H}(f^0)) - Id$. We will see later that we can assume that $\|S(t_0)\mathbf{H}(f^0)\|_{2,r_0} < 1 - \rho_0/r_0$ which allows us to use the estimate (7.17). Moreover, using the property of the smoothing operator (7.3) with $p = q = l + 2s$ or $\lceil \frac{l+s+1}{2} \rceil + s$ we get

$$(7.29) \quad \|\hat{\chi}^1\|_{l+s,\rho_0} \leq M\|\chi^1\|_{l+s,\rho_0} \leq \|\mathbf{H}(f^0)\|_{l+2s,r_0} P(\|\mathbf{H}(f^0)\|_{\lceil (l+2s+1)/2 \rceil + s, r_0}).$$

Then, the estimate (7.16) with the relation $l > 6s + 1$, and the property (7.15) give (7.30)

$$\|\chi^1\|_{l+s,\rho_0} \leq M\|\chi^1\|_{l+s,\rho_0} \leq \|\mathbf{H}(f^0)\|_{l+2s,r_0} P(\|f^0\|_{l,r_0}) \leq M\|\mathbf{H}(f^0)\|_{l+2s,r_0},$$

Now we just have to estimate $\|\mathbf{H}(f^0)\|_{l+2s,r_0}$.

We use again the estimate (7.16) and the interpolation inequality (7.5) to obtain

$$(7.31) \quad \begin{aligned} \|\mathbf{H}(f^0)\|_{l+2s,r_0} &\leq \|\zeta(f^0)\|_{l+3s,r_0} P(\|f^0\|_{l,r_0}) + \|f^0\|_{l+3s,r_0} \|\zeta(f^0)\|_{l,r_0} P(\|f^0\|_{l,r_0}) \\ &\leq M(\|\zeta(f^0)\|_{l,r_0}^{\frac{l-3s-1}{l-1}} \|\zeta(f^0)\|_{2l-1,r_0}^{\frac{3s}{l-1}} + \|f^0\|_{l,r_0}^{\frac{l-3s-1}{l-1}} \|f^0\|_{2l-1,r_0}^{\frac{3s}{l-1}} \|\zeta(f^0)\|_{l,r_0}) \\ &\leq M(t_0^{-\frac{l-3s-1}{l-1} + A\frac{3s}{l-1}} + t_0^{-1+A\frac{3s}{l-1}}). \end{aligned}$$

Finally, by (7.23) and (7.21) we get the estimate $\|\hat{\chi}^1\|_{l+s,\rho_0} \leq Mt_0^{-\mu}$ with $-\mu < -4/5 < -1/2$ and, replacing t_0 by a larger number if necessary (independently of f and d), we have $\|\hat{\chi}^1\|_{l+s,\rho_0} < t_0^{-1/2}$. In the same way, we can show that $\|\mathbf{H}(f^0)\|_{l+2s,r_0} \leq t_0^{-4/5}$ (note that, as we said before, it implies that we can assume that $\|S(t_0)\mathbf{H}(f^0)\|_{2,r_0} < 1 - \rho_0/r_0$).

Now, we suppose that the conditions (1_d) ... (5_d) are satisfied for $d \geq 0$ and we study the step $d + 1$.

Proof of (1_{d+1}) : The point (1_{d+1}) can be proved as above. Moreover, in the same way as in (1₀), we can prove that $\|\mathbf{H}(f^d)\|_{l+2s+2,r_d} \leq t_d^{-4/5}$ and $\|\chi^{d+1}\|_{l+s,\rho_d} < t_d^{-1/2}$.

Proof of (2_{d+1}) : According to (7.14) with $\rho_d \leq r_d$, we can write $\|f^{d+1}\|_{l,r_{d+1}} \leq \|f^d\|_{l,r_d} (1 + \|\hat{\chi}^{d+1}\|_{l+s,\rho_d} P(\|\hat{\chi}^{d+1}\|_{l+s,\rho_d}))$. Since $\|\hat{\chi}^{d+1}\|_{l+s,\rho_d} < t_d^{-1/2}$ we can assume, choosing t_0 large enough, that

$$(7.32) \quad \|\hat{\chi}^{d+1}\|_{l+s,\rho_d} P(\|\hat{\chi}^{d+1}\|_{l+s,\rho_d}) \leq \frac{1}{(d+1)(d+3)},$$

and we get

$$(7.33) \quad \|f^{d+1}\|_{l,r_{d+1}} < C \frac{d+1}{d+2} \left(1 + \frac{1}{(d+1)(d+3)}\right) < C \frac{d+2}{d+3}.$$

Proof of (3_{d+1}) : We have $f^{d+1} = \hat{\phi}_{d+1} \cdot f^d$ with $\hat{\phi}_{d+1} = Id + \hat{\chi}^{d+1} = \Phi(S(t_d)\mathbf{H}(f^d))$ thus, (7.12) (with $\rho_d \leq r_d$) gives

$$(7.34) \quad \|f^{d+1}\|_{2l-1,r_{d+1}} \leq \|f^d\|_{2l-1,r_d} P(\|\hat{\chi}^{d+1}\|_{l+s,\rho_d}) + \|\hat{\chi}^{d+1}\|_{2l-1+s,\rho_d} \|f^d\|_{l,r_d} P(\|\hat{\chi}^{d+1}\|_{l+s,\rho_d}).$$

This gives, by (1_d) and (2_d),

$$(7.35) \quad \|f^{d+1}\|_{2l-1,r_{d+1}} \leq M(\|f^d\|_{2l-1,r_d} + \|\hat{\chi}^{d+1}\|_{2l-1+s,\rho_d}).$$

