Structural stability of singular systems under proportional and derivative feedback

M. ISABEL GARCÍA-PLANAS\textsuperscript{1},

\textsuperscript{1} Dpto. MA1, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona. E-mail: maria.isabel.garcia@upc.edu.

\textbf{keywords:} Singular systems, equivalence relation, structural stability

\textbf{Abstract}

We consider triples of matrices \((E, A, B)\), representing singular linear time invariant systems in the form \(E \dot{x}(t) = Ax(t) + Bu(t)\), with \(E, A \in M_{p \times n}(C)\) and \(B \in M_{n \times m}(C)\), under proportional and derivative feedback.

In this work we study the equivalence relation considered, as a Lie group action that permit us to see the equivalence classes as differentiable manifolds. The description of the tangent and normal space to the orbits permits us to give a characterization of the structural stability of triples of matrices, in terms of numerical invariants.

\section{Introduction}

We denote by \(M_{r \times s}(C)\) the space of complex matrices having \(r\) rows and \(s\) columns, and in the case which \(r = s\) we write \(M_r(C)\).

We consider the set \(M\) of triples of matrices \((E, A, B)\) representing singular linear time invariant systems in the form \(E \dot{x}(t) = Ax(t) + Bu(t)\), with \(E, A \in M_{p \times n}(C)\), \(B \in M_{n \times m}(C)\), \((n, m, p > 0)\).

The concept of structural stability, in the qualitative theory of dynamical systems (structurally stable elements being those whose behavior does not change when applying small perturbations) has been widely studied by several authors in control theory (see [6], [4], for example).

In the sequel we identify triples of matrices \((E, A, B)\) with rectangular matrices in the form \((E \ A \ B)\) in order to use matrix expressions, and we will use the following notations.

\(I_n\) denotes the \(n\)-order identity matrix,
Let $2$ Equivalence relation

The triple $(\lambda, E, B)$ corresponds to the finite zeros of the triple and $J$ responds to the finite zeros of the triple and $J$ denotes a nilpotent matrix in its reduced form $N = \text{diag}(N_1, \ldots, N_t)$, $N_i = \begin{pmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{pmatrix} \in M_{n_i}(C)$, $J$ denotes the Jordan matrix $J = \text{diag}(J_1, \ldots, J_t)$, $J_i = \text{diag}(J_{i1}, \ldots, J_{it})$, $J_{ij} = \lambda_i I_{n_i} + N$, $L = \text{diag} = (L_1, \ldots, L_q)$, $L_j = (I_{n_j} \ 0) \in M_{p_j \times (n_j + 1)}(C)$, $R = \text{diag}(R_1, \ldots, R_p)$, $R_n = (0 \ I_{n_j}) \in M_{n_j \times (n_j + 1)}(C)$

2 Equivalence relation

The standard transformations in state and input spaces $x(t) = Px_1(t)$, $u(t) = Ru_1(t)$ premultiplication by an invertible matrix $QE\dot{x}(t) = QAx(t) + Qu(t)$ as well as feedback $u(t) = u_1(t) - Vx(t)$ and derivative feedback $u(t) = u_1(t) - U\dot{x}(t)$, realized over singular systems relate them in the following manner, two systems are related when one can be obtained from the other by means of one, or more, of the transformations considered. In fact, this transformations define an equivalence relation in the corresponding space of triples of matrices in the following manner.

**Definition 1** Let $(E_i, A_i, B_i)$, $i = 1, 2$ be two triples in $M$. Then, $(E_1, A_1, B_1)$ is equivalent to $(E_2, A_2, B_2)$ if and only if there exist invertible matrices $Q \in \text{Gl}(p; C)$, $P \in \text{Gl}(n; C)$, $R \in \text{Gl}(m; C)$, and matrices $U, V \in M_{m \times n}(C)$, such that

$$\begin{pmatrix} E_2 & A_2 & B_2 \end{pmatrix} = Q \begin{pmatrix} E_1 & A_1 & B_1 \end{pmatrix} \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ U & V & R \end{pmatrix} \quad (1)$$

It is easy to check that this relation is an equivalence relation.

