Observer-based Sensor Fault Detectability: About Robust Positive Invariance Approach and Residual Sensitivity *

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Abstract:
This paper considers detectability of deviation of sensors from their nominal behavior for a class of linear time-invariant discrete-time systems in the presence of bounded additive uncertainties. Detectable sensor faults using interval observers are analyzed considering two distinct approaches: invariant-sets and classical fault-sensitivity method. It can be inferred from this analysis that both approaches derive distinct formulations for minimum detectable fault magnitude, though qualitatively similar. The core difference lies in the method of construction of the invariant set offline in the former method and the reachable approximation of the convergence set using forward iterative techniques in the latter. This paper also contributes in giving a formulation for minimum fault magnitudes with invariant sets using an observer-based approach. Finally, an illustrative example is used to compare both approaches.

Keywords:
Fault Detection, Sensor Faults, Dynamic Models, Positive Invariance, Interval Observers.

1. INTRODUCTION

Fault detection and isolation is of paramount importance to deal with anomalous situations of autonomous systems. A well-established solution to sensor fault diagnosis is the use of physical redundancy and simple decision-making schemes (e.g. voting schemes). However, in many safety-critical systems such as aerospace and petrochemical systems, physical sensing redundancy can be cost consuming and increase hardware complexity (Blanke et al., 2006). For this reason, researchers have proposed the use of analytical redundancy where mathematical models describing the system operation are implemented in software (Blanke et al., 2006; Ding, 2008). However, uncertainty remains always present when modelling a system. Fault diagnosis methods that are able to deal with uncertainty are known in the literature under the attribute *robust*. One way to deal with uncertainty is to assume an unknown but bounded description (Puig, 2010). This description is exploited in the design of fault detection criteria checking the boundedness of the state and output estimation error of

observer-based fault diagnosis method. When the property of boundedness is not satisfied, then faults are detected.

Another property that has been recently exploited by robust sensor fault diagnosis methods for the design fault detection criteria was the positive invariance of the tracking or estimation error dynamics or both as in Olaru et al. (2010). Positive invariance along with attractiveness can offer guarantees for the fault detectability and isolability of sensor faults. These guarantees are necessary for enhancing the performance of an active fault tolerant control schemes with respect to system stability and performance Seron et al. (2012). In Xu et al. (2013) there was a first attempt to relate invariant set approaches with interval observers but in the context of state estimation. In the present work, the aim is to pursue these line of developments but looking at the structural detectability by the prism of the two methods.

Several researchers have treated the detectability of faults as a structural property without taking into account uncertainty (Ding, 2008). However, from a realistic point of view, the detectability of faults should be characterized in the presence of uncertainty. In (Kodakkadan et al., 2015), sensor faults were classified as hidden, strongly detectable and weakly detectable based on the separation of robust positively invariant sets where the tracking state error con-

verges. On top of that, the computation of the minimum magnitude of the fault that can be detected in presence of uncertainty is an important diagnosis performance index highlighted by several researchers (Meseguer et al., 2010).

In this paper, two robust approaches are described for the computation of the minimum magnitude of sensor faults affecting a linear time invariant (LTI) system in the presence of uncertainty. The first approach is based on the computation robust positively invariant sets where the residual (output estimation error) generated by a Luenberger observer converges. The second approach is based on the computation of the sensor fault sensitivity of a reachable approximation of the set within which the residual generated by an interval observer converges. The approximation is realized by applying forward iterative techniques. Both approaches characterize the residual set in both healthy and faulty situation and apply set separability conditions for ensuring the sensor fault detection. The differences between the two methods reside principally in the online/offline treatment of the set-theoretic methods.

Section 2 outlines the premise of the work and the assumptions used for the formulation that follows. Section 3 details the sensor fault detectability approach based on robust positive invariance and residual sensitivity approach using a zonotopic interval observer. Section 5 discusses both the qualitative and quantitative features of the two approaches, which are illustrated through a simulation example in Section 6. Section 7 presents the main concluding remarks of this work, along with some future steps.

Notation: $\mathbb{R}^n$ is the n-dimensional Euclidean space with $\|\cdot\|$ their prescribed norm (Euclidean norm for simplicity). The closed convex hull of a set $\mathcal{S}$ will be denoted as $Conv\{\mathcal{S}\}$ and the interval hull with $\square\mathcal{S}$.

