DYNAMICS AND BIFURCATION NEAR THE TRANSITION FROM STABILITY TO COMPLEX INSTABILITY:
An analytical approach using normal forms

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Abstract

We consider a Hamiltonian of three degrees of freedom and a family of periodic orbits with a transition from stability to complex instability, such that there is an irrational collision of the Floquet eigenvalues of opposite sign. We analyze the local dynamics and the bifurcation phenomena linked to this transition. We study the resulting Hamiltonian Hopf-like bifurcation from an analytical point of view by means of normal forms. The existence of a bifurcating family of 2D tori is derived and both cases (direct and inverse bifurcation) are described.

1 Description of the problem and methodology

Let us consider a Hamiltonian with three degrees of freedom and a family of periodic orbits with a transition from stability to complex instability, that is, there is a critical periodic orbit for which a collision of Floquet multipliers of opposite sign at an irrational point -complex instability- takes place.

Our analytical approach is to compute the (formal) normal form, around the critical periodic orbit, up to an arbitrarily high order and to use this normal form to describe the dynamics around this transition. For this computation, we carry out the following process:

- We change the system of coordinates to a suitable one, by means of a symplectic transformation;

- we apply a canonical Floquet transformation to reduce the normal variational equations of the orbit to constant coefficients;

- we complexify the Hamiltonian;

- we describe how to compute, in a some tricky and constructive way, the normal form.

2 Main results

Dealing with the truncated normal form itself and the differential equations associated to it,

- we derive the existence of two families of invariant 2D tori which bifurcate from the critical orbit,

- we identify the coefficient that determines the unfolding type and the stability of the bifurcating tori.

From a suitable parameterization of the 2D tori, we also describe the *effect* of the unfolding on the local phase space in a neighbourhood of the transition:

- Confinement on the direct case, and compression of 3d tori on the inverse one.

- We remark the similarities to the local phase space structure of the Hamiltonian Hopf bifurcation for a two-degree of freedom Hamiltonian.

3   Main related works and applications:

- From a numerical point of view, some studies have been done both for 4D symplectic mappings (see [4], [8] and [12]) and for some particular applications (see for example [5], [9], [10], [12] and [13] -and references therein-, in celestial and galactic dynamics context).

  Concerning analytical studies, the Hamiltonian Hopf bifurcation (a Hamiltonian with two degrees of freedom and a family of equilibrium points with a transition from stability to complex instability) is mainly described in [14]. But, as far as we know, there are only two papers where the bifurcation of an irrational collision was carried out from an analytical point of view: the one by Heggie ([6]), using the Hamiltonian itself but with a previous isoenergetic reduction and a normal form up to order six, and that by Bridges et al. ([3]), where they consider a nonlinear 4D symplectic map, and use the normal form derived in [2] in order to simplify the map. We remark that the theory for existence and stability of periodic orbits in the unfolding of a rational collision is analyzed in [1].

- Our contribution is to present a constructive method (that can be implemented numerically, from a practical point of view) to compute the normal form in order to obtain a good approximation of the relevant invariant objects (periodic orbits, 2D and 3D tori and invariant manifolds) as well as the dynamics around them, for any particular application described by a Hamiltonian.

4   Formulation of the problem

Let $H(\zeta)$ with $\zeta = (\xi, \eta)$, be a real three degree of freedom analytic Hamiltonian, with its associated Hamiltonian system

$$\dot{\zeta} = J_3 \nabla H(\zeta),\tag{1}$$

with J_3 the matrix of the standard canonical 2-form of $\mathbb{R}^6$. Suppose that this system has a non degenerate family of periodic orbits with transition to complex instability, i. e., such that one of its orbits undergoes a collision of Floquet multipliers on the unit circle. Further, denote by $\zeta_0(\theta)$ a 2π-periodic parameterization of this critical orbit, with period $T_0 = 2\pi/\omega_0$.

5   The Jordan structure of the monodromy matrix

If $\mathbf{M}_0(2\pi)$ is the monodromy matrix of $\zeta_0(\theta)$, its Jordan normal form will have a double eigenvalue equal to 1 forming a nontrivial Jordan block (in the generic non degenerate sense), and two double eigenvalues $\lambda \neq \pm 1$, $1/\lambda = \bar{\lambda}$ again in a two nontrivial blocks.