**Theorem 1** ([7]) Let $(E, A, B)$ be a triple. Then, it is equivalent to

$$\begin{pmatrix} E_1 \\ S_E \end{pmatrix}, \begin{pmatrix} A_1 \\ S_A \end{pmatrix}, \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \quad (2)$$

where $(E_1, A_1, B_1)$ is the largest regular subtriple in its Kronecker reduced form (see [7]), concretely

$$(E_1, A_1, B_1) = \begin{pmatrix} I_1 \\ I_2 \\ N_2 \end{pmatrix}, \begin{pmatrix} N_1 \\ J \\ I_3 \end{pmatrix}, \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

The triple $(I_1, N_1, B_1)$ a controllable triple in its Kronecker reduced form, $(I_2, J, 0)$ corresponds to the finite zeros of the triple and $J$ in its Jordan reduced form, $(N_2, I_3, 0)$ corresponds to the infinite zeros of the triple and $N_2$ in its Jordan reduced form. The triple $(S_E, S_A, 0)$ is the strictly singular part of the system in its Kronecker reduced form:

$$\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

A complete system of invariants to obtain the canonical reduced form can be found in [7]. The proof consists of associating the pencil $\lambda \begin{pmatrix} E & B & 0 \end{pmatrix} + \begin{pmatrix} A & 0 & B \end{pmatrix}$ and check that the equivalence relation considered for triples coincides with strict equivalence defined for pencils.
2.1 Equivalence as a Lie group action

Equivalence relation 1 may be seen as induced by the action of the Lie group $\mathcal{G} = \{(Q, P, R, U, V) \mid Q \in \text{Gl}(p; C), P \in \text{Gl}(n; C), R \in \text{Gl}(m; C), U, V \in M_{m \times n}(C)\}$. Using short notations $g = (Q, P, R, U, V) \in \mathcal{G}$ and $x = (E, A, B) \in M$, we define multiplication in $\mathcal{G}$, action of the group $\mathcal{G}$, and equivalence condition (1) as follows

$$g_1 g_2 = (Q_2 Q_1, P_1 P_2, R_1 R_2, U_1 P_2 + R_1 U_2, V_1 P_2 + R_1 V_2) \in \mathcal{G},$$

$$g \circ x = Q \begin{pmatrix} E_1 & A_1 & B_1 \\ 0 & 0 & 0 \\ U & V & R \end{pmatrix} \in M,$$

$$x_2 = g \circ x_1.$$

Multiplication in the group corresponds to successive equivalence transformations: $g_2 \circ (g_1 \circ x) = (g_1 g_2) \circ x$. Unit element of $\mathcal{G}$ has the form $e = (I_p, I_n, I_m, 0, 0)$.

Let us fix a triple $x_0 = (E_0, A_0, B_0) \in M$ and define the mapping

$$\alpha_{x_0}(g) = g \circ x_0.$$  \hfill (4)

The equivalence class of the triple $x_0$ under equivalence relation (1) coincides with the equivalence class of the triple with respect to the action of $\mathcal{G}$, that is the range of the function $\alpha_{x_0}$. It is called the orbit of $x_0$ and denoted by

$$\mathcal{O}(x_0) = \text{Im} \alpha_{x_0} = \{g \circ x_0 \mid g \in \mathcal{G}\}.$$  \hfill (5)

The mapping $\alpha_{x_0}$ is differentiable, and $\mathcal{O}(x_0)$ is an smooth submanifold of $M$.

Let us use the notation $T_e\mathcal{G}$ for tangent space to the manifold $\mathcal{G}$ at the unit element $e$. Since $\mathcal{G}$ is an open subset of $M_p(C) \times M_n(C) \times M_m(C) \times M_{m \times n}(C) \times M_{m \times n}(C)$, we have

$$T_e\mathcal{G} = \{(Q, P, R, U, V) \mid Q \in \text{Gl}(p; C), P \in \text{Gl}(n; C), R \in \text{Gl}(m; C)U, V \in M_{m \times n}(C)\}$$

and, since $M$ is a linear space,

$$T_{x_0}M = M.$$

The Euclidean scalar products in the spaces $M$ and $T_e\mathcal{G}$ considered in this paper are defined as follows

$$\langle x_1, x_2 \rangle_1 = \text{tr}(E_1 E_2^*) + \text{tr}(A_1 A_2^*) + \text{tr}(B_1 B_2^*),$$

where $x_i \in (E_i, A_i, B_i) \in M$,

$$\langle y_1, y_2 \rangle_2 = \text{tr}(Q_1 Q_2^*) + \text{tr}(P_1 P_2^*) + \text{tr}(R_1 R_2^*) + \text{tr}(U_1 U_2^*) + \text{tr}(V_1 V_2^*),$$

where $y_i = (Q_i, P_i, R_i, U_i, V_i) \in T_e\mathcal{G}$,  \hfill (6)

where $A^*$ denotes the conjugate transpose of a matrix $A$ and $\text{tr}$ the trace of the matrix.