A polytopic set is a set with flat boundaries. A polyhedron (or a polyhedral set) in $\mathbb{R}^n$ is a (convex) polytopic set obtained as the intersection of a finite number of closed half-spaces. A bounded polyhedron is also defined as the convex hull of its vertices. The set of vertices of a polytope $P \subset \mathbb{R}^n$ is denoted $\mathcal{V}(P)$.

The Minkowski sum of two sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{R}^n$ will be denoted by $\mathcal{S}_1 \oplus \mathcal{S}_2$.

2. PROBLEM FORMULATION

Consider the family of discrete LTI systems

$$x(k+1) = A\,x(k) + B\,u(k) + g(w(k)), \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, and $u \in \mathbb{R}^m$ is the input (control) signal of the dynamics. The matrix A is the system matrix and B is the input-transfer matrix of appropriate dimensions. The dynamic system is affected by additive disturbances that reside in the set

$$\mathcal{W} = \{g(w) : |g(w)| \leq \bar{z}\}, \qquad (2)$$

where the function $g : \mathbb{R}^n \to \mathbb{R}^n$ gives the shape and $\bar{z}$ imposes the sublevel set.

The set $\mathcal{W}$ can be described in a zonotopic form as

$$\mathcal{W} = \omega^c \oplus H_\omega \mathbf{B}^q, \qquad (3)$$

where ω^c and H_ω are the center and the segments of the set $\mathcal{W}$, respectively. $\mathbf{B}^q$ is a q-dimensional unitary box and $\mathcal{V}(\mathcal{W}) = H_\omega\{\mathcal{V}(\mathbf{B}^q)\}$.

The states of the system (1) are measured by a set of p sensors. Under healthy conditions, the output vector $y \in \mathbb{R}^p$ is described by

$$y(k) = C\,x(k), \qquad (4)$$

where C is the output matrix. When at least one sensor is faulty, the output equation becomes

$$y(k) = C\,x(k) + F(k), \qquad (5)$$

where $F \in \mathbb{R}^p$ represents the fault.

Assumption 1. The pair $\{A, C\}$ is observable.

2.1 Robust Sensor Fault Detectability

In this paper, the fault detection process is based on residual generation using observers. In order to detect faults, we monitor a residual vector r defined as:

$$r(k) = y(k) - C\hat{x}(k). \qquad (6)$$

where $\hat{x}$ is the estimation of the system state generated by the following standard Luenberger observer

$$\hat{x}(k+1) = A\,\hat{x}(k) + B\,u(k) + L(y - C\hat{x}(k)), \qquad (7)$$

where L is the observer gain such that $A_e = A - LC$ is a Schur matrix.

The objective is to determine the range of faults F which can be detected when a set-valued description of the uncertainty as the one in (3) is available in the system description (1) and considering the following assumptions:

Assumption 2. A permanent bias fault affects only one sensor at the time instant k_f; i.e. $F(k) = [0\ 0\ \ldots\ f_j\ \ldots\ 0]^\top$ for $k \geq k_f$.

3. ROBUST POSITIVE INVARIANCE APPROACH

The first approach to characterize the minimum detectable faults is based on the set-invariance approach extending the results presented in (Kodakkadan et al., 2015).

3.1 Robust Positively Invariant Set

The state estimation under healthy conditions, denoted by $\hat{x}_h$ is generated based on (7) with y described by (4).

Given that the observer (7) is stable (i.e. A_e is Schur) and the disturbances are bounded, according to Kolmanovsky and Gilbert (1998) there exists an invariant set denoted by $\mathcal{S}$ that includes the state estimation error under healthy conditions defined as $\tilde{x}_h(k) = x(k) - \hat{x}_h(k)$, which satisfies the following dynamics

$$\tilde{x}_h(k+1) = A_e\,\tilde{x}_h(k) + g(w(k)). \qquad (8)$$

If the error starts inside this set, then it will remain inside. More than that, if the estimation error starts outside the set $\mathcal{S}$, then it will enter in that set after a finite time instant. Formally,

$$\tilde{x}_h \in \mathcal{S} \implies A_e\,\tilde{x}_h + g(w) \in \mathcal{S}\ \forall g(w) \in \mathcal{W}. \qquad (9)$$