Now, using (7.17) with $\lceil \frac{2l-1+s+1}{2} \rceil + s \leq l + 2s$ we have

$$(7.36) \quad \begin{aligned} \|\hat{\chi}^{d+1}\|_{2l-1+s,\rho_d} &\leq \|S(t_d)\mathbf{H}(f^d)\|_{2l-1+2s,r_d} P(\|S(t_d)\mathbf{H}(f^d)\|_{l+2s,r_d}) \\ &\leq \|S(t_d)\mathbf{H}(f^d)\|_{2l-1+2s,r_d} P(\|\mathbf{H}(f^d)\|_{l+2s,r_d}) \quad \text{by (7.3)}. \end{aligned}$$

As we said above, we have the estimate $\|\mathbf{H}(f^d)\|_{l+2s,r_d} \leq t_d^{-4/5}$ then, we can write

$$(7.37) \quad \begin{aligned} \|\hat{\chi}^{d+1}\|_{2l-1+s,\rho_d} &\leq M\|S(t_d)\mathbf{H}(f^d)\|_{2l-1+2s,r_d} \\ &\leq Mt_d^{4s}\|\mathbf{H}(f^d)\|_{2l-1-2s,r_d} \quad \text{by (7.3)} \\ &\leq Mt_d^{4s}(\|\zeta(f^d)\|_{2l-1-s,r_d} P(\|f^d\|_{l,r_d}) \\ &\quad + \|f^d\|_{2l-1-s,r_d} \|\zeta(f^d)\|_{l,r_d} P(\|f^d\|_{l,r_d})) \quad \text{by (7.16)}. \end{aligned}$$

We get $\|\hat{\chi}^{d+1}\|_{2l-1+s,\rho_d} \leq Mt_d^{A+4s}$ and, consequently,

$$(7.38) \quad \|f^{d+1}\|_{2l-1,r_{d+1}} \leq Mt_d^{A+4s}.$$

To finish, since $A = 8s+4$, we have that $\|f^{d+1}\|_{2l-1,r_{d+1}} \leq Mt_d^B$ with $0 < B < 3A/2$ thus, replacing t_0 by a larger number if necessary, we get $\|f^{d+1}\|_{2l-1,r_{d+1}} < t_d^{3A/2} = t_{d+1}^A$.

Proof of (4_{d+1}) : We have (see (7.15))

$$(7.39) \quad \|\zeta(f^{d+1})\|_{2l-1,r_{d+1}} \leq \|f^{d+1}\|_{2l-1,r_{d+1}} P(\|f^{d+1}\|_{l,r_{d+1}}).$$

Using the estimates of $\|f^{d+1}\|_{2l-1,r_{d+1}}$ and $\|f^{d+1}\|_{l,r_{d+1}}$ given above, we obtain $\|\zeta(f^{d+1})\|_{2l-1,r_{d+1}} \leq Mt_d^{A+4s}$, and we conclude as in (3_{d+1}).

Proof of (5_{d+1}) : Recall that we have $f^{d+1} = \hat{\phi}_{d+1} \cdot f^d$ with $\hat{\phi}_{d+1} = \Phi(S(t_d)\mathbf{H}(f^d)) = Id + \hat{\chi}^{d+1}$ and $\phi_{d+1} = \Phi(\mathbf{H}(f^d)) = Id + \chi^{d+1}$.

We can write

$$(7.40) \quad \zeta(\hat{\phi}_{d+1} \cdot f^d) = \zeta(\phi_{d+1} \cdot f^d) + \zeta(\hat{\phi}_{d+1} \cdot f^d - \phi_{d+1} \cdot f^d)$$

On the one hand, by (7.19) with $\rho_d \leq r_d$ we get

$$(7.41) \quad \|\zeta(\hat{\phi}_{d+1} \cdot f^d)\|_{l,r_{d+1}} \leq \|\zeta(f^d)\|_{l+s,r_d}^{1+\delta} Q(\|f^d\|_{l+s,r_d}, \|\chi^{d+1}\|_{l+s,\rho_d}, \|\zeta(f^d)\|_{l+s,r_d}, \|f^d\|_{l,r_d})$$

where Q is a polynomial whose degree τ in the first variable does not depend on l and f . Now, using the interpolation inequality (7.5) with $r = l$ and $p = 2l - 1$ we

get

$$(7.42) \quad \|\zeta(f^d)\|_{l+s, r_d} \leq Mt_d^{-\frac{l-s-1}{l-1} + A\frac{s}{l-1}}$$

$$(7.43) \quad \text{and} \quad \|f^d\|_{l+s, r_d} \leq Mt_d^{A\frac{s}{l-1}}.$$

The inequality (7.42) with (7.21) imply that $\|\zeta(f^d)\|_{l+s, r_d} \leq Mt_d^{-\frac{4}{5}}$.

Then, using points $(1_d) \dots (5_d)$ and the estimate $\|\chi^{d+1}\|_{l+s, \rho_d} < t_d^{-1/2}$ (see the proof of (1_0)), the inequality (7.41) gives

$$(7.44) \quad \|\zeta(\phi_{d+1} \cdot f^d)\|_{l, r_{d+1}} \leq Mt_d^{-(1+\delta)\frac{l-s-1}{l-1} + (1+\delta+\tau)A\frac{s}{l-1}}.$$

Finally, using the technical conditions (7.23), (7.21) and (7.22), we have

$$(7.45) \quad \|\zeta(\phi_{d+1} \cdot f^d)\|_{l, r_{d+1}} \leq Mt_d^{-\mu}$$

where $-\mu < -3/2$ and, replacing t_0 by a larger number if necessary, we obtain $\|\zeta(\phi_{d+1} \cdot f^d)\|_{l, r_{d+1}} < \frac{1}{2}t_d^{-3/2}$.

