Managing demand uncertainty:
Probabilistic selling versus inventory substitution

Abstract

Demand variability is prevailing in the current rapidly changing business environment, which makes it difficult for a retailer that sells multiple substitutable products to determine the optimal inventory. To combat demand uncertainty, both strategies of inventory substitution and probabilistic selling can be used. Although the two strategies differ in operation, we believe that they share a common feature in combating demand uncertainty by encouraging some customers to give up some specific demand for the product to enable demand substitution. It is interesting to explore which strategy is more advantageous to the retailer. We endogenize the inventory decision and demonstrate the efficiency of probabilistic selling through demand substitution. Then we analyze some special cases without cannibalization, and computationally evaluate the profitability and inventory decisions of the two strategies in a more general case to generate managerial insights. The results show that the retailer should adjust inventory decisions depending on products' substitution possibility. The interesting computational result is that probabilistic selling is more profitable with relatively lower product similarity and higher price-sensitive customers, while inventory substitution outperforms probabilistic selling with higher product similarity. Higher demand uncertainty will increase the profitability advantage of probabilistic selling over inventory substitution.

Keywords: OR in marketing; probabilistic selling; opaque selling; inventory substitution; demand uncertainty

1 Introduction

The prevailing variability of the business environment, rapidly evolving technologies, fierce competition, and sophisticated customer demands are increasing the difficulty for firms to determine the optimal inventory under demand uncertainty (Shen et al., 2016). For example, many retailers try to capture market share and meet customers' various demands by carrying a wide variety of products. Usually the products are similar and may be substitutable, e.g., clothes in different colour, beverages

with different flavours, and bags in different patterns. Although increasing product variety can increase the retailer's market size, it would also increase its total inventory, leading to longer inventory cycles and higher safety stock (Rajagopalan, 2013). In addition, any mis-match between inventory and demand, even for a single product, would reduce profit due to the inventory cost or stock-out cost. Uncertain demand for multiple products makes it more arduous to match supply and demand for improving inventory efficiency. Therefore, it is important to effectively manage demand uncertainty when firms seek to benefit from market expansion through increasing product variety.

To address the problem of managing demand uncertainty with multiple substitutable products, the retailer can consider two strategies, namely inventory substitution, which is well known in Operations Research, and probabilistic selling, which is popular in Marketing. Being an effective tool to minimize the mis-match between capacity and demand, inventory substitution uses substitute products to meet demand when stock-out occurs (Mcgillivray and Silver, 1978; Parlar and Goyal, 1984; Ernst and Kouvelis, 1999). Probabilistic selling means the retailer creates an additional probabilistic product with hidden information using existing products (Fay and Xie, 2008). For instance, travel agencies offer probabilistic service products (e.g. hotel rooms, air tickets, package tours etc.) with some information concealed from customers until customers confirm their orders. Online retailers like Tmall.com, Amazon.com, and AgonSwim.com (Fay and Xie, 2010) offer discounted probabilistic products with some attributes, e.g., colour, style, brand etc, unknown to customers until they receive the products. The price-sensitive customers who are indifferent to the attributes would choose to buy discounted probabilistic products.

Although the two strategies are triggered by the need to address different problems in different research fields, we see that both strategies share the same spirit of demand substitution. Specifically, in applying inventory substitution, the retailer substitutes the remaining inventory of one product for another. The customer whose required product is sold out can choose to accept the substitute or not. In applying probabilistic selling, the retailer offers customers an additional lower-priced choice to enhance the demand substitution of specific products with full information by the probabilistic product. Inventory substitution induces insensitive customers and makes use of available inventory (of

a substitute product) at the end stage of selling, while probabilistic selling induces insensitive customers at the beginning stage and then uses available inventory (of either the requisite product or a substitute product) during the selling stage. Consequently, the retailer can substitute products through the demand of insensitive customers to minimize the mis-match between inventory and demand. Although the two strategies differ in operation to hedge against demand uncertainty, they share a common characteristic in combating demand uncertainty by encouraging some customers to give up some specific demand, e.g., colour, pattern etc, for the product to enable demand substitution. Therefore, it is interesting to explore which strategy is more advantageous for the retailer that sells substitutable products with demand uncertainty.

However, despite the popularity of probabilistic selling in marketing research for the purposes of market expansion and price discrimination (Fay and Xie, 2008, 2014), little is known about probabilistic selling as an inventory tool. There is little research on using economic models to analyze the inventory mechanism of probabilistic selling (Fay and Xie, 2011, 2014). The first study that endogenizes the capacity decision is Wu and Wu (2015), which explores opaque selling as a strategy to induce demand postponement. They considered the one-product scenario with stochastic demand from the perspective of an intermediary. While we also study probabilistic selling in the newsvendor setting, we focus on exploring the inventory ability of probabilistic selling from the perspective of a retailer that can manage demand uncertainty through demand substitution.

In this paper we develop a single-period newsvendor model with three products to analyze probabilistic selling with a view to generating insights into using probabilistic selling to manage demand uncertainty. We then compare probabilistic selling with inventory substitution in the special cases without cannibalization. To gain additional insights into the normal situation, we use computational examples to compare the two strategies in terms of overall profit and inventory with considerations of customer transition (reflected by the cannibalization index under probabilistic selling and by the substitution fraction under inventory substitution) and demand uncertainty.

We make two main contributions under this paper: First, this is the first study that captures the inventory decision in probabilistic selling considering the cannibalization effect. Second, this is

the first paper that compares the performance of probabilistic selling and inventory substitution in managing demand uncertainty through demand substitution. The results show that the retailer's inventory decisions depends on products' substitution possibility. The comparison results show that probabilistic selling outperforms inventory substitution with relatively lower product similarity and higher price-sensitive customers, while inventory substitution is more profitable than probabilistic selling when product similarity is higher. Besides, higher demand uncertainty will increase the profitability advantage of probabilistic selling over inventory substitution. Our work enriches the research about probabilistic selling as an inventory management tool. The analytical approach and research findings may help practitioners gain more insight on the capacity of probabilistic selling on combating demand uncertainty, and facilitate their inventory related decision-making.

We organize the rest of this paper as follows: In Section 2 we briefly review the closely related literature. In Section 3 we analyze the substitution patterns and formulate the model under inventory substitution. In Section 4 we derive and analyze the optimality conditions of inventory decisions under probabilistic selling. In Section 5 we compare probabilistic selling and inventory substitution in terms of inventory decision and expected profit for some special cases. In Section 6 we computationally compare the profitability and inventory of the two strategies with considerations of customer transition and demand uncertainty. In Section 7 we conclude the paper, discuss the managerial implications of the research findings, and suggest topics for future research.

2   Literature review

2.1   *Probabilistic selling*

Many literature consider probabilistic selling the same as opaque selling. The forms of opaque selling include "NYOP (Name-Your-Own-Price)", "opaque products", or "variable opaque products" (Fay and Xie, 2008; Post, 2010; Post and Spann, 2012; Spann et al., 2004). Fay and Xie (2008) first defined and analyzed probabilistic selling strategy used by a multi-item retailer to explore the core mechanism of opaque selling. The Marketing and Strategy literature on probabilistic selling focuses on

customer preference heterogeneity (Fay and Xie, 2008, 2010), consumer bounded rationality (Huang and Yu, 2014), price discrimination (Rice et al., 2014), market structure (Zhang et al., 2014) etc to demonstrate the advantage of probabilistic selling over other selling strategies.

