# SYMMETRIES OF HOMOGENEOUS COSMOLOGIES

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We reformulate the dynamics of homogeneous cosmologies with a scalar field matter source with an arbitrary self-interaction potential in the language of jet bundles and extensions of vector fields. In this framework, the Bianchi—scalar field equations become subsets of the second Bianchi jet bundle, $J^2$, and every Bianchi cosmology is naturally extended to live on a variety of $J^2$. We are interested in the existence and behaviour of extensions of arbitrary Bianchi-Lie and variational vector fields acting on the Bianchi variety and accordingly we classify all such vector fields corresponding to both Bianchi classes $A$ and $B$. We give examples of functions defined on Bianchi jet bundles which are constant along some Bianchi models (first integrals) and use these to find particular solutions in the Bianchi total space. We discuss how our approach could be used to shed new light to questions like isotropization and the nature of singularities of homogeneous cosmologies by examining the behaviour of the variational vector fields and also give rise to interesting questions about the 'evolution' and nature of the cosmological symmetries themselves.

## 1.   Introduction

An Equivalence Problem asks whether two geometric objects (e.g., manifolds, metrics, differential equations, Lagrangians, cosmological models etc) are the same under a suitable change of variables. Symmetries of a geometric object are defined as *self-equivalences* of the object and the determination of the symmetry *group* of an object is a special case of the general equivalence problem. Two equivalent geometric objects have isomorphic symmetry groups and, indeed, symmetry plays a central role in equivalence since if the symmetry groups of two objects are not isomorphic (e.g., of different dimensionality), the objects cannot be equivalent.

There are two main approaches to Symmetry Theory or Equivalence Problems, that of Sophus Lie and that of Ellie Cartan. Lie's approach originally attempted to classify all possible Lie groups of transformations on one- or two-dimensional manifolds and has recently attracted a lot of attention [1], especially the newly significant role it plays in problems of differential equations and variational calculus. On the other hand, the *coframe equivalence problem*, which includes all other equivalence problems as particular cases, introduced and algorithmically solved by Cartan, although it initially triggered a lot of activity (mainly due to the effort of Cartan's students), subsequently declined in applications due to its calculational complexity. However, very recently it received renewed attention and was revitalized primarily by Olver (see [2] and references therein). Lie's approach is more "analytic" than Cartan's, which is more geometric in nature and is implemented by using the full machinery of differential forms.

Equivalence problems in gravitation are abundant.

For example, the conformal equivalence [3] between different higher order theories of gravitation and general relativity with an additional field is achieved via a Legendre transformation which is a particular case of contact symmetry. Usually the first step to solving an equivalence problem is the determination of symmetries of the object in question. Since Einstein equations are the Euler-Lagrange equations of the Hilbert action functional, many equivalent problems in relativity are bound to involve *variational* (Noether) symmetries, that is, point or generalized transformations which leave the action functional invariant. Since every variational symmetry of a (variational) problem is a (Lie) symmetry of the associated Euler-Lagrange equations, one may first determine the complete symmetry group of the equations and then decide which of these are variational.

A different kind of equivalence problem appearing in relativity is that of classifying the symmetries of particular relativistic *models*. Although there is to date no systematic attempt to tackle such equivalence problems, some interesting work has been done to determine the symmetries of some of the Bianchi cosmological models. In particular, Capozzielo *et al.* [4] classified the variational symmetries of some of the Bianchi Lagrangians and gave (through the Noether theorem) some first integrals (in particular, those for the Bianchi types I and V).

Since the Lie approach is more 'easily' implemented and is less geometric in nature, in this paper we focus on the implementation of this method to the problem of finding the Lie symmetry groups of all Bianchi cosmologies in vacuum or with a scalar field possessing an arbitrary self-interaction potential. We also give, when possible, the associated first integrals, thus complementing the previous work [4].

The plan of this paper is as follows. Sec. 2 gives a brief introduction to the basic ideas of the geometric theory of symmetries which is fundamental to our

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work. In Sec. 3 we apply the theory of Sec. 2 to homogeneous cosmologies and, in particular, recast the Bianchi dynamics in the language of jet bundles and extensions of vector fields and give a full list of symmetries for the Bianchi-scalar models with an arbitrary self-interaction potential. The use of these results is exemplified in Sec. 4 where we give explicit forms of first integrals and construct solutions in many new cases of Bianchi-scalar cosmologies. We conclude in Sec. 5 by pointing out several possible directions for future research.

## 2. Jet bundles, extensions and variational symmetries

Although in this paper we focus on sets of ordinary differential equations, for the purpose of this section it is very convenient to consider a general set of partial differential equations involving $p$ independent variables $x = (x^1, \ldots, x^p)$ and $q$ dependent variables $u = (u^1, \ldots, u^q)$. The *total space* is the Euclidean space[1] $E = M \times N = \{(x, u) : x \in M \subset \mathbb{R}^p, \ u \in N \subset \mathbb{R}^q\} \subset \mathbb{R}^{p+q}$. We restrict attention only to the most elementary type of symmetry that is, *point symmetries* which are defined to be local diffeomorphisms of $E$ onto itself, $g : E \longrightarrow E : (x, u) \mapsto g(x, u)$

$$(\bar{x}, \bar{u}) = g(x, u) = (X(x, u), \Psi(x, u)) \tag{1}$$

which act pointwise on $E$. The set of all such diffeomorphisms forms a group, the *symmetry group* of the total space $E$. A basic example of a point transformation is constructed by starting with a vector field on $E$ (that is a section of $TE$)

$$v = \xi^i(x, u)\partial_{x^i} + \phi^\alpha(x, u)\partial_{u^\alpha},$$
$$i = 1, \ldots, p, \qquad \alpha = 1, \ldots, q \tag{2}$$

and considering its flow $\exp(tv)$. This defines a 1-parameter group of point transformations of the total space $E$ onto itself. (In what follows, in a slight abuse of language, we refer to a symmetry either as $g$ or as its infinitesimal generator $v$.)

A function $f : M \longrightarrow N$, $u = f(x)$, (*ie* a section of the bundle $E$) is *invariant* under a group of transformations $G$ if its graph $\Gamma_f = \{(x, f(x)) : x \in M\} \subset E$ is a $G$-*invariant* subset, which means that $\forall g \in G$ and $(x, u) \in \Gamma_f$, $g(x, u) \in \Gamma_f$. $\Gamma_f$ is a regular $p$-dimensional submanifold of $E$ although only transverse submanifolds of $E$ give (locally) graphs of functions of the total space (see [2], p. 107). We must distinguish between an invariant function $u = f(x)$ and *an invariant* of $G$: A function $I(x, u)$ on $E$ [2] such

that $v(I) = 0$. This is a direct generalisation, on functions defined on $E$, of the usual concept of *an invariant* on $M$, $f(x)$, which means a real-valued function $f \in C^\infty(M)$ such that $v(f) = 0$ where $v = \xi^i(x)\partial_{x^i}$. (In this case we have a first order, linear, homogeneous partial differential equation $v[u] = 0$ which is solved by the method of characteristics.) In all these cases $G$ is the *symmetry group* and members of $G$ are called *the symmetries of the function*.