On the other hand, using the estimate $\|S(t_d)H(f^d)\|_{l+s, r_d} \leq M\|H(f^d)\|_{l+s, r_d}$ (see (7.3)) with the inequality (7.18) we get

$$(7.46) \quad \begin{aligned} \|\hat{\phi}_{d+1} \cdot f^d - \phi_{d+1} \cdot f^d\|_{l, r_{d+1}} &\leq \|S(t_d)H(f^d) - H(f^d)\|_{l+s, r_d} \|f^d\|_{l+s, r_d} P(\|H(f^d)\|_{l+s, r_d}) \\ &\quad + \|f^d\|_{l+s, r_d} R_{(2)}(\|H(f^d)\|_{l+s, r_d}), \end{aligned}$$

where $R_{(2)}$ is a polynomial with positive coefficients and which contains only terms of degree greater or equal to 2. We said in the proof of the point (1_{d+1}) that $\|H(f^d)\|_{l+s+2, r_d} < t_d^{-4/5}$ and we saw above (see (7.43)) that $\|f^d\|_{l+s, r_d} \leq Mt_d^{A\frac{s}{l-1}}$. Then, we get :

$$(7.47) \quad \|f^d\|_{l+s, r_d} R_{(2)}(\|H(f^d)\|_{l+s, r_d}) \leq Mt_d^{-2 \times \frac{4}{5} + A\frac{s}{l-1}}.$$

Moreover, the property of the smoothing operator (7.4) gives :

$$(7.48) \quad \|S(t_d)H(f^d) - H(f^d)\|_{l+s, r_d} \leq Mt_d^{-2} \|H(f^d)\|_{l+s+2, r_d} \leq Mt_d^{-2 - \frac{4}{5}},$$

which induces

$$(7.49) \quad \|S(t_d)H(f^d) - H(f^d)\|_{l+s, r_d} \|f^d\|_{l+s, r_d} P(\|H(f^d)\|_{l+s, r_d}) \leq Mt_d^{-2 - \frac{4}{5} + A\frac{s}{l-1}}.$$

Finally, the estimate (7.46) gives

$$(7.50) \quad \|\hat{\phi}_{d+1} \cdot f^d - \phi_{d+1} \cdot f^d\|_{l, r_{d+1}} \leq M(t_d^{-2 - \frac{4}{5} + A\frac{s}{l-1}} + t_d^{-2 \times \frac{4}{5} + A\frac{s}{l-1}}).$$

With the condition (7.24) we then obtain :

$$(7.51) \quad \|\hat{\phi}_{d+1} \cdot f^d - \phi_{d+1} \cdot f^d\|_{l, r_{d+1}} \leq Mt_d^{-\nu}$$

where $-\nu < -3/2$ and, replacing t_0 by a larger number if necessary, we obtain $\|\zeta(\phi_{d+1} \cdot f^d)\|_{l, r_{d+1}} < \frac{1}{2}t_d^{-3/2}$.

As a conclusion, we can write

$$(7.52) \quad \|\zeta(f^{d+1})\|_{l, r_{d+1}} \leq \frac{1}{2}t_d^{-3/2} + \frac{1}{2}t_d^{-3/2} = t_{d+1}^{-1}$$

Lemma 7.13 is proved. \square

Proof of Lemma 7.14 : As in the proof of the previous lemma, the letter M_k is a positive constant which does not depend on d and which varies from inequality to inequality. In the same way, P_k is a polynomials with positive real coefficients, which depends only on k and which varies from inequality to inequality.

Proof of (i) : We follow the same method as in the proof of the point (1₀) of the previous lemma. Using the relation $k \geq l \geq 6s + 1$ and the interpolation inequality (7.5) with $r = k$ and $p = 2k - 1$, we can show that for all $d \geq d_k$, we have

$$(7.53) \quad \|\hat{\chi}^{d+1}\|_{k+1+s, \rho_d} \leq M_k (t_d^{-\frac{k-3s-2}{k-1} + A\frac{3s+1}{k-1}} + t_d^{-1 + A\frac{3s+1}{k-1}}) \leq M_k t_d^{-\mu},$$

where $-\mu < -4/5$ (using (7.21)). Thus, there exists $d_{k+1} > d_k$ such that for all $d \geq d_{k+1}$ we have $\|\hat{\chi}^{d+1}\|_{k+1+s, \rho_d} < t_d^{-1/2}$. Note that in the same way, we can also prove that $\|\chi^{d+1}\|_{k+1+s, \rho_d} < t_d^{-1/2}$ and $\|H(f^d)\|_{k+2s+3, r_d} < t_d^{-4/5}$.

Proof of (ii) : For $d \geq d_{k+1}$, we have by (7.14)

$$(7.54) \quad \|f^{d+1}\|_{k+1, r_{d+1}} \leq \|f^d\|_{k+1, r_d} (1 + \|\hat{\chi}^{d+1}\|_{k+1+s, \rho_d} P(\|\hat{\chi}^{d+1}\|_{k+1+s, \rho_d})).$$

In Point (i) we saw that $\|\hat{\chi}^{d+1}\|_{k+1+s, \rho_d} < t_d^{-1/2}$ then, we can assume, replacing d_{k+1} by a larger integer if necessary, that

$$(7.55) \quad \|\hat{\chi}^{d+1}\|_{k+1+s, \rho_d} P(\|\hat{\chi}^{d+1}\|_{k+1+s, \rho_d}) \leq \frac{1}{(d+1)(d+3)},$$

for all $d \geq d_{k+1}$. Now we choose a positive constant \tilde{C}_{k+1} (independent of d) such that $\|f^{d_{k+1}}\|_{k+1, r_{d_{k+1}}} < \tilde{C}_{k+1} \frac{d_{k+1}+1}{d_{k+1}+2}$. We then obtain, as in the proof of Point (2) of the previous lemma, that $\|f^d\|_{k+1, r_{d+1}} < \tilde{C}_{k+1} \frac{d+1}{d+2}$ for any $d \geq d_{k+1}$. Note that \tilde{C}_{k+1} is, a priori, not the constant of statement of the lemma. Later in the proof (see the proof of the point (iii)), we will replace it by a larger one.

