ON BIRKHOFF'S CONJECTURE ABOUT CONVEX BILLIARDS

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Abstract. Birkhoff conjectured that the elliptic billiard was the only integrable convex billiard. Here we prove a local version of this conjecture: any non-trivial symmetric entire perturbation of an elliptic billiard is non-integrable.

1. Introduction

Let us consider the problem of the "convex billiard table": let $C$ be an (analytic) closed convex curve of the plane $\mathbb{R}^2$, parameterized by $\gamma : \mathbb{T} \rightarrow C$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, in such a way that $C$ is traveled counterclockwise. Suppose that a material point moves inside $C$ and collides with $C$ according to the law "the angle of incidence is equal to the angle of reflection". Following Birkhoff [1], this discrete dynamical system can be modeled by an (analytic) a.p.m. in an annulus. More precisely, consider the annulus $A = \{z = (\varphi, v) \in \mathbb{T} \times \mathbb{R} : |v| < |\dot{\gamma}(\varphi)|\}$, where the coordinate $\varphi$ is the parameter on $C$ and $v = |\dot{\gamma}(\varphi)| \cos \vartheta$, with $\vartheta \in (0, \pi)$ the angle of incidence-reflection of the material point. In this way, we obtain a map $T : A \rightarrow A$ given by $(\varphi, v) \mapsto (\Phi, V)$ that models the billiard (see Figure 1).

The function $S : \{\varphi, \Phi \in \mathbb{T}^2 : \varphi \neq \Phi\} \rightarrow \mathbb{R}$ defined by $S(\varphi, \Phi) = |\gamma(\varphi) - \gamma(\Phi)|$ is a generating function of $T$, i.e.:

$$\frac{\partial S}{\partial \varphi}(\varphi, \Phi) = \frac{\langle \gamma(\varphi) - \gamma(\Phi), \dot{\gamma}(\varphi) \rangle}{|\gamma(\varphi) - \gamma(\Phi)|} = -|\dot{\gamma}(\varphi)| \cos \vartheta = -v,$$

$$\frac{\partial S}{\partial \Phi}(\varphi, \Phi) = \frac{\langle \gamma(\Phi) - \gamma(\varphi), \dot{\gamma}(\Phi) \rangle}{|\gamma(\varphi) - \gamma(\Phi)|} = |\dot{\gamma}(\Phi)| \cos \Theta = V,$$

and consequently $T$ is an a.p.m. ($\det DT(\varphi, v) = 1$), and $(\varphi, v)$ are canonical conjugated coordinates.
Figure 1: \( T(\varphi, v) = (\Phi, V) \), where \( v = |\gamma(\varphi)| \cos \vartheta \) and \( V = |\gamma(\Phi)| \cos \Theta \).

**Remark 1.1** The orbits of \( T \) (biinfinite sequences \((z_n)_{n \in \mathbb{Z}} \subset A\) such that \( T(z_n) = z_{n+1} \)) are in one-to-one correspondence with the critical points of the formal series (called the *action*)

\[
W((\varphi_n)_{n \in \mathbb{Z}}) = \sum_{n \in \mathbb{Z}} S(\varphi_n, \varphi_{n+1}),
\]

since these critical points \((\varphi_n)_{n \in \mathbb{Z}} \subset \mathbb{T}\) satisfy the equations

\[
(1) \quad \partial_1 S(\varphi_n, \varphi_{n+1}) + \partial_2 S(\varphi_{n-1}, \varphi_n) = 0, \quad \forall n \in \mathbb{Z}.
\]

Thus, having a generating function allows us to work with only half of the coordinates (the *base* coordinates, i.e., the \( \varphi \)'s). The *fiber* coordinates (i.e., the \( v \)'s) are superfluous.

**Remark 1.2** Let \( C \) and \( C' \) two closed convex curves such that one is the image of the other by a similarity. Then the two associated a.p.m. \( T, T' \), have an equivalent dynamics since the angle of incidence-reflection remains unchanged by the similarity.

The map \( T \) has no fixed points but is geometrically clear that it has periodic orbits of period 2, corresponding to opposite points with the maximum and “minimum” distance between them. In these orbits the angle of incidence-reflection is \( \pi/2 \) and thus \( v = 0 \).
To study the dynamics of these 2-periodic points of \( T \), it is better to consider them as fixed points of \( T^2 \), and study \( T^2 \). But since it is not easy to find the generating function for \( T^2 \), we instead introduce a simplification.