How do we characterize invariant functions? Since the graph $\Gamma_f$ of an invariant function $u = f(x)$ is determined by the vanishing of its components $u^\alpha - f^\alpha(x)$, $\alpha = 1, \ldots, q$, a direct application of the *infinitesimal invariance criterion* (*cf.* [2], p. 65) gives, using (2),

$$0 = v[u^\alpha - f^\alpha(x)] = \phi^\alpha(x, u) - \xi^i(x, u)\frac{\partial f^\alpha}{\partial x^i},$$
$$\alpha = 1, \ldots, q \tag{3}$$

for every infinitesimal generator $v$ of the form (2). We call the $q$-tuple in the right hand sides of (3) *the characteristic* of the vector field (2) and denote it by $Q^\alpha(x, f^{(1)})$. Therefore (3) states that a function $u = f(x)$ defined on the total space $E$ is invariant under a (connected) group of point transformations if and only if it satisfies the equations

$$Q^\alpha(x, f^{(1)}) = 0, \qquad \alpha = 1, \ldots, q. \tag{4}$$

Since in what follows we shall be dealing with symmetries of differential equations and of variational problems, we need to have a definition of symmetry for functions that depend not only on the $x$'s and $u$'s but also *on the derivatives* of the dependent variables $u^\alpha$ and the independent variables $x^i$. This necessarily takes us away from the total space $E$ with the coordinates $(x, u)$ and into the higher-dimensional analogues of $E$ called jet bundles. This will also require an extension of the infinitesimal invariance criterion (3) to deal with such functions.

Therefore we need to develop some aspects (particularly interesting from the point of view of the applications in the next Sections) of the theory of *jet bundles* and *extension of vector fields*. Furthermore, we shall have to recast differential equations in a more geometric language using the notion of a *variety on the jet bundle*. Below, we give a very rapid overview of only those elements of the relevant theory we need, referring to [1] and [2] for a more complete discussion including references.

The notion of a jet bundle is very simple. We may think of it as a space whose coordinates are the $p$ independent variables $x^i$, the $q$ dependent variables $u^\alpha$ and the derivatives of $u^\alpha$, $\alpha = 1, \ldots, q$, of orders 1 up to and including $n$. This is the $n$-th jet bundle of the total space $E = M \times N \subset \mathbb{R}^p \times \mathbb{R}^q$ which we denote by $J^n = J^n E = M \times N^{(n)}$ where $M$ is the (base) space

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[1]In the language of vector bundles $E$ is replaced by a vector bundle over the base space $M$ (see [1]) and $M$ and $N$ are differential manifolds, not necessarily submanifolds of Euclidean space.

[2]Notice that for this definition $x$ and $u$ are not necessarily dependent variables.

of the independent variables and $N^{(n)}$ is the fiber containing the remaining variables (dependent plus their derivatives of orders 1 to $n$). (A more rigorous way is to define $J^n$ as the set of equivalence classes of $C^\infty$ functions wherein two functions are equivalent at $x$ if and only if they are in $n$-th order contact at $x$, meaning that their Taylor polynomials at $x$ of order $n$ are identical.)

Having 'extended' the total space $E$ to the jet bundle $J^n E$, it is natural to define the *n-th extension,* $f^{(n)}$, *of a function* $f : M \longrightarrow N$ to be a section of $J^n$, *ie*, $f^{(n)} : M \longrightarrow N^{(n)}$ is defined by evaluating all partial derivatives of $f$ of order 1 to $n$. The graph of the extended function $f^{(n)}$, $\Gamma_{f^{(n)}} = \{(x, f^{(n)}(x))\}$, will similarly be a $p$-dimensional submanifold of $J^n$. Since the point transformation (1) will also act on the derivatives of functions $f : M \longrightarrow N$, we can define the *induced extended (point) transformation* $g^{(n)} : J^n \longrightarrow J^n$ by

$$\left(\bar{x}, \bar{u}^{(n)}\right) = g^{(n)}\left(x, u^{(n)}\right) \qquad (5)$$

on the $n$-th jet space. This will transform the graphs of *extended* functions giving $g^{(n)}\Gamma_{f^{(n)}} = \Gamma_{g(f)^{(n)}}$ ([2], p. 113).

A smooth real-valued function $F : J^n \longrightarrow \mathbb{R}$ (or on an open subset of $J^n$) is called a *differential function of order* $n$. An $n$-th order differential equation is defined by the vanishing of an $n$-th order differential function. The *total derivative,* $D_{x^i}F$, of a differential function of order $n$ with respect to $x^i$, is an $(n+1)$-th order differential function defined in the usual way. For instance, in the case of one independent variable $x$ and one dependent variable $u$, we have the following formula for the total derivative of $F(x, u^{(n)})$ with respect to $x$:

$$D_x F = F_x + u_x F_u + u_{xx} F_{u_x} + u_{xxx} F_{u_{xx}} + \ldots. \qquad (6)$$

Obviously vector fields on $E$ can also be extended to vector fields on $J^n E$. The *n-th extension of a vector field* $v$ with the characteristic $Q = (Q^1, \ldots, Q^q)$ has the following form (see [1] for a proof):

$$v^{(n)} = \xi^i(x, u)\partial_{x^i} + \phi_J{}^\alpha(x, u^{(j)})\partial_{u_J{}^\alpha}, \qquad (7)$$

where $\phi_J{}^\alpha = D_J Q^\alpha + \xi^i u_{J,i}^\alpha$, with $i = 1, \ldots, p$, $\alpha = 1, \ldots, q$, $|J| = j = 0, \ldots, n$ ($J$ is an obvious multi-index) and summation over repeated indices is implied in the usual way.

A point symmetry of a system of (partial) differential equations is a point transformation $g : E \longrightarrow E$ with the property that, if $u = f(x)$, is a solution, then the transformed function $\bar{u} = \bar{f}(\bar{x})$ is also a solution. Suppose now that we are given a family of differential functions $\Delta_\beta : J^n \longrightarrow R : (x, u^{(n)}) \mapsto \Delta_\beta(x, u^{(n)})$, indexed by $\beta = 1, \ldots, m$. A set of differential equations of order $n$ is defined by the simultaneous vanishing of a given family of differential functions:

$$\Delta_\beta(x, u^{(n)}) = 0. \qquad (8)$$

It is very important to view these equations as defining, or defined by, a *variety*

$$S_\Delta = \{(x, u^{(n)}) : \ \Delta_\beta(x, u^{(n)}) = 0, \ \beta = 1, \ldots, m\}, (9)$$

which is a subset of $J^n$ consisting of all points of $J^n$ which simultaneously satisfy Eqs. (8). Thus in our geometric reformulation a set of differential equations is a *(sub)set of some space*. A *solution* of Eqs. (8) is a function $u = f(x)$ on $E$ such that the graph of its $n^{th}$ extension, $\Gamma_{f^{(n)}}$, lies entirely on $S_\Delta$ (this is translated traditionally as 'the function $u = f(x)$ identically satisfies Eqs. (8)'.

It follows that the defining property of a symmetry of a differential equation discussed above, as a transformation that maps solutions into solutions, can be geometrically reformulated in an elegant manner by simply requiring that

$$g^{(n)}(S_\Delta) \subset S_\Delta. \qquad (10)$$

On the other hand, the infinitesimal invariance criterion, (3), immediately implies that the fundamental condition for $v$ to be a symmetry of the fully regular[3] system of differential equations ([2], p. 179) is

$$v^{(n)}(\Delta_\beta) = 0. \qquad (11)$$

Notice that $v^n$ acts only on those points on $S_\Delta$ which lie on a solution. Eq. (11) typically results in a large, overdetermined, linear system of partial differential equations and its solution requires very lengthy calculations which are usually performed using some of the computer algebra packages available (see below, also [5]).