Proof of (v) : The proof follows the same idea as the proof of Point (5) in the previous lemma, that is why we don't give a lot of details. Consider an integer $d \geq d_{k+1}$. We write obviously

$$(7.56) \quad \zeta(f^{d+1}) = \zeta(\hat{\phi}_{d+1} \cdot f^d) = \zeta(\phi_{d+1} \cdot f^d) + \zeta(\hat{\phi}_{d+1} \cdot f^d - \phi_{d+1} \cdot f^d).$$

On the one hand, by (7.19), the interpolation inequality (7.5) with $r = k$ and $p = 2k - 1$, the condition (7.21), and the estimate $\|\chi^{d+1}\|_{k+1+s, \rho_d} < t_d^{-1/2}$ (see the proof of (i)), we obtain

$$(7.57) \quad \|\zeta(\phi_{d+1} \cdot f^d)\|_{k+1, r_{d+1}} \leq M_k t_d^{-(1+\delta)\frac{k-s-2}{k-1} + (1+\delta+\tau)A\frac{s+1}{k-1}}$$

(recall that τ and δ are introduced in Theorem 7.7). Then, by (7.23) and (7.21), we have $\|\zeta(\phi_{d+1} \cdot f^d)\|_{k+1, r_{d+1}} \leq M_k t_d^{-\mu}$ where $-\mu < -3/2$ and, replacing d_{k+1} by a larger integer if necessary, we obtain $\|\zeta(\phi_{d+1} \cdot f^d)\|_{k+1, r_{d+1}} < \frac{1}{2} t_d^{-3/2}$.

On the other hand, following the same way as in the proof of point (5) in the previous lemma (with the estimate $\|H(f^d)\|_{k+2s+3, r_d} < t_d^{-4/5}$ given in (i)), we can

prove that

$$(7.58) \quad \|\zeta(\hat{\phi}_{d+1} \cdot f^d - \phi_{d+1} \cdot f^d)\|_{l, r_{d+1}} \leq M_k t_d^{-\nu},$$

with $-\nu < -3/2$, and replacing d_{k+1} by a larger integer if necessary, we can write

$$(7.59) \quad \|\zeta(\hat{\phi}_{d+1} \cdot f^d - \phi_{d+1} \cdot f^d)\|_{l, r_{d+1}} \leq \frac{1}{2} t_d^{-3/2}.$$

Finally, we obtain for all $d \geq d_{k+1}$,

$$(7.60) \quad \|\zeta(f^{d+1})\|_{k+1, r_{d+1}} < t_{d+1}^{-1},$$

which gives the result (replacing actually d_{k+1} by $d_{k+1} + 1$).

Proof of (iii) and (iv) : We first write, using the inequality (7.15), for all $d \geq d_{k+1}$,

$$(7.61) \quad \|\zeta(f^d)\|_{2k+1, r_d} \leq \|f^d\|_{2k+1, r_d} P_k(\|f^d\|_{k+1, r_d}).$$

Putting $V_{k+1} := \max(1, T_{2k+1}(\tilde{C}_{k+1}))$ (recall that \tilde{C}_{k+1} was introduced in (ii)), we obtain by the point (ii),

$$(7.62) \quad \|\zeta(f^d)\|_{2k+1, r_d} \leq V_{k+1} \|f^d\|_{2k+1, r_d}.$$

We will use this inequality at the end of the proof.

In the same way as in the proof of (3_d) of the previous lemma, we can show that for all $d \geq d_{k+1}$ we have

$$(7.63) \quad \|f^{d+1}\|_{2k+1, r_{d+1}} \leq M_k (\|f^d\|_{2k+1, r_d} + \|\hat{\chi}^{d+1}\|_{2k+1+s, \rho_d}),$$

with

$$(7.64) \quad \begin{aligned} \|\hat{\chi}^{d+1}\|_{2k+1+s, \rho_d} &\leq M_k \|S(t_d) \mathbf{H}(f^d)\|_{2l+1+2s, r_d} \\ &\leq M_k t_d^{4s+2} \|\mathbf{H}(f^d)\|_{2k-1-2s, r_d} \quad \text{by (7.3)} \\ &\leq M_k t_d^{4s+2} (\|\zeta(f^d)\|_{2k-1-s, r_d} P_k(\|f^d\|_{k, r_d}) \\ &\quad + \|f^d\|_{2k-1-s, r_d} \|\zeta(f^d)\|_{k, r_d} P_k(\|f^d\|_{k, r_d})) \quad \text{by (7.16)}. \end{aligned}$$

We then get the estimate $\|\hat{\chi}^{d+1}\|_{2k+1+s, \rho_d} \leq M_k t_d^{A+4s+2}$, which gives

$$(7.65) \quad \|f^{d+1}\|_{2k+1, r_{d+1}} \leq M_k (\|f^d\|_{2k+1, r_d} + t_d^{A+4s+2}).$$

Now, since $A > 8s + 4$, replacing d_{k+1} by a larger integer if necessary, we can assume that for any $d \geq d_{k+1}$, we have $M_k t_d^{A+4s+2} < \frac{1}{2V_{k+1}} t_d^{3A/2}$ (note that it also implies $M_k < \frac{1}{2V_{k+1}} t_d^{A/2}$). This gives

$$(7.66) \quad \|f^{d+1}\|_{2k+1, r_{d+1}} \leq \frac{1}{2V_{k+1}} t_d^{A/2} \|f^d\|_{2k+1, r_d} + \frac{1}{2V_{k+1}} t_d^{3A/2}.$$

We choose a positive constant C_{k+1} such that

$$(7.67) \quad C_{k+1} > \max\left(1, \tilde{C}_{k+1}, \frac{\|f^{d_{k+1}}\|_{2k+1, r_{d_{k+1}}}}{t_{d_{k+1}}^A}\right).$$

We then have $\|f^{d_{k+1}}\|_{2k+1, r_{d_{k+1}}} < C_{k+1} t_{d_{k+1}}^A$ and, using (7.66) we obtain by induction :

$$(7.68) \quad \|f^d\|_{2k+1, r_d} < \frac{C_{k+1}}{V_{k+1}} t_d^A < C_{k+1} t_d^A,$$

for all $d \geq d_{k+1}$.

