OPTIMAL DISTANCE NETWORKS OF LOW DEGREE FOR PARALLEL COMPUTERS †

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Abstract: We introduce and study a family oi interconnection schemes, the *Midimew networks*, based on circulant graphs of degree 4.
A family of such circulants is determined and shown to be optimal with respect to two distance parameters simultaneously, namely maximum distance and average distance, among all circulants of degree 4. These graphs are regular, point-symmetric, and maximally connected, and one such optimal graph exists for any given number of nodes.
The proposed interconnection schemes consist of mesh-connected networks with wrap-around links, and are isomorphic to the optimal distance circulants previously considered. We demonstrate how to construct one such network for any number of nodes, examine their good properties to build interconnection schemes for multicomputers, and discuss some interesting particular cases.
The problem of routing is also addressed, and a basic algorithm is provided, which is adequate to implement the routing policy required to convey messages, traversing shortest paths between nodes.

Index terms:
Circulant graphs
Graph models
Large message-passing multicomputers
Mesh-connected topologies
Message routing
Minimum distances
Static regular interconnection networks

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I. INTRODUCTION

The study of interconnection networks and their combinatorial properties has been pursued in the last decades with many different goals in mind, and more recently has been encouraged by its applicability to large scale parallel computer systems, mainly as one of the consequences of the growing use of VLSI technologies.

A parallel architecture constitutes a response, based upon the utilization of many cooperating computing elements (even full computers), to the increasing demands for greater computational power, which go beyond the potential of uniprocessor systems.

Paradigms such as message-interchange oriented concurrent programming, and distribution of memory into modules owned by each processing element, must be taken into account in order to achieve higher performance through massive (beyond 100) replication of processing elements. The resulting architectures may be broadly denoted as multicomputer systems [4], where each system node essentially includes processing element(s), some amount of private memory, and the inter-node communication unit.

Several multicomputer architectures have been proposed and designed in the last few years. The Cosmic Cube [25] constitutes the first completed experimental effort which has become the archetype of early operative multicomputers. Commercial successors to this concurrent computer include, among others, the Intel iPSC/1, N-cube/10, and Ametek S/14, which were introduced in 1985. Further developments have given rise to some other machines, such as the Intel iPSC/2 of the Symult Series 2010.

In order to efficiently exploit these parallel systems, one of the important problems to be addressed refers to the interconnection scheme which ties all of the system nodes together. The most essential function of the interconnection network is the allowance of an efficient message interchange between processes which execute on the system nodes. In order to minimize the potential communication bottleneck which appears in the implementation of multicomputer systems, the design of this network then becomes a major topic to be studied.

The characteristics of the interconnection topology directly affect the expected performance measures of the global system. The ideal solution, of providing a direct linking to connect every pair of nodes, is prohibitively expensive as soon as N (number of nodes) becomes large. Therefore other cost-effective schemes should be proposed and evaluated. The total number of links must be reduced, yet providing low communication overhead, as well as allowing simple routing strategies to keep high operational capabilities in presence of links/nodes faults [1].

Consequently, among other important factors which affect the expected system performance, attention must be paid to the amount of extra delay due to the nonexistence of direct links between any pair of nodes, and to the routing procedures to be executed for message passing communication schemes.

One classification of communication subsystems for parallel computers distinguishes three types: 1) link-based schemes, 2) bus-oriented schemes, 3) interconnection network-based schemes [18]. The present work deals with topologies belonging to the first type. We consider structures with dedicated channels available for information transfer between nodes, which require message routing through intermediate nodes. This approach is well suited to high traffic rates and to variable communication patterns.

Combinatorial aspects of networks in multicomputer systems (and also in local area networks, array processors and other parallel computer systems) are often studied by means of graph theory, which is specially well suited as a directly applicable tool to analyze and design link-based topologies. We adopt this approach in the present work.

In our graph models, the vertices represent the nodes of the multicomputer system and the edges represent the bidirectional communication links between nodes. Measures obtained from the graph model allow us to predict the system behavior.

To obtain a high performance in a multicomputer system, the structure of its interconnection scheme must accomplish several conditions. Different performance metrics are commonly used in order to characterize the merits of the proposal. One generally accepted set of network parameters, which is adequate for this purpose, includes the following: total number of links, diameter, average distance, link- and node-connectivity, symmetry, embeddability of algorithms, and extensibility. Let us consider briefly each of these network parameters, related to the properties of the underlying graphs.

The graph degree refers to the maximum number of edges incident with a graph node. We consider regular networks in which this number is the same for all nodes. Making constant and small the graph degree signifies a simplicity for the routing policy, as well as a reduction in the cost of nodes and links.

The graph diameter and average distance are measures related to the delay during transmission of messages, and thus, the performance. To keep the values of these measures as small as possible is a desirable goal in order to obtain a high system throughput.

The edge- and vertex-connectivities refer to the minimum number of edges and vertices, respectively, whose removal results in a disconnected graph. That is, these two properties represent a measure for the degree of robustness exhibited by the network, and so are related to fault-tolerance issues.

The vertex-symmetry property of a graph allows the analysis of its topological characteristics by considering any arbitrary vertex as the reference node. This desirable property makes the network look the same when viewed from any vertex, and is related to the reduction in the complexity of designing distributed routing algorithms, as well as to programmability issues.

Good embeddability of many important parallel computation graphs, such as rings, trees, meshes, and others, allows the topology to attain an efficient matching to the communication structure inherent to many well-known parallel applications. This characteristic also facilitates the use of existing software which has been designed for other topologies.

Finally, the extensibility of a network is an important property which allows a graceful scaling of the network size, without adversely affecting the cost/performance nor the existing interconnection setup, and maintains the node degree as well.

Although the above commented set of parameters may serve to accomplish, to a certain extent, a qualitative comparison of network proposals, it does not constitute a definitive set of good performance criteria. The lack of standard metrics only permits partial quantitative evaluations, which must, in any case, take into account the particular area of applications kept in mind when the system is designed. In this direction, several authors have previously reported different comparative studies of multicomputer networks in order to establish some relative merits among them [1], [2], [24], [23], [29].

Let us outline now more precisely the extent of our paper, and also review other related research works from different authors. We shall address here the problem of appropriately selecting the intercommunication topology used to connect a large number of nodes. Our interest points at one class of circulant graphs with degree 4, as a good compromise with respect to the above mentioned desirable characteristics. This type of graphs was first introduced by Harary [16] and later named as circulant graphs by Boesch and Tindell [8] because they hold a circulant adjacency matrix. As we shall justify at the end of section II, our graphs are maximally connected. Therefore, highly reliable networks may be constructed on their basis, and this fact justifies the interest shown by several authors.

Once the high connectivity of circulant graphs had been asserted, the problem of diameter reduction on certain families of circulant graphs was addressed, in order to minimize the communication delays associated with these topologies. The reader is referred to [28] for an excellent survey on this subject. More recently, in [9] the minimization of diameter in a class of degree 4 circulant graphs is studied, where also reliability topics are discussed. On the other hand, results obtaining simultaneous minimization of diameter and average distance for some circulant graphs of degree 4 have been previously published in [7]. Compound graphs based upon a family of circulant graphs are also proposed in [10] in order to design hierarchical networks for message-passing multicomputers.

Another type of well-known topologies of this kind are the toroidal meshes, which are isomorphic to some circulant graphs. In the ILLIAC IV computer [5], which is one of the first parallel machines successfully designed, underlies a variant of toroidal mesh as interconnection topology. Other subsequent topologies based on circulant graphs are the Twisted Torus [19] and the Double Twisted Torus [27], and all of them are constructed as different types of mesh-connected networks with wrap-around links, family to which the networks obtained in our work belong. This type of mesh networks is of particular interest because they allow a convenient and simple projection onto the plane. This fact provides a practical way to implement these interconnection networks for any large number of nodes, in opposition to other topologies not so well suited for this purpose, as the binary n-cube networks which make necessary the mapping of their n dimensions into a 2D space [11].

The low degree of mesh networks presents a practical advantage over other topologies with greater number of links per node: to achieve a more complete utilization of the network connectivity when programming parallel applications on the resulting architectures [13]. Moreover, meshes with wrap-around links such as those obtained in our work have proved to be topologies which allow an efficient mapping of many regular problems.

Directed graphs of a similar nature have attracted attention of researchers in the last years. Analysis of digraphs showing minimum diameter and/or average distance have been reported [15], [22], [17]. Directed graphs of this kind are commonly denoted as "double loop graphs", or "reliable loop topologies", pointing out their good fault tolerance capabilities. Nevertheless, their scope of application is mainly reduced to local area networks, due to the fact that their links are oriented and do not allow the exploitation of the locality of communications between processes resident in neighbour nodes of the multicomputer system [4].

Now we shall briefly describe the organization of this paper. We pursue to obtain, characterize, and design a class of topologies able to link any number of functional nodes, in order to construct interconnection networks for general-purpose multicomputers exhibiting good cost/performance trade-offs. These topologies are suitable, as well, for a broader scope of other parallel architectures.

Section II deals with circulant graphs of degree 4. A class of such graphs, which is optimal in the sense of simultaneously minimizing diameter and average distance, is identified. We characterize the topology, in a nice manner, by means of only one adjacency pattern parameter which defines the graph for any number of vertices. The optimal distance properties of these graphs are proved in this section.

In order to implement interconnection networks for parallel machines, several practical considerations must be taken into account. For this constructive purpose, the circulant graphs of section II present several disadvantages which are addressed in section III, and which lead us to search for a convenient transformation of these topologies. Consequently, a class of mesh-connected networks with wrap-around links is obtained here, by using an isomorphism which preserves the optimal distance properties. This isomorphism is adequately presented and proved, and yields simple yet precise guidelines to construct networks of this class, for any number of nodes. We call them *Midimew networks*. Particular cases of these networks are discussed, and comments about the number of links in the boundaries are given. Several quantitative comparisons are also provided at the end of this section, in order to establish the suitability of Midimew networks as an alternative to other well known static interconnection structures.

The routing policy is the problem analyzed in section IV, where an algorithm to obtain all the information required to guide messages between Midimew network nodes, is presented. This algorithm serves to obtain descriptions of paths of minimum length, between any pair of source and destination nodes. Besides, the same algorithm computes routing records for other alternative paths, which could be useful to add fault-tolerance capabilities, and to avoid local traffic congestions, if they are integrated into an adaptive global routing mechanism.

Section V closes the paper with the summary of the work and with the main concluding remarks related to it.

II. OPTIMAL CIRCULANT GRAPHS

This section is devoted to study a family of circulant graphs of degree 4 connecting any given number of nodes greater than 2. The definition of this family is given, and two theorems are presented to show that these circulant graphs are optimal in the sense that they have minimum diameter and minimum average distance among all the circulant graphs of degree 4. Some known results are used to show that these graphs are also maximally connected.

The family consists of one graph for each number of nodes. The description of these graphs is as follows. Let $N > 2$ be the number of vertices in the graph. Assume each vertex labeled by an integer from 0 to $N-1$. Then there is an edge from each vertex n, $0 \leq n \leq N-1$, to the four vertices $((n+b) \bmod N)$, $((n-b) \bmod N)$ $((n+(b-1)) \bmod N)$, and $((n-(b-1)) \bmod N)$, where the value of the integer b is $b = \left\lceil \sqrt{\frac{N}{2}} \right\rceil$. Observe that $b > 1$ since $N > 2$. The graphs obtained in this manner are denoted $C_N(b)$. Observe that knowledge of N is enough to compute the parameter b and to set all the edges appropriately.