Let $d\alpha_{x_0} : T_e\mathcal{G} \longrightarrow M$ be the differential of $\alpha_{x_0}$ at the unit element $e$. Using expressions (3) and (4), we find:

$$d\alpha_{x_0}(y) = (E_0 P + Q E_0 + B_0 U, A_0 P + Q A_0 + B_0 V, B_0 R + Q B_0) \in M,$$

where $x = (X, Y, Z, T) \in T_e\mathcal{G}$.

The adjoint linear mapping $d\alpha_{x_0}^* : M \longrightarrow T_e\mathcal{G}$ is defined by the relation

$$\langle d\alpha_{x_0}(y), z \rangle_1 = \langle y, d\alpha_{x_0}^*(z) \rangle_2, \text{ for } y \in T_e\mathcal{G}, \text{ and } z \in M.$$  \hfill (8)
The mappings \(d\alpha_{x_0}\) and \(d\alpha_{x_0}^*\) provide a simple description of the tangent space \(T_{x_0}\mathcal{O}(x_0)\), and its normal complement \((T_{x_0}\mathcal{O}(x_0))^\perp\).

**Theorem 2** The tangent spaces to the orbit and stabilizer of the triple of matrices \(x_0\) and the corresponding normal complementary subspace with respect to \(M\) can be found in the following form

\[
i)\quad T_{x_0}\mathcal{O}(x_0) = \text{Im} d\alpha_{x_0} \subset M, \\
ii)\quad (T_{x_0}\mathcal{O}(x_0))^\perp = \text{Ker} d\alpha_{x_0}^* \subset M.
\]

**Proof.** Assertion i follows from (7). Then assertion ii follows from properties of the adjoint function \(d\alpha_{x_0}^*\) (see [5] for example). \(\square\)

**Proposition 1** Let \(x_0 = (E, A, B) \in M\) be a triple of matrices. Then,

\[
T_{x_0}\mathcal{O}(x_0) = \{(QE + EP + BU, QA + AP + BV, QB + BR) \mid \forall(Q, P, R, U, V) \in \mathcal{G}\}
\]

\[
(T_{x_0}\mathcal{O}(x_0))^\perp = \{(X, Y, Z) \mid EX^* + AY^* + BZ^* = 0, X^*E + Y^*A = 0, X^*B = 0, Y^*B = 0, Z^*B = 0\}
\]

Using Kronecker products and vec operator (see [5] for its definition and properties) we can represent the vectors in \(T(\mathcal{O}(E, A, B))\) as

\[
\begin{pmatrix}
(QE + EP + BU) \\
(QA + AP + BV) \\
(QB + BR)
\end{pmatrix} =
\begin{pmatrix}
E^t \otimes I_p \\
A^t \otimes I_p \\
B^t \otimes I_p
\end{pmatrix} \text{vec}(Q) +
\begin{pmatrix}
I_n \otimes E \\
I_n \otimes A \\
I_m \otimes B
\end{pmatrix} \text{vec}(P) +
\begin{pmatrix}
0 \\
0 \\
I_m \times B
\end{pmatrix} \text{vec}(R) +
\begin{pmatrix}
0 \\
I_n \otimes B \\
0
\end{pmatrix} \text{vec}(V)
\]

In this notation, we may say that the tangent space is the range of following matrix

\[
T_{(E, A, B)} =
\begin{pmatrix}
E^t \otimes I_p & I_n \otimes E & 0 & I_n \otimes B & 0 \\
A^t \otimes I_p & I_n \otimes A & 0 & 0 & I_n \otimes B \\
B^t \otimes I_p & 0 & I_m \otimes B & 0 & 0
\end{pmatrix}
\] (9)

If confusion it is not possible we will write simply \(T\).

As a consequence we have of these statements we have the following obvious result.

**Theorem 3** Let \((E, A, B)\) be a triple of matrices in \(M\) and we consider the matrix \(T\) as in (9). Then

\[
\dim T_{(E, A, B)}\mathcal{O}(E, A, B) = p^2 + n^2 + m^2 + 2nm - \dim(\text{Ker} T),
\] (10)

\[
\dim(T_{(E, A, B)}\mathcal{O}(E, A, B))^\perp = \dim(\text{Ker} T) - (n - p + m)^2 - mp.
\] (11)
The dimension of the orthogonal space is also known as the codimension of the orbit.