Proposition 1 *Under the conditions stated, there exists a symplectic complex basis $\{u_1, v_1, v_2, u_2, w_1, w_2\}$, with respect to the canonical 2-form, such that*

1.

$$
\mathbf{M}_0(2\pi)|_{<u_1,u_2,v_1,v_2,w_1,w_2>} =
\begin{pmatrix}
1 & a & & & & \\
0 & 1 & & & & \\
& & \lambda & \lambda & & \\
& & 0 & \lambda & & \\
& & & & 1/\lambda & 0 \\
& & & & -1/\lambda & 1/\lambda
\end{pmatrix}
\tag{2}
$$

2. If we define

$$
\mathbf{B} =
\begin{pmatrix}
i\dfrac{\widetilde{\omega}_1}{T_0} & \dfrac{1}{T_0} & & \\
0 & i\dfrac{\widetilde{\omega}_1}{T_0} & & \\
& & -i\dfrac{\widetilde{\omega}_1}{T_0} & 0 \\
& & -\dfrac{1}{T_0} & -i\dfrac{\widetilde{\omega}_1}{T_0}
\end{pmatrix},
\tag{3}
$$

with $\lambda = e^{-i\widetilde{\omega}_1/T_0}$, then $\mathbf{B}$ verifies $\mathbf{M}_0(2\pi)|_{<v_1,v_2,w_1,w_2>} = e^{\mathbf{B}T_0}$, i. e. $\mathbf{B}$ is the Floquet matrix of the normal variational equations.

Remark 1 *Let us define*

$$
v_1^* = \frac{v_1 + w_2}{\sqrt{2}}, \quad v_2^* = \frac{v_1 - w_2}{\sqrt{2i}}, \quad w_1^* = \frac{v_2 - w_1}{\sqrt{2}}, \quad w_2^* = \frac{v_2 + w_1}{\sqrt{2i}}.
\tag{4}
$$

Then we have that $\{u_1, u_2, v_1^*, v_2^*, w_1^*, w_2^*\}$ is a symplectic real basis that reduces $\mathbf{M}_0(2\pi)$ to its "real Jordan" normal form

$$\mathbf{M}_0(2\pi)|_{<u_{1,2},v_{1,2}^*,w_{1,2}^*>} = \begin{pmatrix} 1 & a & & & & \\ 0 & 1 & & & & \\ \hline & & \cos\widetilde{\omega}_1 & \sin\widetilde{\omega}_1 & \cos\widetilde{\omega}_1 & \sin\widetilde{\omega}_1 \\ & & -\sin\widetilde{\omega}_1 & \cos\widetilde{\omega}_1 & -\sin\widetilde{\omega}_1 & \cos\widetilde{\omega}_1 \\ & & 0 & 0 & \cos\widetilde{\omega}_1 & \sin\widetilde{\omega}_1 \\ & & 0 & 0 & -\sin\widetilde{\omega}_1 & \cos\widetilde{\omega}_1 \end{pmatrix}. \qquad (5)$$

Moreover, in this basis matrix **B** *becomes:*

$$\mathbf{B} = \begin{pmatrix} 0 & \dfrac{\widetilde{\omega}_1}{T_0} & \dfrac{1}{T_0} & 0 \\ -\dfrac{\widetilde{\omega}_1}{T_0} & 0 & 0 & \dfrac{1}{T_0} \\ \hline 0 & 0 & 0 & \dfrac{\widetilde{\omega}_1}{T_0} \\ 0 & 0 & \dfrac{\widetilde{\omega}_1}{T_0} & 0 \end{pmatrix}. \qquad (6)$$

6 The quadratic part of the Hamiltonian in the adapted coordinates

In order to compute the normal form around the periodic orbit, we assume that we can introduce a suitable system of canonical coordinates to describe a neighbourhood of this orbit. This adapted system of coordinates has to contain an angular variable θ (and its canonical conjugate variable I) to parameterize the whole periodic orbit. Moreover, we have to introduce other four Cartesian coordinates to describe the normal behavior of the orbit. As we have shown in the previous section, these normal coordinates can be chosen (real or complex) in such a way that the normal variational equations of the orbit are reduced to a constant coefficient system with matrix **B**. Let us remark that as it has been done in [7], in some cases these coordinates can be found explicitly in a practical implementation of this normal form methodology.

