BACHELOR'S THESIS

Degree in Mathematics

Title: Symplectic toric manifolds, Delzant Theorem and integrable systems

Author: Robert Cardona Aguilar

Advisor: Eva Miranda Galceran and Cédric Oms

Department: Departament de Matemàtiques

Academic year: 2016-2017

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Facultat de Matemàtiques i Estadística

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

Degree in Mathematics

Bachelor's Degree Thesis

Symplectic toric manifolds, Delzant Theorem and integrable sytems

Robert Cardona Aguilar

Supervised by Eva Miranda and Cédric Oms

Barcelona, June 2017

To my beloved family, that believed in me, gave me advice and supported all my decisions. Special thanks to my tutors Eva and Cédric for their great guidance and good mood.

Abstract

Symplectic geometry is a branch of differential geometry that studies differentiable manifolds equipped with a closed nondegenerate two-form. They became much more interesting when it was observerd that the phase space of some classical dynamical systems, under the hamiltonian formulation, take structure of a symplectic manifold. In this thesis, after a review of all needed background on symplectic geometry, we focus on toric manifolds. It is the case when the symplectic manifold we consider admits an effective smooth action of a torus of exactly half the dimension of the symplectic manifold. We explain Delzant theorem, which classifies all compact toric manifolds and the link of these manifolds with integrable systems, a particular case of dynamical system, through Arnold-Liouville theorem. Finally, under the Severo Ochoa - Introduction to Research program at ICMAT and tutoring of Daniel Peralta, we develop a new proof for the first statement of Arnold-Liouville theorem.

Contents

1 Preliminaries

In this section we will introduce all the necessary background in Symplectic geometry to understand the statement and the proof of Delzant theorem. Most of the content in this section was learned from [3].

1.1 Skew-symmetric bilinear maps

Let V be a vector space over $\mathbb{R}$ of dimension m. A bilinear map $\Omega : V \times V \to \mathbb{R}$ is **skew-symmetric** if $\Omega(u, v) = -\Omega(v, u)$, for all $u, v \in V$.

Theorem 1. *Let Ω be a skew-symmetric bilinear map on V. Then there is a basis $u_1, ..., u_k, e_1, ..., e_n, v_1, ..., v_n$ of V such that*

$$\begin{aligned} &\Omega(u_i, v) = 0, && \forall i \text{ and } \forall v \in V, \\ &\Omega(e_i, e_j) = \Omega(v_i, v_j) = 0, && \forall i, j \text{ and } , \\ &\Omega(e_i, f_j) = \delta_{ij}, && \forall i, j. \end{aligned}$$

The dimension of V can be written dim $V = 2n+k$. This basis is not unique, and in matrix notation we have

$$\Omega(u, v) = \mathbf{u} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id} \\ 0 & -\mathrm{Id} & 0 \end{bmatrix} \mathbf{v}.$$

Proof. Choose a basis $u_1, ..., u_k$ of $U := \{u \in V \mid \Omega(u, v) = 0 \ \forall v \in V\}$. Choose a complementary space W to U in V,

$$V = U \oplus W.$$

Take any nonzero $e_1 \in W$. There is $f_1 \in W$ such that $\Omega(e_1, f_1) \neq 0$. We can assume that $\Omega(e_1, f_1) = 1$. Let

$$\begin{aligned} W_1 &= \text{ span of } e_1, f_1 \\ W_1^\Omega &= \{w \in \Omega | \Omega(w, v) = 0 \text{ for all } v \in W_1\}. \end{aligned}$$

Claim. $W_1 \cap W_1^\Omega = \{0\}$.

Suppose $v = ae_1 + bf_1 \in W_1 \cap W_1^\Omega$.

$$\begin{aligned} 0 &= \Omega(v, e_1) = -b, \\ 0 &= \Omega(v, f_1) = a, \end{aligned}$$

implies $v = 0$.

Claim. $W = W_1 \oplus W_1^\Omega$.

Suppose that $v \in W$ has $\Omega(v, e_1) = c$ and $\Omega(v, f_1) = d$. Then

$$v = \underbrace{(-cf_1 + de_1)}_{\in W_1} + \underbrace{(v + cf_1 - de_1)}_{\in W_1^\Omega}.$$

Now take e_2. There is $f_2 \in W_1^{\Omega}$ such that $\Omega(e_2, f_2) \neq 0$. Assume that $\Omega(e_2, f_2) = 1$ and let W_2 be the span of e_2, f_2. Etc. This iteration stops because dim $V < \infty$. We obtain then

$$V = U \oplus W_1 \oplus ... \oplus W_n$$

where all the summands are orthogonal with respect to Ω, and W_i has basis e_i, f_i with $\Omega(e_i, f_i) = 1$.

The dimension of the subspace U does not depend on the choice of the basis, so

$$k := \dim U \text{ is an invariant of } (V, \Omega).$$

Since $k + 2n = m = \dim V$, we have that

$$n \text{ is an invariant of } (V, \Omega); \;\; 2n \text{ is called the rank of } \Omega.$$

□

Definition. *The map $\tilde{\Omega} : V \to V^*$ is the linear map defined by $\tilde{\Omega}(v)(u) = \Omega(v, u)$.*

Definition. *A skew-symmetric bilienar map Ω is **symplectic** (or nondegenerate) if $\tilde{\Omega}$ is bijective. Then Ω is called a linear symplectic structure on V, and (V, Ω) is called a symplectic vector space.*

As $\tilde{\Omega}$ is bijective, we have by Theorem 1 a basis $e_1, ..., e_n, v_1, ..., v_n$ and we have

$$\Omega(u, v) = [\text{---}\ u\ \text{---}] \begin{bmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix}.$$

In particular, for a vector space to be symplectic, it has to be of even dimension.

1.2 Symplectic structure on manifolds and hamiltonian vector fields

Definition. *Given a manifold M^{2n} of even dimension and a closed non-degenerate 2-form $\omega \in \Omega(M)$, the pair (M^{2n}, w) is called a **symplectic manifold**.*

A diffeomorphism f from (M_1, ω_1) to (M_2, ω_2) is a symplectomorphism if $f^*\omega_2 = \omega_1$.

Example. A very important example of symplectic structure can be constructed in the cotangent bundle $M = T^*X$ of any n-dimensional manifold X. Let $(U, x_1, ..., x_n)$ be a coordinate chart at $x \in X$ with $x_i : U \to \mathbb{R}$ the coordinate facts. The differentials $(dx_1)_x, ..., (dx_n)_x$ form a basis of T_x^*X. That means for $\xi \in T_x^*X$, $\xi = \sum_{i=1}^n \xi_i (dx_i)_x$ for some $\xi_i \in \mathbb{R}$. In particular it induces a map

$$\begin{aligned} T^*U &\longrightarrow \mathbb{R}^{2n} \\ (x, \xi) &\longmapsto (x_1, ..., x_n, \xi_1, ..., \xi_n). \end{aligned}$$

This is a coordinate chart for M. The transition functions on intersections are smooth: given two charts $(U, x_1, ..., x_n, \xi_1, ..., \xi_n)$ and $(U', x_1', ..., x_n', \xi_1', ..., \xi_n')$ then

$$\xi = \sum_{i=1}^{n} \xi_i (dx_i)_x = \sum_{i,j} \xi_i \frac{\partial x_i}{\partial x'_j} (dx'_j)_x = \sum_{i=1}^{n} \xi'_i (dx'_i)_x.$$

Hence M is a $2n$-dimensional manifold. We can define on it a 2-form ω by

$$\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i.$$

Clearly, defining the 1-form

$$\alpha = \sum_{i=1}^{n} \xi_i dx_i,$$

we have that $\omega = -d\alpha$. The form ω is independent from coordinates as a consequence of the following claim.

Claim. *The form α is intrinsically defined.*

Proof. Let $(U, x_1, ..., x_n, \xi_1, ..., \xi_n)$ and $(U', x'_1, ..., x'_n, \xi'_1, ..., \xi'_n)$ be two coordinate charts for the cotangent space. On $U \cap U'$, the two sets of coordinates are related by the change of charts $\xi'_j = \sum_i \xi_i (\frac{\partial x_i}{\partial x'_j})$. Since $dx'_j = \sum_i (\frac{\partial x'_j}{\partial x_i}) dx_i$, we have

$$\alpha = \sum_i \xi_i dx_i = \sum_j \xi'_j dx'_j = \alpha' \qquad \square$$

It is the tautological form, called the Liouville 1-form, and ω is the canonical symplectic form.

An observation that can be done about symplectic manifolds is that they are necessarily orientable. Consider the $2n$-form ω^n: by definition of ω it never vanishes and so defines a volume form in M (equivalent to M being orientable). In the particular case of dimension 2, a volume form is exactly a symplectic form. This leads to another example of class of symplectic manifolds: **orientable surfaces**. Some other examples are

- $(\mathbb{R}^{2n}, \omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i)$,
- $(\mathbb{C}^n, \omega = \frac{i}{2} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i)$,
- $(S^2, \omega = dh \wedge d\theta)$.