Suppose now that \( C \) is symmetric with regard to a point. By the previous remark 1.2, we can assume that this point is the origin. Then it is possible to choose a parameterization \( \gamma \) of \( C \) such that \( \gamma(\varphi + \pi) = -\gamma(\varphi) \) and the 2-periodic orbits are of the form \((\varphi_0, 0), (\varphi_0 + \pi, 0)\), that is, two opposite points over \( C \).

**Remark 1.3** We can consider the variable \( \varphi \) defined modulus \( \pi \) in the symmetric case.

Let now \( R : \mathbb{A} \rightarrow \mathbb{A} \) be the involution \( R(\varphi, v) = (\varphi + \pi, v) \) (\( R^2 \) is the identity). By the symmetry of \( C \), \( T \) and \( R \) commute and it is a commonplace to use this symmetry to convert the 2-periodic points into fixed points. Concretely, we define a new map \( F : \mathbb{A} \rightarrow \mathbb{A} \) by \( F = R \circ T \). Since \( F^2 = T^2 \), the dynamics of \( F \) and \( T \) are equivalent. Moreover, since \( \gamma(\Phi + \pi) = -\gamma(\Phi) \), it is easy to check that

\[
\mathcal{L}(\varphi, \Phi) = S(\varphi, \Phi + \pi) = |\gamma(\varphi) + \gamma(\Phi)|
\]

is a generating function for \( F \), and consequently \( F \) is an a.p.m.

**2. Elliptic Billiards**

The simplest examples of convex curves are the ellipses. It is clear that the case of a circumference is very degenerated for a billiard, since it consists only of 2-periodic orbits. So, let us consider now a non-circular ellipse:

\[
C_0 = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\} = \{ \gamma_0(\varphi) = (a \cos \varphi, b \sin \varphi) : \varphi \in \mathbb{T} \},
\]

with \( a^2 \neq b^2 \). Without loss of generality we can assume that \( a^2 - b^2 = 1 \) (we change the ellipse using a similarity, if necessary). Thus \( a > 1, b > 0 \) and the foci of the ellipse are \((\pm 1, 0)\). Let us denote \( T_0 : \mathbb{A} \rightarrow \mathbb{A} \) the analytic a.p.m. associated to the ellipse \( C_0 \), and \( F_0 = R \circ T_0 \).
It is clear that the points \((0, 0)\) and \((\pi, 0)\) form a 2-periodic orbit for \(T_0\) that corresponds to the vertices \((\pm a, 0)\) of the ellipse, and hence \(z_\infty := (0, 0)\) is a fixed point for \(F_0\). \(((\pi/2, 0)\) is another fixed point for \(F_0\).)

Birkhoff was the first one in noticing that this system is integrable, i.e., there exists an analytic function \(H(\varphi, v)\), called first integral, that preserves the orbits of \(T_0\): \(H \circ T_0 = H\). As a consequence, the curves \(\{H = \text{constant}\}\) are invariant under \(T_0\). Obviously, \(H\) is also a first integral of \(F_0 = R \circ T_0\).

In fact, it is not difficult to check that \(H(\varphi, v) = (\sin^2 \varphi - v^2)/2\) is such a first integral, under the assumption \(a^2 - b^2 = 1\). (See, for instance, [2] for the derivation of \(I = -2H + 1\).)

In addition to the involution \(R\), the ellipse \(C_0\) has another symmetry \(R^* : A \longrightarrow A\) given by \(R^*(\varphi, v) = (\pi - \varphi, v)\). \(R^*\) is also an involution, and moreover \(F_0^{-1} = R^* \circ F_0 \circ R^*\), i.e., \(F_0\) is \(R^*\)-reversible.

The dynamics of \(F_0\), based in the level curves of \(H\) is drawn in Figure 2 where the resemblance with the phase portrait of a pendulum shows up clearly.

Figure 2: Phase portrait of \(F_0\); \(\Gamma^\pm\) are the separatrices of \(F_0\).

The main properties of \(F_0\) are listed in the following Lemma, whose proof can be found in [3].