Then, if we denote by (q_1, q_2, p_1, p_2) the complex normal variables introduced before (plus a suitable scaling), we have that the quadratic part of the Hamiltonian takes the form:

$$H_2 = \omega_0 I + i\omega_1 (q_1 p_1 + q_2 p_2) + q_2 p_1, \qquad (7)$$

where $\omega_1 = \widetilde{\omega}_1/T_0$. If in equation (7) we use the Floquet *real* variables,

then the quadratic part of the Hamiltonian is,

$$H_2 = \omega_0 I + \omega_1 \left(x_1 y_2 - x_2 y_1\right) + \frac{1}{2} \left(y_1^2 + y_2^2\right), \tag{8}$$

where the variables (x_1, x_2, y_1, y_2) are related with the complex ones, $(\boldsymbol{q}, \boldsymbol{p})$ by

$$q_1 = \frac{x_1 - ix_2}{\sqrt{2}}, \quad q_2 = \frac{y_1 - iy_2}{\sqrt{2}}, \quad p_1 = \frac{y_1 + iy_2}{\sqrt{2}}, \quad p_2 = -\frac{x_1 + ix_2}{\sqrt{2}}, \tag{9}$$

relations which follow immediately from the change of basis in (4). Thus,

$$\bar{q}_1 = -p_2, \quad \bar{p}_1 = q_2, \quad \bar{q}_2 = p_1, \quad \bar{p}_2 = -q_1.$$

The symmetries above imply that if we consider the following expansion of the Hamiltonian,

$$H\left(\theta, q_1, q_2, I, p_1, p_2\right) = \sum_{l, m_1, m_2, n_1, n_2, k} h_{l, m_1, m_2, n_1, n_2, k} e^{ik\theta} I^l q_1^{m_1} q_2^{m_2} p_1^{n_1} p_2^{n_2},$$

then we have the following symmetries coming from the complexification

$$\bar{h}_{l, m_1, m_2, n_1, n_2, k} = (-1)^{m_1 + n_2} h_{l, n_2, n_1, m_2, m_1, -k}.$$

7   Normal form at higher order

Proposition 2 *If we assume that there exist certain real constants $C > 0$ and $\tau > 1$, such that the ω_0, ω_1 satisfy the following Diophantine condition*

$$|\boldsymbol{\omega} \cdot \boldsymbol{k}| \geq \frac{C}{|\boldsymbol{k}|^\tau}, \tag{10}$$

where $|\boldsymbol{k}| = |k_0| + |k_1|$ and $\boldsymbol{\omega} \cdot \boldsymbol{k} = k_0 \omega_0 + k_1 \omega_1$, for all k_0, $k_1 \in \mathbb{Z}$; then the (formal) normal form Λ of the Hamiltonian H, in the complex variables introduced in the last section, takes the form

$$\Lambda = \omega_0 I + i\omega_1 \left(q_1 p_1 + q_2 p_2\right) + q_2 p_1 + \mathcal{N} \left(I, q_1 p_2, i\left(q_1 p_1 + q_2 p_2\right)\right), \tag{11}$$

with $\mathcal{N}(A, B, C)$ being a real formal power series which begins with terms of degree two.

SKETCH OF PROOF. The proof of the proposition 2 is done by using the Lie series method to remove in an increasing order the "non-resonant terms" of

the Hamiltonian. Let us note that this order is defined by counting twice the degree in the I variable with respect to $(\boldsymbol{q}, \boldsymbol{p})$.

1. The homological equation to be solved in any step of the normal form process takes the form:

$$\{H_2, G\} + R = N, \tag{12}$$

where R contains the terms to be removed (of a given degree s) by a suitable G, while N stands for the non-removable terms.