Equivalently, the robust invariance implies $\mathcal{S}$ is RPI if and only if $A_e\mathcal{S} \oplus \mathcal{W} \subseteq \mathcal{S}$. To construct the invariant set $\mathcal{S}$, several constructive methods can be employed (See

Kofman (2005)). The ultimate bounds method described in Kofman et al. (2007) is used in this work to be computationally attractive. The Jordan Canonical form of A_e is given by $J = V^{-1}A_e\,V$ with V some invertible matrix and J the diagonal matrix corresponding to the Jordan-Normal form of A_e. According to Kofman et al. (2007), the state estimation error $\tilde{x}(k)$ described by the dynamics in (8) will ultimately converge within the polyhedral RPI set $\mathcal{S}$ defined as

$$\mathcal{S} = \left\{ \tilde{x} \in \mathbb{R}^n : |V^{-1}\,\tilde{x}(k)| \leq (I - |J|)^{-1}\,|V^{-1}|\bar{z}\, + \varepsilon \right\}, \tag{10}$$

with a strictly positive (small) ε. Since the construction of invariant sets using the ultimate bound formulation is symmetric around the origin, there is also an equivalent representation of $\mathcal{S}$ as a zonotopic set as

$$\mathcal{S} = \omega^c \oplus H_s \mathbf{B}^n, \tag{11}$$

where ω^c is the origin and H_s is the generator matrix zonotopic set $\mathcal{S}$, respectively, when $\omega^c = 0$. See Stoican et al. (2011) for further details on zonotopic ultimate bounds.

Under faulty conditions, the state estimation satisfies the following dynamics:

$$\hat{x}(k+1) = A\,\hat{x}(k) + B\,u(k) + L(Cx(k) + F(k) - C\hat{x}(k)), \tag{12}$$

and the state estimation error dynamics in (8) will be

$$\tilde{x}(k+1) = A_e\,\tilde{x}(k) + g(w(k)) - LF(k). \tag{13}$$

Under faulty conditions, the residual can be re-written as

$$r(k) = r_h(k) + r_f(k), \tag{14}$$

where $r_h(k)$ and $r_f(k)$ are the healthy and faulty components of the residuals, respectively.

During healthy working mode of the sensor, precisely when $f_i(k) = 0\ \forall i \in \{1, \ldots, r\}$, $r(k) = r_h(k) = C\tilde{x}_h(k)$ where $\tilde{x}_h$ is described by (8). The residual vector will converge to a limit set related to the RPI set of the state estimation error under healthy conditions, i.e.,

$$\tilde{x}_h(k) \in \mathcal{S} \implies r_h(k) \in C\mathcal{S} = \mathcal{S}_h. \tag{15}$$

Taking into account (15), sensor fault is detected when $r \notin \mathcal{S}_h$.

During faulty working mode of the system (see Assumption 2), the residual given in (14) will converge to a *faulty RPI set* denoted by $\mathcal{S}_f$. A sensor fault f_i is guaranteed to be detectable using the residual $r_i = y_i - C_i\hat{x} = C_ix + f_i$, if the faulty RPI set is separated from the healthy RPI set. Moreover, we can obtain some certain conditions for ensuring the detectability of the fault at the next time instant of the fault occurrence.

Given that $\tilde{x}_h(k) \in \mathcal{S}$ during healthy mode, as shown in (15), since the set $\mathcal{S}$ is centered at the origin, the states of the estimation error described by (13) at the next time instant will move into a translation of the set $\mathcal{S}$ with $-LF$ as denoted by

$$\tilde{x}(k_f + 1) \in \mathcal{S} \oplus (-LF). \tag{16}$$

Based on this, after the occurrence of a fault, the residual for i-th sensor under faulty conditions is defined as

$$r_{f_i}(k_f + 1) = C_i\tilde{x}_f(k_f + 1) + f_i. \tag{17}$$

Then the residual r_i at the next instant after the occurrence of sensor fault belongs to the set

$$\mathcal{S}_{f_i}^*(k_f + 1) = C_i\{\mathcal{S} \oplus -L_if_i\} + f_i, \tag{18}$$

for each sensor i and the corresponding residual r_i with C_i and L_i being the i-th row of C and i-th column of L, respectively.