Now, by (7.62), we have

$$\|\zeta(f^d)\|_{2k+1, r_d} \leq V_{k+1} \frac{C_{k+1}}{V_{k+1}} t_d^A,$$

for all $d \geq d_{k+1}$.

Moreover, the definition of C_{k+1} completes the proof of the point (i), (ii) and (v).

Lemma 7.14 is proved. \square

8. APPENDIX 2 : SOME TECHNICAL RESULTS

This section is devoted to state and prove some technical results we used in the proof of Theorem 4.1.

8.1. The local diffeomorphisms. As we said in section 6, the proof of Theorem 4.1 consists in checking that our situation is a particular case of the ‘‘SCI-context’’ given in the Appendix 1. We then need some estimates on local diffeomorphisms and action of local diffeomorphisms on smooth functions. Most of the properties of the definition of SCI-spaces, SCI-groups and SCI-actions can be found in [10] and [24]. In each of the followings lemmas, if ρ is a positive real number, B_ρ is the closed ball in \mathbb{R}^n of radius ρ and center 0.

We first recall here the following usefull Lemma.

Lemma 8.1. *Let $r > 0$ and $0 < \eta < 1$ be two positive numbers. Consider two smooth maps*

$$f : B_{r(1+\eta)} \rightarrow \mathbb{R}^q \quad \text{and} \quad \chi : B_r \rightarrow \mathbb{R}^n$$

such that $\chi(0) = 0$ and $\|\chi\|_{1,r} < \eta$. Then the composition $f \circ (id + \chi)$ is a smooth map from B_r to \mathbb{R}^n which satisfies the following inequality:

$$(8.1) \quad \|f \circ (id + \chi)\|_{k,r} \leq \|f\|_{k,r(1+\eta)} (1 + P_k(\|\chi\|_{k,r}))$$

where P_k is a polynomial of degree k with vanishing constant term (and which is independent of f and χ).

Moreover, writing for any x

$$(8.2) \quad f(x + \chi(x) + \zeta(x)) - f(x + \chi(x)) = \int_0^1 df(x + \chi(x) + t\zeta(x))(\zeta(x)) dt$$

we get :

Lemma 8.2. *Let $r > 0$ and $0 < \eta < 1$ be two positive numbers. Consider three smooth maps*

$$f : B_{r(1+\eta)} \rightarrow \mathbb{R}^q, \chi \text{ and } \xi : B_r \rightarrow \mathbb{R}^n$$

such that $\chi(0) = \xi(0) = 0$ and $\|\chi\|_{1,r} + \|\xi\|_{1,r} < \eta$. Then we have the estimate

$$(8.3) \quad \|f \circ (Id + \chi + \xi) - f \circ (Id + \chi)\|_{k,r} \leq \|f\|_{k+1, r(1+\eta)} \|\xi\|_{k,r} R(\|\chi\|_{k,r}, \|\xi\|_{k,r}),$$

where R is a polynomial in two variables (which is independent of f , χ and ξ).

In [24] all the diffeomorphisms we used were of type $Id + \chi$ where χ was directly defined by the background. Here as we want diffeomorphisms preserving a given Poisson structure, we work with the flows of Hamiltonian vector fields (these vector fields are “naturally” defined by the context). Even if such a diffeomorphism is of type $Id + \chi$ too, we only have informations on the vector field X defining it. Lemma 8.3 allows to use the estimates given above combined with the estimates of X .

Lemma 8.3. *Let $r > 0$ and $0 < \varepsilon < 1$ be two positive numbers. Consider a smooth vector field X on $B_{r+\varepsilon}$ vanishing at 0 and ϕ^t its flow written $\phi^t = Id + \chi^t$ with $\chi^t(0) = 0$.*

a) *If $\|X\|_{1,r+\varepsilon} < \varepsilon$ then for all $t \in [0, 1]$,*

$$(8.4) \quad \|\chi^t\|_{1,r} \leq C\|X\|_{1,r+\varepsilon},$$

where C is a positive constant independent of X and χ^t .

b) *If $\|X\|_{1,r+\varepsilon} < \varepsilon$ then for all $t \in [0, 1]$ and all $L \geq 2$,*

$$(8.5) \quad \|\chi^t\|_{L,r} \leq \|X\|_{L,r+\varepsilon} P_L(\|X\|_{L,r+\varepsilon}),$$

where P_L is a polynomial independent of X and χ^t with positive coefficients, and $l = \lfloor \frac{L}{2} \rfloor + 1$ ($\lfloor \cdot \rfloor$ denotes the integer part).

Proof. a) If x is in B_r and $t \in [0, 1]$, then we can write

$$(8.6) \quad \chi^t(x) = \phi^t(x) - x = \int_0^t X(\phi^\tau(x)) d\tau.$$

It is clear, since $\|X\|_{1,r+\varepsilon} < \varepsilon$, that (8.6) gives

$$(8.7) \quad \|\chi^t\|_{0,r} \leq \|X\|_{0,r+\varepsilon} \leq \|X\|_{1,r+\varepsilon}.$$

Now for i and j in $\{1, \dots, n\}$ we have

$$(8.8) \quad \begin{aligned} \left| \frac{\partial \chi_i^t}{\partial x_j}(x) \right| &\leq \int_0^t \sum_{k=1}^n \left| \frac{\partial X_i}{\partial x_k}(\phi^\tau(x)) \right| \left| \frac{\partial \phi_k^\tau}{\partial x_j}(x) \right| d\tau \\ &\leq \int_0^t \sum_{k=1}^n \left| \frac{\partial X_i}{\partial x_k}(\phi^\tau(x)) \right| \left| \frac{\partial \chi_k^\tau}{\partial x_j}(x) \right| d\tau + \int_0^t \left| \frac{\partial X_i}{\partial x_j}(\phi^\tau(x)) \right| d\tau \\ &\leq \|X\|_{1,r+\varepsilon} \int_0^t \sum_{k=1}^n \left| \frac{\partial \chi_k^\tau}{\partial x_j}(x) \right| d\tau + \|X\|_{1,r+\varepsilon}, \end{aligned}$$

which gives,

$$(8.9) \quad \sum_{k=1}^n \left| \frac{\partial \chi_k^t}{\partial x_j}(x) \right| \leq n\|X\|_{1,r+\varepsilon} \int_0^t \sum_{k=1}^n \left| \frac{\partial \chi_k^\tau}{\partial x_j}(x) \right| d\tau + n\|X\|_{1,r+\varepsilon}.$$