2. Making the calculations one can see that if F is a monomial of type $F = a_{l,m,n,k} e^{ik\theta} I^l q_1^{m_1} q_2^{m_2} p_1^{n_1} p_2^{n_2}$, then

$$L_{H_2} F := -\{H_2, F\} = \left(\Omega_{m_1, m_2, n_1, n_2, k} + m_1 \frac{q_1}{q_2} - n_2 \frac{p_1}{p_2}\right) F, \tag{13}$$

being, by definition

$$\Omega_{m,n,k} = \Omega_{m_1, m_2, n_1, n_2, k} := i\omega_0 k + i\omega_1 (m_1 + m_2 - n_1 - n_2), \tag{14}$$

then, as in addition, the frequencies ω_0 and ω_1 are rational independent, it can be shown that the *necessary* conditions for a monomial to be resonant are

$$k = 0 \quad \text{and} \quad (\Omega_{m,n,0} = 0 \Leftrightarrow m_1 + m_2 = n_1 + n_2). \tag{15}$$

By discussing the structure of the equation (12), we can see that if the above resonance conditions are not fulfilled by the monomial F, one can remove the corresponding term in the homological equations giving the adequate value to the coefficients $a_{l,m,n,k}$ of the generating function G.

3. By a more careful analysis of the algebraic structure of equation (12), one can see that the possible non-removable terms in the normal form are the ones given by

$$f = \sum_{l,M} \sum_{0 \leq m_1, n_2 \leq M} f_{l, m_1, M-m_1, M-n_2, n_2} I^l q_1^{m_1} q_2^{M-m_1} p_1^{M-n_2} p_2^{n_2}. \tag{16}$$

Introducing the new variables,

$$\left.\begin{array}{ll} X = q_1 p_2, & Y = q_2 p_1, \\ Q = i(q_1 p_1 + q_2 p_2), & P = q_1 p_1 - q_2 p_2 \end{array}\right\} \tag{17}$$

(note that X, Y, Q and P are *real* variables), it can be proved (see [11] for details) that the terms in (16) which can not be removed, written in the variables (17) are

$$\sum_{l,M} \sum_{n=0}^{M} g_{l,n,M-n} I^l X^n Q^{M-n}, \tag{18}$$

with $g_{l,n,M-n} \in \mathbb{R}$. Thus, the normal form is a formal power series in the variables I, $q_1 p_2$ and $i(q_1 p_1 + q_2 p_2)$, with real coefficients.

8   The resonant normal form

As stated in proposition 2 the resonant normal form around the periodic orbit that corresponds to the transition from stability to complex instability is given by:

$$\Lambda(\theta, q_1, q_2, I, p_1, p_2) = \omega_0 I + i\omega_1(q_1 p_1 + q_2 p_2) + q_2 p_1 + \mathcal{N}(I, q_1 p_2, i(q_1 p_1 + q_2 p_2)),$$

where $\mathcal{N}(A, B, C)$ is a real formal power series expansion, beginning at degree 2 with respect to (A, B, C). Nevertheless, in what follows we will discuss the properties of $\mathcal{N}$ as if it were a real analytic function, defined around $A = B = C = 0$. In fact, due to the Diophantine conditions, this is so if $\mathcal{N}$ is truncated at some finite order, as usual in practical computations involving normal forms.

In order to simplify the Hamiltonian equations associated to Λ, we shall introduce the following system of canonical coordinates. We replace (q_1, q_2, p_1, p_2) by:

$$
\begin{aligned}
q_1 &= \sqrt{r}\exp(i\varphi), \\
q_2 &= \sqrt{r}\exp(i\varphi)p_r - \frac{\exp(i\varphi)}{2i\sqrt{r}}p_\varphi, \\
p_1 &= \sqrt{r}\exp(-i\varphi)p_r + \frac{\exp(-i\varphi)}{2i\sqrt{r}}p_\varphi, \\
p_2 &= -\sqrt{r}\exp(-i\varphi),
\end{aligned}
$$

where $\varphi \in \mathbb{S}^1$, and where we remark that $r > 0$, p_r and p_φ are real variables. We can write them as:

$$
\begin{aligned}
r &= -q_1 p_2, \\
p_r &= \frac{q_1 p_1 - q_2 p_2}{2r}, \\
p_\varphi &= i(q_1 p_1 + q_2 p_2).
\end{aligned}
$$