The $(\mathbb{R}^{2n}, \omega_0)$ example is very important: any symplectic manifold (M^{2n}, ω) is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

Theorem (Darboux)**.** *Let (M, ω) be a $2m$-dimensional symplectic manifold and p be any point in M. Then there is a coordinate chart $(U, x_1, ..., x_n, y_1, ..., y_n)$ centered at p such that on U*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

This chart is called a ***Darboux chart****.*

Let's recall now a few concepts about vector fields in a manifold M, and introduce new ones for a symplectic manifold (M, ω). Given a complete vector field $X \in \Gamma(M)$, its flow $\varphi_t : M \longrightarrow M$ is the unique solution for any $p \in M$ of the ODE:

$$\begin{cases} \varphi_0(p) = p \\ \frac{\partial \varphi_t}{\partial t}(p) = X(\varphi_t(p)). \end{cases}$$

The uniqueness and existence of the solution is given by Picard theorem. The family $\{\varphi_t | t \in \mathbb{R}\}$ is then called a one-parameter group of diffeomorphisms of M and denoted

$$\varphi_t = \exp tX.$$

Definition. *The Lie derivative of a differential form α with respect to the vector field X is:*

$$\mathcal{L}_X \alpha = \frac{d}{dt}({\varphi_t}^* \alpha)|_{t=0}.$$

Definition. *A vector field $X \in \Gamma(M)$ is symplectic if $\mathcal{L}_X \omega = 0$.*

Define the interior product or contraction :

$$\begin{aligned} i_X : \Omega^k(M) &\longrightarrow \Omega^{k-1}(M) \\ \alpha &\longmapsto i_X \alpha(X_1, ..., X_{k-1}) \\ &\qquad = \alpha(X, X_1, ..., X_{k-1}). \end{aligned}$$

We can state now the **Cartan's formula**, that relates as follows Lie derivative with the interior product and the exterior derivative d,

$$\mathcal{L}_X \omega = (d \circ i_X + i_X \circ d)\omega. \tag{1}$$

Finally, using Equation (1) and the fact that ω is closed (i.e $d\omega = 0$) we obtain that:

$$\text{X is symplectic} \iff \mathcal{L}_X \omega = 0 \iff i_X \omega \text{ is closed .}$$

Note that $i_X \omega$ is a one-form in M. A particular case of closed forms are exact forms, that is $i_X w = d\beta$ for a certain smooth function β in M. Vector fields X such that $i_X \omega$ is exact are called **hamiltonian** vector fields. In particular, for any smooth function $f : M \longrightarrow \mathbb{R}$ we have by nondegeneracy a unique vector field X_f on M such that $i_{X_f} \omega = df$. It is called the hamiltonian vector field with hamiltonian function f.

Example. Consider the symplectic manifold $(\mathbb{R}^4, \omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$. If we consider for example the function $f = {x_2}^2 + {y_1}^2$ we can compute its corresponding hamiltonian vector field. The derivative of our function is $df = 2x_2 dx_2 + 2y_1 dy_1$. On the other hand, any vector field is of the form

$$X = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + b_1 \frac{\partial}{\partial y_1} + b_2 \frac{\partial}{\partial y_2}, \quad a_1, a_2, b_1, b_2 \in C^\infty(\mathbb{R}^4).$$

Then, the interior product is

$$\begin{aligned} i_X\omega &= \omega(X,\cdot) \\ &= dx_1 \wedge dy_1(X,\cdot) + dx_2 \wedge dy_2(X,\cdot) \\ &= a_1 dy_1 + a_2 dy_2 - b_1 dx_1 - b_2 dx_2. \end{aligned}$$

Imposing $i_X\omega = df$ we obtain that $a_1 = 2y_1, a_2 = 0, b_1 = 0$ and $b_2 = -2x_2$. The hamiltonian vector field with hamiltonian function f is hence

$$X_f = 2y_1 \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial y_2}.$$

1.3 Lie groups and actions

Definition. *G is called a Lie group if G is a smooth manifold and there exist two smooth maps:*

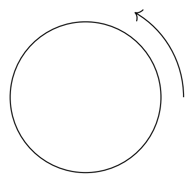
$$\begin{aligned} m : G \times G &\longrightarrow G \\ (x, y) &\longmapsto m(x, y), \end{aligned}$$

and

$$\begin{aligned} i : G &\longrightarrow G \\ x &\longmapsto i(x), \end{aligned}$$

where m is the product and i the inverse giving G a group structure.

Examples. A very simple example of Lie group is $\mathbb{R}$ with addition. Another example of Lie group is the circle S^1 with rotation through θ the standard angle ($mod(2\pi)$). This is equivalent to complex numbers with modulus 1 with multiplication.



Definition. *A Lie action φ of G a Lie group on a manifold M is a map :*

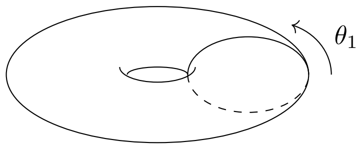
$$\begin{aligned} \varphi : G \times M &\longrightarrow M \\ (g, x) &\longmapsto \varphi(g, x) = g \cdot x, \end{aligned}$$

*such that $e \cdot x = x$ and $g \cdot (h \cdot x) = (g \cdot h) \cdot x$ and φ is smooth. It is **effective** if all element $g \in G \backslash \{e\}$ moves at least one point $p \in M$. It is **free** if e is the only element in G with fixed points.*

Example. We can consider an action of S^1 on T^2 which consist simply on rotating in one of the coordinate angles of the torus

$$\begin{aligned} \varphi : S^1 \times T^2 &\longrightarrow T^2 \\ (\alpha, (\theta_1, \theta_2)) &\longmapsto (\theta_1 + \alpha, \theta_2). \end{aligned}$$

This action is free and effective.



Note that an action can also be written as $\varphi : G \longrightarrow \text{Diff}(M)$. Now given a Lie group G, we denote T_eG (tangent space at neutral element $e \in G$) as $\mathfrak{g}$. For a given $g \in G$ we can consider the smooth function "left multiplication":

$$\begin{aligned} L_g : G &\longrightarrow G \\ h &\longmapsto g \cdot h, \end{aligned}$$

and consider its differential $dL_g|_e : T_eG \longrightarrow T_gG$.

For a given tangent vector $X \in T_eG$ we define the vector field $\tilde{X} \in \Gamma(G)$ as $\tilde{X}_g = dL_g|_e(X)$ and a bracket in $\mathfrak{g}$:

$$\begin{aligned} [\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (X, Y) &\longmapsto ([\tilde{X}, \tilde{Y}]_G)_e, \end{aligned}$$

where $[\cdot,\cdot]_G$ is the usual Lie bracket for vector fields.

Definition. *A **Lie algebra** is a vector space $\mathfrak{g}$ over some field F with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:*

1. *bilinearity,*
2. *antisymmetry: $[x, y] = -[y, x]$, for all $x, y \in \mathfrak{g}$ and*
3. *Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.*

Definition. *With this new defined bracket, $\mathfrak{g}$ has a Lie algebra structure: we call $\mathfrak{g}$ the Lie algebra of the Lie group G.*

Example. A simple example is the case where $G = \mathbb{R}^n$. Then $\mathfrak{g} = T_0\mathbb{R}^n \cong \mathbb{R}^n$. If we take $X \in \mathfrak{g}$, let's check what is $\tilde{X}_g$. By definition $\tilde{X}_g = d(Lg)|_{t=0}(X)$. If we take for example the curve $\gamma(t) = tX$ then

$$\begin{aligned} \tilde{X}_g &= \frac{d}{dt}(Lg \circ \gamma)|_{t=0} \\ &= \frac{d}{dt}(g + tX)|_{t=0} \\ &= X. \end{aligned}$$

As X is fixed, we have $\tilde{X}_g$ constant $\forall g \in G$. Then the bracket of any two tangent vectors X, Y is $[X, Y] = ([\tilde{X}, \tilde{Y}]_G)_0 = (\tilde{X}(\tilde{Y}) - \tilde{Y}(\tilde{X}))_0 = 0$. Last equality stands because constant vector fields commute.

Definition. *An action φ is a **symplectic action** if*

$$\varphi : G \longrightarrow \mathit{Sympl}(M, \omega) \subset \mathit{Diff}(M, \omega),$$

i.e., G acts by symplectomorphisms'.

For the special case where the group acting is $\mathbb{R}$, we have a bijection between complete vector fields in M and smooth Lie actions of $\mathbb{R}$ on M given by:

$$\{\text{complete vector fields on } M\} \longleftrightarrow \{\text{Lie actions of } \mathbb{R} \text{ on } M\}$$
$$X \longmapsto \exp tX$$
$$X_p = \frac{d\varphi_t(p)}{dt}|_{t=0} \longleftarrow \varphi.$$

Claim. *Symplectic complete vector fields are in a one-to-one correspondence with symplectic actions.*

Proof. If the action is symplectic, we have that $\varphi_t^*\omega = \omega\ \forall t$. The associated vector field $X_p = \frac{d\varphi_t(p)}{dt}$ is symplectic because

$$\begin{aligned}\mathcal{L}_X\omega &= \frac{d}{dt}(\varphi_t^* w)|_{t=0} \\ &= \frac{d}{dt}(w) \\ &= 0.\end{aligned}$$

In the other way, given a symplectic vector field X, its associated action is $\varphi_t = \exp tX$. As X is symplectic, we have $\mathcal{L}_X\omega = \frac{d}{dt}(\varphi_t^*\omega)|_{t=0} = 0$. In particular $\varphi_t^*\omega$ is constant. For $t = 0$, we have $\varphi_0 = \omega$ so we conclude that $\varphi_t^*\omega = \omega$ i.e. φ_t is a symplectic action. □

In the special case of hamiltonian vector, we have the following definition:

Definition. *A symplectic action φ of $\mathbb{R}$ or S^1 on (M, w) is hamiltonian if the vector field generated by φ is hamiltonian i.e. $i_X\omega = dH$ for a certain function H in M.*

For the case where G the group acting on M is an n-torus $\mathbb{T}^n$ then an action of G on M is called hamiltonian if each restriction

$$\varphi^i := \varphi|_{i\text{th } S^1 \text{ factor}} : S^1 \longrightarrow Sympl(M, \omega)$$

is hamiltonian in the previous sense.

Example. Let's see a very simple example of hamiltonian action. Consider the action of translating one coordinate in $\mathbb{R}^{2n}$ with $\omega = \sum dx_i \wedge dy_i$.

$$\varphi(t, (x_1, y_1, ..., x_n, y_n)) = (x_1, y_1 + t, ..., x_n, y_n).$$

The associated vector field is $X = \frac{\partial}{\partial y_1}$. This vector field is hamiltonian with hamiltonian function $f = -x_1$. An easy computation checks it:

$$\begin{aligned}i_X\omega &= \omega(X, \cdot) \\ &= \sum dx_i \wedge dy_i(X, \cdot) \\ &= dx_1 \wedge dy_1(\frac{\partial}{\partial y_1}) \\ &= -dx_1 \\ &= d(-x_1).\end{aligned}$$

Example. Consider the S^1 action on T^2 seen on a previous example taking the symplectic form $\omega = d\theta_1 \wedge d\theta_2$,

$$\varphi(\psi, (\theta_1, \theta_2)) \longmapsto (\theta_1 + \psi, \theta_2).$$

The vector field generated by this action is $X = \frac{\partial}{\partial \theta_1}$, then we have

$$\begin{aligned} i_X \omega &= d\theta_1 \wedge d\theta_2(X, \cdot) \\ &= d\theta_2. \end{aligned}$$

This is obviously a closed form, but it is not exact because θ_2 is not globally defined. It is an example where the action is symplectic but not hamiltonian.