Now, Gronwall's Lemma gives

$$(8.10) \quad \sum_{i=1}^n \left| \frac{\partial \chi_i^t}{\partial x_j}(x) \right| \leq (ne^{n\|X\|_{1,r+\varepsilon}t}) \|X\|_{1,r+\varepsilon},$$

for all $t \in [0, 1]$. Using the condition $\|X\|_{1,r+\varepsilon} < \varepsilon$, the point a) follows (with $C = ne^{n\varepsilon}$).

b) To prove this point we write again the trivial relation

$$(8.11) \quad \chi^t(x) = \phi^t(x) - x = \int_0^t X(\phi^\tau(x)) d\tau,$$

and then use an induction on L . We first prove that the inequality holds for $L = 2$. If x in B_r , we write

$$(8.12) \quad \begin{aligned} \frac{\partial^2 \chi_i^t}{\partial x_k \partial x_j}(x) &= \int_0^t \sum_{u,v=1}^n \frac{\partial^2 X_i}{\partial x_v \partial x_u}(\phi^\tau(x)) \frac{\partial \phi_u^\tau}{\partial x_j}(x) \frac{\partial \phi_v^\tau}{\partial x_k}(x) d\tau \\ &+ \int_0^t \sum_{u=1}^n \frac{\partial X_i}{\partial x_u}(\phi^\tau(x)) \frac{\partial^2 \phi_u^\tau}{\partial x_k \partial x_j}(x) d\tau. \end{aligned}$$

It gives by the point $a)$ and the hypothesis $\|X\|_{1,r+\varepsilon} < \varepsilon$,

$$(8.13) \quad \left| \frac{\partial^2 \chi_i^t}{\partial x_k \partial x_j}(x) \right| \leq \|X\|_{2,r+\varepsilon} (1 + C\varepsilon)^2 + \|X\|_{1,r+\varepsilon} \int_0^t \sum_{u=1}^n \left| \frac{\partial^2 \phi_u^\tau}{\partial x_k \partial x_j}(x) \right| d\tau,$$

Note that in this inequality, we have in fact $\frac{\partial^2 \phi_u^\tau}{\partial x_k \partial x_j} = \frac{\partial^2 \chi_u^\tau}{\partial x_k \partial x_j}$. In the same way as in the proof of $a)$, summing these estimates and using the Gronwall Lemma, we obtain

$$(8.14) \quad \sum_{u=1}^n \left| \frac{\partial^2 \phi_u^t}{\partial x_k \partial x_j}(x) \right| \leq n(1 + C\varepsilon)^2 \|X\|_{2,r+\varepsilon} e^{n\|X\|_{1,r+\varepsilon} t} \leq a \|X\|_{2,r+\varepsilon},$$

where a is a positive constant. We then get the expected inequality for $L = 2$.

Now, suppose that $L > 2$ and that the estimates (8.5) are satisfied for all $k = 2, \dots, L-1$. If α is a multiindex in \mathbb{N}^n with $|\alpha| = L$ ($|\alpha|$ is the sum of the components of α), we have for all x in B_r and i in $\{1, \dots, n\}$,

$$(8.15) \quad \frac{\partial^{|\alpha|} \chi_i^t}{\partial x^\alpha}(x) = \int_0^t \frac{\partial^{|\alpha|} (X_i \circ \phi^\tau)}{\partial x^\alpha}(x) d\tau.$$

It is easy to show that

$$(8.16) \quad \frac{\partial^{|\alpha|} (X_i \circ \phi^\tau)}{\partial x^\alpha} = \sum_{1 \leq |\beta| \leq L} \left(\frac{\partial^{|\beta|} X_i}{\partial x^\beta} \circ \phi^\tau \right) A_\beta(\phi^\tau),$$

where $A_\beta(\phi^\tau)$ is of the type

$$(8.17) \quad A_\beta(\phi^\tau) = \sum_{\substack{1 \leq m_i \leq n, |\gamma_i| \geq 1 \\ |\gamma_1| + \dots + |\gamma_{|\beta|} = L}} a_{\gamma m} \frac{\partial^{|\gamma_1|} \phi_{m_1}^\tau}{\partial x^{\gamma_1}} \dots \frac{\partial^{|\gamma_{|\beta|}|} \phi_{m_{|\beta|}}^\tau}{\partial x^{\gamma_{|\beta|}}}$$

where $\phi_{m_1}^\tau$ is the m_1 -component of ϕ^τ and the $a_{\gamma m}$ are nonnegative integers.

If $l = \lfloor \frac{L}{2} \rfloor + 1$ then we can write

$$(8.18) \quad \begin{aligned} \frac{\partial^{|\alpha|} (X_i \circ \phi^\tau)}{\partial x^\alpha} &= \sum_{l < |\beta| \leq L} \left(\frac{\partial^{|\beta|} X_i}{\partial x^\beta} \circ \phi^\tau \right) A_\beta(\phi^\tau) + \sum_{2 \leq |\beta| \leq l} \left(\frac{\partial^{|\beta|} X_i}{\partial x^\beta} \circ \phi^\tau \right) A_\beta(\phi^\tau) \\ &+ \sum_{j=1}^n \left(\frac{\partial X_i}{\partial x_j} \circ \phi^\tau \right) \frac{\partial^{|\alpha|} \chi_j^\tau}{\partial x^\alpha}. \end{aligned}$$