In (2) $\xi^i$ and $\phi^\alpha$ depend only on the total space coordinates $x$ and $u$, and this was assumed by Lie [6]. Subsequently, he extended that dependence to include first or higher order derivatives of the functions $u^\alpha$, $\alpha = 1, \ldots, q$. Such generalized vector fields produce the so called *contact* and *generalized* symmetries. More recently, *nonlocal symmetries* [7, 8], containing *integrals* of the dependent variable(s), have found their use. Thus there is a hierarchical sequence of symmetries: point, contact, generalized, nonlocal.

Once the symmetries of a differential equation have been determined, one may effectively reduce the order of the equation by calculating first integrals. For a set of differential equations, $\Delta_\beta(x, u^{(n)}) = 0$, *a first integral* is a function $I(x, u^{(m)})$ which is constant along solutions, that is

$$D_x I = 0. \qquad (12)$$

This equation is typically a linear partial differential equation and the associated characteristic system provides a means to calculate all first integrals. If all of them can be determined, (8) is said to be integrated.

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[3]This, roughly speaking, means that the system *has* solutions

The complete solution of the (nontrivial) characteristic system of (18) is rarely trivial and one seeks to ease the process of solution by imposing the condition that any first integral be associated with a symmetry of (8), i.e., we require that

$$v^{(n)}(I) = 0, \tag{13}$$

in addition to (8). This equation is also a linear partial differential equation for I and so the number of characteristics is further reduced. Usually, however, the existence of suitable symmetries (not necessarily point ones) equal in number to the order of the system is required, but the cost is the reduction of one first integral for each requirement of invariance under symmetry imposed.

The use of Lie symmetries to determine first integrals can be extremely difficult since the calculations to be performed *after* the symmetries are obtained are usually difficult. The fundamental theorem of Noether [9] provides an attractive alternative. A point (or generalized) transformation is called a *variational (or Noether) symmetry* of the functional

$$A[u] = \int_{\Omega} L(x, u^{(n)}) \mathrm{d}x, \qquad \Omega \subset M \tag{14}$$

if and only if the transformed functional agrees with the original one. The set of all variational symmetries forms a group and this *variational symmetry group* is a symmetry group of the associated Euler-Lagrange equations but not conversely (cf. [2], p. 236). The infinitesimal invariance criterion applied to a (connected) transformation group G gives the basic condition for G to be a variational symmetry group of $A[u]$, namely,

$$v^{(n)}(L) + L \operatorname{div} \xi = 0, \tag{15}$$

for every infinitesimal generator $v$, where $L$ is the Lagrangian of the variational problem and $\xi = (\xi^i)$ is the vector of basic components in (2) (*cf.* [2], p. 236).

As in the case of the Lie symmetries of differential equations, the variational symmetries and consequent variational first integrals impose no requirements on the coefficient functions $\xi$ and $\phi$ apart from differentiability. Variational symmetries can be point, generalized (including contact) or nonlocal. The latter are not of practical use for a local Lagrangian unless the nonlocal terms cancel in such a way that the first integral is local.

Given the ease of computation of variational integrals once the variational symmetries are known, one may wonder at the interest in the more difficult calculation of first integrals using the Lie symmetries of the differential equation. For instance, provided the Hessian of the Lagrangian with respect to the first derivatives is nonsingular, the Euler-Lagrange system is $2n$-dimensional and gives a regular Hamiltonian system. Each variational symmetry reduces the dimension of the system by two. If there exist $n$ independent variational integrals, there are $n$ integrals in involution and the system is integrable according to Liouville's Theorem.

Consequently, variational symmetries are very attractive. However, the calculation of variational symmetries is a closed procedure only in the case of variational *point* symmetries. In the case of generalized symmetries some *ansatz* must be made and there is always a possibility that some symmetry and so its integral will be missed. (There are cases for which variational symmetries are known to be generalized and yet the integral corresponds to a Lie point symmetry. Perhaps the best-known of these are the Laplace-Runge-Lenz components which correspond to generalized variational symmetries linear in the derivatives whereas the corresponding Lie symmetry is point [10].) The fear of incompleteness in the knowledge of variational symmetries in complex systems is a sufficient incentive to supplement the variational method with the Lie method.

In this paper, we restrict ourselves to point symmetries in both the variational and Lie approaches. (The calculation of other types of symmetry is extremely difficult even for scalar ordinary differential equations and often does not give more information than obtained from the point symmetries.) We have remarked that the variational integrals follow easily once the variational symmetries are known. We use the Lie method to supplement the variational approach. Indeed, there are some instances in which we can combine the two methods in the case that a variational integral is invariant under a Lie symmetry, which may or may not be the variational symmetry of the integral. The knowledge of this integral, both Lie and variational, can help one in solving the associated Lagrange system for the other integrals.

Our strategy is now clear and will be described in detail in the next sections. We start by defining the Bianchi total space $E_B$ to be the space including time plus the dependent variables, extend $E_B$ to the second Bianchi jet bundle $J_B^2$ and consider extensions of all basic functions defined on the Bianchi total space (such as the Ricci scalar, *etc.*) to $J_B^2$. In this way the field equations become conditions on $J_B^2$ defining a Bianchi variety, $S_{\Delta,B}$. This is a subset of the second Bianchi jet bundle containing all Bianchi models and consequently the structure of $S_{\Delta,B}$ is of fundamental importance to our work. The point symmetries analysed in the next section have the property that, when extended, they leave $S_{\Delta,B}$ invariant. (This property is also shared by any contact vector fields although in this case (which do not consider in this paper) the Bianchi Lagrangian forms will not be invariants of the Bianchi jet bundle and will have to be modified to include Cartan extensions known as coframes (*cf.* [2]).)

## 3.  Point symmetries of homogeneous cosmologies

In Bianchi homogeneous but anisotropic models the spacetime metric splits so that the spatial (time–dependent) part is given by

$$g_{ab}(t) = \exp(2\lambda) \exp(-2\beta)_{ab}$$

where $\lambda$ plays the role of a time (volume) parameter and $\beta$ is a $3 \times 3$ symmetric, traceless matrix which can be written in a diagonal form with two independent components by introducing the two *anisotropy parameters* $\beta_+$, $\beta_-$:

$$\beta_{ab} = \mathrm{diag}\left(\beta_+, -\frac{1}{2}\beta_+ + \frac{\sqrt{3}}{2}\beta_-, -\frac{1}{2}\beta_+ - \frac{\sqrt{3}}{2}\beta_-\right).$$

The general Lagrangian leading to the full Bianchi–scalar dynamics has the form (see e.g. [11])

$$\mathcal{L} = \mathrm{e}^{3\lambda}\left[R^* + 6\dot{\lambda}^2 - \tfrac{3}{2}\left(\dot{\beta}_1^2 + \dot{\beta}_2^2\right) - \dot{\phi}^2 + 2V\left(\phi\right)\right], \tag{16}$$

where $R^*$ is the Ricci scalar playing the role of a potential term. The *Bianchi* total space is $E_B = \{(t, \lambda, \beta_+, \beta_-, \phi)\} \subset \mathbb{R}^5$. Then the first and second Bianchi jet bundles are $J_B^1 = E_B \times \{(\dot{\lambda}, \dot{\beta}_+, \dot{\beta}_2, \dot{\phi})\} \subset \mathbb{R}^9$ and $J_B^2 = J_B^1 \times \{(\ddot{\lambda}, \ddot{\beta}_+, \ddot{\beta}_-, \ddot{\phi})\} \subset \mathbb{R}^{13}$.