**Lemma 2.1**

a) \(z_\infty = (0, 0)\) is a saddle fixed point of \(F_0\) and \(\text{Spec}[DF_0(z_\infty)] = \{\lambda, \lambda^{-1}\}\), with \(\lambda = (a + 1)(a - 1)^{-1} > 1\).
Moreover, if \( h := \ln \lambda \) the following expressions hold

\[
(3) \quad a = \coth(h/2), \quad b = \cosech(h/2).
\]

b) \( \Gamma^\pm = \{ (\varphi, \pm \sin \varphi) : 0 < \varphi < \pi \} \) are the separatrices of \( F_0 \): if \( z \in \Gamma^\pm \), then \( F^n_0(z) \to z_{\infty} \) when \( n \to \pm \infty \).

c) If \( \sigma^\pm(t) = (\varphi(\pm t), \pm v(t)) \), where \( \varphi(t) = \arccos(\tanh t) = \arcsin(\text{sech} t) \) and \( v(t) = \text{sech} t \), then \( F_0(\sigma^\pm(t)) = \sigma^\pm(t + h) \). In other words, \( \sigma^\pm \) are natural parameterizations of \( \Gamma^\pm \) (with regard to \( F_0 \)).

d) Let \( \Phi(t) = \varphi(t + h) \). Then

\[
(4) \quad \frac{b \sin \varphi(t) + \sin \Phi(t)}{|\gamma_0(\varphi(t)) + \gamma_0(\Phi(t))|} = \text{sech}(t + h/2).
\]

2.1. Entire elliptic billiard perturbations

Birkhoff conjectured that the elliptic billiard was the only integrable convex billiard. Our goal is to see that this is locally true for the symmetric billiards, i.e., any non-trivial symmetric entire perturbation is non-integrable. (Non-trivial perturbation means not reducible to an ellipse.)

Let \( \{ C_\varepsilon \} \) be an arbitrary family of perturbations of the ellipse \( C_0 \), consisting of analytic curves depending on a \( C^2 \) way on \( \varepsilon \) and symmetric with regard to a point \( O_\varepsilon \). Let us denote by \( Q^\pm_\varepsilon \) the two furthest (and opposite) points over \( C_\varepsilon \) with \( Q^\pm_0 = (\pm a, 0) \). Using a similarity that takes \( O_\varepsilon \) and \( Q^\pm_\varepsilon \) to \((0, 0)\) and \((\pm a, 0)\) respectively, the initial family can be put in the following form

\[
(5) \quad C_\varepsilon' = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \varepsilon P(x, y, \varepsilon) = 1 \right\},
\]

where:

I) \( P \) is analytic in \( x, y \) and at least \( C^1 \) in \( \varepsilon \),

II) \( P(x, y, \varepsilon) = P(-x, -y, \varepsilon) \),

III) \( P(a, 0, \varepsilon) = \partial_y P(a, 0, \varepsilon) = 0 \),
or equivalently, like

\[ C_\varepsilon = \{ \gamma(\varphi, \varepsilon) = (a \cos \varphi, \sin \varphi [b + \varepsilon \eta(\varphi, \varepsilon)] : \varphi \in \mathbb{T} \}, \]

where:

i) \( \eta \) is analytic in \( \varphi \) and at least \( C^1 \) in \( \varepsilon \),

ii) \( \eta \) is \( \pi \)-periodic in \( \varphi \).

From III), it follows that

\[ P(a \cos \varphi, b \sin \varphi, \varepsilon) = p(a \cos \varphi, b \sin \varphi, \varepsilon) \sin^2 \varphi, \]

with \( p \) satisfying also I), II). It is easy to check that, in first order in \( \varepsilon \), the relation between \( \eta \) and \( P \) is given by

\[ P(a \cos \varphi, b \sin \varphi, 0) = -2b \eta(\varphi, 0) \sin^2 \varphi. \]

Thus, if \( P(\cdot, \cdot, 0) \) is an entire function in the variables \( x \) and \( y \), the same happens to \( \eta_1 := \eta(\cdot, 0) \). It is clear that if \( \eta_1 = \) constant, \( C_\varepsilon \) is, in first order, a family of ellipses.

**Definition 2.1** Let \( \{C_\varepsilon\} \) be a perturbation of the ellipse \( C_0 \). We say that \( \{C_\varepsilon\} \) is a non-trivial symmetric entire perturbation of the ellipse when it can be put, using similarities, in the form (6) and moreover, \( \eta_1 := \eta(\cdot, 0) \) is a non-constant entire function.