1.4 Moment map

We denote (M, ω) a compact and connected symplectic manifold, G a Lie group, $\mathfrak{g}$ the associated Lie algebra and $\mathfrak{g}^*$ its dual.

Definition. *Given a symplectic action (i.e $\varphi^*\omega = \omega$) of G on M , for $X \in \mathfrak{g}$ **the fundamental vector field** $X^\#$ associated to X is the vector field in M such that its flow is $exp(uX)$.*

Note now that G acts on itself by conjugation:

$$\begin{aligned} G &\longrightarrow \text{Diff}(G) \\ g &\longmapsto \psi_g(a) = g \cdot a \cdot g^{-1}. \end{aligned}$$

As $\psi_g(e) = e$, its differential at e is an invertible linear map :

$$Ad_g = d\psi_g|_e : \mathfrak{g} \longrightarrow \mathfrak{g}.$$

Letting g vary, we obtain the **adjoint representation** of G on $\mathfrak{g}$:

$$\begin{aligned} Ad : G &\longrightarrow GL(\mathfrak{g}) \\ g &\longmapsto Ad_g. \end{aligned}$$

From this we define the **coadjoint representation** Ad^* :

$$\begin{aligned} Ad^* : G &\longrightarrow Gl(\mathfrak{g}^*) \\ g &\longmapsto Ad_g^*, \end{aligned}$$

where Ad_g^* is a linear map that goes from $\mathfrak{g}^*$ to $\mathfrak{g}^*$ defined as (note that $Ad_g^*(\xi)$ is a form that acts on tangent vectors in $\mathfrak{g}$):

$$Ad_g^*(\xi)(X) = \langle \xi, Ad_{g^{-1}}(X) \rangle = \xi(Ad_{g^{-1}}(X)).$$

Definition. *A moment map associated to the action φ is a map*

$$\mu : M \longrightarrow \mathfrak{g}^* \text{ such that:}$$

1) For all $X \in \mathfrak{g}$ we have $\mu(p)(X) := \langle \mu(p), X \rangle$ and $d\mu(X) = i_{X^\#}\omega$ where $X^\#$ is the fundamental vector field generated $\{\exp(tX) | t \in \mathbb{R}\}$

2) μ is equivariant with respect to the given action φ and the coadjoint action Ad^*, that is:

$$\mu \circ \varphi_g = Ad_g^* \circ \mu, \qquad \forall g \in G.$$

There are particular cases (in which we are mainly interested in this thesis) where these conditions can be rephrased.

Case $G = S^1$ or $G = \mathbb{R}$: we have $\mathfrak{g} \cong \mathbb{R}$, $\mathfrak{g}^* \cong \mathbb{R}$,

1) for the generator $X = 1$ of $\mathfrak{g}$, we have $\mu(p)(X) = \mu(p).1$, i.e. $\mu(X) = \mu$, and $X^\#$ is the standard vector field on M generated by S^1. Then $d\mu = i_{X^\#}\omega$.

2) μ is invariant: $\mathcal{L}_{X^\#}\mu = i_{X^\#}d\mu = 0$.

Case $G = T^n$: we have $\mathfrak{g} \cong \mathbb{R}^n$, $\mathfrak{g}^* \cong \mathbb{R}^n$,

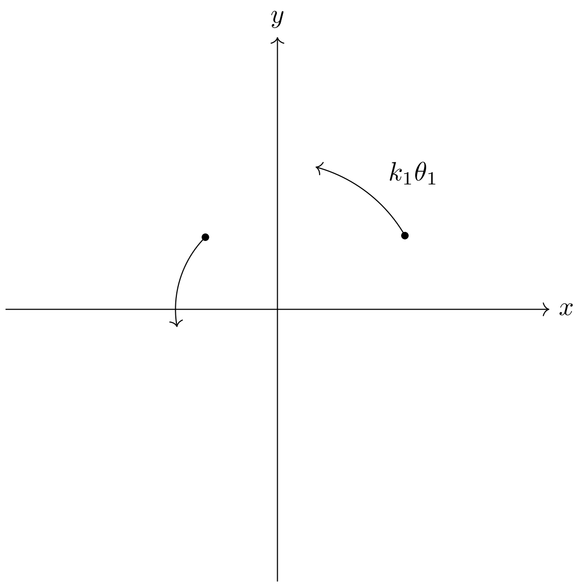
1) For each basis vector X_i of $\mathbb{R}^n$, μ^{X_i} is a hamiltonian function for $X_i^\#$.

2) μ is invariant i.e. $i_{X_j^\#}d\mu_i = 0 \ \forall i, j$.

Example. Let T^n be a n-dimensional torus acting on $\mathbb{C}^n$ by

$$(e^{it_1}, ..., e^{it_n}) \cdot (z_1, ..., z_n) = (e^{it_1 k_1} z_1, ..., e^{it_1 k_n} z_n),$$

where $k_1, ..., k_n \in \mathbb{Z}$ are fixed. For $n = 1$ this corresponds to a rotation in the complex plane with a speed coefficient k_1.



Given a tangent vector in $\mathfrak{g} \cong \mathbb{R}^n$, $X = A_1 \frac{\partial}{\partial t_1}|_p + ... + A_n \frac{\partial}{\partial t_n}|_p$, the associated fundamental vector field is

$$X^\# = A_1 k_1 \frac{\partial}{\partial \theta_1} + ... + A_n k_n \frac{\partial}{\partial \theta_n}.$$

We took polar coordinates in $\mathbb{C}^n$, $(r_1, \theta_1, ..., r_n, \theta_n)$ and standard symplectic form in polar coordinates $\omega = \sum_{i=1}^n r_i dr_i \wedge d\theta_i$. We can check this formula applying change of coordinates to the differential form ω.

$$\begin{aligned} z_i = r_i e^{i\theta_i} &\implies dz_i = e^{i\theta_i} dr_i + i r_i e^{i\theta_i} d\theta_i, \\ \bar{z}_i = r_i e^{-i\theta_i} &\implies d\bar{z}_i = e^{-i\theta_i} dr_i - i r_i e^{-i\theta_i} d\theta_i. \end{aligned}$$

Then

$$\begin{aligned} \tfrac{i}{2} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i &= \tfrac{i}{2} \sum_{i=1}^{n} (e^{i\theta_i} dr_i + i r_i e^{i\theta_i} d\theta_i) \wedge (e^{-i\theta_i} dr_i - i r_i e^{-i\theta_i} d\theta_i) \\ &= \tfrac{i}{2} \sum_{i=1}^{n} -2 i r_i dr_i \wedge d\theta_i \\ &= \sum_{i=1}^{n} r_i dr_i \wedge d\theta_i. \end{aligned}$$

We can compute now the interior product of the fundamental vector field

$$\begin{aligned} i_{X^\#}\omega &= -\sum_{i=1}^{n} A_i k_i r_i dr_i \\ &= -\tfrac{1}{2} \sum_{i=1}^{n} A_i k_i d(r_i^2). \end{aligned}$$

The moment map is

$$\begin{aligned} \mu : \mathbb{C}^n &\longrightarrow \mathbb{R}^n \\ (z_1, ..., z_n) &\longmapsto -\tfrac{1}{2}(k_1|z_1|^2, ..., k_n|z_n|^2) \quad (+\text{constant}) \end{aligned}$$

since

$$\begin{aligned} d\mu(X) &= d(\langle \mu(p), X \rangle) \\ &= d(-\tfrac{1}{2}(A_1 k_1 |z_1|^2, ..., A_n k_n |z_n|^2)) \\ &= -\tfrac{1}{2} \sum_{i=1}^{n} A_i k_i d(r_i^2). \end{aligned}$$

Example. Take now the complex projective space $\mathbb{CP}^n$. It is defined as the projectivization of $\mathbb{C}^{n+1}$.

$$\mathbb{CP}^n := (\mathbb{C}^{n+1} \backslash \{0\})/\mathbb{C}^* = S^{2n+1}/S^1$$

where $\mathbb{C}^*$ acts of $\mathbb{C}^{n+1} \backslash \{0\}$ by component wise multiplication:

$$\lambda(z_0, ..., z_n) = (\lambda z_0, ..., \lambda z_n).$$

Spheres are seen as unit-norm elements of $\mathbb{C}^{n+1}$ and $\mathbb{C}$. This space has symplectic structure (see [12]) which is in fact a consequence of $\mathbb{CP}^n$ being a Kähler manifold (a certain type of differential manifolds with additional structures). The symplectic form is called **Fubini-Study form** and is given by

$$\omega_{FS} = \frac{i}{2|z|^4} \sum_{j,k=1}^{n} |z_j|^2 dz_k \wedge d\bar{z}_k - \bar{z}_j z_k dz_j \wedge d\bar{z}_k.$$

This is in fact a 2-form on $\mathbb{C}^{n+1}\backslash\{0\}$, and ω_{FS} is its pullback to the quotient manifold $\mathbb{CP}^n$.