For instance, let $N = 24$. The corresponding value $b = 4$ is easily computed, and therefore from each vertex n there are edges to the four vertices $((n \pm 4) \bmod 24)$ and $((n \pm 3) \bmod 24)$. The resulting circulant graph $C_{24}(4)$ can be seen in figure 1.

The circulant graphs $C_N(b)$ are node-symmetric. The node symmetry allows us to analyze distance properties by considering any arbitrary vertex as initial node; to simplify the calculations we choose node 0.

In order to prove the optimality, in the above stated sense, of this family of circulant graphs, let us establish some additional notation. For integers a and a', we denote by $[a, a']$ the interval of all the integers n with $a \leq n \leq a'$. V is $[0, N-1]$, which we identify with the set of vertices of the graph.

We also define the set D_k as:

$$D_k = \{(x, y) \in \mathbb{Z}^2 \mid |x| + |y| \leq k\}$$

where x and y are integers, and the accessing function f_N from $\mathbb{Z}^2$ into V as:

$$f_N(x, y) = xb + y(b-1) \bmod N$$

The value given by the accessing function f_N on (x, y) is the node reached from node 0 after x many b-hops and y many $(b-1)$-hops in the graph $C_N(b)$. The set D_k is the domain that should be considered for f_N to find out all those nodes that are within distance k from node 0.

To proceed with the demonstration, we need a few preliminary lemmas.

Lemma 1. $b = \left\lceil \sqrt{\frac{N}{2}} \right\rceil$ if and only if $2(b-1)^2 < N \leq 2b^2$.

Proof. Solve for N in $b - 1 < \sqrt{\frac{N}{2}} \leq b$. ■

Lemma 2. For each k, the cardinality of D_k is $2k^2 + 2k + 1$. Therefore, a circulant graph of degree 4 and diameter k cannot have more than $2k^2 + 2k + 1$ nodes.

Proof. For each x with $-k \leq x \leq k$ there are $2(k - |x|) + 1$ valid values for y to have $(x, y) \in D_k$. Thus, the cardinality of D_k is

$$\sum_{i=-k}^{k} 2(k - |i|) + 1 = 2k^2 + 2k + 1$$ ■

We show next some lemmas regarding surjectiveness and injectiveness properties of the functions $(x, y) \mapsto xb + y(b-1)$ and $z \mapsto z \bmod N$ on various domains.

Lemma 3. The function $(x, y) \mapsto xb + y(b-1)$ from D_b to $[-b^2, b^2]$ is surjective.

Proof. First observe that the indicated function indeed maps all the elements of D_b into $[-b^2, b^2]$, since

$$|xb + y(b-1)| = |(x+y)(b-1) + x| \leq (|x| + |y|)(b-1) + |x| \leq b(b-1) + b = b^2$$

We will show that no element of the range is the image of more than two different preimages, and that exactly $2b$ are the image of two preimages. Since the cardinality of D_b is $2b^2 + 2b + 1$ by lemma 2, we will obtain that there must be $2b^2 + 1$ different images and hence every element of $[-b^2, b^2]$ is an image.

Assume that $xb + y(b-1) = x'b + y'(b-1)$ for $(x, y), (x', y') \in D_b$. A simple operation yields $(y - y')(b-1) = (x' - x)b$, whose solutions are all of the form $x' - x = d(b-1)$ and $y - y' = db$ since b and $b-1$ are relatively prime. Summing the last two equalities and taking absolute values yields $|d(2b-1)| \leq 2b$, which implies $|d| \leq 1$ since $b > 1$. The solution for $d = 0$ is the trivial one, $x = x'$ and $y = y'$, and the two other solutions $d = \pm 1$ are the same by renaming (x, y) and (x', y'). Thus, at most two different pairs are mapped to the same image.

Without loss of generality we can assume $d = 1$, and therefore

$$\begin{aligned} y' &= y - b \\ x' &= x + (b-1) \end{aligned} \qquad (*)$$

Since $|y'| \leq b$, we have $y \geq 0$. Let us count the number of potential repetitions for each value of y. For $y = b$, the only value of x so that $(x, y) \in D_b$ is $x = 0$, and thus $y = b$ can only contribute one repetition (the pair $(x, y) = (0, b)$ and the pair $(x', y') = (b-1, 0)$). Similarly, for $y = 0$, we get $y' = -b$, which implies $x' = 0$, and we have only one repetition (the pair $(x, y) = (-(b-1), 0)$ and the pair $(x', y') = (0, -b)$). Each of the remaining $b - 1$ values of y will contribute exactly two repetitions. Indeed, for each y with $1 \leq y \leq b-1$, the equations $(*)$ yield a unique value $y' = y - b < 0$. By the definition of D_b, we have $|x| \leq b - |y|$ and $|x'| \leq b - |y'|$, and further operation yields $y - b \leq x \leq y - b + 1$, and therefore only two values of x can be chosen. The total number of repeated pairs is $1 + 2(b-1) + 1 = 2b$ as was to be shown. ■

Lemma 4. The function $(x, y) \mapsto xb + y(b-1)$ from D_{b-1} into $[-(b-1)^2 - (b-1), (b-1)^2 + (b-1)]$ is a bijection.

Proof. Follows the same guidelines of the previous proof. The fact that the image of D_{b-1} is included in $[-(b-1)^2 - (b-1), (b-1)^2 + (b-1)]$ is shown analogously. The cardinality of both sets is the same, and therefore it is enough to show that the function is injective. To show that there are no repetitions, proceed as in the proof of the previous lemma to find that pairs (x, y) and (x', y') mapping to the same image must fulfill $x' - x = d(b-1)$ and $y - y' = db$. However, now summing the equalities and taking absolute values yields $|d(2b-1)| \leq 2(b-1)$, since the domain is now D_{b-1}; this equation has only one solution, $d = 0$, and therefore the two pairs are the same. Thus the function is a bijection. ■

Lemma 5. The function $(x, y) \mapsto xb + y(b-1)$ from D_{b-2} into $[-(b-1)^2 - (b-1), (b-1)^2 + (b-1)]$ is injective.

Proof. Follows directly from the previous lemma, since the restriction of an injective function to a smaller domain is also injective. ■

Additionally, we need to characterize the surjectiveness and the injectiveness of the function $z \mapsto z \bmod N$.

Lemma 6. For every integer a, the function $z \mapsto z \bmod N$ from $[-a, a]$ to V is:

(a) surjective if $N \leq 2a + 1$

(b) injective if $N \geq 2a + 1$

Proof.

(a) The image of $[-a, -1]$ coincides with the image of $[N - a, N - 1]$, and thus the image of $[-a, a]$ coincides with the image of the union $[0, a] \cup [N - a, N - 1]$. If $N \leq 2a + 1$ then $[0, a] \cup [N - a, N - 1]$ includes all of $[0, N - 1]$. Since the function is the identity on that last range, it is surjective.

(b) Again the image of $[-a,-1]$ coincides with the image of $[N-a, N-1]$. The function is the identity on the intervals $[0,a]$ and $[N-a, N-1]$, and they are disjoint. Therefore the function is injective. ∎

We are now ready to show that the circulant graph $C_N(b)$ has minimum diameter and minimum average distance. We discuss diameter first.

Theorem 1. For every $N > 2$ and for $b = \left\lceil \sqrt{\frac{N}{2}} \right\rceil$, the diameter of $C_N(b)$ is optimal among all circulant graphs of degree 4.

Proof. We consider two disjoint cases.
Case 1: $2(b-1)^2 < N \leq 2(b-1)^2 + 2(b-1) + 1$. We show that the diameter is $b-1$; this is optimal by lemma 2. We must show that at most $b-1$ hops allow one to reach any node from node 0. Then the result follows since $C_N(b)$ is node-symmetric. The accessing function is $f_N : D_{b-1} \mapsto V$, $f_N(x,y) = (xb + y(b-1)) \bmod N$. The function mapping (x,y) into $xb+y(b-1)$ is surjective onto $[-(b-1)^2-(b-1), (b-1)^2+(b-1)]$ by lemma 4. By the conditions on N and lemma 6(a), the function mapping $xb + y(b-1)$ into $(xb + y(b-1)) \bmod N$ is surjective. Then the function $f_N : D_{b-1} \mapsto V$ is the composition of two surjective functions and is therefore surjective. Thus all nodes can be reached in at most $b-1$ hops.
Case 2: $2(b-1)^2 + 2(b-1) + 1 < N \leq 2b^2$. We show that the diameter is b; again this is optimal by lemma 2. The argument is analogous to case 1. The accessing function from node 0 is $f_N : D_b \mapsto V$, $f_N(x,y) = (xb + y(b-1)) \bmod N$. The function $(x,y) \mapsto xb+y(b-1)$ is surjective onto $[-b^2, b^2]$ by lemma 3, and the $\bmod N$ function on this domain is surjective by the conditions on N and lemma 6(a). Therefore the function $f_N : D_b \mapsto V$ is again surjective, being the composition of two surjective functions. All nodes can be reached in at most b hops. ∎

Now we discuss the average distance. The optimal value for this parameter is obtained when the number of vertices at maximum distance from node 0 is as small as possible, i.e. when all the vertices at a distance strictly smaller than the diameter are different.

Theorem 2. For every $N > 2$ and for $b = \left\lceil \sqrt{\frac{N}{2}} \right\rceil$, the average distance of $C_N(b)$ is optimal among all circulant graphs of degree 4 having optimal diameter.

Proof. We show the following claim: all the nodes at a distance smaller than the diameter are different. Again we consider two disjoint cases.
Case 1: $2(b-1)^2 < N \leq 2(b-1)^2 + 2(b-1) + 1$. Lemma 5 shows that the function $(x,y) \mapsto xb + y(b-1)$ from D_{b-2} into $[-(b-1)^2 - (b-1), (b-1)^2 + (b-1)]$ is injective. By the conditions on N and lemma 6(b), the function $\bmod N$ on this domain is also

injective. Thus $f_N(x,y) = xb + y(b-1) \bmod N$ from D_{b-2} into V is a composition of injective functions and must be also injective.

Case 2: $2(b-1)^2 + 2(b-1) + 1 < N \leq 2b^2$. Similarly, from lemma 4, the function $(x,y) \mapsto xb+y(b-1)$ from D_{b-1} to $[-(b-1)^2-(b-1), (b-1)^2+(b-1)]$ is a bijection and therefore is injective, and by the conditions on N and lemma 6(b) the function mod$N$ on this domain is also injective. Thus $f_N(x,y) = xb + y(b-1) \bmod N$ from D_{b-2} into V is again injective. ∎

In summary, by considering jointly theorems 1 and 2, we have shown that the family of circulant graphs $C_N(b)$ as defined above, indeed minimize both the diameter and the average distance for any value $N > 2$, and are optimal in this sense compared to any other circulant graph of degree 4.

The value of the diameter k follows from theorem 1, and can be expressed, as a function of N, by:

$$k = \begin{cases} b-1 & \text{if } N \leq 2b^2 - 2b + 1 \\ b & \text{if } N > 2b^2 - 2b + 1 \end{cases}$$

The value of the average distance $\bar{k}$ is formally derived in [7]; it also follows from theorem 2, and can be expressed, as a function of N, by:

$$\bar{k} = k\left[1 - \frac{2(k^2-1)}{3(N-1)}\right]$$

To end this section, we discuss the connectivity properties of $C_N(b)$. A complete characterization of the circulants with maximum vertex connectivity appears in theorem 1 of [8]. From this result, a theorem of Wang follows, which directly shows that a connected circulant in which the hops differ by 1 always has maximum vertex connectivity (see the discussion at the end of section 4 in [8]). This result clearly applies to our graphs $C_N(b)$, which hence are maximally vertex-connected. On the other hand, a theorem of Mader quoted in [8] guarantees that every connected vertex-symmetric graph has maximum edge-connectivity. This applies of course to $C_N(b)$. As a consequence, the fault tolerance capabilities of the graphs $C_N(b)$ are optimal among all graphs of degree 4.