Let \( T_\varepsilon \) be the map in the annulus associated to the billiard in \( C_\varepsilon \), where the perturbation considered is a symmetric entire one and let \( F_\varepsilon = R \circ T_\varepsilon \). If \( \varepsilon \) is small enough, \( C_\varepsilon \) is an analytic convex closed curve, and thus \( \{F_\varepsilon\}_{0 \leq \varepsilon \leq 1} \) is a family of analytic a.p.m. with generating function \( L_\varepsilon(\varphi, \Phi, \varepsilon) = |\gamma(\varphi, \varepsilon) + \gamma(\Phi, \varepsilon)| \) that can be written as \( L_\varepsilon(\varphi, \Phi, \varepsilon) = L_0(\varphi, \Phi) + \varepsilon L_1(\varphi, \Phi) + O(\varepsilon^2) \), where

\[ L_0(\varphi, \Phi) = |\gamma_0(\varphi, \varepsilon) + \gamma_0(\Phi, \varepsilon)|, \]

\[ L_1(\varphi, \Phi) = \frac{b \sin \varphi + \sin \Phi}{|\gamma_0(\varphi) + \gamma_0(\Phi)|} \left[ \sin \varphi \eta_1(\varphi) + \sin \Phi \eta_1(\Phi) \right]. \]

For \( \varepsilon \) small enough, the origin \( z_\infty \) is again a hyperbolic point of \( F_\varepsilon \). This point \( z_\infty \) lies in the intersection of the invariant curves \( \mathcal{W}_z, \mathcal{W}_u \), such that any \( (z_n)_{n \in \mathbb{Z}} \) orbit in the manifold \( \mathcal{W}_z^u \) tends to
at an exponential rate as \( n \to +\infty, -\infty \). These invariant curves will not longer coincide for \( \varepsilon \neq 0 \). But for \( \varepsilon \) small enough, there exist open sets \( \mathcal{U}^{u,u} \) in \( \mathcal{W}^{u,u}_\varepsilon \) and analytic functions \( \Psi^{u,u}_\varepsilon : \mathcal{I}^u \to \mathbb{R} \) such that \( \mathcal{U}^{u,u} = \text{Graph}(\Psi^{u,u}_\varepsilon) \), with \( \mathcal{I} := \mathcal{I}^s \cap \mathcal{I}^u \) of size a little bit less than \( \pi \). Let us see that the primitives \( \psi^{u,u}_\varepsilon \) of the functions \( \Psi^{u,u}_\varepsilon \) have a nice interpretation, crucial to the derivation of the Melnikov potential.

**Proposition 2.1** Let \( \psi^{u,u}_\varepsilon : \mathcal{W}^{u,u}_\varepsilon \to \mathbb{R} \) be the analytic functions defined by

\[
\psi^{u}_\varepsilon(z) = -\sum_{n \geq 0} \mathcal{L}_\varepsilon(\varphi^n, \varphi^{n+1}_\varepsilon), \quad \psi^{u}_\varepsilon(z) = \sum_{n < 0} \mathcal{L}_\varepsilon(\varphi^n, \varphi^{n+1}_\varepsilon),
\]

where, if \( z^{u,u} \in \mathcal{W}^{u,u}_\varepsilon \), \( z^{u,u} = (\varphi^n, \nu^n) = F^n_\varepsilon(z^{u,u}) \) for \( n \in \mathbb{Z} \). Then:

1. \( \psi^{u}_\varepsilon(z_\infty) = 0 \).

2. \( \psi^{u,u}_\varepsilon(\varphi) = \psi^{u,u}_\varepsilon(z^{u,u}), \forall \varphi \in \mathcal{I}^{u,u}, z^{u,u} = (\varphi, \Psi^{u,u}_\varepsilon(\varphi)) \).

This result is a direct consequence of the variational principle (1). Moreover, the properties 1. and 2. determine completely the functions \( \psi^{u,u}_\varepsilon \).

To measure the separation between invariant curves, we will use the Melnikov potential, i.e., a primitive of the Melnikov function, that is nothing else that the action \( \sum \mathcal{L}(\varphi_n, \varphi_{n+1}, \varepsilon) \) evaluated on the unperturbed separatrices \( \Gamma := \Gamma^+ \cup \Gamma^- \) up to the first order in \( \varepsilon \).