Take the following $T^1 = S^1$ action on $\mathbb{CP}^1$:

$$e^{it_1} \cdot [z_0, z_1] = [z_0, e^{it_1} z_1],$$

Lets compute the moment map of this action. The action is in fact induced by the one considered before on C^1, with $k_1 = 1$. Taking the affine chart $U_0 = \{[z_0, z_1] \in \mathbb{CP}^1 \ \ |z_0 \neq 0\}$, the Fubiny-Study form is given by [3]:

$$\omega_{FS} = \frac{dx \wedge dy}{(x^2 + y^2 + 1)^2}.$$

Where we the coordinates are $\frac{z_1}{z_0} = z = x + iy$. First, we compute the vector field associated to the action. In polar coordinates it is trivial, since we have the action written like this:

$$\varphi : (t_1, (r, \theta)) \longmapsto (r, \theta + t_1).$$

We have then that $\frac{d(f \circ \varphi_t)}{dt}|_{t=0} = \frac{\partial f}{\partial \theta}$. We deduce that the vector field in polar coordinates is $X_{polar} = \frac{\partial}{\partial \theta}$. In particular, changing coordinates to cartesian, we obtain:

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

To find the moment map, we have to impose the condition $i_X \omega_{FS} = d\mu$.

$$\begin{aligned} i_X \omega_{FS} &= \frac{dx \wedge dy}{(x^2 + y^2 + 1)^2}(X, \cdot) \\ &= \frac{dx \wedge dy}{(x^2 + y^2 + 1)^2}(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \cdot) \\ &= -\frac{xdx}{(x^2 + y^2 + 1)^2} - \frac{ydy}{(x^2 + y^2 + 1)^2}. \end{aligned}$$

Taking into account that $x^2 + y^2 + 1 = |z_1|^2 + |z_0|^2 = |z|^2$ where we took $z_0 = 1$ we deduce that

$$\mu[z_0, z_1] = -\frac{1}{2}(\frac{|z_1|^2}{|z|^2}).$$

We can check easily that this map satisfies the condition in our chart U_0:

$$\begin{aligned} d\mu &= -\frac{1}{2}d(\frac{x^2 + y^2}{x^2 + y^2 + 1}) \\ &= -\frac{1}{2}(\frac{2x(x^2 + y^2 + 1) - (x^2 + y^2)2x}{(x^2 + y^2 + 1)^2}dx \\ &+ \frac{2y(x^2 + y^2 + 1) - (x^2 + y^2)2y}{(x^2 + y^2 + 1)^2}dy \\ &= -\frac{1}{2}(\frac{2x}{(x^2 + y^2 + 1)^2}dx + \frac{2y}{(x^2 + y^2 + 1)^2}) \\ &= -\frac{xdx}{(x^2 + y^2 + 1)^2} - \frac{ydy}{(x^2 + y^2 + 1)^2}. \end{aligned}$$

This example generalizes to $\mathbb{CP}^n$ and the action of T^n:

$$(e^{it_1}, ..., e^{it_n}) \cdot [z_0 : z_1 : ... : z_n] = [z_0 : e^{it_1} z_1 : ... : e^{it_1} z_n],$$

and has moment map

$$\mu[z_0, ..., z_n] = -\frac{1}{2}(\frac{|z_1|^2}{|z|^2}, ..., \frac{|z_n|^2}{|z|^2}).$$

1.5 Symplectic blow-up and Hirzebruch surfaces

Fiber bundles.[11] Let's define a topological construction called a fiber bundle that we will use in this section and later on too. The idea of fiber bundle is a space that is locally a product space, but not necessarily globally. Formally, it is the following.

Definition. *A **fiber bundle** is a structure (E, B, π, F) where E, B and F are topological spaces and $\pi : E \to B$ is a continuous surjection satisfying the following condition, called the local triviality condition. We will call B the base space, E the total space and F the fiber. The map π is called the projection map(or bundle projection). We have that for every $x \in E$, there is an open neighbourhood U in B of $\pi(x)$ such that there is a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that the following diagram commutes (i.e. π agrees with the projection onto the first factor):*

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "proj_1"] \\
U &
\end{tikzcd}$$

where $proj_1$ is the natural projection.

Symplectic blow up. The blowing-up operation in complex geometry consists in replacing a point in a space by the space of complex tangent lines through that point. This local operation can be explicitly written. The blow-up of $\mathbb{C}^n$ is as a set

$$\tilde{\mathbb{C}}^n = \mathbb{C}^n \sqcup \mathbb{CP}^{n-1},$$

with $\mathbb{CP}^{n-1}$ the space of tangent lines through the origin and called the **exceptional divisor**. As a manifold, it is a submanifold of $\mathbb{C}^n \times \mathbb{CP}^{n-1}$ described by

$$\tilde{\mathbb{C}}^n = \{((z_1, ..., z_n), [v_1, ..., v_n]) \mid (z_1, ..., z_n) \in [v_1, ..., v_n]\}.$$

It can also be described as

$$\tilde{\mathbb{C}}^n = \{([p], z) \mid p \in \mathbb{C}^n \backslash \{0\}, z = \lambda p \text{ for some } \lambda \in \mathbb{C}\}.$$

There is a result on these manifolds in general, we have

Theorem. *Let X be a complex manifold of dimension n. Then the blow up of X at one point is an n-dimensional complex manifod $\tilde{X}$ which is diffeomorphic to $X \# \bar{\mathbb{CP}}^n$.*

In order to generalize this construction to symplectic manifolds (M, ω) of dimension $2n$, we need to make sure the resulting manifold still haves a symplectic structure.

Definition. *A blow-up symplectic form on* $\tilde{\mathbb{C}}^n$ *is a* $U(n)$*-invariant symplectic form* ω *such that the difference* $\omega - \beta^*\omega_0$ *is compactly supported, where* $w_0 = \frac{i}{2}\sum_{i=1}^n dz_i \wedge d\bar{z}_i$ *the standard symplectic form in* $\mathbb{C}^n$*.*

Two blow-up symplectic forms are called equivalent if one is the pullback of the other by a $U(n)$-equivariant diffeomorphism of $\tilde{\mathbb{C}}^n$. Let $\Omega^\epsilon(\epsilon > 0)$ be the set of all blow-up symplectic forms on $\tilde{\mathbb{C}}^n$ whose restriction to the exceptional divisor is $\epsilon\omega_{FS}$, where ω_{FS} is the Fubini-Study form on $\mathbb{CP}^n$. An **ϵ-blow-up** of $\mathbb{C}^n$ at the origin is a pair $(\tilde{\mathbb{C}}^n, \omega)$ with $\omega \in \Omega^\epsilon$.

Now let (M, ω) be a $2n$-dimensional symplectic manifold. By Darboux theorem, for each point $q \in M$ there exists a chart $(U, z_1, ..., z_n)$ centered at q with image in $\mathbb{C}^n$ where

$$\omega|_U = \frac{i}{2}\sum_{i=1}^n dz_i \wedge d\bar{z}_i.$$

It is shown that for ϵ small enough, we can perform an ϵ-blow up of M at q without changing the symplectic structure outside a small neightbourghood of q. The blow-up is given by the blow up of $\mathbb{C}^n$ at the origin through the chart that exists by Darboux theorem. The resulting manifold is the ϵ-blow-up of M at q.

Example (Hirzebruch surfaces)**.** We present here a quite technical construction a family of surfaces, which is interesting for further examples on Delzant construction. From now on, denote $\tilde{\mathbb{C}}^n (= \mathbb{C}^n \sqcup \mathbb{CP}^{n-1})$ as L. Let $\mathbb{P}(L \oplus \underline{\mathbb{C}})$ the projectivization of the direct sum of L with a trivial complex line bundle. Consider the map

$$\begin{aligned} \beta \ : \ \mathbb{P}(L \oplus \mathbb{C}) &\longrightarrow \mathbb{CP}^n \\ ([p], [\lambda p, w]) &\longmapsto [\lambda p, w] \end{aligned}$$

Notice that β maps the exceptional divisor

$$E := \{([p], [0 : ... : 0 : 1]) \ | \ [p] \in \mathbb{CP}^{n-1}\} \cong \mathbb{CP}^{n-1}$$

to the point $[0 : ... : 0 : 1] \in \mathbb{CP}^n$, whereas in the complement it is a diffeomorphism

$$S := E^C = \{([p], [\lambda p : w] \ | \ [p] \in \mathbb{CP}^{n-1}, \lambda \in \mathbb{C}^*, w \in \mathbb{C}\} \cong \mathbb{CP}^n \backslash \{[0 : ...0 : 1]\}.$$

Therefore, we can see $\mathbb{P}(L \oplus \mathbb{C})$ as smoothly replacing the point $[0 : ... : 0 : 1]$ in $\mathbb{CP}^n$ by a copy of $\mathbb{CP}^{n-1}$. For $n = 2$ this is known as the **first Hirzebruch surface**.

Hirzebruch surfaces are in fact a family of algebraic surfaces over the complex numbers of dimension 4 in $\mathbb{R}$, or 2 in $\mathbb{C}$. We present its general construction, as they will appear in examples of Delzant construction of toric manifolds.

Identify S^1 with the unit complex numbers in $\mathbb{C}$. Notice that the 3-sphere, $S^3 = \{(z_1, z_2) \ | \ |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2$, admits a free S^1 action defined by

$$\begin{aligned} \varphi : S^1 \times S^3 : &\longrightarrow S^3 \\ (\lambda, (z_1, z_2)) &\longmapsto (\lambda z_1, \lambda z_2) \end{aligned}$$

The quotient of this action is ${S^3}/{S^1} \cong \mathbb{CP}^1$. For any integer $n \in \mathbb{Z}$ define the complex line bundle $L_n \to \mathbb{CP}^1$ whose total space is the following quotient of $S^3 \times \mathbb{C}$

$$L_n := {(S^3 \times \mathbb{C})}/{\sim_n}, \quad (x,z) \sim_n (\lambda x, \lambda^n z) \ \forall \lambda \in S^1.$$

We map $L_n \to \mathbb{CP}^1$ as $[x,z] \mapsto [x] \in {S^3}/{S^1} \cong \mathbb{CP}^1$.

Definition. *For $n \in \mathbb{Z}$ define the Hyrzebruch surface $H_n := \mathbb{P}(L_{-n} \oplus \mathbb{C})$.*

Hirzebruch stated also a classification theorem:

Theorem. *For the smooth manifolds H_n*

$$H_n \cong H_m \text{ diffeomorphic } \iff n = m \text{ mod } 2,$$

and

$$H_n \cong_{\mathbb{C}} H_m \text{ complex diffeomorphic } \iff n = m.$$

From a non complex point of view, the only possible manifolds are $H_0 \cong S^2 \times S^2$ and $H_1 \cong \mathbb{CP}^2 \# (-\mathbb{CP}^2)$, where $\#$ is the connected sum and $-\mathbb{CP}^2$ is $\mathbb{CP}^2$ with opposite orientation.