The sensor fault f_i is guaranteed to be detected using the residual r_i in one time step (Seron et al. (2008)) if and only if $\mathcal{S}_h \cap \mathcal{S}_f(k_f + 1) \neq \emptyset$. A sufficient condition is $\mathcal{S}_{h_i} \cap \mathcal{S}_{f_i}(k_f + 1) \neq \emptyset$ for at least one residual r_i with $i \in \{1, \ldots, p\}$. In other words, two sets are separated if their projections over at least one of the coordinates are separated. The projections of $\mathcal{S}_h$ and $\mathcal{S}_f$ over the i-th axis are described (in terms of intervals) as

$$\mathcal{S}_{h_i} = \left[\min_{j \in \{1, \ldots, 2^n\}} \{C_i\,v_j{}^*\}, \max_{j \in \{1, \ldots, 2^n\}} \{C_i\,v_j{}^*\} \right], \tag{19}$$

$$\mathcal{S}_{f_i} = \left[\min_{j \in \{1, \ldots, 2^n\}} \{C_i\,\{v_j{}^* - L_if_i\} + f_i\}, \right.$$
$$\left. \max_{j \in \{1, \ldots, 2^n\}} \{C_i\,\{v_j{}^* - L_if_i\} + f_i\} \right]. \tag{20}$$

If $\mathcal{S}_{h_i} \cap \mathcal{S}_{f_i} \neq \emptyset$ holds, (28) or (29) should be separated for at least one axis $i\{1, \ldots, p\}$. If the minimum of the interval of $\mathcal{S}_{h_i}$ is greater than the maximum of the interval of $\mathcal{S}_{f_i}$ (and vice-versa), the minimum guaranteed detectable additive sensor faults f_i in one time step is described for the i-th sensor by

$$f_i > \min_{i \in \{1, \ldots, r\}} \left\{ |\{1 - C_iL_i\}|^{-1} \left\{ \max_{j \in \{1, \ldots, 2^n\}} \{C_i\,v_j{}^*\} - \right. \right.$$
$$\left. \left. \min_{j \in \{1, \ldots, 2^n\}} \{C_i\,v_j{}^*\} \right\} \right\},$$

$$f_i < \max_{i \in \{1, \ldots, r\}} \left\{ |\{1 - C_iL_i\}|^{-1} \left\{ \min_{j \in \{1, \ldots, 2^n\}} \{C_i\,v_j{}^*\} - \right. \right.$$
$$\left. \left. \max_{j \in \{1, \ldots, 2^n\}} \{C_i\,v_j{}^*\} \right\} \right\}, \tag{21}$$

where $v_j{}^* \in \mathcal{V}(\mathcal{S})$, with $v_j{}^*$ being the vertices of $\mathcal{S}$, or the j-th row of $\mathcal{V}(\mathcal{S})$. Since the magnitude of the projection on one axis can be represented by the magnitude of the vector spanning that projection which is twice the infinite norm, then

$$\left\{ \max_{j \in \{1, \ldots, 2^n\}} \{C_i\,v_j{}^*\} - \min_{j \in \{1, \ldots, 2^n\}} \{C_i\,v_j{}^*\} \right\} = 2\|\mathcal{V}(C_i\,\mathcal{S})\|_\infty, \tag{22}$$

where $C_i\mathcal{S}$ will be the projection of the set in $\mathcal{S}$ on the i-the axis and $\mathcal{V}(C_i\,\mathcal{S}) = H_{s_i}$ with H_s defined in (11).

By combining (21) and (22), the minimum detectable sensor fault satisfies

$$f_i > \min_{i \in \{1, \ldots, r\}} \left\{ |1 - C_iL_i|^{-1}\{2\|H_{s_i}\|_\infty\} \right\},$$

$$f_i < \min_{i \in \{1, \ldots, r\}} \left\{ |1 - C_iL_i|^{-1}\{-2\|H_{s_i}\|_\infty\} \right\}. \tag{23}$$

Relaxing the specification of one time instant detection of faults, we can obtain certain conditions for characterizing the minimum detectable fault based on the asymptotic stable behavior of the faulty working mode.

Ignoring the effects of the disturbance in the system, during the steady state of the faulty operation mode of the system, the state estimation error as described in (13) will converge to

$$\tilde{x}(k_\infty) = (I - A + LC)^{-1}(-LF), \qquad (24)$$

as $\tilde{x}(k+1) = \tilde{x}(k)$ and $(I - (A - LC))\tilde{x}(k_\infty) = -LF$ with k_∞ denoting the time to converge into the steady state. Note that the inverse of the matrix $(I - A + LC)$ exists since the matrix $(A - LC)$ is Schur by design.