**Proposition 2.2** Let \( L : \Gamma \to \mathbb{R} \) be the function defined by

\[
L(z) = \sum_{n \in \mathbb{Z}} \mathcal{L}_1(\varphi_n, \varphi_{n+1}), \quad z_n = (\varphi_n, \nu_n) = F^n_0(z).
\]

Then:

(i) \( L \) is well-defined (the sum in (10) is absolutely convergent), analytic and \( F^0 \)-invariant (i.e., \( L \circ F^0 = L \)).

(ii) If \( L \) is not identically constant and \( \varepsilon \) is small enough, the invariant curves \( \mathcal{W}^{s}_\varepsilon \), \( \mathcal{W}^{u}_\varepsilon \) intersect along a finite contact homoclinic orbit of \( F_\varepsilon \), and \( F_\varepsilon \) is non-integrable.
Proof of the Proposition. We only sketch here the proof. For more details, see [4].

(i) The sum is absolutely convergent since any orbit in the separatrix $\Gamma$ tends to $z_{\infty}$ at an exponential rate as $|n| \to \infty$ and $L_1(0,0) = 0$. Thus $L$ is well-defined and analytic. Moreover, a shift in the index of the sum does not change its value, so the $F_0$-invariance of $L$ is proved.

(ii) Our measure of the distance between the perturbed asymptotic manifolds is the difference of fiber coordinates $(v’s)$

$$
\Delta_{\ast}(\varphi, \varepsilon) := \frac{d\psi_{\ast, \ast}^u(\varphi)}{d\varphi} - \frac{d\psi_{\ast, \ast}^u(\varphi)}{d\varphi} = \varepsilon M_\ast(\varphi) + O(\varepsilon^2).
$$

$M_\ast$ gives in first order in $\varepsilon$ the distance between invariant curves, and thus can be considered as a Melnikov function.

For a primitive $\delta_\ast$ of $\Delta_{\ast}$ with respect to $\varphi$, $\Delta_{\ast} := d\delta_\ast/d\varphi$, we have the following expression:

$$
\delta_{\ast}(\varphi, \varepsilon) := \psi_{\ast, \ast}^u(\varphi) - \psi_{\ast, \ast}^u(\varphi) = \sum_{n \in \mathbb{Z}} L_\varepsilon \left( \varphi^{o[n]}_n, \varphi^{o[n]}_{n+1} \right),
$$

where $(\varphi^{n, n}_n, v^{n, n}_n) = F_\varepsilon(\varphi, d\psi_{\ast, \ast}^u(\varphi)/d\varphi)$, $o(n) = s$, if $n \geq 0$ and $o(n) = u$, if $n < 0$.

Let $L_\ast : [0, \pi] \to \mathbb{R}$ the analytic function defined by $L_\ast(\varphi) := L(\varphi, d\psi_0(\varphi)/d\varphi)$, where $\psi_0(\varphi) = -\cos \varphi$. Using the uniform approximations of $W^{s,u}_\varepsilon$ by $\Gamma$ and the variational principle, it turns out that

$$
\delta_{\ast}(\varphi, \varepsilon) = \varepsilon L_\ast(\varphi) + O(\varepsilon^2).
$$

Thus, $M_\ast = dL_\ast/d\varphi$, and now it is clear that if $L$ is not identically constant, the same will happen to $L_\ast$. Finally, one can see that $M_\ast$ is not zero, analytic, and has a zero of finite order, using the fact that $L_\ast \circ \varphi$ is $h$-periodic, where $\sigma(t) = (\varphi(t), v(t))$. Consequently, for $\varepsilon$ small enough, the invariant curves $W_{s}^u$, $W_{s}^u$ intersect along a finite contact homoclinic orbit of $F_\ast$. By a result of Cushman [5], $F_\ast$ is non-integrable.

From now on, we consider only $\Gamma^+$. Using the parameterization provided by (2.1), we can write $\Gamma^+ = \{ \sigma^+(t) = (\varphi(t), v(t)) \}$, and consider the Melnikov potential

$$
L(t) = \sum_{n \in \mathbb{Z}} f(t + hn), \text{ with } f(t) = L_1(\varphi(t), \varphi(t + h)),
$$
where for simplicity of notation, we write $L$ instead of $L_\ast \circ \sigma$, with $L_1$ as given in equation (8). Using formula (4), we arrive at an equivalent expression for $f$:

\begin{equation}
 f(t) = \text{sech}(t + \frac{h}{2})[\text{sech}(t)\eta_1(\varphi(t)) + \text{sech}(t + h)\eta_1(\varphi(t + h))].
\end{equation}

### 2.2. Non-integrable billiards

The aim of this paper is to prove the following result.