1.6 First result on topological invariants in symplectic manifolds

As always when considering certain type of manifolds, an interesting field of study is to look for their topological invariants. We state here the first immediat invariant that is found.

Claim. *A compact symplectic manifold (M, ω) has a non-trivial $H^2(M)$, the second cohomology group.*

Proof. One of the consequences of De Rham theorem, a very important result in smooth manifold theory, is that the dimension of $H^n_{DR}(M^{2n}, \mathbb{R})$ is the same as the dimension of $H^n(M, \mathbb{Z})$. Lets show $H^2_{DR}(M, \mathbb{R})$ is not trivial in a symplectic manifold.

As a first observation, note that since ω is closed, so is ω^k, for all $k \in 2, ..., n$. Suppose now $[\omega] = [0]$, i.e. there exists a one-form such that $\omega = d\alpha$. Then using the properties of the exterior derivative and Stokes theorem we have

$$\int_M \omega^n = \int_M d\alpha \wedge \omega^{n-1} = \int_M d(\alpha \wedge \omega^{n-1}) = \int_{\partial M} \alpha \wedge \omega^{n-1} = 0.$$

Last equality stands because M has no boundary because it is compact. But this is a contradiction since ω^n is a volume form so $\omega^n > 0$ which implies $\int_M \omega^n > 0$. We conclude that $[\omega] \neq [0]$ and hence $H^2(M, \mathbb{Z}) \neq 0$. □

Corollary. *The sphere S^{2n} is not symplectic for $n \geq 2$.*

Proof. We know that S^{2n} is compact and that $H^2(S^{2n}) = 0$ for all $n \geq 2$. Hence, S^{2n} is not symplectic. □

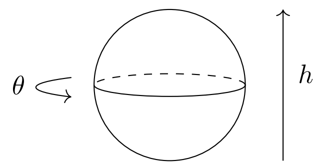
2 Delzant theorem

After all the preliminaries, we can go on with the study of Delzant theorem. In a first place, we state Delzant theorem and show the Delzant construction, which is the most interesting part of the theorem proof. It proves the surjectivity of the bijection that is stated in the theorem. After looking at a few particular examples of construction, we also see (omitting some non-interesting high technical lemma's proofs) the proof of the injectivity in Delzant theorem.

2.1 Delzant Polytopes and Toric manifolds

Definition. *A quaternion $(M^{2n}, \omega, T^n, \mu)$, with (M^{2n}, ω) a symplectic manifold and μ a moment map for the action of T^n, is a* ***toric manifold****.*

Example. A very simple example of manifold which is toric is the sphere S^2 with the standard symplectic form $\omega = d\theta \wedge dh$.



Consider the rotation which is an action of $T^1 = S^1$

$$\begin{aligned} \psi : S^1 \times S^2 &\longrightarrow S^2 \\ (\varphi, (\theta, h)) &\longmapsto (\theta + \varphi, h). \end{aligned}$$

Its moment map is $\mu(\theta, h) = h$. To check this, take a tangent vector in $\mathfrak{g} \cong \mathbb{R}$, $X = a\frac{\partial}{\partial x}$. the associated fundamental vector field is

$$X^{\#} = a\frac{\partial}{\partial \theta}.$$

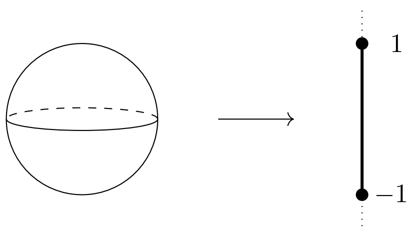
We have then

$$\begin{aligned} i_{X^{\#}}\omega &= d\theta \wedge dh(a\tfrac{\partial}{\partial \theta}, \cdot) \\ &= (a)dh. \end{aligned}$$

And lastly

$$\begin{aligned} d\mu^X &= d(\langle \mu(p), X \rangle) \\ &= d((a)h) \\ &= (a)dh. \end{aligned}$$

The image of the moment map is $[-1, 1]$.



An interesting lemma about toric actions is the following.

Lemma 2. *Let T^n be the n-dimensional torus and φ an effective Lie action of T^n on a manifold M. Then:*
1) φ has $n+1$ fixed points,
2) dim$(M) \geq 2n$, if not φ cannot be effective.

In the particular case where the dimension of M is $2n$ and the manifold is toric, the strong result previous to Delzant theorem is:

Theorem 3 (Guillemin – Sternberg ; Atiyah)**.**
Let $(M^{2n}, \omega, T^n, \mu)$ be a toric manifold with M compact and connected. Then:
1) the level sets of μ are connected,
2) $\mu(M)$ is convex,
3) $\mu(M)$ is the convex hull of fixed points images by the action with moment map μ.

The image $\mu(M)$ of the moment map is called the **moment polytope**

Example. Consider the T^3-action on $\mathbb{CP}^3$ defined as:

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_2}) \cdot [z_0, z_1, z_2, z_3] = [z_0, e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{i\theta_3} z_3],$$

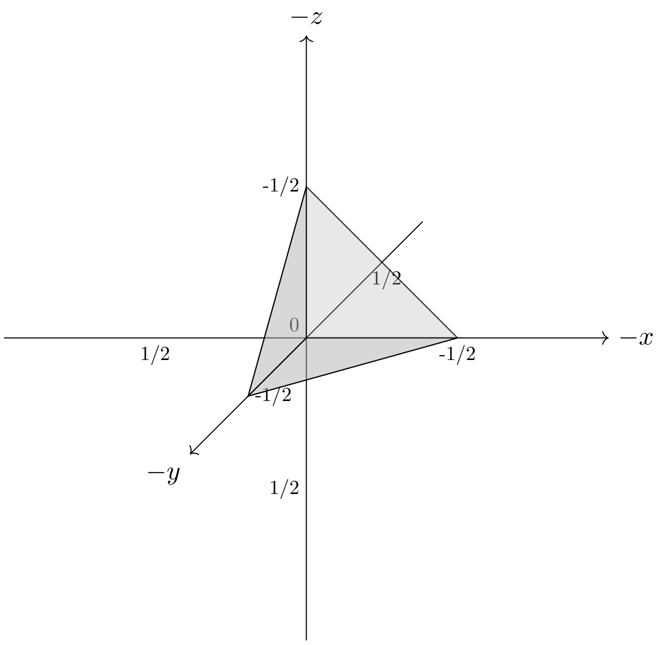
that has moment map (seen in section 1.4)

$$\mu[z_0, z_1, z_2, z_3] = -\frac{1}{2}\left(\frac{|z_1|^2}{|z|^2}, \frac{|z_2|^2}{|z|^2}, \frac{|z_3|^2}{|z|^2}\right).$$

The image of its fixed points are

$$\begin{aligned}
[1,0,0,0] &\longmapsto (0,0,0)\\
[0,1,0,0] &\longmapsto (-\tfrac{1}{2},0,0)\\
[0,0,1,0] &\longmapsto (0,-\tfrac{1}{2},0)\\
[0,0,0,1] &\longmapsto (0,0,-\tfrac{1}{2}).
\end{aligned}$$

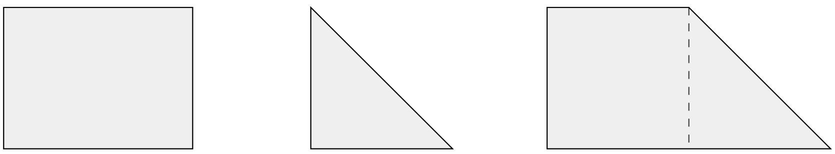
The convex hull of these points define the 3-dimensional polytope which is by Theorem 5 the image of the moment map.



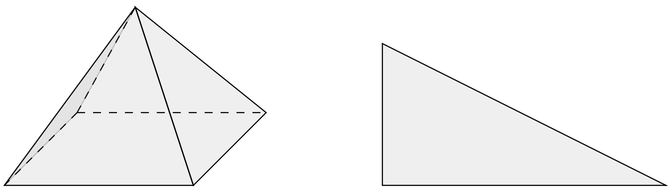
What types of polytopes can be obtained through the image of a moment map? As we will see in Delzant theorem, only the following ones.

Definition. *A polytope in $\mathbb{R}^n$ is a **Delzant polytope** if it is:*
1) simple : n edges meeting at each vertex,
2) rational : edges meeting at vertex p are of form $A_i : p+tu_i, t \geq 0, u_i \in \mathbb{Z}^n$,
3) smooth: for each vertex, the corresponding $u_i's$ can be chosen to be a basis of $\mathbb{Z}^n$.

Examples of Delzant polytopes :



Examples of non-Delzant polytopes :

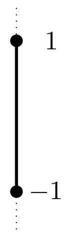


For the non-Delzant polytopes, the first example is not simple and the second one is not smooth.

Algebraic description A facet of a polytope is a $(n-1)$-dimensional face. Let n be the dimension of a Delzant polytope Δ and d its number of facets. A lattice vector $v \in \mathbb{Z}^n$ is **primitive** if it cannot be written as $v = ku$ with $k \in \mathbb{Z}$, $u \in \mathbb{Z}^n$ and $|k| > 1$. Let $v_i, i = 1, ..., d$ be the primitive outward-pointing normal vectors of the facets. Then we can describe algebraically the Delzant polytope as an intersection of halfspaces

$$\Delta = \{x \in (\mathbb{R}^n)^* | \langle x, v_i \rangle \leq \lambda_i, i = 1, ..., d\} \text{ for some } \lambda_i \in \mathbb{R}.$$

Example. For the toric manifold that we saw before, S^2, the moment polytope is this one.



In this case, the algebraic definition of this polytope would be

$$\Delta = \{x \in \mathbb{R} \mid \langle x, (1) \rangle \leq 1, \langle x, (-1) \rangle \leq 0\}.$$

That is $v_1 = (1)$, $v_2 = (-1)$, $\lambda_1 = 1$ and $\lambda_2 = 0$. We will see more examples in the particular constructions of Delzant toric manifolds.