Therefore, if $\tilde{x}(k) \in \mathcal{S}$ during healthy mode, as implied from (15), which is centered at the origin, then

$$\tilde{x}(k) \in \mathcal{S} \oplus (I - A + LC)^{-1}(-LF). \qquad (25)$$

and hence the residual r_i corresponding to each sensor as in (6) during steady state of the permanent faulty working mode, denoted by $r_i^f(k)$, enters and stays in a set such that $r_i^f(k) \in \mathcal{S}_{f_i}$ with

$$\mathcal{S}_{f_i} = C_i\{\mathcal{S} \oplus -\Phi^0 L_i f_i\} + f_i, \qquad (26)$$

where $\Phi^0 = (I - A + LC)^{-1}$.

Theorem 1. Given the dynamical system (8) and any initial condition with an associated RPI set $\mathcal{S}$ as in (10), the minimum guaranteed detectable additive sensor faults f_i satisfies

$$f_i > \min_{i \in \{1,\dots,r\}} \left\{ |1 - C_i \Phi^0 L_i|^{-1} \{2\|H_{s_i}\|_\infty\} \right\},$$
$$f_i < \max_{i \in \{1,\dots,r\}} \left\{ |1 - C_i \Phi^0 L_i|^{-1} \{-2\|H_{s_i}\|_\infty\} \right\}. \qquad (27)$$

Proof. During the permanent faulty mode of the sensor, the residual $r_f(k)$, will converge to an invariant set $\mathcal{S}_f$. Sensor faults are guaranteed to be detected if $\mathcal{S}_{h_i} \cap \mathcal{S}_{f_i} \neq \emptyset$ for at least one i. Two sets are separated if their projections on at least one of the coordinates are separated. The projections of $\mathcal{S}_h$ and $\mathcal{S}_f$ on the i-th axis are described (as intervals) as

$$\mathcal{S}_{h_i} = \left[\min_{j \in \{1,\dots,2^n\}} \{C_i\, v_j^*\}, \max_{j \in \{1,\dots,2^n\}} \{C_i\, v_j^*\} \right], \qquad (28)$$

$$\mathcal{S}_{f_i} = \left[\min_{j \in \{1,\dots,2^n\}} \{C_i\, \{v_j^* - \Phi^0 L_i f_i\} + f_i\}, \right.$$
$$\left. \max_{j \in \{1,\dots,2^n\}} \{C_i\, \{v_j^* - \Phi^0 L_i f_i\} + f_i\} \right]. \qquad (29)$$

If the minimum of the interval of $\mathcal{S}_{h_i}$ is greater than the maximum of the interval of $\mathcal{S}_{f_i}$ (and vice-versa), sensor fault detection can be guaranteed. Following similar reasoning presented for the one step detection, Theorem 1 will be satisfied. Similarly, the minimum detectable fault will be the most sensitive one of all the residuals to that particular sensor fault and hence the proof is completed as we obtain the results stated in Theorem 1 ∎

4. INTERVAL OBSERVER APPROACH

The second approach to characterize the minimum detectable faults is based on the set-invariance approach extending the results presented in Pourasghar et al. (2016).

4.1 Interval observers

The set that includes the system state of (1) at every time instant can be bounded using an zonotopic interval observer of the form (Xu et al., 2013)

$$\hat{\mathcal{X}}(k+1) = (A - LC)\hat{\mathcal{X}}(k) \oplus \{Bu(k)\} \oplus \{Ly(k)\} \oplus \mathcal{W},$$
$$\hat{\mathcal{Y}}_i(k) = C_i\hat{\mathcal{X}}(k), \qquad (30)$$

where $\hat{\mathcal{X}}$ and $\hat{\mathcal{Y}}_i$ are the estimated state sets and predicted output sets for the i-th sensor, respectively. As in the case of observer (7), the gain L is selected such that the matrix $(A - LC)$ is Schur.