However, limited attention has been paid to examining the inventory mechanism of probabilistic selling. Fay and Xie (2011) regarded probabilistic selling as a new mechanism for inventory management in the presence of demand uncertainty despite that the seller is committed to buyers before it has the opportunity to acquire more information. Focusing on the impact of the timing of the assignment of the probabilistic product, Fay and Xie (2014) demonstrated the advantage of probabilistic selling in improving inventory utilization. Nevertheless, the uncertainty in the above studies concerns the probability that one product is more popular than another, and they use the "scenario-based" approach to represent uncertainty (Gupta and Maranas, 2003) rather than the "distribution-based" approach. Just as Rice et al. (2014) pointed out that little research has shown the effectiveness of probabilistic selling when the seller is uncertain about the total category demand rather than the relative popularity of a specific item.

Different from the above literature, we model the demand as normally distributed with a mean and a standard deviation, which is widely used in OM research. Few studies have considered demand uncertainty and recognized the benefit of probabilistic products in increasing inventory efficiency with fixed capacity in the study field of revenue management (Gallego and Phillips, 2004; Gönsch and Steinhardt, 2013). However, they don't endogenize inventory decision in their research. Then Wu and Wu (2015) considered stochastic demand in their single-product inventory model, and integrated demand postponement and opaque selling from the perspective of a travel intermediary. They showed that postponement of delivery allows the firm to use less safety stock to combat demand uncertainty. Different from Wu and Wu (2015), we explore probabilistic selling as an inventory mechanism from the perspective of a retailer selling multiple alternative products. We base the work of Zhang et al. (2016), which used simulation to explore the demand substitution and demand reshaping effects of probabilistic selling. Different from that, we characterize and analyze the optimal inventory decision in probabilistic selling. Furthermore, we focus on comparing probabilistic selling and inventory substitution, both of

which use demand substitution to hedge against demand uncertainty.

2.2 *Inventory substitution*

There is a large body of work on inventory management with substitutable demand. The substitution phenomenon has been widely investigated considering various substitution patterns. The substitution can be led by the supplier, which is common in the airline industry (Vulcano et al., 2012). It can also be led by the customer that is willing to buy a substitute product when their preferred product is out of stock (Parlar and Goyal, 1984; Ernst and Kouvelis, 1999; Baris and Selcuk, 2013; Ye, 2014). The substitution scenarios considered in existing research include two products with one-way or two-way substitution (Mcgillivray and Silver, 1978; Parlar and Goyal, 1984), three products with partial substitution (Ernst and Kouvelis, 1999), and an arbitrary number of products with demand substitution (Netessine and Rudi, 2003; Wang and Parlar, 1994). According to the probability of customers willing to accept substitution, some research considers total substitution (i.e., the probability is equal to 1) (Mcgillivray and Silver, 1978; Pasternack and Drezner, 1991) or constant substitution (i.e., the probability is between 0 and 1) (Parlar and Goyal, 1984; Ernst and Kouvelis, 1999). Some studies assume that the revenue received for a product is independent of the substitution, while others assume that the substitution will incur a performance-related cost (Pasternack and Drezner, 1991). Shah and Avittathur (2007) examined cannibalization considering the downward substitution pattern with a standard product and its customized extensions. We consider partial substitution with cost as Parlar and Goyal (1984) and Pasternack and Drezner (1991) in our paper.

3 Inventory decision under inventory substitution

3.1 *Notation and Assumption*

We consider a retailer that sells two specific products, indexed i, j=1,2 (it is assumed that $i \neq j$). The retailer purchases a quantity Q_i^t of product i and a quantity Q_j^t of product j at the same fixed unit cost $c > 0$, and sells them at price p. The clearance price is s. The stock-out penalty is 0. We

assume that the demand D_i (D_j) is normally distributed with mean u_i (u_j) and standard deviation σ_i (σ_j). Let $f(x_i)$ $(f(x_j))$ and $F(x_i)$ $(F(x_j))$ be the probability density function and cumulative density function of D_i (D_j), respectively. In addition, let $f(x_i, x_j)$ be the joint probability density function of the demand for the products. When the retailer adopts neither probabilistic selling nor inventory substitution, the optimal inventory decision for each product is just the optimal inventory decision for the single-product newsvendor model, i.e., the optimal order quantities $Q_i^{s^*}$ and $Q_j^{s^*}$ are determined by the following equations:

$$F(Q_i^{t^*}) = \frac{p-c}{p-s}, \tag{1}$$

$$F(Q_j^{t^*}) = \frac{p-c}{p-s}. \tag{2}$$

Now we consider the case where the retailer adopts inventory substitution and assume that only a fraction r_s of the unsatisfied customers that face stock-out will accept the substitution (Parlar and Goyal, 1984). Assume that substitution incurs a cost t per unit (Pasternack and Drezner, 1991). We also suppose that $p - t > s$ to make sure that the retailer can benefit from substitution. Substitution occurs when the demand for product i (j) exceeds its supply while the demand for product j (i) is less than its supply (i.e., the substitution paths in Figure 1). After substitution, the total demand for product i may or may not be satisfied.
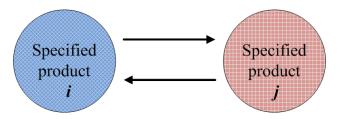

Figure 1: Substitution paths in adopting inventory substitution.

3.2 *The optimal inventory solution*

The expected profit is given in Eq.(3), which comprises the revenue, the savage cost, and the acquisition cost. (Q_i^s, Q_j^s) are the two inventory decisions that jointly maximize the expected profit. The demand for product $i(j)$ comes from the original demand and the substitution demand when demand of product $j(i)$ excess its supply. Therefore, the revenue under inventory substitution comes

from satisfying both the original demand (i.e. $\min(D_j, Q_j^s)$ and $\min(D_i, Q_i^s)$ and the substitution demand (i.e. $\min\left[(Q_i^s - D_i)^+, r_s(D_j - Q_j^s)^+\right]$ and $\min\left[r_s(D_i - Q_i^s)^+, (Q_j^s - D_j)^+\right]$).

$$E(Q_i^s, Q_j^s) = E \begin{cases} p\min(D_i, Q_i^s) + p\min(D_j, Q_j^s) - c(Q_i^s + Q_j^s) \\[2mm] +(p-t)\min\left[(Q_i^s - D_i)^+, r_s(D_j - Q_j^s)^+\right] \\[2mm] +(p-t)\min\left[r_s(D_i - Q_i^s)^+, (Q_j^s - D_j)^+\right] \\[2mm] +s\left[(Q_i^s - D_i)^+ + (Q_j^s - D_j)^+ - r_s(D_i - Q_i^s)^+ - r_s(D_j - Q_j^s)^+\right]. \end{cases} \tag{3}$$

Pasternack and Drezner (1991) have shown the concave property of the expected profit. So the optimal inventory decisions can be determined by applying the first-order condition to the expected total profit function. We characterize the optimal order quantities $(Q_i^{s^*}, Q_j^{s^*})$ in Eq.(4), which is similar to the results in Rudi et al. (2001).