The Euler-Lagrange equations for (16) can be considered as Eqs. (8) with $m = 4$ and $n = 2$. Explicitly (8) becomes

$$\ddot{\lambda} + \frac{3}{2}\dot{\lambda}^2 + \frac{3}{8}\left(\dot{\beta}_1^2 + \dot{\beta}_2^2\right) + \frac{1}{4}\dot{\phi}^2$$
$$- \frac{1}{12}\mathrm{e}^{-3\lambda}\frac{\partial}{\partial\lambda}\left(\mathrm{e}^{3\lambda}R^*\right) - \frac{1}{2}V\left(\phi\right) = 0,$$
$$\ddot{\beta}_1 + 3\dot{\beta}_1\dot{\lambda} + \frac{1}{3}\frac{\partial R^*}{\partial\beta_1} = 0$$
$$\ddot{\beta}_2 + 3\dot{\beta}_2\dot{\lambda} + \frac{1}{3}\frac{\partial R^*}{\partial\beta_2} = 0$$
$$\ddot{\phi} + 3\dot{\phi}\dot{\lambda} + V' = 0. \tag{17}$$

For Bianchi class A models the Ricci scalar $R^*$ as a function on a four-dimensional hypersurface of the Bianchi total space $E_B$ has the explicit form

$$R^* = -\frac{1}{2}\mathrm{e}^{-2\lambda}\left[N_1{}^2\,\mathrm{e}^{4\beta_1}\right.$$
$$+ \mathrm{e}^{-2\beta_1}\left(N_2\,\mathrm{e}^{\sqrt{3}\beta_2} - N_3\,\mathrm{e}^{-\sqrt{3}\beta_2}\right)^2$$
$$\left. - 2N_1\,\mathrm{e}^{\beta_1}\left(N_2\,\mathrm{e}^{\sqrt{3}\beta_2} + N_3\,\mathrm{e}^{-\sqrt{3}\beta_2}\right)\right]$$
$$+ \frac{1}{2}N_1N_2N_3\left(1 + N_1N_2N_3\right), \tag{18}$$

and for class B

$$R^* = 2a^2\,\mathrm{e}^{-2\lambda}\left(3 - \frac{N_2N_3}{a^2}\right)\mathrm{e}^{\beta} \tag{19}$$

with

$$\beta = \frac{2}{3a^2 - N_2N_3}\left(N_2N_3\beta_1 + \sqrt{-3a^2N_2N_3}\beta_2\right), \tag{20}$$

where $a, N_1, N_2, N_3$ are the usual classification constants. For the subsequent symmetry analysis we set

$$u = \mathrm{e}^{\lambda}, \qquad v = \mathrm{e}^{\beta_1}, \qquad w = \mathrm{e}^{\sqrt{3}\beta_2}. \tag{21}$$

In what follows we list the relevant symmetries of different Bianchi models in four cases according to different matter couplings:

**Case 1: Vacuum.** It is obvious that the potential and $\phi$ terms are missing in the $u$ equation and also there is no $\phi$ equation. Hence we have a set of three equations with three dependent variables $u$, $v$ and $w$. This is the easiest case of all the models and, once having examined it, one is able to make inferences about the forms of the symmetries in the subsequent cases.

**Case 2: Scalar field with zero potential.** In this case we have a set of four equations with four dependent variables $u, v, w$ and $\phi$. The potential term is missing in the $u$ and $\phi$ equations.

**Case 3: Constant potential.** The equations have the same form as in Case 2 plus a constant potential term in the $u$ equation.

**Case 4: Arbitrary potential.** It is the most general case. We have a set of four equations with four dependent variables. Moreover, in the $u$ equation there is a potential function and in the $\phi$ equation we have a derivative of this potential with respect to $\phi$.

As we shall see, extra symmetries arise in cases 2–4. This is because the first three equations in (23) have $\phi$ as an ignorable coordinate in cases 2 and 3 and so this is a symmetry of the fourth equation in (23). In case 1, $\phi$ is not an argument of the coefficient functions.

We also note that the actual calculation of Lie symmetries in this paper use, to a certain extent, the package LIE (cf. [12]) which has been around for twenty years. The equation and type of symmetry sought are fed into an input file. The program computes the determining equations and then attempts to solve them. Usually, however, this proves impossible and in this case the operator can intervene manually.

### Bianchi Type I

This is the easiest case since the Ricci scalar is zero.

**Case 1:** The Noether symmetries are

$$\partial_t, \qquad v\partial_v, \qquad w\partial_w, \qquad v\log w\partial_v - 3w\log v\partial_w.$$

The additional Lie symmetries are $t\partial_t$, $ut\partial_u + \frac{3}{2}t^2\partial_t$ and $u\partial_u$.

**Case 2:** The extra variable $\phi$ gives three additional symmetries which are also Noether symmetries:

$$\partial_t, \quad v\partial_v, \quad w\partial_w, \quad \partial_\phi, \quad v\log w\,\partial_v - 3w\log v\,\partial_w,$$
$$v\phi\partial_v - \tfrac{3}{2}\log v\partial_\phi, \qquad w\phi\partial_w - \tfrac{1}{2}\log w\partial_\phi.$$

The additional Lie symmetries are

$$t\partial_t, \qquad ut\partial_u + \tfrac{3}{2}t^2\partial_t \ \text{ and } \ u\partial_u.$$

**Case 3:** The extra term of the constant potential $V(\phi) = C$ affects the results and we obtain the following Noether symmetries:

$$\partial_t, \quad v\partial_v, \quad w\partial_w, \quad \partial_\phi, \quad v\log w\partial_v - 3w\log v\partial_w,$$
$$v\phi\partial_v - \tfrac{3}{2}\log v\partial_\phi, \quad w\phi\partial_w - \tfrac{1}{2}\log w\partial_\phi,$$

and the Lie symmetries

$$u\partial_u, \quad \mathrm{e}^{\sqrt{3}Ct}\partial_t + \mathrm{e}^{\sqrt{3}Ct}u\partial_u, \quad \mathrm{e}^{\sqrt{3}Ct}\partial_t - \mathrm{e}^{\sqrt{3}Ct}u\partial_u.$$

**Case 4:** Since this is the most general case, the number of symmetries is obviously reduced. The Noether symmetries are

$$\partial_t, \qquad v\partial_v, \qquad w\partial_w, \qquad v\log w\partial_v - 3w\log v\partial_w,$$

and the additional Lie symmetry is $u\partial_u$.

### Bianchi Type II

The Ricci scalar takes the form $R^* = -\mathrm{e}^{(4\beta_1 - 2\lambda)/2}$.

**Case 1:** The system has the Noether symmetries $\partial_t$ and $w\partial_w$. The additional Lie ones are $t\partial_t + u\partial_u$ and $-2t\partial_t + v\partial_v$.

**Case 2:** We obtain the additional Noether symmetry $\partial_\phi$ and the additional Lie symmetry $w\phi\partial_w + \tfrac{1}{2}\log w\partial_\phi$.