We are now in conditions to state Delzant theorem, which uses Delzant polytopes to classify compact symplectic toric manifolds.

Theorem (Delzant, 1990)**.** *The moment polytope Δ determines the toric manifold. We have the bijection*

$$\frac{\textit{compact symplectic toric manifolds}}{T^n\textit{-equivariant symplectomorphisms}} \longleftrightarrow \frac{\textit{Delzant polytopes}}{\textit{translations}}.$$

2.2 Delzant construction and examples

The description of Delzant polytopes and the Delzant construction are described in [3]. The injectivity proof is detailed in [7]. The goal of this section is to show that for every given Delzant polytope, we can construct a toric manifold such that its moment polytope is exactly that polytope.

In order to state a result that we need for the construction, we can define a generalization of a toric manifold to any Lie group G.

Definition. *Let (M, ω) be a symplectic manifold, G a Lie group acting on it and μ is a moment map. Then the vector (M, ω, G, μ) is a hamiltonian G-space.*

The previous result that we will need for the construction is the following

Theorem 4 (Marsden-Weinstein-Meyer)**.** *Let (M, w, G, μ) be a hamiltonian G-space for a compact Lie group G. Let $i : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. G acts freely on $\mu^{-1}(0)$. Then*

1. *the orbit space $M_{red} = \mu^{-1}(0)/G$ is a manifold,*
2. *$\pi : \mu^{-1}(0) \to M_{red}$ is a principal G-bundle, and*
3. *there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega = \pi^*\omega_{red}$.*

Let Δ be a Delzant polytope with d facets. Note that $d > n$. As seen before, we can describe it as:

$$\Delta = \{x \in (\mathbb{R}^n)^* | \langle x, v_i \rangle \leq \lambda_i, i = 1, ..., d\} \text{ for some } \lambda_i \in \mathbb{R}.$$

Let $\{e_i; i = 1, ..., d\}$ be the standard basis of $\mathbb{R}^d$. Consider

$$\begin{aligned} \tilde{\pi} : \mathbb{R}^d &\longrightarrow \mathbb{R}^n \\ e_i &\longmapsto v_i. \end{aligned}$$

Claim. *$\tilde{\pi}$ maps $\mathbb{Z}^d$ to $\mathbb{Z}^n$.*

Proof. Because of Δ being Delzant, we have that at a vertex p, the edge vectors $u_1, ..., u_n$ form a basis for $(\mathbb{Z}^n)^*$. We may assume it is the standard basis without loss of generality. Then the corresponding normal vectors to each edge meeting at p are also a basis of $\mathbb{Z}^n$. □

Therefore, $\tilde{\pi}$ induces to a map between tori.

$$\begin{tikzcd} \mathbb{R}^d \arrow[r, "\tilde{\pi}"] \arrow[d, "\pi_1"'] & \mathbb{R}^n \arrow[d, "\pi_2"] & \\ T^d \arrow[r, "\pi"] & T^n \arrow[r] & 0 \end{tikzcd}$$

Let

$$\begin{aligned} N &= Ker(\pi) \\ \mathfrak{n} &= \text{ Lie algebra of } N. \end{aligned}$$

Then we have an exact sequence

$$0 \xrightarrow{\tilde{\pi}} N \xrightarrow{i} T^d \xrightarrow{\pi} T^n \longrightarrow 0$$

that induces (differentiating) to an exact sequence of Lie algebras

$$0 \xrightarrow{\tilde{\pi}} \mathfrak{n} \xrightarrow{i} \mathbb{R}^d \xrightarrow{\pi} \mathbb{R}^n \longrightarrow 0$$

and its dual sequence

$$0 \xrightarrow{\tilde{\pi}} (\mathbb{R}^n)^* \xrightarrow{\pi^*} (\mathbb{R}^d)^* \xrightarrow{i^*} \mathfrak{n}^* \longrightarrow 0.$$

We now consider $\mathbb{C}^d$ with the symplectic form $\omega = \frac{i}{2} \sum dz_k \wedge d\bar{z}_k$ and the standard T^d hamiltonian action

$$(e^{i\theta_1}, ..., e^{i\theta_d}) \cdot (z_1, ..., z_d) = (e^{i\theta_1} z_1, ..., e^{i\theta_d} z_d).$$

The moment map is

$$\mu : \mathbb{C}^d \longrightarrow (\mathbb{R}^d)^*$$
$$\mu(z_1, ..., z_d) \longmapsto -\tfrac{1}{2}(|z_1|^2, ..., |z_d|^2) + \text{constant}$$

and we choose the constant to be $(\lambda_1, ..., \lambda_d)$ from the algebraic description of our polytope. The subtorus N acts with moment map

$$i^* \circ \mu : \mathbb{C}^d \longrightarrow \mathfrak{n}^*.$$

Let $Z = (i^* \circ \mu)^{-1}(0)$ be the zero-level set.

Claim. *The set Z is compact and N acts freely on it.*

Proof. We will show that $\mu(Z) = \Delta'$ where Δ' is the image of Δ by $\pi^* : (\mathbb{R}^n)^* \longrightarrow (\mathbb{R}^d)^*$. Assuming it and using the following claim the proof is immediate.

Claim. *The moment map μ is proper in this particular case.*

Proof. The preimage of a compact subset K in $(\mathbb{R}^d)*$ will have its norm limited by values at the boundary of K. Hence, as the preimage of a closed set is closed, we have that it $\mu^{-1}(K)$ is closed and bounded so it is compact. This is true because $\mathbb{C}^d$ is, as a Banach space, isomorphic to $\mathbb{R}^{2d}$. □

As Δ' is compact and μ is proper it will follow that Z is compact.

Lemma 5. *Let $y \in (\mathbb{R}^d)^*$. Then:*

$$y \in \Delta' \iff y \text{ is in the image of } Z \text{ by } \mu.$$

Proof of the lemma. The given y is in the image of Z by μ if:

1. y is in the image of μ;
2. $i^* y = 0$.

Being in the image of μ means $y_i = \lambda_i - \frac{1}{2}|z_i|^2, \forall i \iff y_i - \lambda_i \leq 0, \forall i \iff y_i \leq \lambda_i, \forall i$. Because we had an exact sequence, $i^* y = 0 \iff y = \pi^*(x)$ for some $x \in (\mathbb{R}^n)^*$. Now:

$$\begin{aligned} y \in Im(Z) &\iff \langle \pi^*(x), e_i \rangle \leq \lambda_i, \forall i \\ &\iff \langle x, \pi(e_i) \rangle \leq \lambda_i, \forall i \\ &\iff x \in \Delta. \end{aligned}$$

Hence, $y \in Im(Z) \iff y \in \pi^*(\Delta) = \Delta'$. Finally we have Z image of a compact Δ' by a proper map μ : Z is compact.

□

We want to show also that N acts freely on Z to apply reduction theorem. Recall that an action is free if

$$g \cdot x = x \;\; \forall x \in X \implies g = e.$$

An equivalent condition is that all stabilizers are trivial:

$$G_x = \{g \in G \mid g \cdot x = x\} = \{e\}, \ \forall x \in X.$$

Let F be a face of Δ' with $dim(F) = n - r$. F is then characterized as a subset of Δ' by r equations:

$$\langle y, e_i \rangle = \lambda_i, \ \ i = i_1, ..., i_r.$$

Denote it $F = F_I$ with $I = (i_1, ..., i_r)$. Let $z = (z_1, ..., z_d) \in Z$.

$$\begin{aligned} z \in \mu^{-1}(F_I) &\iff \mu(z) \in F_I \\ &\iff \langle \mu(z), e_i \rangle = \lambda_i, \ \forall i \in I \\ &\iff -\tfrac{1}{2}|z_i|^2 + \lambda_i = \lambda_i, \ \forall i \in I \\ &\iff z_i = 0, \ \forall i \in I. \end{aligned}$$

Observe that T^d acts on Z, and the stabilizer of z so that $\mu(z) \in F_I$ is

$$(T^d)_I = \{(e^{i\theta_1}, ..., e^{i\theta_d}) \mid e^{i\theta_j} = 1, \forall j \notin I\}.$$

In order to show that N acts freely on Z, consider the worst case scenario of points $z \in Z$ whose stabilizer under the action of T^d is as large as possible. That is when $F_I = y$ is a vertex of Δ'. Then y satisfies:

$$\langle y, e_i \rangle = \lambda_i, \ \ i \in I = \{i_1, ..., i_n\}.$$

Lemma 6. *Let $z \in Z$ be such that $\mu(z)$ is a vertex of Δ'. Then the map $\pi : T^d \longrightarrow T^n$ maps $(T^d)_I$ bijectively on T^n.*

If this is true, we have that in the worst case the stabilizer of z intersects N in the trivial group.

Proof of the lemma. Suppose $y = \mu(z)$ is a vertex of Δ'. We can renumber the indices such that

$$I = (1, ..., n).$$

Hyperplans meeting at y are

$$\langle y', e_i \rangle = \lambda_i, \ \ i = 1, ..., n.$$

The set $\pi(e_1), ..., \pi(e_n)$ is basis of $\mathbb{Z}^n$. Thus, π is bijective. $\square$

This proves that n acts freely in the worst case points of Z, and hence in general. It is the worst case since the other stabilizers $N_{z'}$ for $z' \in Z$ are contained in stabilizers for points z mapped to vertices. This is clear because for a z' different from a vertex will have less vanishing components than a vertex. $\square$

We have now that Z is a compact submanifold of $\mathbb{C}^d$ of dimension:

$$\dim_{\mathbb{R}}(Z) = 2d - (d - n) = d + n.$$

Applying the Marsden-Weinstein-Meyer reduction theorem we have that:

1. $M_\Delta = {}^{Z}/_{N}$ is a manifold and
2. there is a symplectic form ω_Δ satisfying $i^*\omega = \pi^*\omega_\Delta$.