The estimated state sets $\hat{\mathcal{X}}$ obtained by (30) can be expressed using center and segments form of a zonotope as

$$\hat{\mathcal{X}}(k+1) = \hat{x}^c(k+1) \oplus \hat{H}_x(k+1)\mathbf{B}^{r_x}, \qquad (31)$$

with

$$\hat{x}^c(k+1) = (A - LC)\hat{x}^c(k) + Bu(k) + Ly(k),$$
$$\hat{H}_x(k+1) = \left[(A - LC)\hat{H}_x(k)\ H_\omega \right]. \qquad (32)$$

Likewise, the predicted output set $\hat{\mathcal{Y}}_i$ can be expressed using center and segments form as

$$\hat{\mathcal{Y}}_i(k) = \hat{y}_i^c(k) \oplus \hat{H}_{y_i}(k)\mathbf{B}^{r_y}, \qquad (33)$$

with

$$\hat{y}_i^c(k) = C_i\hat{x}^c(k),$$
$$\hat{H}_{y_i}(k) = \left[C_i\hat{H}_x(k) \right], \qquad (34)$$

where $\hat{y}^c$ and $\hat{H}_y$ denote the center and the segments of the output prediction set $\hat{\mathcal{Y}}_i$, respectively.

Then, the residual set in healthy conditions ($f_i = 0$) can be generated by means of the difference between the output measurement and predicted output set that is denoted as

$$\mathcal{R}_i(k) = r_i(k) \oplus H_i(k)\mathbf{B}^{r_r}, \qquad (35)$$

where r_{h_i} and H_{h_i} denote the center and the segments of the residual set $\mathcal{R}_i$, respectively. Both center and segments can be determined considering non-faulty conditions as follows

$$r_{h_i}(k) = y_i(k) - \hat{y}_i^c(k)$$
$$= (I - P(q^{-1}))y_i(k) - N(q^{-1})u(k), \qquad (36)$$
$$H_{h_i}(k) = \left[-C_i\Phi H_\omega \right],$$

where q^{-1} is the shift operator, $N(q^{-1}) = C_i\Phi B$, $P(q^{-1}) = C_i\Phi L$ and $\Phi = \left(q^{-1}I - A + LC \right)^{-1}$.

Theorem 2. Given the dynamical system (8) and the healthy residual set $\mathcal{R}_h$ considering uncertainty bounds (35), the minimum guaranteed detectable additive sensor faults f_i for the ith sensor in steady state is given by

$$f_i > \min_{i \in \{1,\dots,r\}} \left\{ |1 - C_i \Phi^0 L_i|^{-1} \{2\|H_{h_i}(k_\infty)\|_1\} \right\},$$
$$f_i < \min_{i \in \{1,\dots,r\}} \left\{ |1 - C_i \Phi^0 L_i|^{-1} \{-2\|H_{h_i}(k_\infty)\|_1\} \right\}. \qquad (37)$$

Proof. The bounds of the interval hull of the healthy residual in (35) in steady state can then be given by

$$\mathcal{Q}_{\min_i} = \min\{\square\mathcal{R}_{h_i}(k)\} = \{r_{h_i}(k_\infty) - \|H_{h_i}(k_\infty)\|_1\},$$
$$\mathcal{Q}_{\max_i} = \max\{\square\mathcal{R}_{h_i}(k)\} = \{r_{h_i}(k_\infty) + \|H_{h_i}(k_\infty)\|_1\}, \qquad (38)$$

where $\mathcal{Q}_{\max_i}$ and $\mathcal{Q}_{\min_i}$ are upper and lower bounds of the obtained residual set in the healthy mode in steady state, respectively. Sensor faults are detected when

$$r_i \notin \mathcal{R}_{h_i}. \qquad (39)$$

where

$$r_i = (I - P(q^{-1}))y_{h_i}(k) - N(q^{-1})u(k) + (I - P(q^{-1}))G_f f_i(k) \qquad (40)$$

Thus, the sensitivity of the residual to the sensor fault is given by

$$S_{f_i} = (I - P(q^{-1}))G_f, \qquad (41)$$

where S_{f_i} denotes the sensitivity of the residual to the given fault. Hence, in steady state $(q = 1)$, the sensitivity (41) can be rewritten as

$$S_{f_i} = 1 - C_i \Phi^0 L_i. \qquad (42)$$

Therefore, the residual can be formulated based on the sensitivity as

$$r_i(k_\infty) = r_{h_i}(\infty) + S_{f_i} f_i. \qquad (43)$$

Thus, the minimum detectable fault correponds to the one that has a magnitude that brings the residual r_i out of its healthy interval set $\mathcal{R}_{h_i}$ that according to (Meseguer et al., 2010) is achieved if

$$S_{f_i} f_i < \mathcal{Q}_{\min_i} \implies f_i < (S_{f_i})^{-1} \mathcal{Q}_{\min_i},$$
$$S_{f_i} f_i > \mathcal{Q}_{\max_i} \implies f_i > (S_{f_i})^{-1} \mathcal{Q}_{\max_i}.$$