**Case 3:** In this case the model has the Noether symmetries $\partial_t$, $w\partial_w$ and $\partial_\phi$, and the additional Lie symmetry $w\phi\partial_w + \tfrac{1}{2}\log w\partial_\phi$.

**Case 4:** In this case we obtain $\partial_t$ and $w\partial_w$ as Noether symmetries and $u\partial_u + \tfrac{1}{2}v\partial_v$ is a Lie symmetry.

### Bianchi Type III

The Ricci scalar is $R^* = 8\,\mathrm{e}^{-2\lambda}\,\mathrm{e}^{\frac{1}{2}(\sqrt{3}\beta_2 - \beta_1)}$. The symmetry $\partial_t$ is always a Noether one and $\partial_\phi$ is also Noether for the cases where it exists.

**Case 1:** The Lie point symmetries are

$$\partial_t, \quad t\partial_t + 4v\partial_v, \quad u\partial_u + 4w\partial_w, \quad u\partial_u - 4v\partial_v.$$

**Case 2:** In this case we obtain the following Lie point symmetries:

$$\partial_t, \quad t\partial_t + 4v\partial_v, \quad u\partial_u + 4w\partial_w, \quad u\partial_u - 4v\partial_v,$$
$$\partial_\phi, \qquad v\phi\partial_v + w\phi\partial_w - \tfrac{1}{2}\left(3\log v + \log w\right)\partial_\phi.$$

**Case 3:** The Lie point symmetries are

$$\partial_t, \qquad u\partial_u + 4w\partial_w, \qquad \partial_\phi, \qquad u\partial_u - 4v\partial_v.$$

**Case 4:** Arbitrary potential. We obtain the same symmetries as in the previous case apart from the $\partial_\phi$ symmetry, which is lost.

### Bianchi Type V

The Ricci scalar has the form $R^* = 6\,\mathrm{e}^{-2\lambda}$.

**Case 1:** We obtain the Noether symmetries $\partial_t$, $v\partial_v$ and $w\partial_w$. The additional Lie symmetries are $t\partial_t + u\partial_u$ and $v\log w\partial_v - 3w\log v\partial_w$.

**Case 2:** We find the additional Lie point symmetries

$$2v\phi\partial_v - 3\log v\partial_\phi, \qquad 2w\phi\partial_w - \log w\partial_\phi$$

plus the Noether symmetry $\partial_\phi$.

**Case 3:** We obtain the Noether symmetries

$$\partial_t, \qquad v\partial_v, \qquad w\partial_w, \qquad \partial_\phi.$$

The additional Lie symmetries are

$$v\log w\partial_v - 3w\log v\partial_w, \qquad v\phi\partial_v - \tfrac{3}{2}\log v\partial_\phi,$$
$$w\phi\partial_w - \tfrac{1}{2}\log w\partial_\phi.$$

**Case 4:** The additional Lie point symmetries are

$$v\partial_v, \qquad w\partial_w, \qquad v\ln w\partial_v - 3w\ln v\partial_w.$$

There is only one Noether symmetry, $\partial_t$.

### Bianchi Type VI, class A

The Ricci scalar has the form

$$R^* = -\tfrac{1}{2}\,\mathrm{e}^{-2\lambda}\left[\mathrm{e}^{4\beta_1} + \mathrm{e}^{-2(\beta_1 - \sqrt{3}\beta_2)} + 2\,\mathrm{e}^{\beta_1 + \sqrt{3}\beta_2}\right].$$

The only Noether symmetries are $\partial_t$ and $\partial_\phi$ in the cases where it exists.

**Case 1:** The Lie symmetries are

$$\partial_t, \quad t\partial_t - \tfrac{1}{2}v\partial_v - \tfrac{3}{2}w\partial_w, \quad u\partial_u + \tfrac{1}{2}v\partial_v + \tfrac{3}{2}w\partial_w.$$

**Case 2:** There are the same Lie point symmetries as in the previous case.

**Case 3:** We find the following Lie point symmetries:

$$\partial_t, \qquad u\partial_u + \tfrac{1}{2}v\partial_v + \tfrac{3}{2}w\partial_w, \qquad \partial_\phi.$$

**Case 4:** The system has the Lie symmetry $u\partial_u + \tfrac{1}{2}v\partial_v + \tfrac{3}{2}w\partial_w$.

## Bianchi Type VI, class B

The Ricci scalar takes the form

$$R^* = 2\left(3a^2+1\right)\mathrm{e}^{-2\lambda}\exp\left[\frac{2}{3a^2+1}\left(\sqrt{3}a\beta_2 - \beta_1\right)\right].$$

This model has only the standard Noether symmetries. If we apply Program `LIE` to this system, a number of difficulties appear. We set

$$\omega = w^{2a/(3a^2+1)}, \qquad n = v^{2/(3a^2+1)} \tag{22}$$

and further simplify the system by defining the two constants

$$B = \tfrac{1}{2}(3a^2+1), \qquad C = 1/(3a^2). \tag{23}$$

**Case 1:** In this case we obtain the following Lie point symmetries:

$$\partial_t, \qquad t\partial_t + u\partial_u,$$
$$-\tfrac{1}{2}u\partial_u + n\partial_n, \qquad \tfrac{1}{2}u\partial_u + \omega\partial_\omega.$$

**Case 2:** The Lie point symmetries are

$$\partial_t, \qquad t\partial_t + u\partial_u, \qquad -\tfrac{1}{2}u\partial_u + n\partial_n,$$
$$\tfrac{1}{2}u\partial_u + \omega\partial_\omega, \qquad \partial_\phi,$$
$$n\phi\partial_n + \omega\phi\partial_\omega - \tfrac{3}{2}B^2 C\log\omega\ \partial_\phi - \tfrac{3}{2}B^2\log n\ \partial_\phi.$$

**Case 3:** In this case we find the following results:

$$\partial_t, \qquad -\tfrac{1}{2}u\partial_u + n\partial_n, \qquad \tfrac{1}{2}u\partial_u + \omega\partial_\omega\partial_\phi,$$
$$n\phi\partial_n + \omega\phi\partial_\omega - \tfrac{3}{2}B^2 C\log\omega\partial_\phi - \tfrac{3}{2}B^2\log n\partial_\phi.$$

**Case 4:** We obtain the symmetries

$$\partial_t, \qquad \tfrac{1}{2}u\partial_u + n\partial_n, \qquad \tfrac{1}{2}u\partial_u + \omega\partial_\omega.$$

The only Noether symmetries are the usual ones.

## Bianchi Type VII

In this model the Ricci scalar takes the form

$$R^* = -\tfrac{1}{2}\mathrm{e}^{-2\lambda}\left(\mathrm{e}^{4\beta_1} + \mathrm{e}^{-2\left(\beta_1 - \sqrt{3}\beta_2\right)} - 2\,\mathrm{e}^{\beta_1 + \sqrt{3}\beta_2}\right).$$

**Case 1:** There are three Lie point symmetries:

$$\partial_t, \qquad t\partial_t - \tfrac{1}{2}v\partial_v - \tfrac{3}{2}w\partial_w, \qquad u\partial_u + \tfrac{1}{2}v\partial_v + \tfrac{3}{2}u\partial_u \tag{24}$$

**Case 2:** We have the additional symmetry $\partial_\phi$.

**Case 3:** The symmetries of the system are

$$\partial_t, \qquad u\partial_u + \tfrac{1}{2}v\partial_v + \tfrac{3}{2}w\partial_w, \qquad \partial_\phi.$$

**Case 4:** We lose the $\partial_\phi$ symmetry of the previous case. In all cases the only Noether symmetries are the standard ones.