Claim. *For a certain μ', $(M_\Delta, \omega_\Delta, T^n, \mu')$ is a toric manifold of $dim(M_\Delta) = 2n$ and $\mu'(M_\Delta) = \Delta$.*

Proof. We have that given a $z \in Z$ its stabilizer with respect to the T^d-action is $(T^d)_I$. N acts freely on Z so

$$(T^d)_I \cap N = \{e\}$$

In the worst case scenario, z is a vertex of Δ'. There is in any case the inverse map $\pi^{-1} : T^n \longrightarrow (T^d)_I$ and thus the exact sequence

$$0 \longrightarrow N \longrightarrow T^d \longrightarrow T^n \longrightarrow 0$$

splits. Then we have $T^d = N \times T^n$ acting on $(M_\Delta, \omega_\Delta)$. The moment map is

$$\mu : \mathbb{C}^d \longrightarrow (\mathbb{R}^d)^* = \mathfrak{n}^* \oplus (\mathbb{R}^n)^*.$$

Let $j : Z \hookrightarrow \mathbb{C}^d$ be the inclusion map, and let

$$pr_1 : (\mathbb{R}^d)^* \longrightarrow \mathfrak{n} \quad \text{and} \quad pr_2 : (\mathbb{R}^d)^* \longrightarrow (\mathbb{R}^n)^*$$

be the projections maps. The map

$$pr_2 \circ \mu \circ j : Z \longrightarrow (\mathbb{R}^n)^*$$

is constant on N-orbits because it is projected to the T^n component. Thus there exists a map μ'

$$\mu' : M_\Delta \longrightarrow (\mathbb{R}^n)^*,$$

such that

$$\mu' \circ p = pr_2 \circ \mu \circ j.$$

The image of μ' is equal to the image of $pr_2 \circ \mu \circ j$. We showed earlier that $\mu(Z) = \Delta'$. We have that $pr_2 \circ \pi^* = id$ and thus

$$Im(\mu') = pr_2(\Delta') = pr_2 \circ \pi^*(\Delta) = \Delta.$$

In conclusion, the image of M_Δ is the required polytope. $\square$

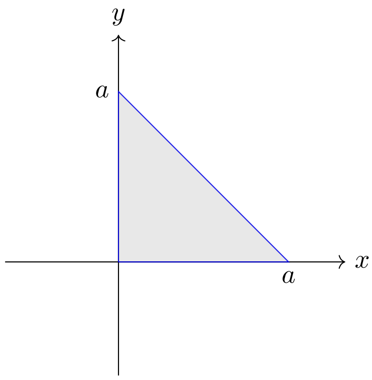
Before going into the examples of Delzant construction, we can state a result that will help us identifying some of the constructed manifolds.

Theorem 7. *Take a Delzant polytope in $\mathbb{R}^n$ with a vertex p with primitive edge vectors $u_1, ..., u_n$ at p. Consider a new polytope obtained by chopping off the corner and replacing it by n new vertices:*

$$p + \epsilon u_j, \;\; j = 1, ..., n,$$

where ϵ is a small positive real number. Then this new polytope is Delzant and the corresponding toric manifold is the ϵ-symplectic blow-up at p of the original one.

Example. Consider an isosceles triangle:



Which can be described algebraically as:

$$\begin{aligned}\Delta &= \{x \geq 0, y \geq 0, x + y \leq a\} \\ &= \{\langle \boldsymbol{x}, v1 \rangle \leq 0, \langle \boldsymbol{x}, v_2 \rangle \leq 0, \langle \boldsymbol{x}, v_3 \rangle \leq a\},\end{aligned}$$

with $v_1 = (-1, 0), v_2 = (0, -1), v_3 = (1, 1)$ and $\boldsymbol{\lambda} = (0, 0, a)$. We have 3 facets in $\mathbb{R}^2$ so our map is:

$$\begin{aligned}\pi : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (1, 0, 0) &\longmapsto (-1, 0) \\ (0, 1, 0) &\longmapsto (0, -1) \\ (0, 0, 1) &\longmapsto (1, 1).\end{aligned}$$

This map can be quotiented to a T^3 map to T^2 and its kernel is $N = Ker(\pi) = \{(e^{i\theta}, e^{i\theta}, e^{i\theta}) \mid \theta \in [0, 2\pi]\}$. Consider now $\mathbb{C}^3$ with symplectic form $\omega = \frac{i}{2} \sum_{k=1}^{3} dz_k \wedge d\bar{z}_k$ and standard hamiltonian action of T^3

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot (z_1, z_2, z_3) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{i\theta_3} z_3).$$

As we already know its moment map (with $\boldsymbol{\lambda}$ as additive constant) is

$$\mu(z_1, z_2, z_3) = -(|z_1|^2, |z_2|^2, |z_3|^2) + (0, 0, a).$$

We can compute the dual of the differential of the inclusion map of N in T^3

$$\begin{aligned}i : N &\longrightarrow T^3 \\ x &\longmapsto (x, x, x).\end{aligned}$$

Its differential has matrix

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

So the matrix of the dual of the differential, that we denote i^*, is the transpose

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

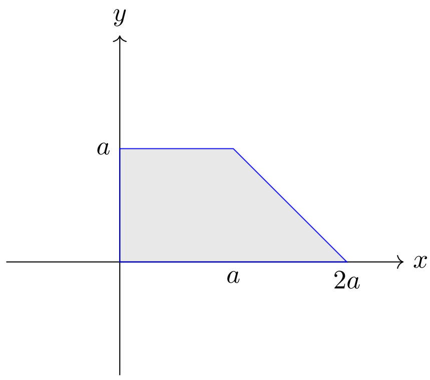
In coordinates, this is $i^*(x_1, x_2, x_3) = x_1 + x_2 + x_3$. Finally we compute the zero-level set

$$(i^* \circ \mu)^{-1}(0) = \{(z_1, z_2, z_3) \mid |z_1|^2 + |z_2|^2 + |z_3|^2 = a\} \cong S^5$$

. The reduced space, which is the sought manifold, is then

$$(i^* \circ \mu)^{-1}(0)\,/_N \cong S^5\,/_{S^1} \cong \mathbb{CP}^2$$

Example. Lets try now a trapezoid:



Which can be described algebraically as:

$$\begin{aligned}\Delta &= \{x \geq 0, y \geq 0, y \leq a, x + y \leq 2a\} \\ &= \{\langle \boldsymbol{x}, v_1 \rangle \leq 0, \langle \boldsymbol{x}, v_2 \rangle \leq 0, \langle \boldsymbol{x}, v_3 \rangle \leq a, \langle \boldsymbol{x}, v_4 \rangle \leq 2a\},\end{aligned}$$

with $v_1 = (-1, 0), v_2 = (0, -1), v_3 = (1, 0), v_4 = (1, 1)$ and $\boldsymbol{\lambda} = (0, 0, a, 2a)$. We have 4 facets in $\mathbb{R}^2$ so our map is:

$$\begin{aligned}\pi : \mathbb{R}^4 &\longrightarrow \mathbb{R}^2 \\ (1, 0, 0, 0) &\longmapsto (-1, 0) \\ (0, 1, 0, 0) &\longmapsto (0, -1) \\ (0, 0, 1, 0) &\longmapsto (1, 0). \\ (0, 0, 0, 1) &\longmapsto (1, 1).\end{aligned}$$

This map can be quotiented to a T^4 map to T^2 and its kernel is $N = Ker(\pi) = \{(e^{i\theta} + e^{i\varphi}, e^{i\varphi}, e^{i\theta}, e^{i\varphi}) \mid \theta, \varphi \in [0, 2\pi]\} \cong T^2$. Consider now $\mathbb{C}^4$ with symplectic form $\omega = \frac{i}{2} \sum_{k=1}^{4} dz_k \wedge d\bar{z}_k$ and standard hamiltonian action of T^4

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}) \cdot (z_1, z_2, z_3, z_4) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{i\theta_3} z_3, e^{i\theta_4} z_4).$$

As we already know its moment map (with $\boldsymbol{\lambda}$ as additive constant) is

$$\mu(z_1, z_2, z_3, z_4) = -(|z_1|^2, |z_2|^2, |z_3|^2, |z_4|^2) + (0, 0, a, 2a)$$

We can compute the dual of the differential of the inclusion map of N in T^4,

$$\begin{aligned} i : N &\longrightarrow T^4 \\ (x, y) &\longmapsto (x + y, y, x, y). \end{aligned}$$

Its differential has matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

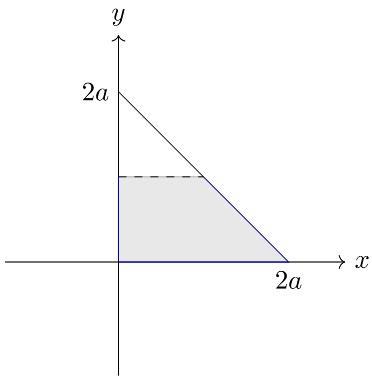
So the matrix of the dual of the differential, that we denote i^*, is the transpose

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In coordinates $i^*(x_1, x_2, x_3, x_4) = (x_1 + x_3, x_1 + x_2 + x_4)$. Finally we compute the zero-level set

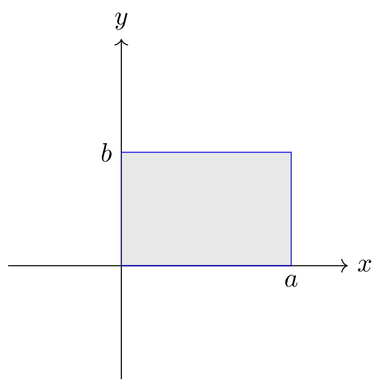
$$(i^* \circ \mu)^{-1}(0) = \{(z_1, z_2, z_3, z_4) \ | \ |z_1|^2 + |z_3|^2 = a, |z_1|^2 + |z_2|^2 + |z_4|^2 = 2a\}$$

In order to understand what is the manifold $M = {(i^* \circ \mu)^{-1}(0)} / {T^2}$ we have to recall Theorem 9 . Notice that our trapezoid is in fact a isosceles triangle (with $2a$ sides) whose top vertex has been chopped off as in the construction of Theorem 9.