Assuming that the mean of the healthy residuals over the steady state to be zero (i.e., $r_{h_i}(k_\infty) = 0$) and the worst-case disturbance affecting the system (making use of (38) and (42)), the minimum detectable sensor faults are given by (38). ∎

5. DISCUSSION

This paper scrutinizes and attempts to bridge the approaches for minimum fault detectability undertaken by the classical framework of defining fault sensitivity using interval observers and the relatively new invariant set approach. Given the construction of the defining set mapping the stable model behavior (the zonotopic convergence set in steady state and the state estimation error invariant set in the two approaches respectively), the effect of minimal fault that can be detected remains the same in both the approaches. The invariant set method, (obtained using a conservative ultimate bound approach in the presented example in Section 6) gives weaker minimal detectable fault compared to the sensitivity approach.

The reason for this difference is that the construction of the invariant set using ultimate bounds will lead to a larger set as the offline construction is independent of an active input signal. However, in the sensitivity approach, the determination of the convergence set of the residual using zonotopic iterations, even though increases the complexity significantly in each iteration (the segment matrix grows), taking into account the active input signal, approximates the set to a smaller region than the former. The penalty the sensitivity method has to pay is the use of online monitoring and computation resources while the former generates a conservative set in the broader sense.

The asymptotic convergence of the set-based observer leads to a *tight zonotopic approximation of the mRPI* while the invariance based method provides offline a *tight invariant approximation of the minimal RPI set* which can be represented (loosely) in a zonotopic form.

Although the results from (27) and (37) obtained from both approaches are comparable and are quite similar, there are few minor differences. Equation (27) uses the infinity norm of the interval vector of the projection of the ultimately bounded RPI zonotopic set while (37) uses one-norm of interval vector of the minimum convergence zonotopic set reached in steady state using active monitoring of the evolution of the dynamics. This will also lead to different zonotopic generator matrices and therefore the comparison is not qualitatively exact, but still gives a fair idea of these two approaches.

However, the guarantees of detectability in one time step and nullifying the effect of reinjection of faulty residuals into the active healthy set is advantageous from the invariant-set perspective. Nevertheless, the sensitivity approach is able to handle faults and their anticipation even during the transitory of the system dynamics as well as during the faulty operations where as the invariant set approach, though computationally attractive, gives information only in the steady state. The invariant-set method guarantees detection of faults in the next time instant of the occurrence of the fault while the sensitivity method might also detect the same but does not give those guarantees.
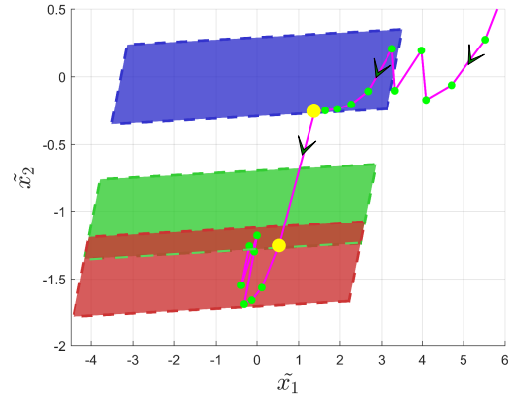

Fig. 1. Set $\mathcal{S}$(blue), $\{\mathcal{S} \oplus -LF\}$ (green) and $\{\mathcal{S} \oplus -\Phi^0 LF\}$(red) for a fault magnitude $F = 2$. The trajectory of the state estimation error dynamics is shown by the magenta line with $\tilde{x}(k_f)$ and $\tilde{x}(k_f + 1)$ shown as yellow dots.