## Bianchi Type VIII

The Ricci scalar is

$$R^* = -\tfrac{1}{2}\mathrm{e}^{-2\lambda}\left[\mathrm{e}^{4\beta_1} + \mathrm{e}^{-2\beta_1}\left(\mathrm{e}^{\sqrt{3}\beta_2} + \mathrm{e}^{-\sqrt{3}\beta_2}\right)^2 \right.$$
$$\left. - 2\,\mathrm{e}^{\beta_1}\left(\mathrm{e}^{\sqrt{3}\beta_2} - \mathrm{e}^{-\sqrt{3}\beta_2}x\right)\right].$$

**Case 1:** The symmetries of the system are $\partial_t$ and $t\partial_t + u\partial_u$.

**Case 2:** We obtain the additional symmetry $\partial_\phi$.

**Case 3:** The system has only the two symmetries $\partial_t$ and $\partial_\phi$.

**Case 4:** The only symmetry is $\partial_t$.

## Bianchi Type IX

The Ricci scalar is

$$R^* = -\tfrac{1}{2}\mathrm{e}^{-2\lambda}\left[\mathrm{e}^{4\beta_1} + \mathrm{e}^{-2\beta_1}\left(\mathrm{e}^{\sqrt{3}\beta_2} - \mathrm{e}^{-\sqrt{3}\beta_2}\right)^2 \right.$$
$$\left. - 2\,\mathrm{e}^{\beta_1}\left(\mathrm{e}^{\sqrt{3}\beta_2} + \mathrm{e}^{-\sqrt{3}\beta_2}x\right)\right] + 1.$$

In Cases 1 and 4 the system has the single symmetry $\partial_t$. The other two cases have the additional symmetry $\partial_\phi$.

We now pass on to giving some examples of first integrals associated with the symmetries obtained, postponing their discussion till Section V.

## 4. Applications

In this section we use the general theory developed in Sec. 1 for obtaining first integrals corresponding to symmetries for Bianchi Types I, III and V and integrate to find some solutions. It is of interest that the expressions for the integrals are almost the same for Cases 2 and 3. We also describe how the existence of an additional variational symmetry restricts the possible forms the potential can take to simple exponentials.

### 4.1. Examples of first integrals

#### 4.1.1. Bianchi Type I

**Case 1:** The results are given collectively in Table 1 where

$$f_1(u) = \frac{1}{16}\left(I_1^2 + I_2^2\right)\left[(\log u)^3 \right.$$
$$\left. + \frac{1}{2}(\log u)^2 + \frac{1}{6}\log u + \frac{1}{36}\right]u^{-6}.$$

Using $I_3$, we find

$$\dot{u} = \sqrt{g_1(u)} \quad \Longleftrightarrow \quad t - t_0 = \int \frac{\mathrm{d}u}{\sqrt{g_1(u)}} \tag{25}$$

Table 1: First integrals for the Bianchi model type I in Case 1 where there is no matter.

| Symmetry | Integral |
|---|---|
| $v\partial_v$ | $I_1 = u^3\dot{v}/v$ |
| $w\partial_w$ | $I_2 = u^3\dot{w}/w$ |
| $v\log w\partial_v - 3w\log v\partial_w$ | $I_3 = \frac{1}{2}(\log u)^3\frac{\dot{u}^2}{u} - f_1(u)$ |

Table 2: First integrals for the Bianchi model type I in the case where there is matter but no potential.

| Symmetry | Integral |
|---|---|
| $v\partial_v$ | $I_1 = u^3\dot{v}/v$ |
| $w\partial_w$ | $I_2 = u^3\dot{w}/w$ |
| $\partial_\phi$ | $I_3 = u^3\dot{\phi}$ |
| $v\log w\ \partial_v$ | $I_4 = \frac{u^3}{I_3}\left[\frac{\dot{u}^2}{u} - \frac{1}{4}I_1 - \frac{1}{6}I_3^2 u^{-6}\right]$ |
| $-3w\log v\ \partial_w$ | $I_5 = t - \alpha\mathrm{arcsinh}\frac{u^3+C}{\beta}$ |

where

$$g_1(u) = \frac{2u^2 I_3}{(\log u)^3} + \frac{2u^2 f_1(u)}{(\log u)^3}. \tag{26}$$

We formally invert (25) to find $u(t)$. The variables $v$ and $w$ follow from the integrals $I_1$ and $I_2$, respectively. Hence we have

$$v = \exp\int\frac{I_1}{u^3}\mathrm{d}t, \qquad w = \exp\int\frac{I_2}{u^3}\mathrm{d}t.$$

We note that in this case we find the solution using only three integrals since they are separable Noetherian integrals.

**Case 2:** Setting

$$\alpha = \frac{2}{3\sqrt{K}}, \qquad \beta = \left[\frac{2}{3}\frac{I_3^2}{K^2}\left(K - 6I_4^2\right)\right]^{1/2},$$

$$C = \frac{2I_3 I_4}{K}, \qquad K = \frac{I_1^2 + I_2^2}{I_3^2},$$

we obtain the integrals listed in Table 2. (Note that $I_5$ may have arccosh instead of arcsinh, as we find after the calculation of the Lie integrals.) Integrating, we obtain the following solutions:

$$u = [\beta\sinh(t - I_5) - C]^{1/3},$$

$$v = \exp\int\frac{I_1}{u^3}\mathrm{d}t$$

$$= \exp\left\{I_1\frac{1}{\sqrt{\beta^2 + C^2}}\right.$$

$$\left.\times\log\frac{\beta\tanh[(t - I_5)/2] - \beta + \sqrt{\beta^2 + C^2}}{-C\tanh[(t - I_5)/2] - \beta - \sqrt{\beta^2 + C^2}}\right\},$$

$$w = \exp\int\frac{I_2}{u^3}\mathrm{d}t$$

$$= \exp\left\{I_2\frac{1}{\sqrt{\beta^2 + C^2}}\right.$$

$$\left.\times\log\frac{\beta\tanh[(t - I_5)/2] - \beta + \sqrt{\beta^2 + C^2}}{-C\tanh[(t - I_5)/2] - \beta - \sqrt{\beta^2 + C^2}}\right\}.$$

**Case 3:** The integrals have the same form as in the previous case, the only difference being in the expression for $\beta$ where an extra term due to the potential is present,

$$\beta = \left[\frac{2}{3}\frac{I_3^2}{K^2}\left(K - 6I_4^2\right) - \frac{C}{2K}\right]^{1/2}. \tag{27}$$

### 4.1.2.  Bianchi Type III

The integrals in Cases 2 and 3 are essentially the same. The integration becomes more involved in Case 3.