**Theorem 2.1** Let $\{C_\varepsilon\}$ be any non-trivial symmetric entire perturbation of an ellipse. Then the billiard in $C_\varepsilon$ is non-integrable for $0 < |\varepsilon| \ll 1$.

**Proof of the Theorem.** By proposition 2.2, we only have to prove that $L$ is non constant. To this end, we will use the following result.

**Lemma 2.2** In a complex neighbourhood $D$ of $t = \pi i/2$, there exists an analytic function $G$ such that $L(t)$ reads as

\[ L(t) = \frac{2a}{b}\text{sech}(t + h/2)\text{sech}(t - h/2)\eta_1(\varphi(t)) + G(t). \]

Accepting for the moment this lemma, and using the fact that $\text{sech}(t + h/2)\text{sech}(t - h/2)$ is analytic non zero on $t = \pi i/2$, it turns out that $t = \pi i/2$ is a singular point of $L(t)$ if and only if the same happens to $\eta_1(\varphi(t))$. But by hypothesis, $\eta_1$ is a $\pi$-periodic entire function, and by lemma 2.1, $\sin \varphi(t) = \text{sech}(t)$ and $\cos \varphi(t) = \tanh(t)$ have simple poles at $t = \pi i/2$ and no more singularities on $\Im t = \pi/2$. Since $\eta_1$ is non constant, $t = \pi i/2$ is a singular point of $\eta_1(\varphi(t))$ and consequently $L(t)$ is not a constant.

**Proof of Lemma 2.2.** Looking at the expression (11) of $f$, and using the fact that $\eta_1$ is entire, the only possible singularities of $f$ with $\Im t = \pi/2$ are $t = \pi i/2$, $t = \pi i/2 - h$, and $t = \pi i/2 - h/2$. Nevertheless, $t = \pi i/2 - h/2$ is a removable singularity, since it is a simple pole of $\text{sech}(t + h/2)$, and also a zero of $\text{sech}(t)\eta_1(\varphi(t)) + \text{sech}(t + h)\eta_1(\varphi(t + h))$. So we can write

\[ L(t) = \sum_{n \in \mathbb{Z}} f(t + nh) = f(t - h) + f(t) + G_1(t). \]
where \( G_1(t) = \sum_{n \neq 0, 1} f(t + nh) \) is analytic on \( t = \pi i/2 \), and

\[
\begin{align*}
f(t - h) + f(t) &= \text{sech}(t - h/2) \left[ \text{sech}(t - h) \eta_1(\varphi(t - h)) + \text{sech}(t) \eta_1(\varphi(t)) \right] \\
&\quad + \text{sech}(t + h/2) \left[ \text{sech}(t) \eta_1(\varphi(t)) + \text{sech}(t + h) \eta_1(\varphi(t + h)) \right] \\
&= \left( \text{sech}(t - h/2) + \text{sech}(t + h/2) \right) \text{sech}(t) \eta_1(\varphi(t)) + G_2(t),
\end{align*}
\]

where \( G_2(t) = \text{sech}(t - h/2) \text{sech}(t - h) \eta_1(\varphi(t - h)) + \text{sech}(t + h/2) \times \text{sech}(t + h) \eta_1(\varphi(t + h)) \) is analytic on \( t = \pi i/2 \).

And finally, using the formula \( \text{sech}(t - h/2) + \text{sech}(t + h/2) = 2 \cosh(h/2) \text{sech}(t - h/2) \text{sech}(t + h/2) \cosh(t) \), and relations (3), the lemma follows with \( G(t) = G_1(t) + G_2(t) \).

\[\square\]

**Remark 2.1** The same proof works for the point \( t = \frac{3\pi i}{2} \) (instead of \( t = \frac{\pi i}{2} \)). The assumption of entire function on \( \eta_1 \) has only been used to ensure that for \( t_p = \frac{\pi i}{2} \) or \( t_p = \frac{3\pi i}{2} \), \( \eta_1(\varphi(t)) \) has an isolated singular point at \( t_p \) but is analytic on \( t_p + nh \) for \( n \neq 0 \).

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