$$\begin{cases} F(Q_i^{s^*}) = \dfrac{p - c + (p - t - s)R(Q_i^{s^*}, Q_j^{s^*})}{p - s} \\[4mm] F(Q_j^{s^*}) = \dfrac{p - c + (p - t - s)T(R_i^{s^*}, Q_j^{s^*})}{p - s}, \end{cases} \tag{4}$$

where

$$R(Q_i^{s^*}, Q_j^{s^*}) = \int_0^{Q_i^s} \int_{Q_j^{s^*} + (Q_i^{s^*} - D_i^t)/r_s}^{\infty} f(x_i, x_j) dx_j dx_i - r_s \int_0^{Q_j^{s^*}} \int_{Q_i^{s^*}}^{Q_i^{s^*} + (Q_j^{s^*} - D_j^t)/r_s} f(x_i, x_j) dx_i dx_j,$$

$$T(Q_i^{s^*}, Q_j^{s^*}) = \int_0^{Q_j^{s^*}} \int_{Q_i^{s^*} + (Q_j^{s^*} - D_j^t)/r_s}^{\infty} f(x_i, x_j) dx_j dx_i - r_s \int_0^{Q_i^{s^*}} \int_{Q_j^{s^*}}^{Q_j^{s^*} + (Q_i^{s^*} - D_i^t)/r_s} f(x_i, x_j) dx_i dx_j.$$

The first term of $R(Q_i^{s^*}, Q_j^{s^*})$ raises Q_i^s due to the possibility that the excess inventory of product i may not meet the substitution demand for product j, while the second term lowers Q_i^s because the substitution demand for product Q_i^s can be satisfied with the excess inventory of product Q_j^s. The same observation holds for $T(Q_i^{s^*}, Q_j^{s^*})$.

4 Inventory decision under probabilistic selling

4.1 *Notation and assumption*

Under probabilistic selling, the offer of the probabilistic product indexed k may cannibalize the specific product market (Granados et al., 2010; Post and Spann, 2012). So, given the cannibalization effect, the observed demand distribution needs to be revised as (D_i^p, D_j^p, D_k^p) in Section 4.2. As before, the retailer has to purchase quantities Q_j^d, Q_i^d, and Q_k^d to meet the demands for the specific products i, j, and the probabilistic product k, respectively. The quantity Q_k^d is a mix of products i and j. If we assume that the proportion of product i in the mix is r, then the retailer has to order $Q_i^p = Q_i^d + r Q_k^d$ of product i and $Q_j^p = Q_j^d + (1-r) Q_k^d$ of product j.

Following Fay and Xie (2014), Jerath et al. (2010), and Wu and Wu (2015) in operationalizing probabilistic selling, we assume that probabilistic selling postpones the delivery of the probabilistic product with regular price to meet the substitution demand for a specific product sold at a higher price. The retailer obtains revenue p for each specific product and p_0 ($p > p_0 > s$) for each probabilistic product sold. Since the consumer of the probabilistic product pays a lower price, they would accept uncertainty about product availability and postponement of product delivery. Figure 2 shows the sequence of events.
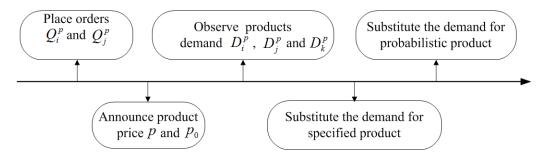

Figure 2: The sequence of events.

4.2 *Revised demand distribution*

We consider the cannibalization effect on the specific products, which means that the demand for the probabilistic product D_k^p consists of two parts: the demand that switches from the specific

products to the probabilistic product, and the new market expansion demand D_k induced by the low-priced probabilistic product. We assume that D_k is normally distributed with mean u_k and standard deviation σ_k. The demand D_i and D_j, and the demand $D_i(D_j)$ and the new market expansion demand D_k are correlated with ρ_{ij} and $\rho_{ik}(\rho_{jk})$, respectively. Let a_i (a_j) be the cannibalization index of the demand for the specific product i (j) $(0 \leq a_i(a_j) \leq 1)$, which is independent of D_k. The observed demand for the probabilistic product is given by $D_k^p = a_i D_i + a_j D_j + D_k$.

The observed demands D_i^p and D_j^p are different from the original demands D_i and D_j in traditional selling. It is important to define the demand relationships between traditional selling and probabilistic selling, for we will compare the two selling strategies with respect to the inventory decision and expected profit. Following Eynan and Fouque (2003), and Hsieh (2011), we characterize the distribution parameters of the observed demands D_i^p, D_j^p, and D_k^p under probabilistic selling as follows:

$$u_i^p = (1 - a_i)u_i,$$

$$u_j^p = (1 - a_j)u_j,$$

$$\sigma_i^p = (1 - a_i)\sigma_i,$$

$$\sigma_j^p = (1 - a_j)\sigma_j,$$

$$u_k^p = a_i u_i + a_j u_j + u_k,$$

$$\sigma_k^p = \sqrt{a_i^2 \sigma_i^2 + a_j^2 \sigma_j^2 + \sigma_k^2 + 2a_i \rho_{ik} \sigma_i \sigma_k + 2a_j \rho_{jk} \sigma_j \sigma_k + 2a_i a_j \rho_{ij} \sigma_i \sigma_j}.$$

Because the demand for the probabilistic product includes a part of the original demands for the specific products, the demand correlation after cannibalization should be updated as follows:

$$\rho_{ik}^* = \frac{a_i \sigma_i + a_j \rho_{ij} \sigma_j + \rho_{ik} \sigma_k}{\sigma_k^p}, \tag{5}$$

$$\rho_{jk}^* = \frac{a_j \sigma_j + a_i \rho_{ij} \sigma_i + \rho_{jk} \sigma_j \sigma_k}{\sigma_k^p}. \tag{6}$$

So we define the joint probability density function of the demand for the products under probabilistic selling $f(x_i, x_j, x_k)$ as $f^*(x_i, x_j, x_k)$ after cannibalization.
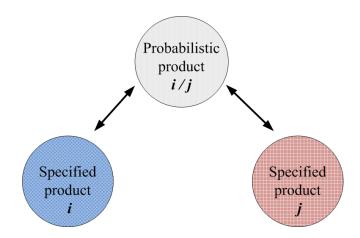
4.3 Substitution pattern

Figure 3: The substitution pattern under probabilistic selling.

Probabilistic selling encourages substitution between the specific products and the probabilistic product when stock-out occurs. As shown in Figure 3, different from inventory substitution, there is no direct substitution between the specific products. The substitution between the specific products occurs through the probabilistic product. Besides, the offer of probabilistic selling may increase total product sales.

Furthermore, there are two stages of substitution owing to postponement of product delivery under probabilistic selling. In the first stage of substitution, substitution occurs to meet the demands for the higher-priced specific products. For instance, if the demand realization of either specific product i (j) exceeds its available inventory Q_i^d (Q_j^d), the retailer can select the popular product i (j) from the probabilistic product inventory Q_k^d to meet the high-priced demand first. In the second stage of substitution, if the demand for the probabilistic product exceeds its remaining inventory after the first stage of substitution while the specific products are available, either of the specific products can serve as a substitute. There is no possibility that both specific products i and j are out of stock when the demand for the probabilistic product can be fully satisfied.