**Case 1:** With the notations

$$\alpha = -2\log u - \frac{1}{2}\log v + \frac{1}{2}\log w,$$

$$\beta = \sqrt{3}\log v + \frac{1}{\sqrt{3}}\log w,$$

$$\gamma = \log u, \tag{28}$$

the symmetry $v\partial_v + w\partial_w$ transforms to $\partial_\beta$ with the associated Lagrange system

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}\alpha}{0} = \frac{\mathrm{d}\beta}{1} = \frac{\mathrm{d}\gamma}{0} = \frac{\mathrm{d}\dot{\alpha}}{0} = \frac{\mathrm{d}\dot{\beta}}{0} = \frac{\mathrm{d}\dot{\gamma}}{0}. \tag{29}$$

The characteristics of this system are

$$p = t, \qquad u = \alpha, \qquad w = \gamma,$$
$$x = \dot{\alpha}, \qquad y = \dot{\beta}, \qquad z = \dot{\gamma}. \tag{30}$$

Using these characteristics, we obtain

$$\frac{\mathrm{d}p}{1} = \frac{\mathrm{d}u}{x} = \frac{\mathrm{d}w}{z} = \frac{\mathrm{d}x}{\frac{3}{4}(x^2 + y^2) - 4\,\mathrm{e}^u}$$

$$= \frac{\mathrm{d}y}{-3xy} = \frac{\mathrm{d}z}{-\frac{3}{8}(x^2 + y^2 + 4xz + 8z^2) + \frac{2}{3}\,\mathrm{e}^u}$$

and thus

$$\frac{\mathrm{d}u}{x} = \frac{\mathrm{d}y}{-3xy} \iff I_1 = \mathrm{e}^{3\alpha}\dot{\beta}, \tag{31}$$

$$\frac{\mathrm{d}u}{x} = \frac{\mathrm{d}x}{\frac{3}{4}(x^2 + y^2) - 4\,\mathrm{e}^u}$$

$$\iff I_2 = \frac{\dot{\alpha}^2}{2\alpha^{3/2}} - f(\alpha), \tag{32}$$

$$\frac{\mathrm{d}p}{1} = \frac{\mathrm{d}u}{x} \iff I_3 = t - \int\frac{\mathrm{d}\alpha}{\dot{\alpha}}, \tag{33}$$

where

$$f(\alpha) = \frac{3}{4}I_1^2 \int \alpha^{-3/2}\,\mathrm{e}^{-6\alpha}\mathrm{d}\alpha + 4\int \alpha^{-\frac{3}{2}}\,\mathrm{e}^{\alpha}\mathrm{d}a. \quad (34)$$

(Note that $\beta(t)$ follows from the quadrature of (31).)

The associated Lagrange's system for the symmetry $\partial_t$ is

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}u}{0} = \frac{\mathrm{d}v}{0} = \frac{\mathrm{d}\gamma}{0} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}z}{0} \quad (35)$$

and gives the characteristics

$$u = \alpha, \qquad v = \beta, \qquad w = \gamma,$$
$$x = \dot{\alpha}, \qquad y = \dot{\beta}, \qquad z = \dot{\gamma}.$$

From the system

$$\frac{\mathrm{d}u}{x} = \frac{\mathrm{d}v}{y} = \frac{\mathrm{d}w}{z} = \frac{\mathrm{d}x}{\frac{3}{4}(x^2+y^2)-4\,\mathrm{e}^u}$$
$$= \frac{\mathrm{d}y}{-3yx} = \frac{\mathrm{d}z}{-\frac{3}{8}(x^2+y^2+4xz+8z^2)+\frac{2}{3}\,\mathrm{e}^u}$$

we have

$$\frac{\mathrm{d}u}{x} = \frac{\mathrm{d}z}{-\frac{3}{8}(x^2+y^2+4xz+8z^2)+\frac{2}{3}\,\mathrm{e}^u},$$
$$0 = \frac{\mathrm{d}z}{\mathrm{d}u} + \frac{3z^2}{x} + \frac{3}{2}z + \frac{3}{8}\frac{x^2+y^2}{x} - \frac{2}{3}\frac{\mathrm{e}^u}{x} \quad (36)$$

wherein we already have $x(u)$ and $y(u)$ from (32) and (31), respectively. Eq. (36) is a Riccati equation. On using the generalized Riccati transformation [13]

$$z = f\,\omega'(u)/\omega(u), \quad (37)$$

we obtain the second-order equation

$$0 = f\frac{\omega''}{\omega} - fu\frac{\omega'^2}{\omega} + f'\frac{\omega'}{\omega} + \frac{3}{x}f^2\frac{\omega'^2}{\omega}$$
$$+ \frac{3}{8}\frac{x^2+y^2}{x} - \frac{2}{3}\frac{\mathrm{e}^u}{x} \quad (38)$$

and, setting $f = x/3$ to remove the $\omega'^2$ terms, we eventually obtain

$$0 = \omega'' + \left(\frac{x'}{x} + \frac{3}{2}\right)\omega' + \left(\frac{9}{8}\frac{x^2+y^2}{x^2} - 2\frac{\mathrm{e}^u}{x}\right)\omega, \quad (39)$$

which is a linear second-order equation in $\omega$ as a function of $u$. So we deduce $\dot{\gamma}$ as a function of $\alpha$,

$$\frac{\mathrm{d}u}{x} = \frac{\mathrm{d}w}{z} \iff I_4\gamma - \int\frac{\dot{\gamma}}{\dot{\alpha}}\mathrm{d}a. \quad (40)$$

Hence

$$t - t_0 = \int\frac{\mathrm{d}a}{\sqrt{g(a)}}\beta = \int\frac{I_2}{\mathrm{e}^{3\alpha}}\mathrm{d}t, \quad (41)$$

so that this case is formally reduced to a quadrature and the solution of a linear second-order differential equation.

Table 3: First integrals for Bianchi type V model in the first case where there is no matter

| Symmetries | Integrals |
|---|---|
| $\partial_t$ | $I_1 = u^3\dot{v}/v$ |
| | $I_2 = u^3\dot{w}/w$ |
| $v\partial_v$ | $I_3 = \frac{1}{2}(\log u)^3\dfrac{\dot{u}^2}{u} - f_1(u)$ |

**Case 2:** In this case the integrals have the same form. The only difference is that $\gamma$ depends on $\phi$. From the symmetry $\partial_t$ we have the extra integral

$$I_4 = \mathrm{e}^{3\gamma}\dot{\phi} \quad (42)$$

which can be solved to find

$$\phi = \int I_4 e^{-3\gamma}\mathrm{d}t. \quad (43)$$

**Case 3:** We obtain exactly the same integrals as in the previous case. We note that the second-order derivative of $\gamma$ depends on the constant potential.

### 4.1.3. Bianchi Type V

We consider here only the problem of finding the Lie first integrals for Bianchi V in the first three cases, delaying the last case to the next subsection. In Case 1 the integrals have the same expressions as those met in the Bianchi I (with a different form of $f_1$ and hence different $g_1$s).