Here, (r, φ) are new positions, and (p_r, p_φ) are the corresponding conjugate momenta. If we re-write Λ in these new coordinates, it takes the form:

$$\Lambda(\theta, \varphi, r, I, p_\varphi, p_r) = \omega_0 I + \omega_1 p_\varphi + r p_r^2 + \frac{p_\varphi^2}{4r} + \mathcal{N}(I, -r, p_\varphi).$$

The corresponding Hamiltonian equations are:

$$
\begin{aligned}
\dot{\theta} &= \omega_0 + \partial_1 \mathcal{N}(I, -r, p_\varphi), \\
\dot{\varphi} &= \omega_1 + \frac{p_\varphi}{2r} + \partial_3 \mathcal{N}(I, -r, p_\varphi), \\
\dot{r} &= 2rp_r, \\
\dot{I} &= 0, \\
\dot{p}_\varphi &= 0, \\
\dot{p}_r &= -p_r^2 + \frac{p_\varphi^2}{4r^2} + \partial_2 \mathcal{N}(I, -r, p_\varphi).
\end{aligned}
$$

We have that I and p_φ are first integrals of the resonant normal form. Hence, by taking fixed values of $I \equiv I^0$ and $p_\varphi \equiv p_\varphi^0$, we can reduce the Λ to a one degree-of-freedom Hamiltonian system, given by

$$
\Lambda_0(r, p_r) = r p_r^2 + \frac{(p_\varphi^0)^2}{4r} + \mathcal{N}(I^0, -r, p_\varphi^0).
$$

We can compute:

- 2-dimensional tori of Λ as fixed points of Λ_0.

- 3-dimensional tori of Λ as periodic orbits of Λ_0.

The fixed points of such reduced system are given by $p_r = 0$, and r as a solution of

$$
\frac{(p_\varphi^0)^2}{4r^2} + \partial_2 \mathcal{N}(I^0, -r, p_\varphi^0) = 0. \tag{19}
$$

In order to parameterize the solutions of (19), we introduce a new (dependent) variable N defined as

$$
N = \partial_2 \mathcal{N}(I, -r, p_\varphi).
$$

Then, let us remark that to give sense to equation (19), we have to restrict $N \leq 0$ and $r \geq 0$. Thus, we can parameterize the solutions of (19) as

$$
\begin{aligned}
p_\varphi^\pm(r, N) &= \pm 2r\sqrt{-N}, \tag{20} \\
p_r^\pm(r, N) &= 0, \\
N &= \partial_2 \mathcal{N}(I^\pm(r, N), -r, \pm 2r\sqrt{-N}), \tag{21}
\end{aligned}
$$

where $I^\pm$ as function of (r, N) are obtained from the implicit function theorem if we assume that

$$
\partial_{2,1} \mathcal{N}(0, 0, 0) \neq 0,
$$

Remark 2 *It seems as if we had two different families of fixed points of Λ_0 from (21), and hence, two different families of "bifurcated" 2-dimensional tori from Λ, but we remark that when we put $N = 0$, then $I^+(r, 0) = I^-(r, 0)$. In fact the sign $\pm$ is only associated to the choice of the sign of p_φ in (20).*

Finally, the frequencies of the quasi-periodic motion on the 2-dimensional tori computed are given by

$$
\begin{aligned}
\dot{\theta} &= \Omega_0^\pm \equiv \omega_0 + \partial_1 \mathcal{N}(I^\pm(r, N), -r, \pm 2r\sqrt{-N}), \\
\dot{\varphi} &= \Omega_1^\pm \equiv \omega_1 \pm \sqrt{-N} + \partial_3 \mathcal{N}(I^\pm(r, N), -r, \pm 2r\sqrt{-N}).
\end{aligned}
$$

9 Determination of the stability of the bifurcated 2-dimensional tori

The eigenvalues of the linearized system of Λ_0 around the fixed points computed before are:

$$
\pm \sqrt{4N - 2r\partial_{2,2}\mathcal{N}(I^\pm(r, N), -r, \pm 2r\sqrt{-N})}.
$$

- If $\partial_{2,2}\mathcal{N}(0, 0, 0) > 0$, then the eigenvalues are purely imaginary for small values of $r \geq 0$ and $N \leq 0$, both not simultaneously zero (case of a direct bifurcation).