4.4 The optimal inventory decision

The decision variables in the inventory model are (Q_i^p, Q_j^p) rather than (Q_i^d, Q_j^d, Q_j^k). Thus it suffices to characterize the second stage of substitution. Specifically, we present these cases and their

Table 1: Classification scheme for all the possible cases and their corresponding occurring probabilities

Case	i	j	k	Probability
1	Yes	Yes	Yes	$\Pr(D_i^p < Q_i^p, D_j^p < Q_j^p, D_k^p < (Q_i^p + Q_j^p - D_i^p - D_j^p))$
2	Yes	Yes	No	$\Pr(D_i^p < Q_i^p, D_j^p < Q_j^p, D_k^p > (Q_i^p + Q_j^p - D_i^p - D_j^p))$
3	Yes	No	Yes	$\Pr(D_i^p < Q_i^p, D_j^p > Q_j^p, D_k^p < (Q_i^p - D_i^p))$
4	Yes	No	No	$\Pr(D_i^p < Q_i^p, D_j^p > Q_j^p, D_k^p > (Q_i^p - D_i^p))$
5	No	Yes	Yes	$\Pr(D_i^p > Q_i^p, D_j^p < Q_j^p, D_k^p < (Q_i^p - D_i^p))$
6	No	Yes	No	$\Pr(D_i^p > Q_i^p, D_j^p < Q_j^p, D_k^p > (Q_i^p - D_i^p))$
7	No	No	No	$\Pr(D_i^p > Q_i^p, D_j^p > Q_j^p, D_k^p > (Q_i^p - D_i^p))$

corresponding probabilities of occurrence in Table 1. For instance, Case 3 means that product j is out of stock, while product i has excess inventory, and the excess inventory is sufficient to cover the demand for the probabilistic product. The expected profit, which includes the revenue, the savage cost, and the acquisition cost, is given as follows:

$$
E(Q_i^p, Q_j^p) = E \begin{cases} p\min(D_i^p, Q_i^p) + p\min(D_j^p, Q_j^p) - c(Q_i^p + Q_j^p) \\ +p_0 \min\left[D_k, [(Q_i^p - D_i^p)^+ + (Q_j^p - D_j^p)^+]\right] \\ +s\left[(Q_i^p - D_i^p)^+ + (Q_j^p - D_j^p)^+ - D_k\right]^+. \end{cases} \tag{7}
$$

Proposition 1. If the distribution function of the demand is continuous and differentiable, then the expected profit function is concave in (Q_i^p, Q_j^p).

Proof. See the Appendix.

It can be recognized from Eq.(7) and Table 1 that the modelling of product i and product j are symmetrical. We just analyze the inventory decision of one product and the analysis of the other product is similar. Differentiating the expected total profit once, we obtain the expected value of a marginal unit of product i as follows:

$$
\begin{aligned}
\frac{\partial E(Q_i^p, Q_j^p)}{\partial Q_i^p} &= p(1 - Pr(D_i^p < Q_i^p)) \\
&+ p_0[\Pr(D_i^p < Q_i^p, D_j^p > Q_j^p, D_k^p > Q_i^p - D_i^p) \\
&+ \Pr(D_i^p < Q_i^p, D_j^p < Q_j^p, D_k^p > Q_i^p + Q_j^p - D_i^p - D_j^p)] \\
&+ s[\Pr(D_i^p < Q_i^p) - \Pr(D_i^p < Q_i^p, D_j^p > Q_j^p, D_k^p > Q_i^p - D_i^p) \\
&- \Pr(D_i^p < Q_i^p, D_j^p < Q_j^p, D_k^p > Q_i^p + Q_j^p - D_i^p - D_j^p)] - c.
\end{aligned} \tag{8}
$$

The first term of Eq.(8) means that any additional inventory of product i will result in an incremental sales except when there is excess inventory of product i ($D_i^p < Q_i^p$). The retailer can still benefit from the marginal unit of product Q_i^p by satisfying the demand for the probabilistic product (which may happen whenever the demand for product j can be satisfied), yielding p_0. The third term means that, if the inventory of product i exceeds its demand and the demand for the probabilistic product can also be satisfied, the marginal unit of product i is only worth its salvage value s. To simplify the notation, we re-arrange Eq.(8) and characterize the optimal order quantities.

Proposition 2. The optimal order quantities $(Q_i^{p^*}, Q_j^{p^*})$ under probabilistic selling can be expressed as

$$
\begin{cases}
F(Q_i^{p*}) &= \dfrac{p-c+(p_0-s)G(Q_i^{p^*},Q_j^{p^*})}{p-s} \\[2mm]
F(Q_j^{p^*}) &= \dfrac{p-c+(p_0-s)N(Q_i^{p^*},Q_j^{p^*})}{p-s},
\end{cases}
\tag{9}
$$

where

$$
\begin{aligned}
G(Q_i^{p^*},Q_j^{p^*}) = &\int_0^{Q_i^{p^*}} \int_0^{Q_j^{p^*}} \int_{Q_i^{p^*}+Q_j^{p^*}-D_i^p-D_j^p}^{\infty} f^*(x_i,x_j,x_k)dx_i dx_j dx_k \\
&+ \int_0^{Q_i^{p^*}} \int_{Q_j^{p^*}}^{\infty} \int_{Q_i^{p^*}-D_i^p}^{\infty} f^*(x_i,x_j,x_k)dx_i dx_j dx_k,
\end{aligned}
$$

$$
\begin{aligned}
N(Q_i^{p^*},Q_j^{p^*}) = &\int_0^{Q_i^{p^*}} \int_0^{Q_j^{p^*}} \int_{Q_i^{p^*}+Q_j^{p^*}-D_i^p-D_j^p}^{\infty} f^*(x_i,x_j,x_k)dx_i dx_j dx_k \\
&+ \int_{Q_i^p}^{\infty} \int_0^{Q_j^{p^*}} \int_{Q_j^{p^*}-D_j^p}^{\infty} f^*(x_i,x_j,x_k)dx_i dx_j dx_k.
\end{aligned}
$$

We see that the optimal order quantities are adjustments of the solution for the newsvendor model given in Eq.(1) and Eq.(2). Specifically, $G(Q_i^{p^*},Q_j^{p^*})$ raises Q_i^p due to the possibility of substituting for the probabilistic product. Substitution occurs in two cases: one is the case where product i has excess inventory while product j is out of stock, the other is the case where both specific products i and j have excess inventory, while the probabilistic product is out of stock. Similarly, $N(Q_i^{p^*},Q_j^{p^*})$

raises Q_j^p due to the substitution ability of product j. The implication of this proposition is that the retailer should hold more inventory of the products that have greater possibilities to substitute the other products. Thus the retailer can make incremental profit from the substitute product.

Compared with inventory substitution, the inventory decision under probabilistic selling is influenced by the price of the probabilistic product. Next, we analytically derive some properties of the effect of p_0 on the optimal inventory decision.