III. MIDIMEW NETWORKS

In the light of the discussion presented in the previous section, it is clear that the graphs $C_N(b)$ are interesting in that they minimize both the diameter and the average distance, and exhibit homogeneity and high connectivity; however, planar design and embedding of parallel algorithms could be substantially improved if a regular way of laying them out is found, such as a mesh with wrap-around links.

We shall describe in this section a method that yields a transformation of these graphs into a mesh-connected network with wrap-around links, preserving all the properties enjoyed by the $C_N(b)$ graphs. We call this class of mesh-connected topologies **Midimew networks** (Minimum Distance Mesh with Wrap-around links). The resulting topology highly simplifies the task of solving the aforementioned design problems, since the processing elements not in the boundaries only need communication with the four physically nearest neighbors, and the corresponding design yields a planar representation having the minimum number of crossing links. On the other hand, the near-neighbor connectivity is strongly suggested by several kinds of computational problems such as those of image processing and numerical applications. Consequently, the treatment of these types of programs results in a very convenient mapping on a mesh-connected graph.

We present in this section a Midimew network isomorphic to each of the graphs $C_N(b)$, and satisfying the indicated requirements. As a consequence, a simple methodology for the systematic construction of Midimew networks is obtained. Also, the rectangular Midimew networks are characterized, and furthermore it is shown that exactly two square Midimew networks exist.

Let $N > 2$ be fixed, and b as in the previous section. We will use the following additional notation:

$$r = \left\lceil \frac{N}{b} \right\rceil b - N \qquad 0 \le r < b$$

$$h = b + r = \left(b \left\lceil \frac{N}{b} \right\rceil + b \right) - N$$

$$v = \left\lceil \frac{N}{b} \right\rceil - r = N - (b-1) \left\lceil \frac{N}{b} \right\rceil$$

The following lemmas will be used to prove the existence of an isomorphism between $C_N(b)$ graphs and Midimew networks. Their proofs are reasonably straightforward and therefore omitted here.

Lemma 7. For any two integers i, j with $0 \le i < h$ and $0 \le j < v$, it holds that $0 \le i(b-1) + jb < 2N$.

Lemma 8. For every integer d, if $N-(v-d)b$ is a multiple of $(b-1)$ then d is also a multiple of $(b-1)$.

Now we prove our main theorem in this section. Two additional definitions are required.

$$G_0 = \{(i,j) \mid 0 \le i < h \text{ and } 0 \le j < v\}$$
$$G = \{(i,j) \mid r \le i < h \text{ and } 0 \le j < v\} \cup \{(i,j) \mid 0 \le i < r \text{ and } 0 \le j < b-1\}$$

The relation between G_0 and G is depicted in figure 2.

Theorem 3. The function $f: G \mapsto V$ defined by $f(i,j) = (i(b-1)+jb) \bmod N$ is a bijection.

Proof. First observe that the cardinality of G is:

$$(h-r)v + r(b-1) =$$
$$b(\left\lceil \frac{N}{b} \right\rceil - r) + r(b-1) = b\left\lceil \frac{N}{b} \right\rceil - r = N$$

Therefore it suffices to show that f is injective. To this purpose we will show that:

(*i*) No more than two pairs (i,j) and (p,q) from G_0 are mapped to the same image, and

(*ii*) If two pairs (i,j) and (p,q) from G_0 are mapped to the same image then one of them lies out of G.

Let (i,j) and (p,q) be two different pairs giving the same value:

$$(i(b-1)+jb) \equiv (p(b-1)+qb) \pmod N$$

Without loss of generality, we assume that $(i(b-1)+jb) \le (p(b-1)+qb)$ and, furthermore, in case of equality we assume $i \le p$. By lemma 7, either:

(a) $(i(b-1)+jb) = (p(b-1)+qb)$, or

(b) $(i(b-1)+jb)+N = (p(b-1)+qb)$.

Claim: Case (b) does not hold.

To prove it, assume (b) true; we will derive a contradiction. From (b) it follows that

$$(p-i)(b-1) = N-(q-j)b$$

Therefore $N-(q-j)b$ must be a multiple of $(b-1)$. The smallest value of $(p-i)$ is found when $(q-j)$ is as large as possible. Since $(q-j) \le q < v$, we have that $(q-j) = v-d$ for some $d > 0$. By lemma 8, d must be a multiple of $(b-1)$, and therefore d is at least $(b-1) \ge 1$, which implies that $(p-i)(b-1)$ is at least $N-(v-(b-1))b$. Substituting the value of v it turns out that the smallest value of $(p-i)$ is h, which contradicts the range of p and i. Therefore case (b) does not hold, and this proves the claim.

Thus case (a) holds, and it is equivalent to

$$(p-i)(b-1) = (j-q)b$$

Since $(b-1)$ and b are relatively prime, all the solutions of this equation are of the form

$$(p-i) = ab, \qquad (j-q) = a(b-1)$$

where a ranges over the integers. The convention that $p \geq i$ gives $a \geq 0$. Let us find an upper bound on a. Since $p < h$, $i \geq 0$, $j < v$, and $q \geq 0$, we have $ab = (p-i) < h$ and $a(b-1) = (j-q) < v$. Therefore $ab + a(b-1) < h + v$. Substituting h and v by their values, and using lemma 1 in the previous section and the fact that $b > 1$, a bound of $a < 2$ is obtained by straightforward manipulation. This proves our first statement: the inverse image of each element of V contains at most two pairs, corresponding to the only admissible values for a, 0 and 1. Value 0 gives the trivial solution $p = i$, $j = q$, discarded by the assumption that the pairs were different. The remaining value gives $i = p - b$ and $j = q + b - 1$. From the inequalities $0 \leq i$, $p < h$, $0 \leq q$, and $j < v$, we derive:

$$\begin{aligned} 0 \leq i < h - b = r \\ b - 1 \leq j < v \end{aligned} \qquad (**)$$

Thus, if (i,j) is a pair for which there exists another different pair (p,q) giving the same result, i.e. $(i(b-1) + jb) = (p(b-1) + qb)$, then i and j must satisfy the properties (**). But, by definition, these properties are precisely those that prevent the pair (i,j) to belong to G. The theorem is proved. ∎

Now we present our interpretation of the theorem, in order to obtain systematic guidelines to construct Midimew networks. Transform the set G into a grid by setting at most four edges from each pair (i,j) to those of their immediate neighbors $(i \pm 1, j)$ and $(i, j \pm 1)$ that belong to G; we identify the set G with the graph obtained in this manner, which is denoted also G. Consider the function f given in theorem 3, which maps pairs from G onto the vertices of the circulant graph $C_N(b)$ in a bijective manner. Observe that the following holds:

$$f(i+1,j) = (f(i,j) + (b-1)) \bmod N$$
$$f(i-1,j) = (f(i,j) - (b-1)) \bmod N$$
$$f(i,j+1) = (f(i,j) + b) \bmod N$$
$$f(i,j-1) = (f(i,j) - b) \bmod N$$

Thus, if two pairs of G are joined by an edge of the grid then their images under f are joined by an edge of the circulant graph. Since f is bijective, this shows that the grid G is isomorphic to a subgraph of $C_N(b)$.

A graph isomorphic to $C_N(b)$ can be obtained by setting some wrap-around connections corresponding to the edges of $C_N(b)$ missing in $f(G)$. Once these wrap-around connections are established, if done in the appropriate manner, the extended grid will be isomorphic to $C_N(b)$.

Let us discuss how to set the wrap-around edges on G to obtain a graph isomorphic to $C_N(b)$. Each of the nodes in the bottom border (i.e. those elements of G with $j = 0$) have to be connected to some node in the top border (which can be at two different heights); similarly, each node in the right border must be connected to a node in the left border (which can be in two different columns). To maintain the isomorphism, care must be taken that each wrap-around edge added in this way corresponds to an edge of $C_N(b)$.

These edges must be set as follows:

A) Node $(i, 0)$ of G must be connected to the top node of the column $(i + r) \bmod h$.

B) Node $(h - 1, j)$ of G must be connected to the leftmost node of the row $(j + b - 1) \bmod v$.

Edges described under A) correspond to a "$-b$" edge in $C_N(b)$. The top node of the indicated column is $((i + r) \bmod (b + r), v - 1)$ if such node exists, otherwise it is $((i + r) \bmod (b + r), b - 2)$. Moreover, the first case holds when $i \leq b$, and then $(i + r) \bmod (b + r)$ is plainly $(i + r)$, while the second case holds when $i > b$, and in this case $(i + r) \bmod (b + r)$ becomes $i - b$. All this is easy to show from the definition of G (a glance to figures 3 and 4 may help). To prove that these edges are correct, it is enough to check the following easy identities:

$$f(i, 0) - b = \begin{cases} f(i + r, v - 1) & \text{if } (i + r, v - 1) \in G \\ f(i - b, b - 2) & \text{otherwise} \end{cases}$$

Edges described under B) correspond to a "$+(b - 1)$" edge in $C_N(b)$. The leftmost node of the indicated row is $(0, (j + b - 1) \bmod v)$ if such node exists, otherwise it is $(r, (j + b - 1) \bmod v)$. Moreover, the first case holds when $j + b - 1 \geq v$, and then $(j + b - 1) \bmod v$ becomes $j + b - 1 - v$, while the second case holds when $j + b - 1 < v$, and in this case $(j + b - 1) \bmod v$ is plainly $j + b - 1$. See the definition of G and the figures to check all these facts. The correctness of these edges can be checked also easily by proving the following identities:

$$f(h - 1, j) + b - 1 = \begin{cases} f(0, (j + b - 1) - v) & \text{if } (0, (j + b - 1) - v) \in G \\ f(r, (j + b - 1)) & \text{otherwise} \end{cases}$$

This argument shows how to extend the grid G with wrap-around edges, obtaining an isomorphism with the circulant graph $C_N(b)$. The extended network therefore shares all the graph-theoretic properties of $C_N(b)$, and thus minimizes simultaneously the diameter and the average distance for the given number of nodes N. It also exhibits the high connectivity and the other desirable properties of the circulant graph described in section II, and is therefore suitable as a good alternative to other proposed static interconnection networks, as will be further examined at the end of this section.

Systematic construction of Midimew networks

As a consequence of the results presented up to now in this section, we describe next an algorithm for systematically constructing Midimew networks for any given number of nodes $N > 2$.

Step 1: Computing parameters. Find the following quantities:

$$b = \left\lceil \sqrt{\frac{N}{2}} \right\rceil \qquad r = \left\lceil \frac{N}{b} \right\rceil b - N$$

$$h = b + r \qquad v = \left\lceil \frac{N}{b} \right\rceil - r$$

Step 2: Design a rectangular grid of horizontal dimension h and vertical dimension v.

Step 3: If $r \neq 0$ and $r \neq \lceil \frac{N}{b} \rceil - b + 1$, then discard from the drawn grid a leftmost upper rectangle of horizontal dimension r and vertical dimension $v - b + 1$.

Step 4: Establish the wrap-around links: Connect each node $(i, 0)$ to the top node of the column $(i + r) \bmod h$, and connect each node $(h - 1, j)$ to the leftmost node of the row $(j + b - 1) \bmod v$.