- If $\partial_{2,2}\mathcal{N}(0, 0, 0) < 0$, then we have a transition from stability to instability characterized by:

$$
4N - 2r\partial_{2,2}\mathcal{N}(I^\pm(r, N), -r, \pm 2r\sqrt{-N}) = 0,
$$

(case of an inverse bifurcation). This expression allows to write r as function of N:

$$
r \equiv r^\pm(N), \text{ with } r^\pm(0) = 0.
$$

Both cases are illustrated in figures 1 and 2. The plots therein are obtained taking only the terms of degree two in $\mathcal{N}$ (fourth order in the Hamiltonian). In both figures, the vertical axis corresponds to $p_\varphi = 0$ (stable tori in the direct bifurcation and unstable in the inverse case); and the horizontal axis contains the periodic orbits, $r = 0$ (stable if $I < 0$ and unstable if $I > 0$).
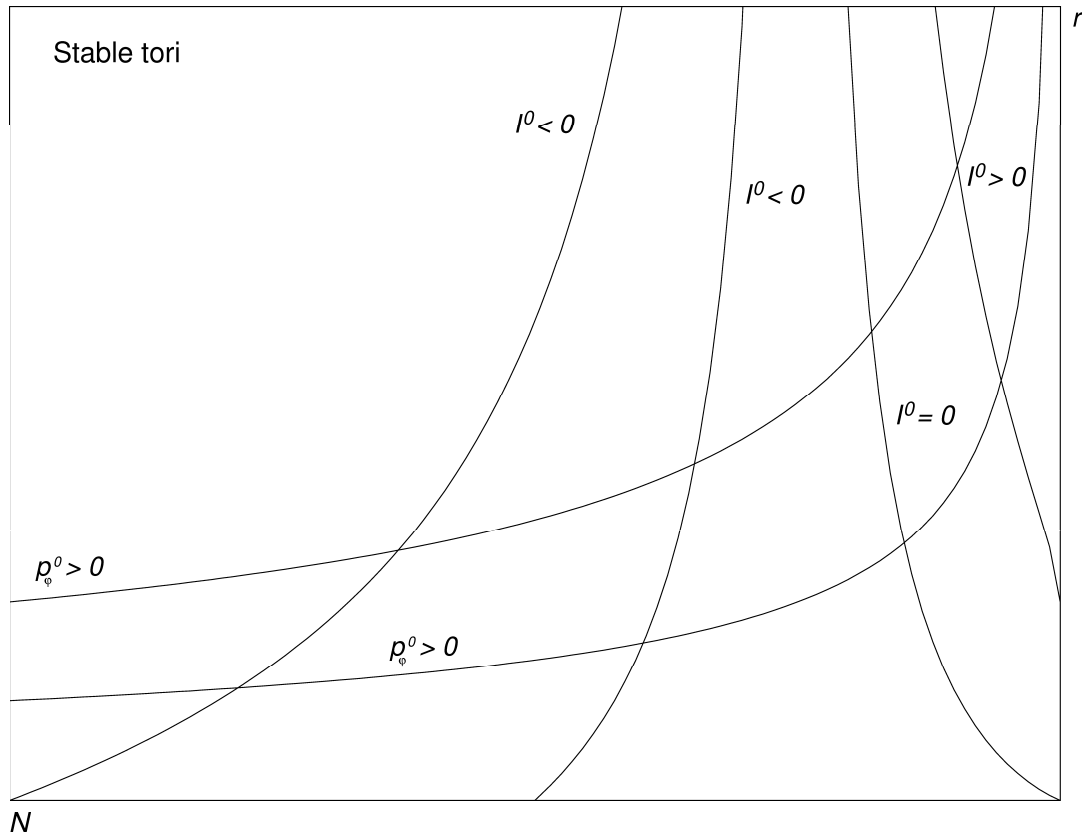