Proposition 3. When the price of the probabilistic product increases, the retailer should adjust its inventory decision depending on the substitution possibility. Specifically,

a) The retailer should order more product j and less product i ($\frac{\partial E(Q_i^{p*})}{\partial p_0} < 0$, $\frac{\partial E(Q_j^{p*})}{\partial p_0} > 0$) if the substitution possibility of product i is sufficiently small ($G < \frac{b^*}{d^*}N$).

b) The retailer should order more product i and less product j ($\frac{\partial E(Q_i^{p*})}{\partial p_0} > 0$, $\frac{\partial E(Q_j^{p*})}{\partial p_0} < 0$) if the substitution possibility of product i is sufficiently large ($G > \frac{a^*}{c^*}N$).

c) The retailer should order more of both products i and j ($\frac{\partial E(Q_i^{p*})}{\partial p_0} > 0$, $\frac{\partial E(Q_j^{p*})}{\partial p_0} > 0$) if the substitution possibility is moderate ($\frac{b^*}{d^*}N < G < \frac{a^*}{c^*}N$),

where a^*, b^*, c^*, and d^* denote the value of $\frac{\partial E(Q_i^p,Q_j^p)^2}{\partial^2 Q_i^p}$, $\frac{\partial E(Q_i^p,Q_j^p)^2}{\partial Q_i^p \partial Q_j^p}$, $\frac{\partial E(Q_i^p,Q_j^p)^2}{\partial Q_j^p \partial Q_i^p}$, and $\frac{\partial E(Q_i^p,Q_j^p)^2}{\partial^2 Q_j^p}$ at (Q_i^{p*},Q_j^{p*}), respectively. Besides, $\frac{a^*}{c^*} > 1$ and $\frac{b^*}{d^*} < 1$.

Proof. See the Appendix.

In the classic single-product newsvendor model, the optimal inventory (i.e. Eq.(1) and Eq.(2)) increases with the price. However, when it comes to the two-product setting of probabilistic selling. How does the retailer adjust the optimal inventory decision when the price of the probabilistic product increases? Proposition 3 states that, if there is an increase in the price of the probabilistic product, the retailer should order more of one product and less of the other when the difference of their substitution possibilities is large (e.g. $G < \frac{b^*}{d^*}N$, $G > \frac{a^*}{c^*}N$). And, when the difference is not large, the retailer should increase the inventory of both products (e.g. $\frac{b^*}{d^*}N < G < \frac{a^*}{c^*}N$). That means that the retailer should always order more product with higher substitution possibility. However, whether increase the inventory of product with lower substitution possibility or not depends on the difference of the two

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product's substitution possibility.

5 Comparisons for some special cases

Comparing the two strategies is difficult when the substitution fraction r_s, and the cannibalization indices a_i and a_j are non-zero. Therefore, we first compare the two strategies under some special cases, and then conduct computational studies in the next part to compare the two strategies in general. In practice, the cannibalization index a in probabilistic selling may become zero when customers' price sensitivity is low or product differentiation is very large (Granados et al., 2010; Post and Spann, 2012). The substitution fraction r_s may become zero when product differentiation is very large.

Case 1: $a_i = a_j = 0$, $D_k = 0$ and $r_s{=}0$. Inventory substitution is equivalent to probabilistic selling as they have the same optimal order quantity and expected profit, and neither strategy can improve the profit of the retailer.

This case may arises when the product differentiation is too large in a saturated market. In this case, both inventory substitution and probabilistic selling fail to generate additional profit from substitutable demand, and the introduction of low-priced products cannot attract new demand in this market. Thus, both strategies reduce to the classical newsvendor model.

Case 2: $a_i = a_j = 0$, $D_k \neq 0$ and $r_s = 0$. Probabilistic selling outperforms inventory substitution as it yields a higher profit with a higher inventory level.

In this case, neither the customers in probabilistic selling nor in inventory substitution accept a substitute when the product differentiation is too large. However, there are some new customers enticed to buy discounted products under probabilistic selling. This case can be explained by similar arguments in Post and Spann (2012), and Anderson (2009). Therefore, inventory substitution reduces to the classical newsvendor model and probabilistic selling can increase the profit by market expansion. However, the optimal inventory level under probabilistic selling is higher than that under inventory substitution because both $Q_i^{p*} > \frac{p-c}{p-s}$ and $Q_j^{p*} > \frac{p-c}{p-s}$ when $G(Q_i^{p^*}, Q_j^{p^*}) > 0$ and $N(Q_i^{p^*}, Q_j^{p^*}) > 0$.

Case 3: $a_i = a_j = 0$, $D_k \neq 0$ and $r_s \neq 0$. Inventory substitution requires more customers willing to accept the substitute product than probabilistic selling to achieve the same profit at the same marginal profit from the substitute product.

In this case, the comparison of the two strategies depends on the demand and marginal profit obtained from the substitute product. When the size of the new market expansion demand under probabilistic selling is the same as the number of customers that face a stock-out situation and accept another product, the retailer has the same amount of discounted product sales under the two selling strategies. However, probabilistic selling can increase the high-priced product sales through substitution while inventory substitution cannot. Therefore, probabilistic selling outperforms inventory substitution when it can attract an equal number of customers willing to accept a substitute. Besides, because the retailer offers the probabilistic product in the first selling stage rather than the second under inventory substitution, it has the potential to secure more demand for the low-priced products.

6   Computational studies

In this section we consider a more general case where the cannibalization index $a_i = a_j \neq 0$ and $r_s \neq 0$. The main question that drives the design of the computational studies is under what conditions probabilistic selling outperforms inventory substitution, and vice versa. Specifically, we explore the effects of the customer transfer coefficient and demand uncertainty on the optimal profit and inventory under both strategies. The customer transfer coefficient is reflected by the substitution fraction r_s under inventory substitution and the cannibalization index $a_i(a_j)$ under probabilistic selling. The difference is that one is positive transfer induced by price and the other is negative transfer forced by the stock-out of products.

We assume that the original demands D_i^t and D_j^t are equal, which are normally distributed with parameters that satisfy the assumptions: mean $u_i(u_j) = 100$, standard deivation $\sigma_i(\sigma_j) = \sigma = [20, 30, 40, 50]$, the initial correlation coefficient $\rho_{ij} = 0$, $p = 40$, $c = 20$, $s = 10$, $a_i = a_j = a \in [0, 1]$,

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$t = 2$, and $r_s \in [0, 1]$. In order to focus on the substitution effect of the two strategies, we assume that there is no new market expansion demand under probabilistic selling (e.g., $D_k = 0$ when the original market is saturated). For simplicity, we use "PS" and "IS" to denote probabilistic selling and inventory substitution, respectively.

6.1   The effect of the customer transfer coefficient

As shown in Figure 4, with different price discounts (i.e., 95%, 90%, and 85%), the results reveal the same trend that probabilistic selling achieves the highest expected profit at a relatively small customer transfer coefficient, while the expected profit under inventory substitution increases with the customer transfer coefficient. These two observations are consistent with the results in Zhang et al. (2016) and Rajaram and Tang (2001), respectively.
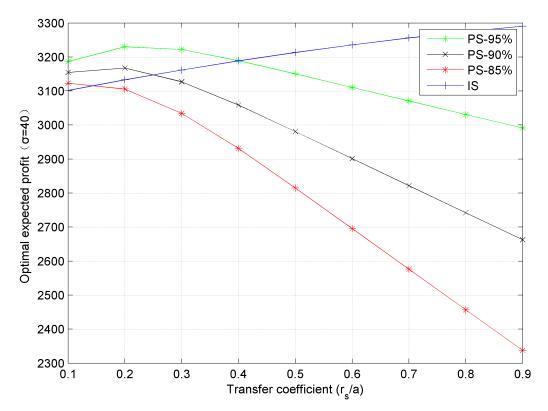

Figure 4: Expected profit of inventory substitution and probabilistic selling with different price discounts.