Taking into account that the rank of a matrix coincides with the number of non-zero singular values of the matrix, we can give the following characterization of the codimension of the orbit.

**Corollary 1** Let \((E, A, B)\) be a triple of matrices in \(M\). Then

\[
\text{codim } O(E, A, B) = \text{the number of zero singular values of } T.
\]

(12)

### 3 Structural stability

In this Section we will recall the definition of structural stability, according to that appearing in the paper by Willems (see [8]), as well as equivalent conditions.

Let \(X\) be a topological space and consider an equivalence relation defined on it.

**Definition 2** An element \(x \in X\) is structurally stable if and only if there exists an open neighbourhood \(U \subset X\) of \(x\) such that all the elements in it are equivalent to \(x\).

Let us assume that \(X\) is a differentiable manifold and the equivalence relation in \(X\) is that induced by the action of a Lie group \(G\) which acts on \(X\), giving rise to orbits which are also differentiable manifolds.

Let us denote by \(T_xO(x)\) the tangent space in \(x \in X\) to the orbit of \(x\), \(O(x)\), and consider any Hermitian product in \(X\).

**Proposition 2** Under the assumptions above, the following conditions are equivalent:

a) \(x \in X\) is structurally stable,

b) \(\dim O(x) = \dim X\),

c) \(\dim T_xO(x) = \dim X\),

d) \(\dim T_xO(x) ^\perp = 0\).

**Proof.** An element \(x \in X\) is structurally stable if and only if there exists an open neighbourhood contained in its orbit. Thus its orbit should be an open submanifold and therefore its dimension equal to \(\dim X\). The last statement follows from the fact that

\[
\dim T_xO(x) + \dim T_xO(x)^\perp = \dim X.
\]

\(\square\)

In our particular set-up we have the following result.

**Theorem 4** A triple \((E, A, B)\) is structurally stable if and only if the matrix \(T\), has full rank.

Taking into account the homogeneity of the orbits, a triple is structurally stable if and only if each triple in the orbit is structurally stable. Then we can consider a triple in its canonical reduced form in order to obtain a characterization of structural stability related with the structural invariants.

**Lemma 1** A necessary condition for structural stability of a triple \((E, A, B)\) is that the matrix \(B\) has full rank.
Proof. For the lower semicontinuity of the rank. □

From now on, we assume that the matrix $B$, has full rank.

Theorem 5 A triple $(E, A, B)$ is structurally stable

1. for $m \geq n, p$ or $n \geq m > p$, if and only if $\text{rank} B = p$.

2. for $n = p - m$ there are not stable triples.

3. for $n > p - m$, if and only if there are $m$ blocks $L_1$, $\ell - s$ blocks $L_{\ell_1}$ and $s$ $L_{\ell_1+1}$ where $n = (p - m)c + \ell$, and $p - m = \ell\ell_1 + s$.

4. for $n < p - m$, if and only if there are $m$ blocks $L_1$, $\ell - s$ blocks $L_{\ell_1}$ and $s$ $L_{\ell_1+1}$ where $p - m = nc + \ell$, and $n = \ell\ell_1 + s$.

5. for $n = p$, $m > 0$, if and only if there are not continuous invariants, nor infinite zeroes, and row minimal indices, $\text{rank} B = r = \min\{p, m\}$, there are $r$ column minimal indices of order 1, and $r$ column minimal indices equals or differing in only one unity.

Proof. It suffices to compute $\text{rank} T_{(E, A, B)}$ for an equivalent triple in its canonical reduced form and to observe that if we consider a triple in the form (2), we have that

$$ \text{rank} T = \text{rank} \begin{pmatrix} E_1^t \otimes I_p & I_r \otimes B & 0 \\ A_1^t \otimes I_p & I_r \otimes A & 0 \\ B_1^t \otimes I_p & 0 & I_m \otimes B \end{pmatrix} + \text{rank} \begin{pmatrix} S_E^t \otimes I_p & I_s \otimes B & 0 \\ S_A^t \otimes I_p & 0 & I_s \otimes B \end{pmatrix}, $$

$(r + s = p, E_1, A_1 \in M_r(C), S_E, S_A \in M_s(C))$. □

Example 1 Let $(E, A, B)$ be a triple in $M$ with $E = I_2$, $A = N \in M_2(C)$ and $B = e_2^t$. In this case we have that $n > p - m$ and there are two blocks $L_1$ (notice that the associate pencil is equivalent to $\lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$).

Moreover if we compute the rank of matrix $T$, we have

$$ \text{rank} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 10. $$
Bibliography