6. ILLUSTRATIVE EXAMPLE

Let the dynamical model in (1) be defined as follows:

$$A = \begin{bmatrix} 0.9067 & -0.0687 \\ 0.0104 & 0.7933 \end{bmatrix}, B = \begin{bmatrix} 0.0272 \\ 0.3127 \end{bmatrix}, C = [0 \ 1]. \quad (44)$$

The disturbances are bounded such that $|g(w)| \leq 0.2$. The observer gain L is placed such that

$$A_e = \begin{bmatrix} 0.9067 & -0.3909 \\ 0.0104 & 0.2933 \end{bmatrix}. \qquad (45)$$

The Jordan decomposition of A_e is formed as $J = V^{-1} A_e V$ with

$$J = \begin{bmatrix} 0.9000 & 0.0000 \\ 0.0000 & 0.3000 \end{bmatrix}, V = \begin{bmatrix} 0.9999 & 0.5416 \\ 0.0171 & 0.8407 \end{bmatrix}, \qquad (46)$$

and the invariant set $\mathcal{S}$ can be constructed using (10).

Following the invariant-set-based approach and deriving the guaranteed detectability conditions from (23), the minimum faults that are detected in one step are

$$|F| > 1.4034, \qquad (47)$$

for the selected C and observer gain $L = [0.3222\ 0.5000]^\top$. Relaxing the conditions of the one-step detection, the minimum detectable fault now becomes

$$|F| > 0.9824, \qquad (48)$$

The invariant set for the error dynamics $\mathcal{S}$(blue), the set where the dynamics will go at $r_i(k_f + 1) \in \{\mathcal{S} \oplus -LF\}$ and the *invariant faulty set* $\{\mathcal{S} \oplus -\Phi^0 LF\}$ for a fault magnitude $F = 2$ are depicted in Fig. 1. Inferring from the above results, for fault magnitude satisfying $\mathcal{S}_{h_i} \cap \mathcal{S}_{f_i} = \emptyset$ but not $\mathcal{S}_{h_i} \cap \mathcal{S}_{f_i}^* \neq \emptyset$ will guarantee its detection after asymptotic stability of the faulty behavior but not in one time step after its occurrence as shown in Fig. 2, i.e., when the magnitude of fault is greater than 0.9824 but less than 1.4034, fault is guaranteed to be detected but not in one time instant.
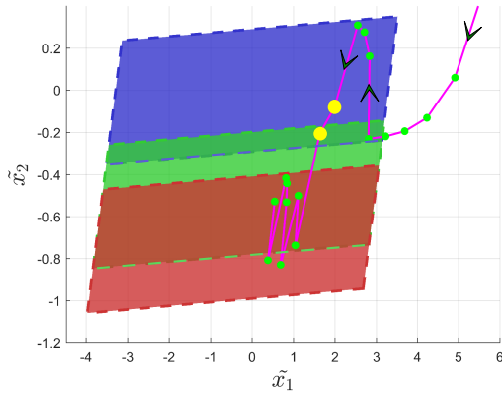

Fig. 2. Set $\mathcal{S}$(blue), $\{\mathcal{S} \oplus -LF\}$ (green) and $\{\mathcal{S} \oplus -\Phi^0 LF\}$(red) for a fault magnitude $F = 0.99$. The movement of the state estimation error from the occurrence of fault shown by the magenta line with $\tilde{x}(k_f)$ and $\tilde{x}(k_f + 1)$ shown as yellow dots.

On the other hand, in the case of interval obsersers using the sensitivity approach, fault detectability analysis is carried out at the steady state of the system for comparative homogeneity. Similarly, same C matrix and observer gain L are used. The sensitivity of the residual with respect to the fault parameter F as in (42) is

$$S_{f_i} = 0.2856, \text{with } \Phi = \begin{bmatrix} 10.0904 & -5.6313 \\ 0.1485 & 1.3322 \end{bmatrix}. \qquad (49)$$

In steady state, the minimal detectable fault can be obtained as

$$|F| > 0.8290. \qquad (50)$$

7. CONCLUSION

This paper presented the comparison between two approaches (invariant sets and interval observers) in finding the minimal detectable sensor faults when observers are used for fault detection. This paper extends a comparative study regarding state estimation, with an application to fault detection and minimal faults characterization that were not addressed in the previous works. The difference is reasoned qualitatively and a quantitative comparison with an illustrative example has also been presented. Future work shall include incorporating the advantages of both approaches handling multiple concomitant faults and also input-state constraints.

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