The efficiency of demand substitution under probabilistic selling increases with a smaller customer transfer coefficient (if the index is too small, the buffering effect of the probabilistic product for demand substitution becomes insignificant), while being restricted at larger customer transfer coefficient. One

reason is that profit decreases with demand correlation when demand is multivariate normal (Netessine and Rudi, 2003). It can be obtained from Eqs 5-6 that the correlation between the newly revised demand would increase as more customers switch from product i (j) to k. The positively correlated demands of products i, j, and k result in high possibilities of large and small substitution demands simultaneously. The probability that the retailer substitutes specific products for the probabilistic product, or vice versa when stock-out occurs is relatively small. Another reason is that more customers will switch to buying the probabilistic product that yields a lower profit margin, which can harm the retailer's profit. When the profit that demand substitution brings cannot offset the lower sales of the specific products, profit improvement will decrease. Therefore, probabilistic selling is most advantageous when the customer transfer coefficient is large enough to enable substitution, but not so large that very few consumers will buy the high-priced products.

As shown in Figure 4, the expected profit under probabilistic selling with a fixed customer transfer coefficient would decrease with price discount. This means that probabilistic selling requires some customers that have high price sensitivity to be attracted by the product with a small discount. Otherwise, when only few customers are attracted by a large discount, inventory substitution would be more advantageous than probabilistic selling.

6.2 *Comparison of the expected profit*

In this section we compare the performance of the two strategies when customers are price-sensitive. We take the fixed discount 95% as an example and define $PS_e - IS_e$ as the difference in the optimal expected profit under the two strategies. A positive value means that probabilistic selling is more advantageous; otherwise, inventory substitution strategy is more advantageous. We draw a colour map as shown in Figure 5 to facilitate analysis of strategy selection. We colour a positive value in red and a negative one in blue. From the computational results, we make the following observations:

Observation 1: With a relatively small customer transfer coefficient under probabilistic selling and inventory substitution, the former is more profitable. Inventory substitution outperforms probabilistic selling when the transfer coefficients under the two strategies are very high.

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(a) σ=20
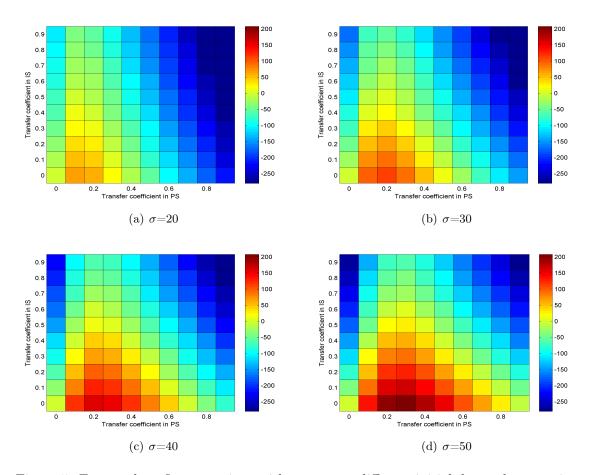
(b) σ=30

(c) σ=40

(d) σ=50

Figure 5: Expected profit comparison with respect to different initial demand uncertainty

Observation 2: Probabilistic selling is more advantageous than inventory substitution at higher demand uncertainty.

From Figures 4 and 5, we see that probabilistic selling can greatly improve profit with a smaller customer transfer coefficient than inventory substitution. For example, probabilistic selling can achieve a higher profit with $a = 0.1$ when σ=20, while inventory substitution requires $r_s = 0.4$ to achieve the same profit. When σ=30, probabilistic selling can achieve a higher profit with $a = 0.2$, while inventory substitution requires $r_s = 0.5$. However, when the customer transfer coefficient is larger, the advantage of probabilistic selling diminishes while inventory substitution can still bring more profit to the retailer. The customer transfer coefficient a under probabilistic selling mainly depends on customers' price sensitivity and product differentiation. Probabilistic selling requires that some customers are sensitive to price to be attracted to buy the probabilistic product (Zhang et al., 2016; Fay and Xie, 2008). At the same time, product differentiation should be large enough to avoid too much transfer (Post, 2010). Inventory substitution mainly depends on product differentiation. And the more customers that will

accept another product are, the more sales the retailer can get in the second selling stage.

Therefore, lower product differentiation is necessary for inventory substitution to be advantageous. Just as shown in Figure 5, when product differentiation is very low, and the customer transfer coefficient a and r_s are high, inventory substitution can bring more profit to the retailer. Therefore, the implication for the retailer is as follows: Adopting a proper selling strategy to manage demand uncertainty depends on customer characteristics and product differentiation. If the specific products have great similarity, inventory substitution is more advantageous, while relatively lower product similarity and higher price-sensitive customers can bring more profit to the retailer that adopts probabilistic selling. This observation is consistent with reality that probabilistic selling is common in third-party intermediary platforms which sells various products from different vendors, e.g., a seller may use inventory substitution to sell double-bed rooms and twin-bed rooms in one specific hotel, and may use probabilistic selling to sell rooms belonging to different hotels (e.g., Hotwire.com).
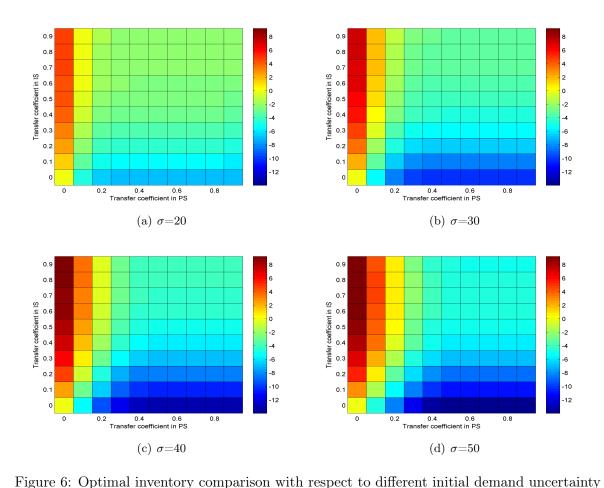
Observation 2 is obvious. The red-coloured area increases with demand uncertainty. The application range for adopting probabilistic selling is much wider when demand uncertainty is larger. Therefore, probabilistic selling is a more promising strategy to combat demand uncertainty.

6.3 *Comparison of the optimal inventory decision*

We define $PS_v - IS_v$ as the difference between the optimal total inventory under the two strategies. A negative value means that the retailer would hold less inventory when implementing probabilistic selling. We draw a colour map as shown in Figure 6, in which we colour the positive values in red and the negative values in blue. From the results, we make the following observation.

Observation 3: Probabilistic selling is more advantageous than inventory substitution in reducing inventory under most circumstances.

As shown in Figure 6, the blue-coloured area is very large. The inventory level under probabilistic selling is usually lower than that under inventory substitution. Combined with Figure 5, we find that when probabilistic selling outperforms inventory substitution in terms of yielding a higher profit, its optimal inventory is always lower than that under inventory substitution. The only exception is when

Figure 6: Optimal inventory comparison with respect to different initial demand uncertainty

$a = 0.1$ and $r_s = 0.3$ with σ=50. In contrast, when inventory substitution is more profitable, its optimal inventory is usually higher than that under probabilistic selling. Therefore, if it is optimal for the retailer to adopt probabilistic selling in a specific environment, it can usually obtain a higher profit with lower inventory than inventory substitution.