The Midimew network is constructed. The networks in figures 3 and 4 have been obtained by this algorithm, and the nodes have been labeled according to the function f of theorem 3. Notice that the mesh of figure 3 is isomorphic to the circulant graph of figure 1.

Rectangular and square cases

It can be seen from figure 2 that fully rectangular Midimew networks are obtained in each of the following cases:

1/ When $b + r = b$, i.e. $r = 0$. This case occurs when N is a multiple of b, and then the dimensions of the network are $h = b$ and $v = N/b$. For each b, exactly four such full rectangles appear, with sizes $v = 2b - 3$, $v = 2b - 2$, $v = 2b - 1$, and $v = 2b$, corresponding to $N = 2b^2 - 3b$, $N = 2b^2 - 2b$, $N = 2b^2 - b$, and $N = 2b^2$.

2/ When $b - 1 = \lceil \frac{N}{b} \rceil - r$, i.e. $r = \lceil \frac{N}{b} \rceil - b + 1$. This case occurs when $N = (\lceil \frac{N}{b} \rceil + 1)(b - 1)$, and then the dimensions of the network are $h = N/(b - 1)$

and $v = b - 1$. For each b, exactly one such full rectangle appears, namely with $h = 2b - 1$, that is, when $N = 2b^2 - 3b + 1$.

Remark that, out of the $4b - 2$ Midimew networks existing for each value of $b > 2$ ($N > 8$), only 5 of them are rectangular. On the other hand, only two square Midimew networks exist, corresponding to the cases where $N = 4$ or $N = 9$. This can be easily seen from the fact that a square Midimew can be obtained only if $b = N/b$ or $b - 1 = N/(b - 1)$, and that the second equality never holds. Some manipulation shows that the first case implies that $N < 4\sqrt{N} - 2$. Solving for $\sqrt{N}$ yields that $N < 12$, and thus the only perfect squares fulfilling this condition are 4 and 9. Figure 5 presents the square Midimew with $N = 9$.

A generally desirable characteristic is to achieve high density meshes (large number of nodes) yet minimizing the number of wrap-around links, since they may raise some implementation difficulties. Square meshes with wrap-around links present an advantage over similar networks with rectangular or "L" shapes, due to the fact that the square is the quadrangular geometric figure which minimizes the perimeter for a given area. The number of wrap-around links in Midimew networks is equal to the value of the network semiperimeter, and the area corresponds to the total number of nodes.

Midimew networks are designed with the criterion of minimization of distances in mind. Furthermore, the isomorphism between $C_N(b)$ graphs and Midimew networks has been chosen looking for the minimization of the resulting topology perimeter. Figure 6 shows the percentage of wrap-around links with respect to the total number of network links needed to construct Midimew networks, as a function of the number of nodes N. The same figure also shows this numerical relation for the case of 2-D torus networks.

In connection with this point, it is worthwhile remarking that the authors have introduced and presented in [6] another class of square meshes with wrap-around links, which are only defined for N equal to an even power of 2, exhibiting a good trade-off between distances and number of peripheral links. These topologies are based on another different class of circulant graphs of degree 4, characterized by adjacency patterns of the following form: each node n, $0 \leq n \leq N - 1$, is connected to the four neighbours:

$$(n + (\frac{\sqrt{N}}{2} - 1)) \bmod N$$

$$(n - (\frac{\sqrt{N}}{2} - 1)) \bmod N$$

$$(n + \sqrt{N}) \bmod N$$

$$(n - \sqrt{N}) \bmod N$$

For these $\sqrt{N} \times \sqrt{N}$ networks, the resulting values of the diameter k and average distance $\bar{k}$ are:

$$k = 3\sqrt{N}/4$$

$$\bar{k} = \frac{(23N/8 - 1)\sqrt{N}}{6(N-1)}$$

Figure 7 shows an example of a network of that class for $N = 64$. Further study of this class is now in course.

In order to allow an estimation of the suitability of Midimew networks, we provide here a table with relevant topological properties (see table 1), which compares several well-known static interconnection networks (2-D and 3-D mesh, torus, hypercube) and the hereby proposed alternative (Midimew).

The hypercube topology constitutes, up to the moment, the most popular interconnection scheme which has been used to implement the first generation multicomputers. Many important scientific and technological efforts have served to establish the computational model, as well as the main applicability areas, of the early message-passing multicomputers, mostly based on the hypercube topology. Nevertheless, such networks present some important drawbacks, namely their difficult scalability and planar implementation, and the far from complete utilization of their high connectivity properties. For this reasons, among others, hypercube networks are nowadays leaving room to other different topologies exhibiting lower connectivity, but which lead to more feasible 2-D or 3-D implementations, and which allow a greater utilization of the network resources. Besides, topologies able to increment the system bandwidth (by using wider paths) and to facilitate the system programmability, are desirable. Thus, we consider a realistic choice to include in the comparison provided by table 1 the 2-D and 3-D meshes and toruses, together with the Midimew structures.

Since there is no single standard measure to directly evaluate and compare the performance of these interconnection networks, no attempt is made here to establish any ranking identifying which one is the best. We believe instead that some valuable information may be drawn from several commonly accepted parameters which serve as relative merit figures; we shall consider here the product of diameter and degree, the average distance, and the message density, whose values are shown in figures 8, 9, 10, and 11, as functions of the number of nodes.

Figure 8 illustrates the comparison of all the networks listed in table 1 with respect to a typical cost measure (see, for example, [14], [10]), i.e. the product of diameter and degree. Figure 9 depicts with greater clarity the behavior of the same function for a number of nodes up to around 2200. Present medium-grain multicomputers fall in this range, see [4].

In figure 10, the values of average distance for these networks (except hypercubes) can be seen. On the other hand, figure 11 shows the curves of message-density values, another frequently used metric [1] which serves also as a good comparison parameter. Message density can be defined as the product of average distance times number of nodes, divided by the total number of links.

Figures 12a and 12b show a quantification of the improvement obtained in the average distance of Midimew networks, compared to 2-D toruses and 2-D meshes respectively. In figure 12c, 2-D and 3-D toruses and meshes are also compared. Finally, in order to complete this analysis, the improvement in message density of Midimew networks versus 2-D meshes can be seen in figure 13.

IV. ROUTING SCHEME

The problem of message routing between nodes in a Midimew network is partially addressed in this section, providing the basis to solve it. We present and prove correct here an algorithm taking as input data the numbers identifying the source and destination nodes, and computing as outputs two routing records. One of them describes a set of shortest paths between the input nodes; the second describes another set of alternative paths.

Each routing record is constituted by two signed integers x, y. Their absolute values $|x|$, $|y|$ represent the distance between the input nodes in terms of the number of hops through each kind of link, i.e. $|x|$ indicates the number of times the b hop must be followed and $|y|$ indicates the number of times the $b-1$ hop must be followed. The signs determine the orientation to be chosen for the hops when the path is constructed.

The routing records furnished by the algorithm are to be used by another routing module which would decide the exact path to be followed on the basis of external conditions such as node or link faults or local traffic congestion. In this sense, the algorithm for generating routing records is conceived to be integrated into an adaptive multipath routing system.

Let us proceed to describe some facts needed to prove our algorithm correct. Since the graph is undirected, any shortest path joining any two nodes can be used for communication in either direction. Thus our problem is to find the routing record corresponding to the shortest paths joining any two nodes s and t, with $s,\ t \in V$. Without loss of generality we can assume that $t \geq s$. We must find two integers x and y that minimize the sum $|x|+|y|$ under the condition that

$$0 \leq m = t - s = xb + y(b-1) < N$$

Taking into account the properties of the graphs considered here, it can be seen that we can restrict our study to the values $0 \leq m \leq N \operatorname{div} 2$, because the values $N \operatorname{div} 2 < m$ can be reduced to these by simply changing the signs of x and y after operating with $(N-m)$.

The general problem is, thus, to obtain solutions to equation

$$m = xb + y(b-1) \qquad (***)$$

with m, x, y, and b integers, and $b > 1$.

If (x_0, y_0) is a solution, then $m = x_0 b + y_0(b-1)$. Since b and $b-1$ are relatively prime, any other solution can be expressed as $(x_0 - \alpha, y_0 + \beta)$ imposing the condition

$\alpha b = \beta(b-1)$, which leads to $\alpha = n(b-1)$ and $\beta = nb$ with $n \in \mathbb{Z}$. Consequently, any solution to equation (***) belongs to the family $(x_0 - n(b-1), y_0 + nb)$.

A particular solution can be straightforwardly obtained in the form

$$\begin{aligned} x_0 &= Q + R \\ y_0 &= -R \end{aligned} \quad \text{with} \quad \begin{aligned} Q &= m \text{ div } b \\ R &= m \bmod b \end{aligned}$$

taking into account that equation (***) can be rewritten as

$$m = (x+y)b + (-y)$$

We conclude that any solution is in the form

$$\begin{aligned} x_n &= Q + R - n(b-1) \\ y_n &= -R + nb \end{aligned} \quad \text{with } n \text{ integer.}$$

Shortest path solution

Now we impose the condition that $P_n = |x_n| + |y_n| \geq 0$ be minimum, so that we minimize the distance to be traversed. Recall that theorem 1 guarantees that an optimal path always exists with $P_n \leq b$.

Theorem 4. Values $n < 0$ and $n > 1$ never yield shortest path solutions.

Proof . We prove the following two claims:
Claim 1. $P_n > P_0$ for every $n < 0$.
Claim 2. $P_n > b$ for every $n > 1$.

From these two claims the theorem follows.

Proof of claim 1. It is straightforward by checking that $|x_n| > |x_0|$ and $|y_n| > |y_0|$ for any $n < 0$.

Proof of claim 2. It is not difficult to see that the following inequalities hold:

$$Q + R \leq 2(b-1)$$
$$R < b$$
$$Q + 2R \leq 3(b-1)$$

Then, for any $n > 1$, we have

$$P_n = n(b-1) - (Q+R) + nb - R = n(2b-1) - (Q+2R)$$

which yields $P_n > b$ by algebraic manipulation. This completes the proof. ∎

Consequently, theorem 4 guarantees that, in any case, only values $n = 0$ and $n = 1$ may yield shortest path solutions. Now we must derive the conditions required to choose the optimal solution as P_0 or P_1. That is, we determine the solution (x_0, y_0) or (x_1, y_1) to be the best routing record. Recall that $P_0 = |x_0| + |y_0| = x_0 - y_0 = Q + 2R$ and $P_1 = |x_1| + |y_1| = |Q + R - (b-1)| + b - R$.

Theorem 5. $P_0 < P_1$ if and only if $y_0 = 0$ or $x_0 < y_1$.

Proof . We consider two disjoint cases, depending on the value of $Q + R$, and later combine them into the statement.

Case 1. $Q + R \geq b - 1$. Then $P_1 = Q + 1$, and $P_0 < P_1$ if and only if $R = 0$. In this case, P_0 represents the shortest paths because $P_0 = Q \leq b$.

Case 2. $Q + R < b - 1$. Then

$$P_1 = (b-1) - (Q+R) + b - R = (2b-1) - (Q+2R) = (2b-1) - P_0$$

and $P_0 < P_1$ if and only if $(Q+2R) < b$. In this case P_0 is the routing record associated to the optimal solution because $P_0 = Q + 2R < b$ but $P_1 = 2b - 1 - (Q + 2R) \geq b$.