When $l < |\beta| \leq L$, all the $|\gamma_i|$ in the sum (8.17) defining $A_\beta(\phi^\tau)$ are smaller than l . On the other hand, when $1 < |\beta| \leq l$, then in each product in the expression (8.17) of $A_\beta(\phi^\tau)$ there is at most one factor $\frac{\partial^{|\gamma|} \phi_m^u}{\partial x^\gamma}$ with $L > |\gamma| > l$ (the others have $|\gamma| \leq l$). Therefore, using the point *a*) with the hypothesis $\|X\|_{1,r+\varepsilon} < \varepsilon$ and the induction hypothesis we obtain, for all x in B_r ,

$$(8.19) \quad \left| \frac{\partial^{|\alpha|} \chi_i^t}{\partial x^\alpha}(x) \right| \leq \|X\|_{L,r+\varepsilon} q(\|X\|_{l,r+\varepsilon}) + \|X\|_{1,r+\varepsilon} \int_0^t \sum_{j=1}^n \left| \frac{\partial^{|\alpha|} \chi_j^\tau}{\partial x^\alpha}(x) \right| d\tau,$$

where q is a polynomial independent of X and χ^t with positive coefficients. We then conclude as in the case $L = 2$ (via Gronwall's Lemma). \square

Finally, we show the following result corresponding to inequality (7.18).

Lemma 8.4. *Let $r > 0$ and $0 < \eta < 1$ be two positive numbers. There exists a real number $\alpha > 0$ such that if f , g_1 and g_2 are three smooth functions on $B_{r(1+\eta)}$ verifying $\|g_1\|_{2,r(1+\eta)} < \alpha\eta$ and $\|g_2\|_{2,r(1+\eta)} < \alpha\eta$ and if we denote by ϕ_1 (resp. ϕ_2) the time-1 flow of the Hamiltonian vector field X_{g_1} (resp. X_{g_2}) of the function g_1 (resp. g_2) with respect to a Poisson structure, then we have the estimate :*

$$\begin{aligned} \|f \circ \phi_1 - f \circ \phi_2\|_{k,r} &\leq \|g_1 - g_2\|_{k+1,r(1+\eta)} \|f\|_{k+1,r(1+\eta)} P(\|g_1\|_{k+1,r(1+\eta)}) \\ &\quad + \|f\|_{k+2,r(1+\eta)} R_{(2)}(\|g_1\|_{k+2,r(1+\eta)}, \|g_1\|_{k+2,r(1+\eta)}) \end{aligned}$$

where P and $R_{(2)}$ are polynomials with real positive coefficients (independent of f , g_1 and g_2) and $R_{(2)}$ contains only terms of degree greater or equal to two.

Proof. Let x be an element of the closed ball B_r . If ϕ_1^t and ϕ_2^t design the flows of the vector fields X_{g_1} and X_{g_2} , we can write :

$$\begin{aligned} f(\phi_1(x)) - f(\phi_2(x)) &= f(\phi_1(x)) - f(x) + f(x) - f(\phi_2(x)) \\ (8.20) \quad &= \int_0^1 (X_{g_1} \cdot f) \circ \phi_1^t(x) dt - \int_0^1 (X_{g_2} \cdot f) \circ \phi_2^t(x) dt \\ &= \int_0^1 \{g_1, f\} \circ \phi_1^t(x) dt - \int_0^1 \{g_2, f\} \circ \phi_2^t(x) dt \\ &= \int_0^1 \{g_1 - g_2, f\} \circ \phi_1^t(x) dt + \int_0^1 \{g_2, f\} \circ \phi_1^t(x) dt \\ &\quad - \int_0^1 \{g_2, f\} \circ \phi_2^t(x) dt. \end{aligned}$$

Now, using the same argument as above with $\{g_2, f\}$ instead of f , we get

$$\begin{aligned} (8.21) \quad f(\phi_1(x)) - f(\phi_2(x)) &= \int_0^1 \{g_1 - g_2, f\} \circ \phi_1^t(x) dt \\ &\quad + \int_0^1 \left(\int_0^t \{g_1, \{g_2, f\}\} \circ \phi_1^\tau(x) d\tau \right) dt \\ &\quad - \int_0^1 \left(\int_0^t \{g_2, \{g_2, f\}\} \circ \phi_2^\tau(x) d\tau \right) dt. \end{aligned}$$

Now, we choose the real number $\alpha > 0$ in order to make sure that we can apply correctly Lemma 8.3 and Lemma 8.1 which depend on small conditions.

Finally, applying Lemmas 8.1 and 8.3, we then obtain the result (remark that we have, for example, $\|\{g_2, f\}\|_{k,r(1+\eta)} \leq C\|g_2\|_{k+1,r(1+\eta)}\|f\|_{k+1,r(1+\eta)}$, and also $\|X_{g_1}\|_{k,r(1+\eta)} \leq M\|g_1\|_{k+1,r(1+\eta)}$ where C and M are positive constants independent of f , g_1 and g_2). \square

8.2. Momentum maps. Consider a momentum map $\lambda : M \rightarrow \mathfrak{g}^*$ with respect to the Poisson structure Π . We saw in section 5.1 that we can associate to λ a Chevalley-Eilenberg complex $C^\bullet(\mathfrak{g}, C^\infty(M))$, with differential δ , and homotopy operator h . If μ is another momentum map with respect to the same Poisson structure then we can see the difference $\mu - \lambda$ as an 1-cochain in the complex. We then define $\phi^t = Id + \chi^t$ the flow of the Hamiltonian vector field $X_{h(\mu-\lambda)}$ with respect to the Poisson structure and $\phi = \phi^1$ the time-1 flow.