7   Conclusions

By offering the low priced probabilistic product to induce some customers to buy a flexible product, the retailer can substitute demand when stock out occurs to hedge against the demand uncertainty. Both probabilistic selling and inventory substitution strategy share the common feature in combating demand uncertainty through demand substitution. Therefore, our paper focuses on analyzing and comparing the efficiency of the two strategies. In this paper we first develop a single-period newsvendor model with three products to analyze probabilistic selling with a view to generating insights into

using probabilistic selling to manage demand uncertainty. We then compare probabilistic selling with inventory substitution in the special cases without cannibalization. To gain additional insights into the normal situation, we use computational examples to compare the two strategies in terms of overall profit and inventory with considerations of customer transition and demand uncertainty.

While both inventory substitution and probabilistic selling can induce demand from product-insensitive customers to achieve demand substitution, they differ in that probabilistic selling allows customers to accept uncertainty voluntarily at a discounted price rather than forcing them to accept another product like that under inventory substitution. The computational results show that probabilistic selling will bring more profit to the retailer when selling products with relatively lower similarity to higher price-sensitive customers, and it is more profitable to use inventory substitution to sell products with high similarity. Besides, higher demand uncertainty increases the profitability of probabilistic selling over inventory substitution.

The research of exploring the inventory mechanism of probabilistic selling compared with inventory substitution has theoretical and practical significance. The paper enriches the research about probabilistic selling as an inventory management tool to combat demand uncertainty. According to the conclusions of this manuscript, the retailer can choose the efficient strategy considering product differentiation, customer characteristics, and level of demand uncertainty. This has significant practical implications for the retailer that sells multiple products as follows: First, under probabilistic selling, the retailer should not be afraid of cannibalization because a proper degree of cannibalization can benefit the retailer in terms of yielding a higher expected profit. When the price of the probabilistic product increases, the retailer should always order more inventory of the product with higher substitution possibility. Second, if the retailer sells the substitute product with lower product similarity to price-sensitive customers, it is advised to use probabilistic selling to achieve a higher profit, and order less inventory than inventory substitution in most cases. On the other hand, inventory substitution is the better choice for the retailer when the product similarity is higher.

We assume in this study that the price of the probabilistic product is an exogenous variable. Future research may extend our work by combining the pricing and inventory decisions. It is also

worth considering PS in a supply chain setting (Shen et al., 2017; Minner, 2003). For example, it is interesting to explore the conditions under which a retailer's probabilistic selling will benefit the supplier, the retailer, and both.

Acknowledgments

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Appendix. Proposition 1

The expected profit function is

$$E(Q_i{}^p, Q_j{}^p) =$$

$$\int_0^{Q_i^p} \int_0^{Q_j^p} \int_0^{Q_i^p + Q_j^p - D_i^p - D_j^p} [pD_i^p + pD_j^p + p_0 D_0^p + s(Q_i^p + Q_j^p - D_i^p - D_j^p - D_0^p)] f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ \int_0^{Q_i^p} \int_0^{Q_j^p} \int_{Q_i^p + Q_j^p - D_i^p - D_j^p}^{\infty} [pD_i^p + pD_j^p + p_0(Q_i^p + Q_j^p - D_i^p - D_j^p)] f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ \int_0^{Q_i^p} \int_{Q_j^p}^{\infty} \int_0^{Q_i^p - D_i^p} [pD_i^p + pQ_j^p + p_0 D_0 + s(Q_i^p - D_i^p - D_0)] f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ \int_0^{Q_i^p} \int_{Q_j^p}^{\infty} \int_{Q_i^p - D_i^p}^{\infty} [pD_i^p + pQ_j^p + p_0(Q_i^p - D_i^p)] f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ \int_{Q_i^p}^{\infty} \int_0^{Q_j^p} \int_0^{Q_j^p - D_j^p} [pQ_i^p + pD_j^p + p_0 D_0 + s(Q_j^p - D_j^p - D_0)] f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ \int_{Q_i^p}^{\infty} \int_0^{Q_j^p} \int_{Q_j^p - D_j^p}^{\infty} [pQ_i^p + pD_j^p + p_0(Q_j^p - D_j^p)] f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ \int_{Q_i^p}^{\infty} \int_{Q_j^p}^{\infty} \int_0^{\infty} [pQ_i^p + pQ_j^p] f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$- c(Q_i^p + Q_j^p).$$

So

$$\frac{\partial E(Q_i^p, Q_j^p)}{\partial Q_i^p} =$$

$$s \int_0^{Q_i^p} \int_0^{Q_j^p} \int_0^{Q_i^p + Q_j^p - D_i^p - D_j^p} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p_0 \int_0^{Q_i^p} \int_0^{Q_j^p} \int_{Q_i^p + Q_j^p - D_i^p - D_j^p}^{\infty} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ s \int_0^{Q_i^p} \int_{Q_j^p}^{\infty} \int_0^{Q_i^p - D_i^p} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p_0 \int_0^{Q_i^p} \int_{Q_j^p}^{\infty} \int_{Q_i^p - D_i^p}^{\infty} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p \int_{Q_i^p}^{\infty} \int_0^{Q_j^p} \int_0^{Q_j^p - D_j^p} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p \int_{Q_i^p}^{\infty} \int_0^{Q_j^p} \int_{Q_j^p - D_j^p}^{\infty} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p \int_{Q_i^p}^{\infty} \int_{Q_j^p}^{\infty} \int_0^{\infty} f^*(x_i, x_j, x_k) dx_i dx_j dx_i dx_j dx_k$$

$$- c,$$

$$\frac{\partial E(Q_i^p, Q_j^p)}{\partial Q_j^p} =$$

$$s \int_0^{Q_i^p} \int_0^{Q_j^p} \int_0^{Q_i^p + Q_j^p - D_i^p - D_j^p} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p_0 \int_0^{Q_i^p} \int_0^{Q_j^p} \int_{Q_i^p + Q_j^p - D_i^p - D_j^p}^{\infty} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p \int_0^{Q_i^p} \int_{Q_j^p}^{\infty} \int_0^{Q_i^p - D_i^p} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p \int_0^{Q_i^p} \int_{Q_j^p}^{\infty} \int_{Q_i^p - D_i^p}^{\infty} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ s \int_{Q_i^p}^{\infty} \int_0^{Q_j^p} \int_0^{Q_j^p - D_j^p} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p_0 \int_{Q_i^p}^{\infty} \int_0^{Q_j^p} \int_{Q_j^p - D_j^p}^{\infty} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p \int_{Q_i^p}^{\infty} \int_{Q_j^p}^{\infty} \int_0^{\infty} f^*(x_i, x_j, x_k) dx_i dx_j dx_i dx_j dx_k$$

$$- c,$$

$$\frac{\partial E(Q_i^p, Q_j^p)^2}{\partial^2 Q_i^p} =$$

$$(s - p_0) \int_0^{Q_i^p} \int_0^{Q_j^p} f^*(x_i, x_j, Q_i^p + Q_j^p - D_i^p - D_j^p) dx_i dx_j dx_k$$

$$+ (s - p) \int_0^{Q_j^p} \int_0^{Q_j^p - D_j^p} f^*(Q_i^p, x_j, x_k) dx_j dx_k$$