So far, we have proved that P_0 is the shortest paths solution if and only if

$$[(R = 0) \wedge Q + R \geq b - 1) \vee (Q + 2R < b \wedge Q + R < b - 1)]$$

Now this condition can be rewritten in a simpler manner since it reduces to $(R = 0 \vee Q + 2R < b)$ through straightforward boolean manipulations. Finally, as $R = -y_0$ and $Q + 2R = x_0 - y_1 + b$, the proof of the theorem is completed. ∎

Theorems 4 and 5 allow us to design a simple algorithm to generate the shortest paths routing record. The algorithm uses the previous results to find the values x_s, y_s of the routing record assuming $t \geq s$ and $m \leq N$ div 2, and deals with the remaining cases by changing the signs of the solution. The algorithm is as follows:

```
{ Algorithm to obtain the routing record x_s, y_s for shortest paths }
{ An alternative routing record is also obtained, x_ns, y_ns }
{ Data: s source node, t destination node, N number of nodes, b = ⌈√(N/2)⌉ }
BEGIN
    m := ABS(t - s);
    IF t <= s THEN sign := -1 ELSE sign := 1;
    IF m > N DIV 2 THEN BEGIN
        sign := -sign;
        m := N - m
        END;
    y0 := -(m MOD b);
    x0 := (m DIV b) - y0;
    y1 := b + y0;
    x1 := x0 - (b - 1);
    IF (y0 = 0 OR x0 < y1) THEN BEGIN
        y_s := y0; y_ns := y1;
        x_s := x0; x_ns := x1
        END
    ELSE BEGIN
        y_s := y1; y_ns := y0;
        x_s := x1; x_ns := x0;
        END;
    y_s := y_s * sign; y_ns := y_ns * sign;
    x_s := x_s * sign; x_ns := x_ns * sign;
END.
```

Using the routing records generated by this algorithm, the implemented global routing mechanism shall convey messages to their respective destination through some shortest path.

For these and other related topologies, a total number of $\binom{|x_s|+|y_s|}{|x_s|}$ different shortest paths exist [7]. The global routing policy must decide which one is chosen, at a given time, accordingly with the possible conditions of node/link faults, and presence of local traffic congestions in the network.

The algorithm to generate routing records which has been presented above, provides all the information required to implement any one among the set of all possible shortest paths. In addition, another alternative routing record is produced which can be used to improve the adaptive capabilities. In a forthcoming work, the authors will present a more sophisticated algorithm to generate both optimal and suboptimal routing records. The paths in the second alternative obtained shall be edge and node disjoint from the paths associated to the optimal routing record, and yet shall minimize the distances under these disjointness condition.

The utilization of this approach in connection with deadlock-free adaptive routing strategies, as those presented in [21], should provide a global routing scheme able to guarantee a high throughput, to diffuse local traffic congestions, and to allow the definition and use of redundant paths for fault-tolerance.

V. CONCLUSIONS

Let us now summarize the main results obtained, as well as present a brief discussion of several related considerations.

In first place we have characterized a family of optimal circulant graphs with degree 4, denoted here as $C_N(b)$. The optimality refers to the simultaneous minimization of diameter and average distance for all circulant graphs of degree 4. For any arbitrary number N of nodes, one such graph exists, and the only design parameter needed to define each $C_N(b)$ is $b = \left\lceil \sqrt{\frac{N}{2}} \right\rceil$. Besides the distances optimization, these graphs exhibit homogeneity and maximum connectivity, and all these desirable characteristics provide the ability to use them as the basis to construct interconnection networks suitable for parallel architectures.

The resulting networks obtained in this paper, named Midimew networks, preserve all the convenient properties of the $C_N(b)$ graphs. The obtention of Midimew networks is accomplished by using an adequate graph isomorphism, which has been described and demonstrated in a subsequent section. The interest of such a transformation is justified because Midimew networks are better suited than circulant graph topologies as they show practical advantages when the interconnection topology is implemented, mainly 1) the ability to attain easier planar implementation with VLSI technology, and 2) greater simplicity to program the parallel architectures. Both features are of major importance in the development of current parallel computer systems. As a matter of fact, many well-known parallel algorithms are available which exhibit regular communication patterns, and are, consequently, adequate to be mapped on 2D mesh topologies like Midimew networks.

On the other side, our aim is to design topologies to be implemented in general-purpose machines. Present programming environments for multicomputers, like CE/RK [26] and Cantor [3] include dynamic allocation of processes to processing elements. Fine-grain multicomputers have shown that a random process allocation scheme can achieve high system throughput. Mosaic [4] is a good example of this. Thus, it is of fundamental importance, in this context, to minimize distance-related parameters in order to obtain a reduction in the communication latency due to the interconnection network.

Moreover, small degree networks allow the implementation of parallel channels, with a number of links sufficient for the required information bandwidth, to keep a lower communication delay overhead. 2D or 3D meshes are, in this aspect, better than hypercubes [11], provided that adequate routing techniques are implemented [12]. Easy expandability and greater ability to make a complete use of the network connectivity are, furthermore, other reasons for the growing appearance of mesh topologies in recent second generation multicomputer systems, such as the iPSC/2 and the Symult series 2010.

Different variations on mesh-like topologies are also present in some local area networks, as the Manhattan LAN [20], which uses a class of mesh with wrap-around links.

Altogether, it seems clear to us that the family of Midimew networks constitute, from a strictly topological point of view, a good candidate to apply compound graphs techniques on them, in order to design new hierarchical networks adequate to message-passing architectures. An example of the application of these techniques to obtain the network topology for a parallel system is the design of DOOM, currently under development [10].

Finally, we have addressed in this paper the problem of message routing in Midimew networks, with the purpose of giving a more complete operational environment. A low-complexity algorithm to generate routing records, which may serve as the basis for a multipath global routing scheme, is provided. The output of this algorithm directly gives the information about a set of paths with minimum length, and also about another set of non-shortest paths. Thus, it makes possible the implementation of adaptive routing strategies, which are useful when the presence of node/link failures or local traffic congestions is considered.

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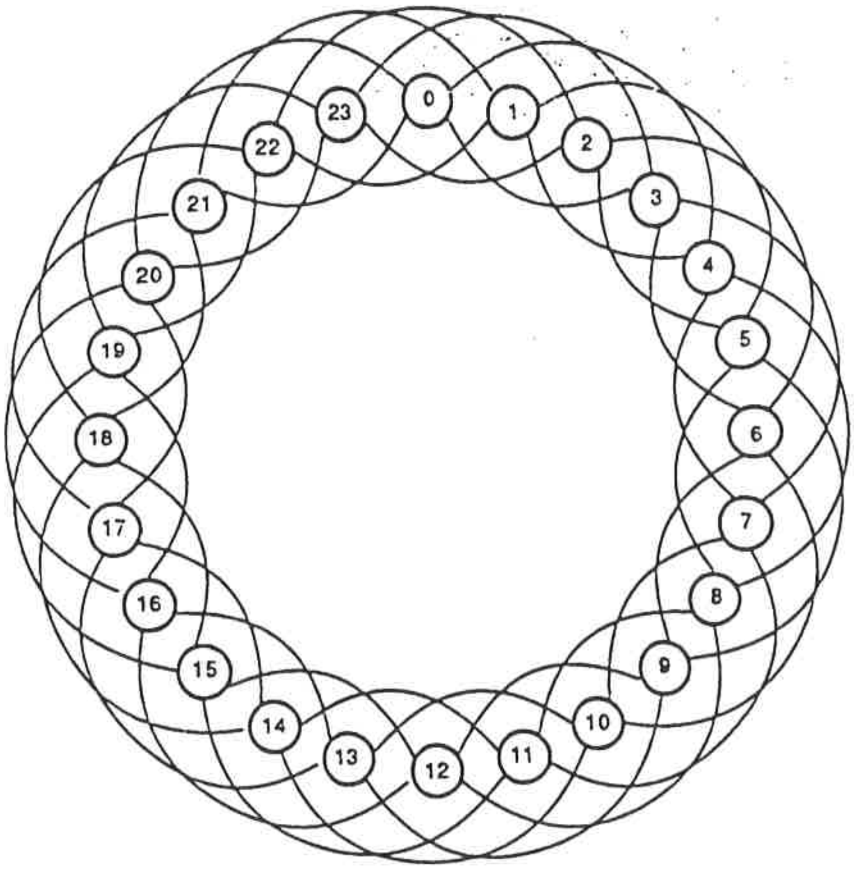

Figure 1. Circulant graph $C_N(b)$ with $N = 24$ and $b = 4$.

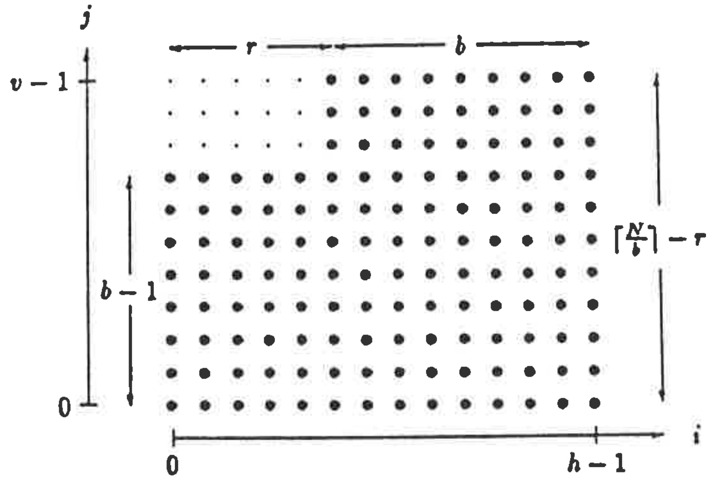

Figure 2. Relationship between G and G_0.

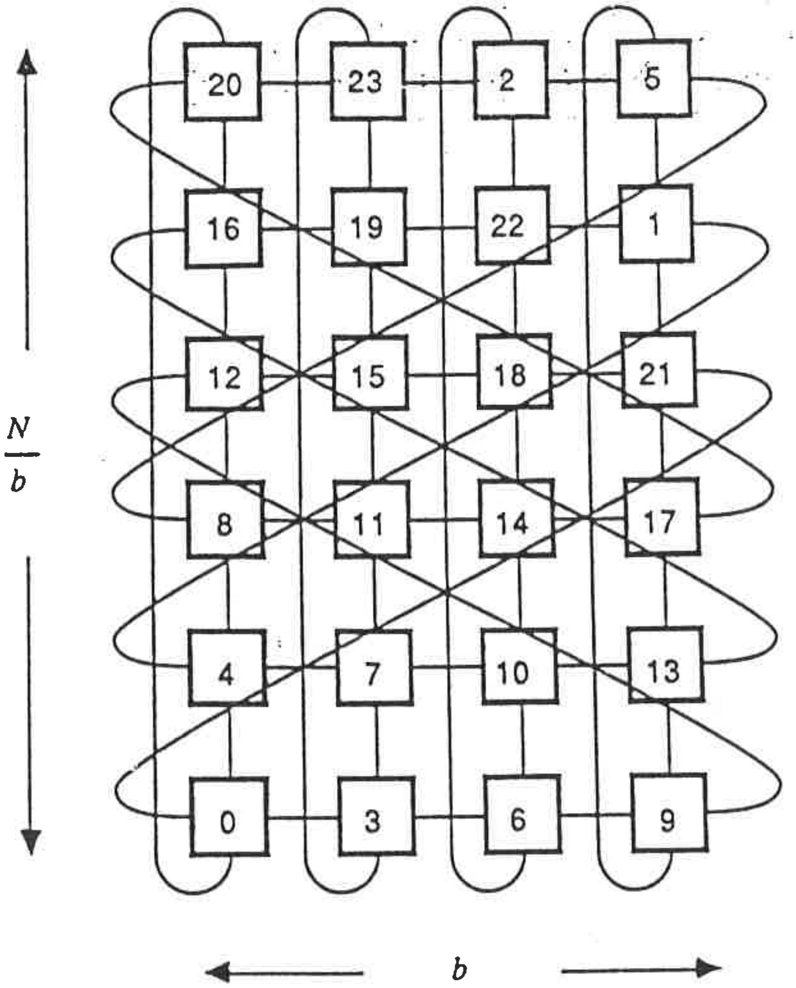

Figure 3. Midimew network with $N = 24$ and $b = 4$ $(r = 0)$.