Figure 1: Direct case. The stable tori unfold on the unstable side. The hyperbolas obtained from (20) show families of 2D tori with a fixed value of $p_\varphi = p_\varphi^0$, whereas each "vertical" curve correspond to a fixed $I = I^0$.
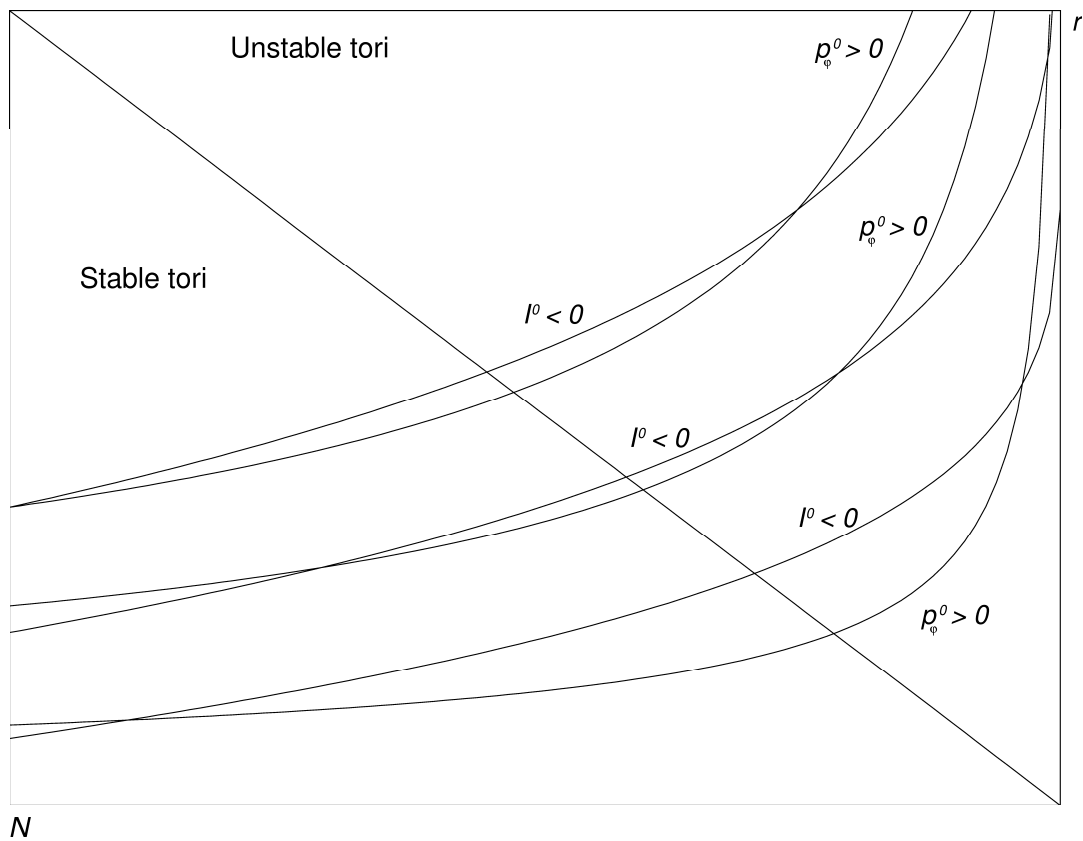
Figure 2: Inverse case. The unstable tori unfold on the stable side. The hyperbolas and the other curves represent families of 2D-tori with $p_\varphi = p_\varphi^0$ and with $I = I^0$. The "transversal line" separates the unstable an stable tori region.

10   Future work

- Theoretical approach.

  1. Bounds on the remainder of the normal form.
  2. Proof of the existence of bifurcated 2D tori (in Cantor sets) for the complete (non-integrable) Hamiltonian.

- Application for a particular example: numerical implementation of the normal form methodology around a transition orbit of the RTBP.

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