$$+ (s - p_0) \int_0^{Q_i^p} \int_{Q_j^p}^{\infty} f^*(x_i, x_j, Q_i^p - D_i^p) dx_i dx_j$$

$$+ (p_0 - p) \int_{Q_j^p}^{\infty} \int_0^{\infty} f^*(Q_i^p, x_j, x_k) dx_j dx_k$$

$$+ (p_0 - p) \int_0^{Q_2} \int_{Q_2 - D_2}^{\infty} f^*(Q_i^p - D_i^p, x_j, x_k) dx_j dx_k \leq 0,$$

$$\frac{\partial E(Q_i^p, Q_j^p)^2}{\partial^2 Q_j^p} =$$

$$(s - p_0) \int_0^{Q_i^p} \int_0^{Q_j^p} f^*(x_i, x_j, Q_i^p + Q_j^p - D_i^p - D_j^p) dx_i dx_j dx_k$$

$$+ (s - p) \int_0^{Q_i^p} \int_0^{Q_i^p - D_i^p} f^*(x_i, Q_j^p, x_k) dx_i dx_k$$

$$+ (s - p_0) \int_{Q_i^p}^{\infty} \int_0^{Q_j^p} f^*(x_i, x_j, Q_j^p - D_j^p) dx_i dx_j$$

$$+ (p_0 - p) \int_{Q_i^p}^{\infty} \int_0^{\infty} f^*(x_i, Q_j^p, x_k) dx_i dx_k$$

$$+ (p_0 - p) \int_0^{Q_i^p} \int_{Q_i^p - D_i^p}^{\infty} f^*(x_i, Q_j^p, x_k) dx_i dx_k \leq 0.$$

In addition,

$$\frac{\partial E(Q_i^p, Q_j^p)^2}{\partial Q_i^p \partial Q_j^p} = \frac{\partial E(Q_i^p, Q_j^p)^2}{\partial Q_j^p \partial Q_i^p} =$$

$$(s - p_0) \int_0^{Q_i^p} \int_0^{Q_j^p} f^*(x_i, x_j, Q_i^p + Q_j^p - D_i^p - D_j^p) dx_i dx_j dx_k$$

Therefore, the Hessian Matrix

$$\begin{vmatrix} \frac{\partial E(Q_i^p, Q_j^p)^2}{\partial^2 Q_i^p} & \frac{\partial E(Q_i^p, Q_j^p)^2}{\partial Q_i^p \partial Q_j^p} \\ \frac{\partial E(Q_i^p, Q_j^p)^2}{\partial Q_j^p \partial Q_i^p} & \frac{\partial E(Q_i^p, Q_j^p)^2}{\partial^2 Q_j^p} \end{vmatrix} \geq 0.$$

We have the result.

Appendix. Proposition 3

The optimal order quantities must satisfy the following equations

$$
\begin{cases}
\frac{\partial E(Q_i^p, Q_j^p)}{\partial Q_i^p} &= p - (p-s)F(Q_i^{p*}) + (p_0 - s)G(Q_i^{p*}, Q_j^{p*}) - c = 0, \\
\frac{\partial E(Q_i^p, Q_j^p)}{\partial Q_j^p} &= p - (p-s)F(Q_j^{p*}) + (p_0 - s)N(Q_i^{p*}, Q_j^{p*}) - c = 0.
\end{cases}
$$

Differentiating the above results with respect to p_0 yields

$$
\begin{cases}
a^* \frac{\partial E(Q_i^{p*})}{\partial p_0} + b^* \frac{\partial E(Q_j^{p*})}{\partial p_0} &= -G(Q_i^{p*}, Q_j^{p*}), \\
c^* \frac{\partial E(Q_i^{p*})}{\partial p_0} + d^* \frac{\partial E(Q_j^{p*})}{\partial p_0} &= -N(Q_i^{p*}, Q_j^{p*}).
\end{cases}
$$

where a^*, b^*, c^*, and d^* denote the values of $\frac{\partial E(Q_i^p, Q_j^p)^2}{\partial^2 Q_i^p}$, $\frac{\partial E(Q_i^p, Q_j^p)^2}{\partial Q_i^p \partial Q_j^p}$, $\frac{\partial E(Q_i^p, Q_j^p)^2}{\partial Q_j^p \partial Q_i^p}$, and $\frac{\partial E(Q_i^p, Q_j^p)^2}{\partial^2 Q_j^p}$ at (Q_i^{p*}, Q_j^{p*}), respectively.

Then we get

$$
\begin{cases}
\frac{\partial E(Q_i^{p*})}{\partial p_0} &= \frac{b^* N(Q_i^{p*}, Q_j^{p*}) - d^* G(Q_i^{p*}, Q_j^{p*})}{a^* d^* - b^* c^*}, \\
\frac{\partial E(Q_j^{p*})}{\partial p_0} &= \frac{c^* G(Q_i^{p*}, Q_j^{p*}) - a^* N(Q_i^{p*}, Q_j^{p*})}{a^* d^* - b^* c^*}.
\end{cases}
$$

There are four cases to consider as follows:

Case 1 If $\{c^*G > a^*N \ \& \ b^*N < d^*G\}$, then $\{G < \frac{a^*}{c^*}N \ \& \ G < \frac{b^*}{d^*}N \ \& \ \frac{a^*}{c^*} > 1 \ \& \ \frac{b^*}{d^*} < 1\}$. Therefore, we can deduce that $\frac{\partial E(Q_i^{p*})}{\partial p_0} < 0$ and $\frac{\partial E(Q_j^{p*})}{\partial p_0} > 0$ with $G < \frac{b^*}{d^*}N$.

Case 2 If $\{c^*G < a^*N \ \& \ b^*N > d^*G\}$, then $\{G > \frac{a^*}{c^*}N \ \& \ G > \frac{b^*}{d^*}N \ \& \ \frac{a^*}{c^*} > 1 \ \& \ \frac{b^*}{d^*} < 1\}$. Therefore, we can deduce that $\frac{\partial E(Q_i^{p*})}{\partial p_0} > 0$ and $\frac{\partial E(Q_j^{p*})}{\partial p_0} < 0$ with $G > \frac{a^*}{c^*}N$.

Case 3 If $\{c^*G > a^*N \ \& \ b^*N > d^*G\}$, then $\{G < \frac{a^*}{c^*}N \ \& \ G > \frac{b^*}{d^*}N \ \& \ \frac{a^*}{c^*} > 1 \ \& \ \frac{b^*}{d^*} < 1\}$. Therefore, we can deduce that $\frac{\partial E(Q_i^{p*})}{\partial p_0} > 0$ and $\frac{\partial E(Q_j^{p*})}{\partial p_0} > 0$ with $\frac{b^*}{d^*}N < G < \frac{a^*}{c^*}N$.

Case 4 If $\{c^*G < a^*N \ \& \ b^*N < d^*G\}$, then $\{G > \frac{a^*}{c^*}N \ \& \ G < \frac{b^*}{d^*}N \ \& \ \frac{a^*}{c^*} > 1 \ \& \ \frac{b^*}{d^*} < 1\}$. There is no intersection set. Therefore, we can deduce that $\frac{\partial E(Q_i^{p*})}{\partial p_0} < 0$ and $\frac{\partial E(Q_j^{p*})}{\partial p_0} < 0$ can not coexist.