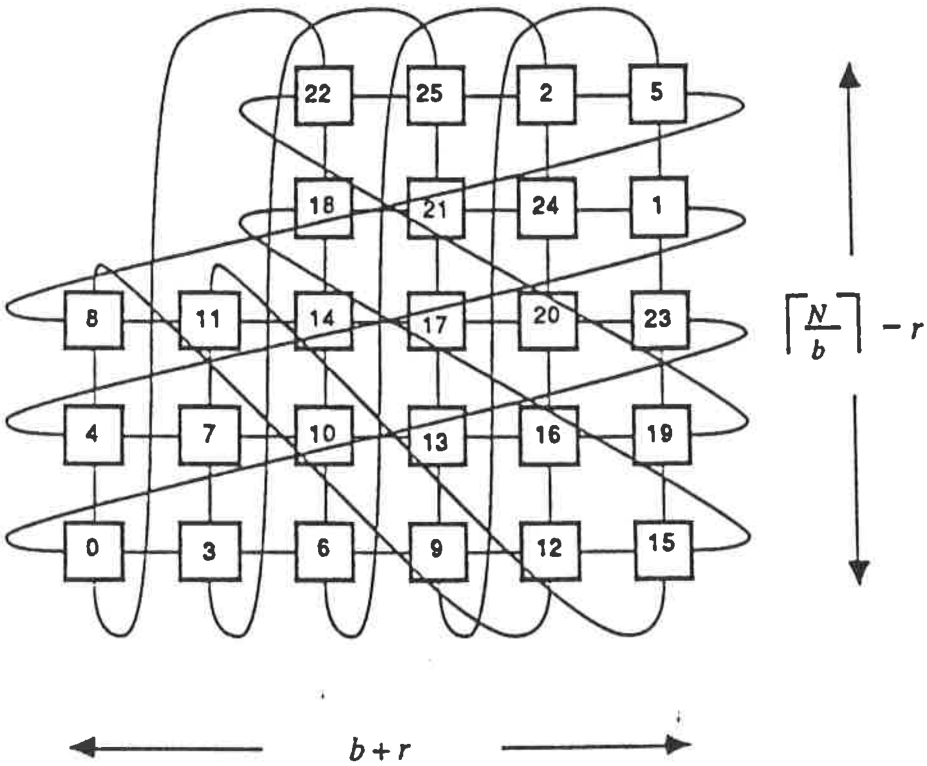

Figure 4. Midimew network with $N = 26$ and $b = 4$ $(r = 2)$.

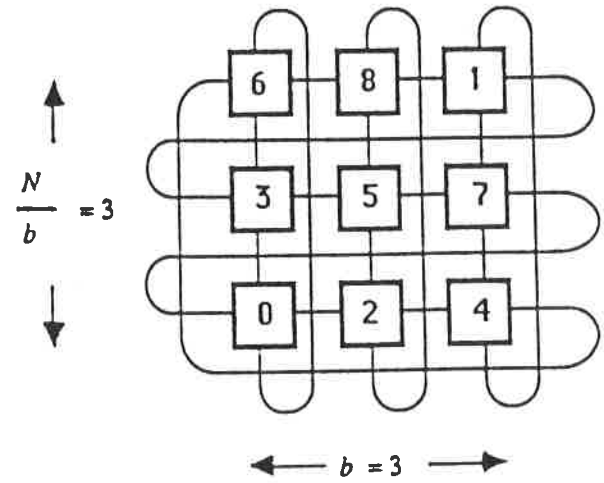

Figure 5. Square Midimew with $N = 9$.

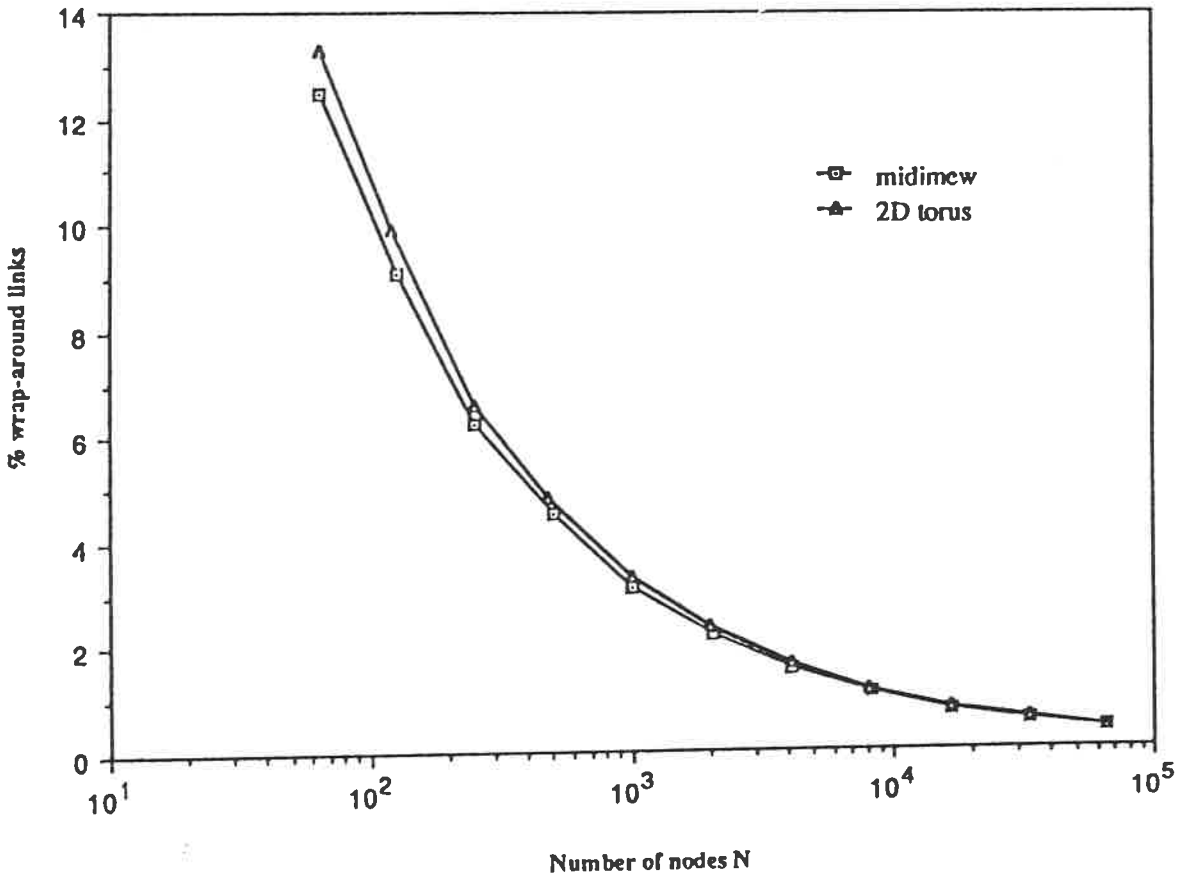

Figure 6. Number of wrap-around links (% of the total)

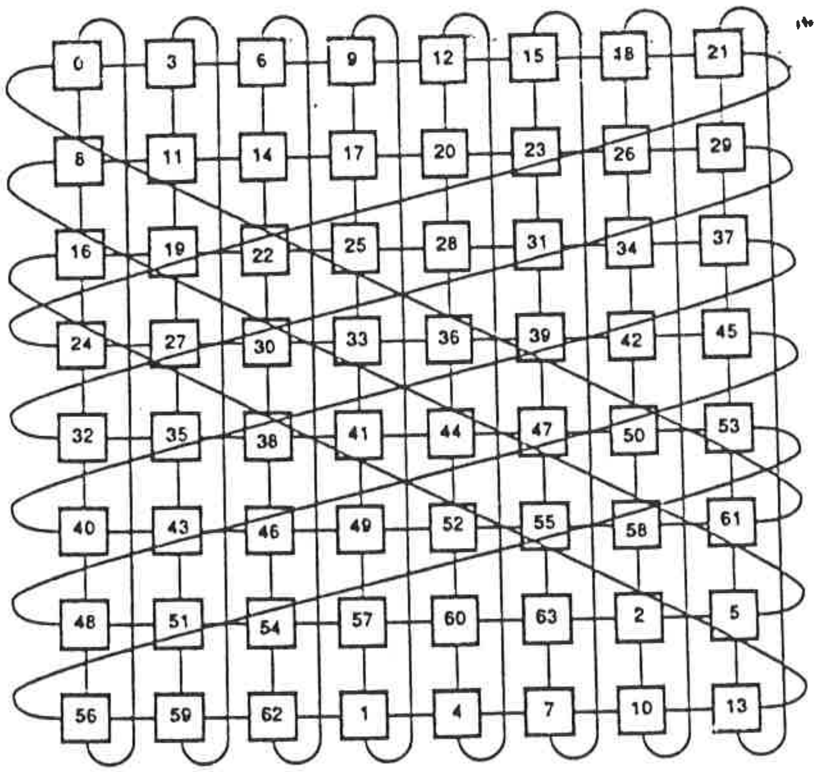

Figure 7. Square mesh for $N = 64$ with hops 3 and 8.