**Case 1:** The integrals are listed in Table 3 where

$$f_1(u) = \frac{1}{4}\left[(\log u)^3 + \frac{3(\log u)^2}{2} + \frac{3\log u}{2} + \frac{3}{4}\right]u^{-2}$$
$$- \frac{1}{48}\left(3I_1^2 + I_2^2\right)$$
$$\times \left[(\log u)^3 + \frac{1}{2}(\log u)^2 + \frac{1}{6}\log u + \frac{1}{36}\right]u^{-6}.$$

Using the integral $I_3$, we find

$$\dot{u} = \sqrt{g_1(u)} \iff t - t_0 = \int\frac{\mathrm{d}u}{\sqrt{g_1(u)}} \quad (44)$$

where

$$g_1(u) = \frac{2u^2 I_3}{(\log u)^3} + \frac{2u^2 f(u)}{(\log u)^3}. \quad (45)$$

From (44) we can deduce $u(t)$. The variables $v$ and $w$ follow from the integrals $I_1$ and $I_2$, respectively. Hence we have

$$v = \exp\int\frac{I_1}{u^3}\mathrm{d}t, \qquad w = \exp\int\frac{I_2}{u^3}\mathrm{d}t. \quad (46)$$

Table 4: First integrals for Bianchi type V model in the case where there is matter but no potential

| Symmetries | Integrals |
|---|---|
| $\partial_t$ | $I_1 = u^3 \dot{v}/v$ |
| | $I_2 = u^3 \dot{w}/w$ |
| | $I_3 = u^3 \dot{\phi}$ |
| $v\partial_v$ | $I_4 = \dfrac{1}{2}(\log u)^3 \dfrac{\dot{u}^2}{u} - f_2(u)$ |

(This solution uses only three integrals since they are separable.)

**Case 2:** The integrals are shown in Table 4 where

$$f_2(u) = \frac{1}{4}\left[(\log u)^3 + \frac{3(\log u)^2}{2} + \frac{3\log u}{2} + \frac{3}{4}\right]u^{-2}$$

$$-\frac{1}{48}\left(3I_1^2 + I_2^2 + 2I_3^2\right)$$

$$\times \left[(\log u)^3 + \frac{1}{2}(\log u)^2 + \frac{1}{6}\log u + \frac{1}{36}\right]u^{-6}. \quad (47)$$

Using the integral $I_4$, we see that

$$t - t_0 = \int \frac{\mathrm{d}u}{\sqrt{g_2(u)}}, \quad (48)$$

where

$$g_2(u) = \frac{2u^2 I_3}{(\log u)^3} + \frac{2u^2 f_2(u)}{(\log u)}^3. \quad (49)$$

On the other hand, $I_1$ and $I_2$ give

$$v = \exp\int \frac{I_1}{u^3}\mathrm{d}t, \qquad w = \exp\int \frac{I_2}{u^3}\mathrm{d}t. \quad (50)$$

Furthermore, using $I_3$, we obtain

$$\phi = \exp\int \frac{I_3}{u^3}\mathrm{d}t. \quad (51)$$

We conclude that this case is reduced to quadratures.

**Case 3:** The integrals have exactly the same form as in the previous case with a new function $f_3(u)$ replacing $f_2(u)$. We have

$$f_3(u) = \frac{1}{4}\left[(\log u)^3 + \frac{3(\log u)^2}{2} + \frac{3\log u}{2} + \frac{3}{4}\right]u^{-2}$$

$$+\frac{1}{8}C(\log u)^4 - \frac{1}{48}\left(3I_1^2 + I_2^2 + 2I_3^2\right)\left[(\log u)^3\right.$$

$$\left. +\frac{1}{2}(\log u)^2 + \frac{1}{6}\log u + \frac{1}{36}\right]u^{-6}. \quad (52)$$

### 4.2. Counteracting the symmetry breaking potential

In this subsection we focus on Case 4. Note that, for instance, the vector field $t\partial_t$ is a symmetry for Cases 1 and 2 but not for Cases 3 and 4 and this is obviously due to the potential couplings. In order to "fix" this problem we consider an additional term which "kills" any potential terms present in the $\lambda$ and $\phi$ equations and becomes a symmetry for the other equations of the system.

Consider the vector field

$$v = t\partial_t + a\partial_\phi, \quad (53)$$

where $a$ is a constant. Note that this is a symmetry for (17b) and (17c) since the presence of the additional term does not affect these two equations. If we apply the new symmetry to (17a) and (17d), we expect to find restrictions on the form of the potential.

Applying the second extension of (53) to (17a), we find that the potential must satisfy the constraint

$$V'' + \frac{2}{a}V' = 0 \quad (54)$$

and similarly from (17d) we deduce that

$$V' + \frac{2}{a}V = 0, \quad (55)$$

giving immediately

$$V = K\,\mathrm{e}^{-2\phi/a}. \quad (56)$$

Recall that the potential terms are responsible for reducing the number of symmetries in Case 2. We now consider the general point symmetry of Case 2 for which there is no potential,

$$v = (A + Bt)\partial_t + (C + Dt)\partial_\lambda$$
$$+ (E + F\phi + G\beta_2)\partial_{\beta_1} + (H + I\phi - G\beta_1)\partial_{\beta_2}$$
$$+ \left(J - \frac{3}{2}F\beta_1 - \frac{3}{2}I\beta_2\right)\partial_\phi \quad (57)$$

and apply its second extension to the set of equations where the potential is unrestricted. Eq. (17a) implies that $D = F = I = 0$, which means that from the ten initially possible symmetries we lose three, the ones which correspond to the three vanishing constants. Also (17d) gives

$$V = K\,\mathrm{e}^{a\phi}, \quad (58)$$
$$0 = Ja + 2B, \quad (59)$$

so that one obtains six independent constants as expected (in Case 4 of this Type we found five independent symmetries). ????

A conclusion from this argument is that when one applies the general form of the symmetry, one ends up with the same symmetries as in Case 4 plus an additional one but with a restricted potential.

Our method can indeed be used in more general situations. More general Bianchi cosmologies have extra terms in the Euler-Lagrange equations (17) and one is forced to introduce a symmetry with an extra term. On considering the expression of the Ricci scalar in both classes and following the same reasoning as for Bianchi Type I, we are led to the symmetry

$$v = t\partial_t + a\partial_\phi + \partial_\lambda. \tag{60}$$

If we take the second extension of (60), we conclude that for both classes $A$ and $B$ the potential is an exponential function of $\phi$ and has exactly the same form as in Bianchi Type I.

## 5. Conclusions

The geometric reformulation of homogeneous cosmologies and the subsequent applications discussed in the paper allow some more general conclusions to be drawn regarding the dynamics of these cosmologies. All models have in common the trivial Noether symmetry $\partial_t$ since all systems are autonomous. Case 2 of most of the models has the common symmetry $\partial_\phi$. The symmetries that one usually obtains in Cases 2 and 3 are combinations of the symmetries in the other two cases. Only Bianchi Type I in the case where the potential is constant gives a symmetry which involves the constant potential. This cannot occur in other models. Since Types VIII and IX are known to be the most complex, one does expect this to be reflected in the calculation of their symmetries groups and this was indeed the case as we obtained only the symmetries $\partial_t$ and $\partial_\phi$.

There are several directions in which one could extend research in this field. Firstly, the symmetry group calculations performed here can be extended to include other matter fields. Indeed, an interesting project could be to consider the effects of incorporating a perfect fluid, electromagnetic fields, *etc.*, on the number and nature of the symmetries found here. A detailed analysis of the nature and number of contact symmetries possibly present in homogeneous cosmologies would be also very welcome.

The well-known dynamical aspects of Bianchi models must in some sense be reflected in the symmetry groups discussed here. How do our symmetries behave as these models expand? One does expect that neither their number nor their nature remains intact asymptotically towards or away from singularities. Besides, some models are known to isotropise. Do their *symmetries* evolve towards precisely those of the associated FRW models? We conjecture that they do, but this is a problem for the future.

### Acknowledgement

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