Lemma 8.5. *Let $r > 0$ and $0 < \eta < 1$ be two positive numbers. With the notations above, we have the two following properties :*

a) *For any positive integer k we have*

$$(8.22) \quad \|\delta(\mu - \lambda)\|_{k,r} \leq C\|\mu - \lambda\|_{k+1,r}^2,$$

where C is a positive constant independent of μ and λ .

b) *There exists a constant $\alpha > 0$ such that if $\|\mu - \lambda\|_{s+2,r(1+\eta)} < \alpha\eta$, then we have, for any positive integer k :*

$$(8.23) \quad \|\mu \circ \phi - \lambda\|_{k,r} \leq \|\mu - \lambda\|_{k+s+2,r(1+\eta)}^2 P(\|\mu - \lambda\|_{k+s+1,r(1+\eta)})$$

where P is a polynomial with positive coefficients, independent of μ and λ .

Proof. Let us consider a basis $\{\xi_1, \dots, \xi_n\}$ of the Lie algebra \mathfrak{g} and the real numbers c_{ij}^p defined by $[\xi_i, \xi_j] = \sum_{p=1}^n c_{ij}^p \xi_p$. In this proof, we adopt for instance the notation λ_i , for $\lambda(\xi_i)$ or $\xi_i \circ \lambda$.

We first prove the point a). In order to simplify, we denote by $f = \mu - \lambda$. By definition of the differential δ (see section 5.1), we have :

$$(8.24) \quad \begin{aligned} \delta f(\xi_i \wedge \xi_j) &= \xi_i \cdot f(\xi_j) - \xi_j \cdot f(\xi_i) - f([\xi_i, \xi_j]) \\ &= \{\lambda_i, f_j\} - \{\lambda_j, f_i\} - \sum_{p=1}^n c_{ij}^p f_p. \end{aligned}$$

It allows us to write the following equality :

$$(8.25) \quad \begin{aligned} \{f_i, f_j\} &= \{\mu_i, \mu_j\} - \{\lambda_i, \mu_j\} - \{\mu_i, \lambda_j\} + \{\lambda_i, \lambda_j\} \\ &= \{\mu_i, \mu_j\} - \delta f(\xi_i \wedge \xi_j) - \sum_{p=1}^n c_{ij}^p f_p - \{\lambda_i, \lambda_j\} \end{aligned}$$

Now, since λ and μ are momentum maps, we have

$$(8.26) \quad \{\mu_i, \mu_j\} = [\xi_i, \xi_j] \circ \mu = \sum_{p=1}^n c_{ij}^p \mu_p$$

and also $\{\lambda_i, \lambda_j\} = \sum_{p=1}^n c_{ij}^p \lambda_p$.

Therefore, we obtain :

$$(8.27) \quad \delta f_d(\xi_i \wedge \xi_j) = -\{f_i, f_j\}.$$

Finally, we just write the following estimates :

$$(8.28) \quad \|\delta f\|_{k,r} \leq n(n-1)\|\Pi\|_{k,r}\|f\|_{k+1,r}^2,$$

where Π is the Poisson structure considered.

Now, we prove the point *b*) of the lemma. Let x be in the closed ball B_r , we first write :

$$(8.29) \quad \mu \circ \phi(x) - \lambda(x) = \mu \circ \phi(x) - \lambda \circ \phi(x) + \lambda \circ \phi(x) - \lambda(x).$$

Now, by the definition of δ , we have for each $i \in \{1, \dots, n\}$:

$$(8.30) \quad \begin{aligned} \lambda_i \circ \phi(x) - \lambda_i(x) &= \int_0^1 (X_{h(\mu-\lambda)} \cdot \lambda_i) \circ \phi^t(x) dt \\ &= \int_0^1 \{h(\mu-\lambda), \lambda_i\} \circ \phi^t(x) dt \\ &= - \int_0^1 \delta h(\mu-\lambda)_i \circ \phi^t(x) dt. \end{aligned}$$

We know (see Lemma 5.6) that

$$(8.31) \quad \mu - \lambda = \delta h(\mu - \lambda) + h(\delta(\mu - \lambda)),$$

therefore, we get

$$(8.32) \quad \lambda_i \circ \phi(x) - \lambda_i(x) = - \int_0^1 (\mu - \lambda)_i \circ \phi^t(x) dt + \int_0^1 h(\delta(\mu - \lambda))_i \circ \phi^t(x) dt.$$

Finally, injecting this equality in (8.29), we get

$$(8.33) \quad \begin{aligned} \mu_i \circ \phi(x) - \lambda_i(x) &= (\mu - \lambda)_i \circ \phi(x) - \int_0^1 (\mu - \lambda)_i \circ \phi^t(x) dt \\ &\quad + \int_0^1 h(\delta(\mu - \lambda))_i \circ \phi^t(x) dt \\ &= \int_0^1 \left(\int_t^1 (X_{h(\mu-\lambda)} \cdot (\mu - \lambda)_i) \circ \phi^\tau(x) d\tau \right) dt \\ &\quad + \int_0^1 h(\delta(\mu - \lambda))_i \circ \phi^t(x) dt \\ &= \int_0^1 \left(\int_t^1 \{h(\mu - \lambda), (\mu - \lambda)_i\} \circ \phi^\tau(x) d\tau \right) dt \\ &\quad + \int_0^1 h(\delta(\mu - \lambda))_i \circ \phi^t(x) dt. \end{aligned}$$

Now, the equality (8.33) combined with Lemma 5.6, Lemma 8.1 and the point *a*) of this lemma give :

$$(8.34) \quad \|\mu \circ \phi - \lambda\|_{k,r} \leq \|\mu - \lambda\|_{k+s+2,r(1+\eta)}^2 P(\|\chi^t\|_{k,r}),$$

where P is a polynomial with positive coefficients and which does not depend on μ and λ . Finally, we conclude using Lemma 8.3.

Note that we know, using Lemma 5.6, that $\|X_{h(\mu-\lambda)}\|_{1,r(1+\eta)} \leq M\|\mu-\lambda\|_{2+s,r(1+\eta)}$ where M is a positive constant independent of μ and λ . The condition we gave in

the statement of *b*) ($\|\mu - \lambda\|_{s+2,r(1+\eta)} < \alpha\eta$) is to make sure that we can apply correctly Lemma 8.3 and Lemma 8.1 which depend on small conditions.

□

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