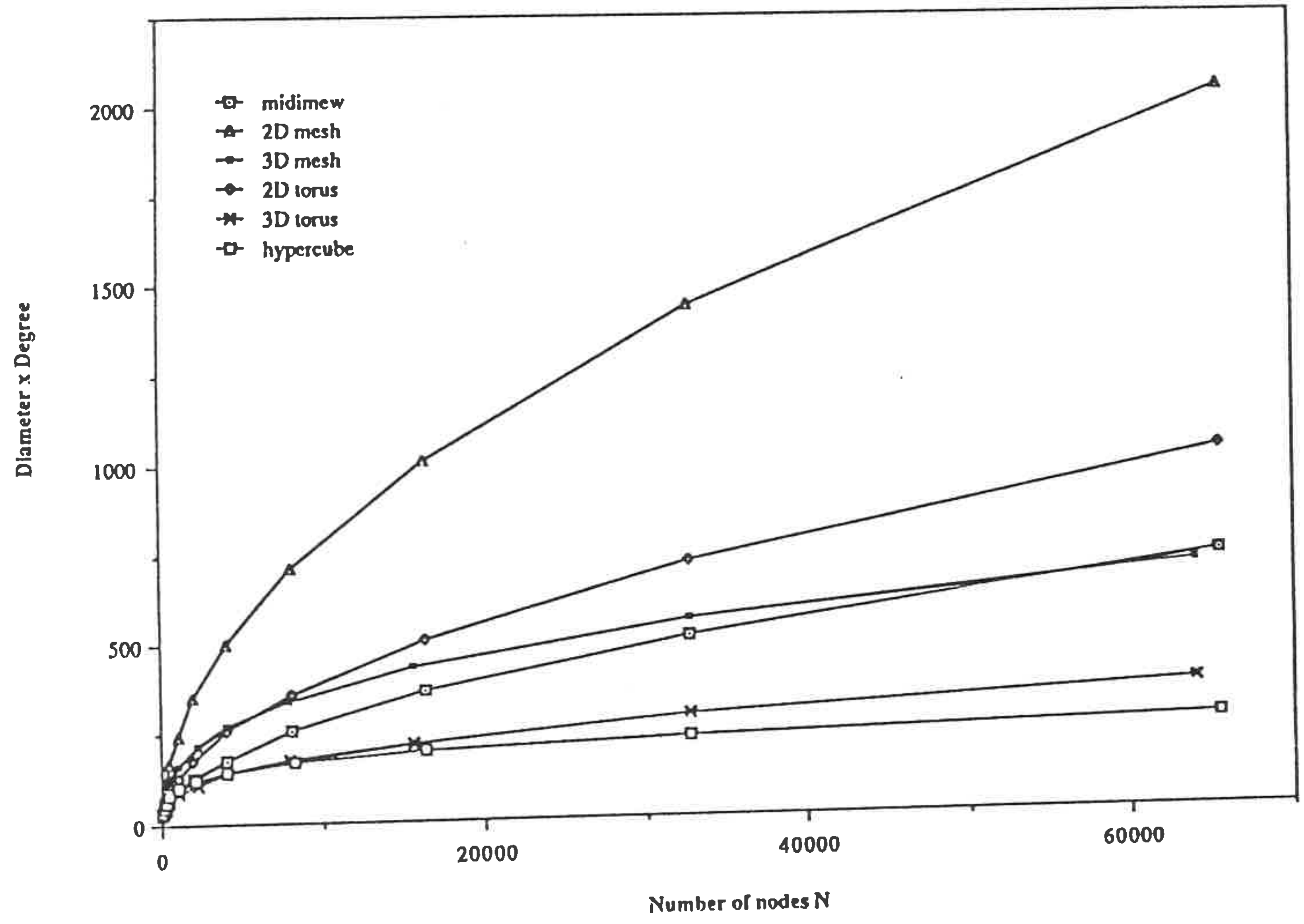

Figure 8. Diameter x Degree as a function of N

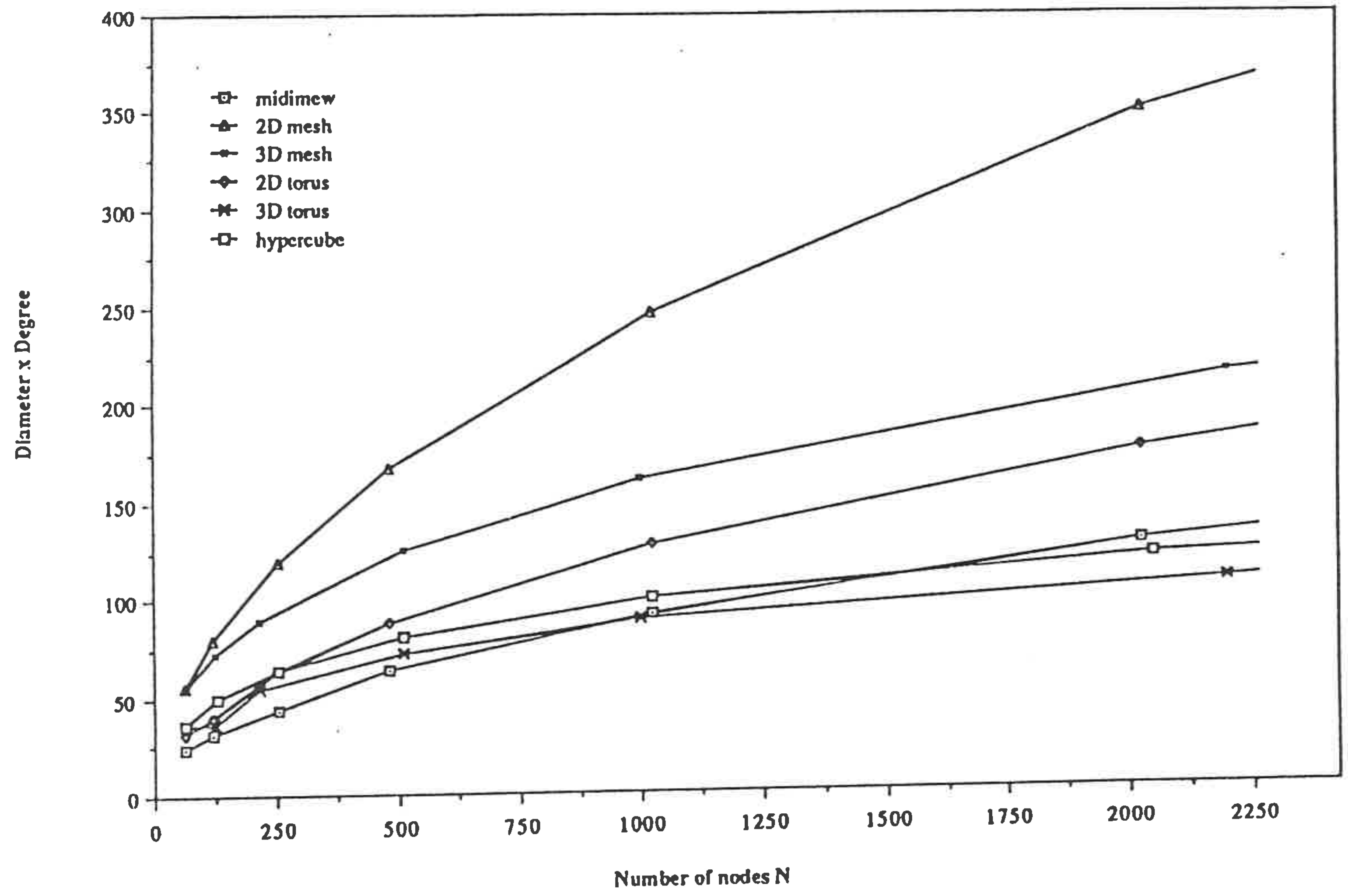

Figure 9. Diameter x Degree as a function of N (N < 2250)

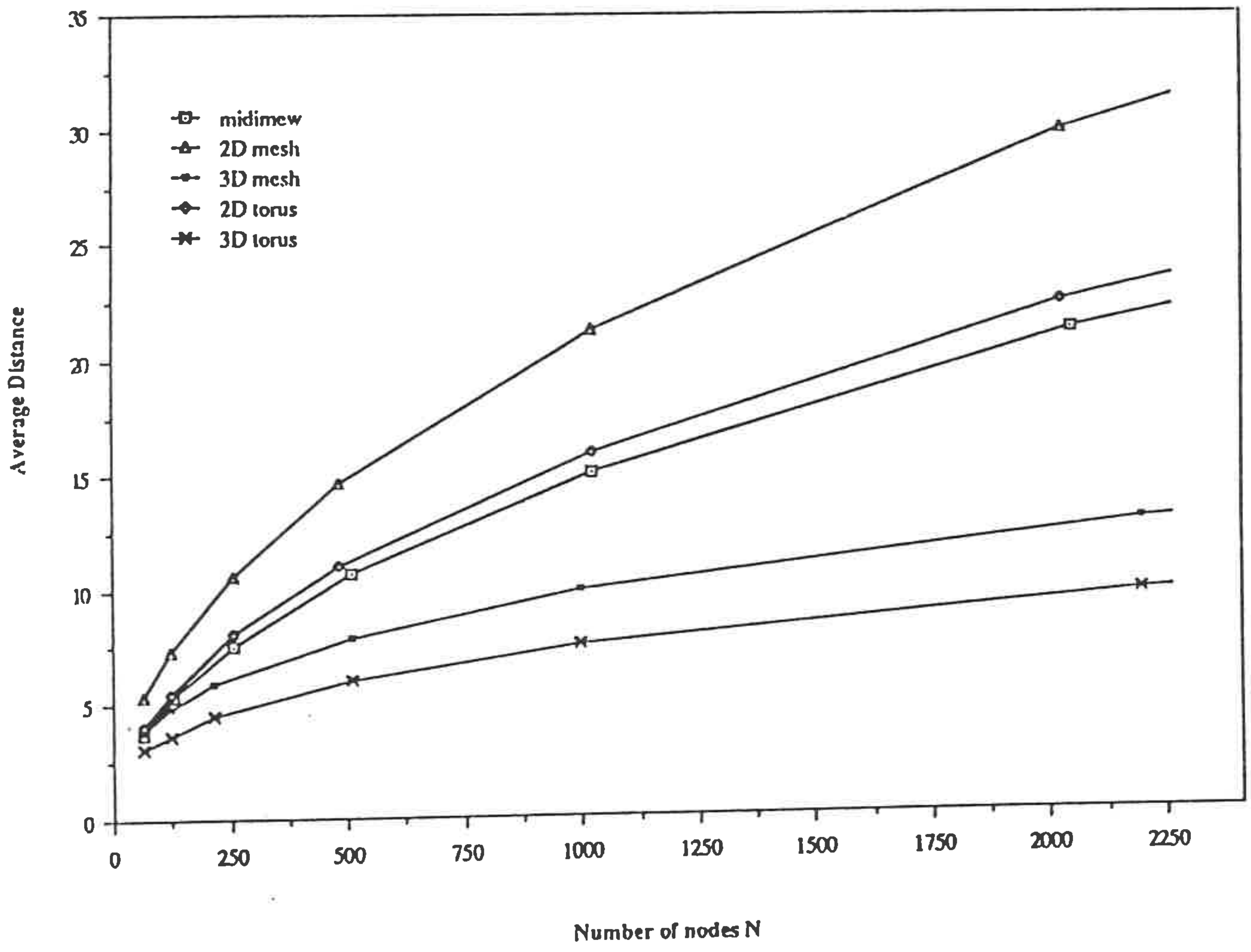

Figure 10. Average Distance as a function of N (N < 2250)

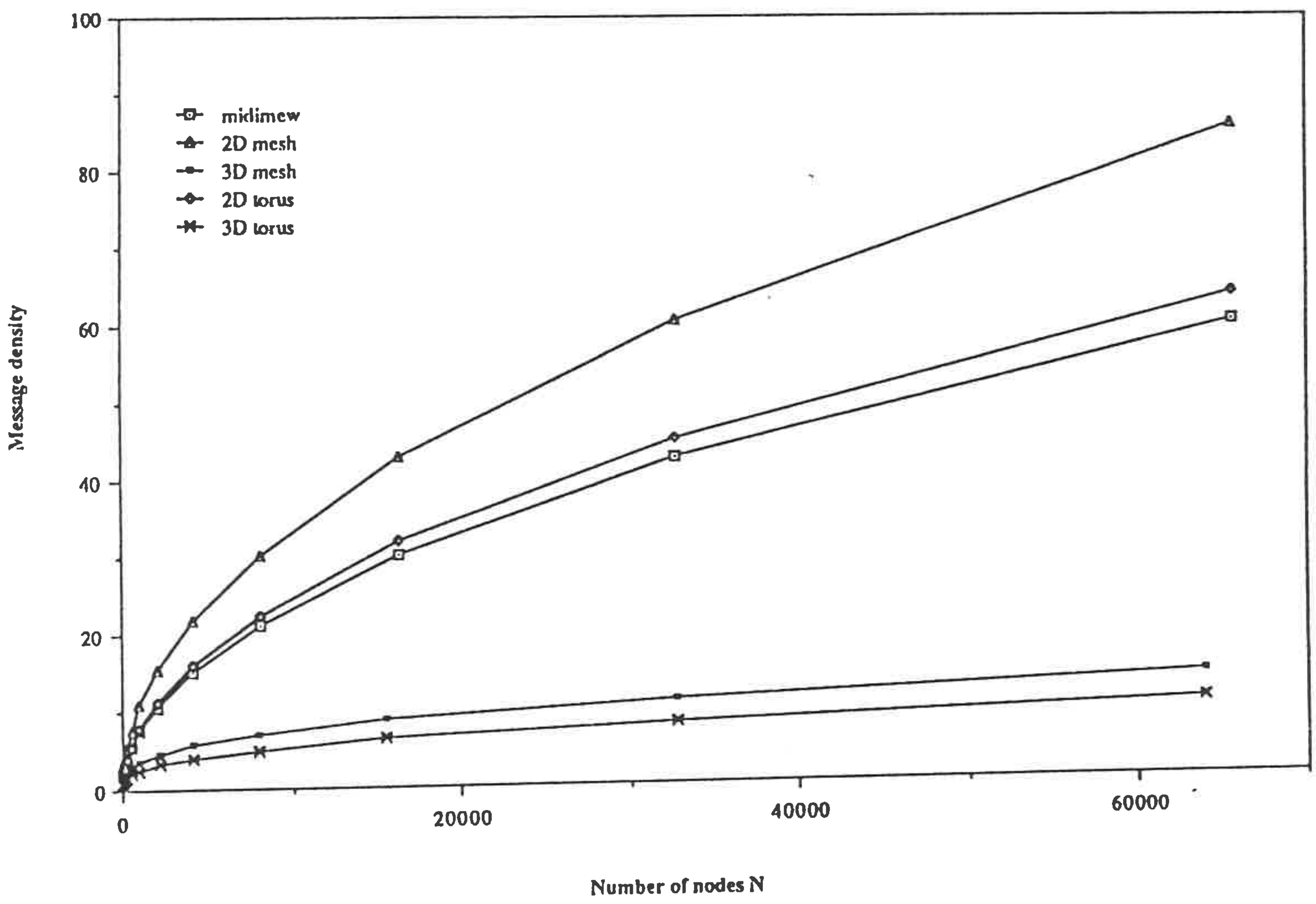

Figure 11. Message density as a function of N

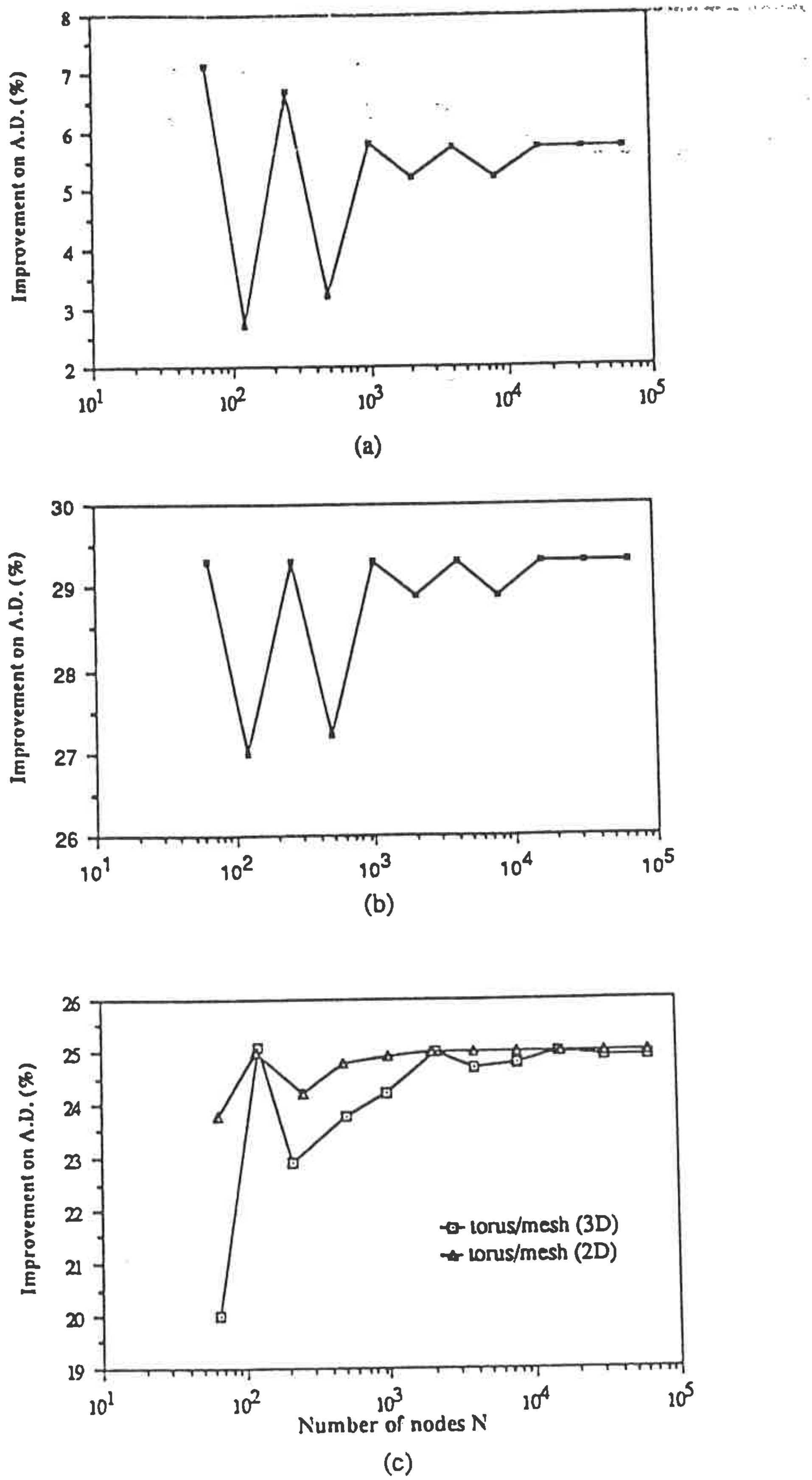

Figure 12. Improvements in Average Distance:
(a) Midimew versus 2D torus
(b) Midimew versus 2D mesh
(c) Torus versus mesh (2D & 3D)

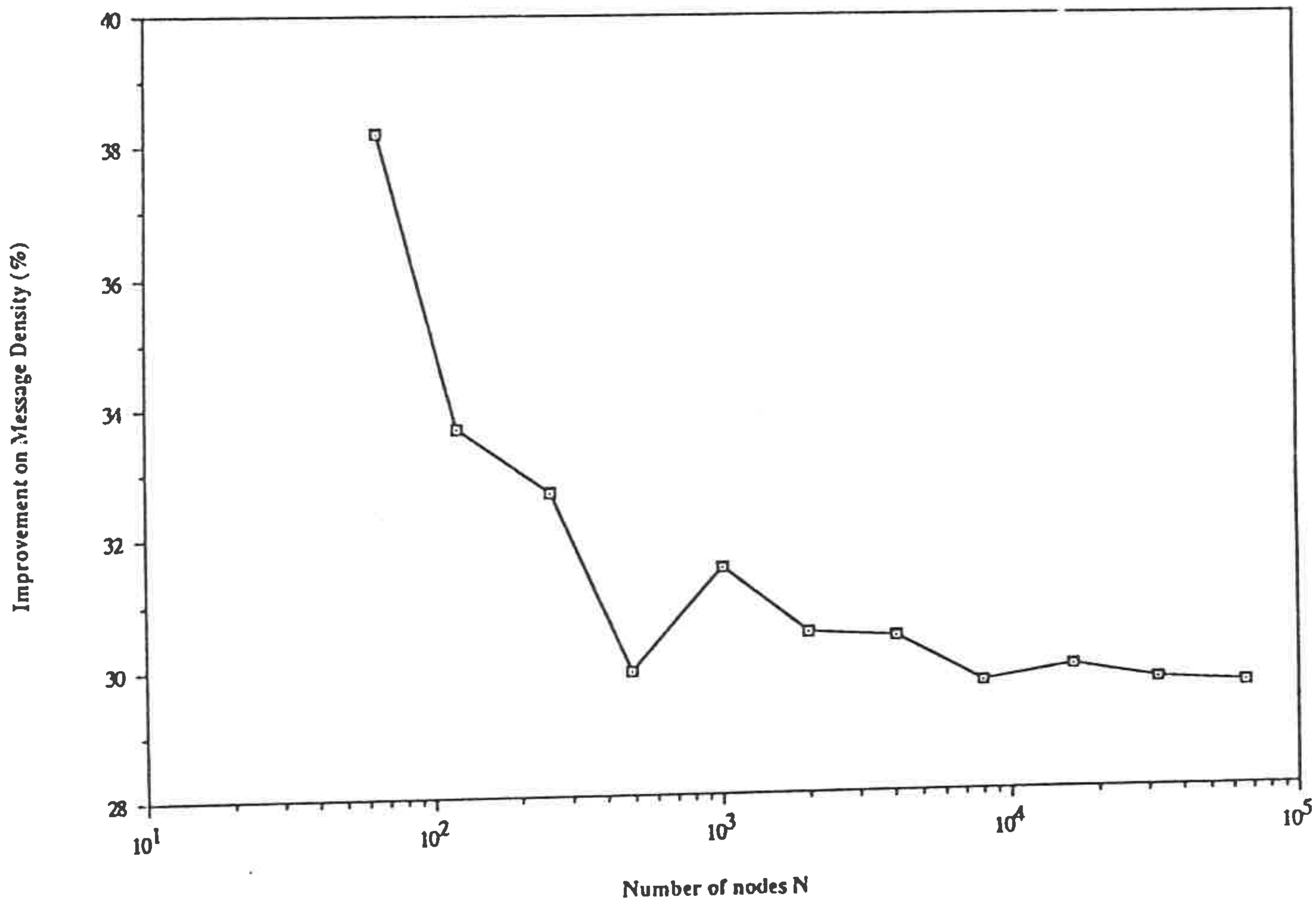

Figure 13. Improvement in Message Density. Midimew versus 2D mesh.

network structure	notes about structure	no. of nodes	no. of links	degree	diameter	average distance	node-symmetry and regularity
TORUS	W = no. nodes/dim. D = no. dim.	W^D	DW^D	$2D$	$D\lfloor \frac{W}{2} \rfloor$	$D\lfloor \frac{W}{2} \rfloor \lfloor \frac{W+1}{2} \rfloor \frac{W^{D-1}}{W^D-1}$	YES
2-D MESH	dimensions: p, q	pq	$2pq - p - q$	4	$p + q - 2$	$\frac{p+q}{3}$	NO
3-D MESH	dimensions: p, q, r	pqr	$3pqr - pq - pr - qr$	6	$p + q + r - 3$	$\frac{pqr(p+q+r)-pq-pr-qr}{3(pqr-1)}$	NO
HYPERCUBE	D = no. dim.	2^D	$D2^{D-1}$	D	D	$\frac{D2^{D-1}}{2^D-1}$	YES
MIDIMEW	$b = \left\lceil \sqrt{\frac{N}{2}} \right\rceil$	N	$2N$	4	$k = b$ or $k = b - 1$	$\frac{k-2(k^3-k)}{3(N-1)}$	YES

Tabla 1. Topological properties of some interconnection networks

K 3
$\frac{3k(n-1) - 2(k^3-k)}{3(N-1)}$