So in fact M is the symplectic blow-up of $\mathbb{CP}^2$ at the fixed point which image through moment map is the top vertex. This is exactly the first Hirzebruch surface H_1 as defined in section 1.5.

Example. Consider a rectangle:



Which can be described algebraically as:

$$\begin{aligned}\Delta &= \{x \geq 0, y \geq 0, x \leq a, y \leq b\} \\ &= \{\langle \boldsymbol{x}, v1\rangle \leq 0, \langle \boldsymbol{x}, v_2\rangle \leq 0, \langle \boldsymbol{x}, v_3\rangle \leq a, \langle \boldsymbol{x}, v_4\rangle \leq b\},\end{aligned}$$

with $v_1 = (-1, 0), v_2 = (0, -1), v_3 = (1, 0), v_4 = (0, 1)$ and $\boldsymbol{\lambda} = (0, 0, a, b)$. We have 4 facets in $\mathbb{R}^2$ so our map is:

$$\begin{aligned}\pi : \mathbb{R}^4 &\longrightarrow \mathbb{R}^2 \\ (1, 0, 0, 0) &\longmapsto (-1, 0) \\ (0, 1, 0, 0) &\longmapsto (0, -1) \\ (0, 0, 1, 0) &\longmapsto (1, 0). \\ (0, 0, 0, 1) &\longmapsto (0, 1).\end{aligned}$$

This map can be quotiented to a T^4 map to T^2 and its kernel is $N = Ker(\pi) = \{(e^{i\theta}, e^{i\varphi}, e^{i\theta}, e^{i\varphi}) \mid \theta, \varphi \in [0, 2\pi]\}$. Consider now $\mathbb{C}^4$ with symplectic form $\omega = \frac{i}{2}\sum_{k=1}^{4} dz_k \wedge d\bar{z}_k$ and standard hamiltonian action of T^4

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}) \cdot (z_1, z_2, z_3, z_4) = (e^{i\theta_1}z_1, e^{i\theta_2}z_2, e^{i\theta_3}z_3, e^{i\theta_4}z_4).$$

As we already know its moment map (with $\boldsymbol{\lambda}$ as additive constant) is

$$\mu(z_1, z_2, z_3, z_4) = -(|z_1|^2, |z_2|^2, |z_3|^2, |z_4|^2) + (0, 0, a, b)$$

We can compute the dual of the differential of the inclusion map of N in T^4

$$\begin{aligned}i : N &\longrightarrow T^3 \\ (x, y) &\longmapsto (x, y, x, y).\end{aligned}$$

Its differential has matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

So the matrix of the dual of the differential, that we denote i^*, is the transpose

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

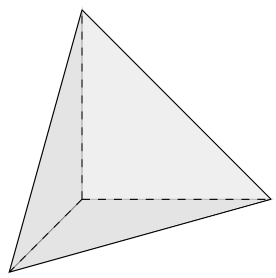
In coordinates $i^*(x_1, x_2, x_3, x_4) = (x_1 + x_3, x_2 + x_4)$. Finally we compute the zero-level set

$$(i^* \circ \mu)^{-1}(0) = \{(z_1, z_2, z_3, z_4) \mid |z_1|^2 + |z_3|^2 = a, |z_2|^2 + |z_4|^2 = b\} \cong S^3 \times S^3$$

. The reduced space, which is the sought manifold, is then

$$(i^* \circ \mu)^{-1}(0) \,/_{N} \cong S^3 \times S^3 \,/_{T^2} \cong S^2 \times S^2$$

Example. The isosceles triangle can be generalized to any dimension. For dimension 3 we can draw it



The generalized polytope can be described algebraically as:

$$\Delta = \{x_i \geq 0 \ \ \forall i = 1, ..., n; \Sigma_{i=1}^{n} x_i \leq a\},$$

with $v_i = -e_i \ \forall i, v_n + 1 = (1, ..., 1)$ and $\boldsymbol{\lambda} = (0, ..., 0, a)$. We have $n + 1$ facets in $\mathbb{R}^n$. If $\{\tilde{e}_i, i = 1, ..., n + 1\}$ is the canonical basis of $\mathbb{R}^{n+1}$ our map is:

$$\begin{aligned} \pi : \mathbb{R}^{n+1} &\longrightarrow \mathbb{R}^n \\ \tilde{e}_i &\longmapsto -e_i \ \ \forall i = 1, ..., n \\ \tilde{e}_{n+1} &\longmapsto (1, ..., 1) \end{aligned}$$

This map can be quotiented to a T^{n+1} map to T^n and its kernel is $N = Ker(\pi) = \{(e^{i\theta}, ..., e^{i\theta}) \mid \theta \in [0, 2\pi]\} \cong S^1$. Consider now $\mathbb{C}^{n+1}$ with symplectic form $\omega = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$ and standard hamiltonian action of T^{n+1}

$$(e^{i\theta_1}, ..., e^{i\theta_{n+1}}) \cdot (z_1, ..., z_{n+1}) = (e^{i\theta_1} z_1, ..., e^{i\theta_{n+1}} z_{n+1}).$$

As we already know its moment map (with $\boldsymbol{\lambda}$ as additive constant) is

$$\mu(z_1, ..., z_{n+1}) = -(|z_1|^2, ..., |z_{n+1}|^2) + (0, ..., 0, a).$$

We can compute the dual of the differential of the inclusion map of N in T^n,

$$\begin{aligned} i : N &\longrightarrow T^n \\ x &\longmapsto (x, ..., x). \end{aligned}$$

Its differential has matrix

$$\begin{bmatrix} 1 & \dots & 1 \end{bmatrix}.$$

So the matrix of the dual of the differential, that we denote i^*, is the transpose

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

In coordinates $i^*(x_1, ..., x_{n+1}) = x_1 + ... + x_{n+1}$. Finally we compute the zero-level set

$$(i^* \circ \mu)^{-1}(0) = \{(z_1, ..., z_{n+1}) \mid |z_1|^2 + ... + |z_{n+1}|^2 = a\} \cong S^{2n+1}.$$

The reduced space, which is the sought manifold, is then

$$(i^* \circ \mu)^{-1}(0) \big/ N \cong S^{2n+1} \big/ S^1 \cong \mathbb{CP}^n.$$

2.3 Injectivity of Delzant theorem

The goal of this subsection is to prove the injectivity in Delzant theorem.

Theorem 8. *Let M_1 and M_2 be two symplectic manifolds of dimension $2n$ and T^n a torus of dimension n acting on both manifolds by hamiltonian effective actions. Let μ_i be the corresponding moment maps. If $\mu_1(M_1) = \mu_2(M_2)$, there exist a symplectic diffeomorphism T^n-equivariant φ from M_1 to M_2 that makes commutative the following diagram.*

$$\begin{tikzcd} M_1 \arrow[r, "\varphi"] \arrow[d, "\mu_1"'] & M_2 \arrow[d, "\mu_2"] \\ \mu_1(M_1) \arrow[r, leftrightarrow, "Id"] & \mu_2(M_2) \end{tikzcd}$$

The proof of the theorem leans on the following lemma, which is a consequence of some Atiyah theorems.

Lemma 9. *Let (M, ω) be a symplectic manifold of dimension $2n$ and μ the moment map of an effective action of torus T^n on M. Then:*

- μ is the quotient map of torus action,

- for all $z \in \mu(M)$, the manifold $\mu^{-1}(z)$ is a torus of dimension equal to the face of $\mu(M)$ containing z and

- the stabilizer group of a point p in M is the connex group whose Lie algebra is the annihilator in $\mathfrak{g}$ of face containing $\mu(p)$.

Let x be a point of $\mu(M)$; its preimage is an stabilizer orbit. The stabilizer group of points in $\mu^{-1}(x)$ is a direct factor so this submanifold is of trivial normal bundle. By Weinstein, we know that the symplectic normal bundle of an isotropic submanifold caracterizes (up to symplectomorphism) it's neighbourhood. If F is any facet of $\mu(M)$, the interior of F being convex, μ is a trivial bundle of $\Omega_F = \mu^{-1}(\text{int } F)$ into int F and Ω_F is a deformation retract of $\mu^{-1}(x)$. It is then a symplectic submanifold of M with trivial normal bundle. A semilocal equivariant version in a neighbourhood of Ω_F of Darboux theorem shows the following lemma:

Lemma 10. *Let $B(\epsilon)$ be a ball of center 0 and radius ϵ in $\mathbb{C}$. Let K be an open convex set relatively compact on the interior of F, a face of Δ.*

We equip $(\mathbb{R}/\mathbb{Z})^i \times K \times B(\epsilon)^{n-i}$ with the symplectic form :

$$\sigma = \sum_{1 \leq j \leq i} d\alpha_j \wedge da_j + \sum_{i+1 \leq k \leq n} dx_k \wedge dy_k,$$

where α are the coordinates on $\mathbb{R}/\mathbb{Z}$, a coordinates on F and $z = x + iy$ coordinates in $B(\epsilon)$.

Then there exists an symplectic isomorphism of a neighbourhood of reciprocal image of K on $(\mathbb{R}/\mathbb{Z})^i \times K \times B(\epsilon)^{n-i}$ transforming the action of T^n in the action of $(\mathbb{R}/\mathbb{Z})^n$ defined as

$$\begin{aligned}&(\theta_1, ..., \theta_n) \cdot (\alpha_1, ..., \alpha_i; a_1, ..., a_i; z_{i+1}, ..., z_n) = \\ &(\alpha_1 + \theta_1, ..., \alpha_i + \theta_i; a_1, ..., a_i; e^{2i\pi\theta i+1} z_{i+1}, ..., e^{2i\pi\theta_n} z_n).\end{aligned}$$

Its moment map being:

$$J(\alpha_1, ..., \alpha_i; a_1, ..., a_i; z_{i+1}, ..., z_n) = \mu + (a_1, ..., a_i; |z_{i+1}|^2, ..., |z_n|^2).$$

Keep the same notations and denoting $S^{2n-2i-1}(\epsilon)$ the boundary of ball $B(\epsilon)^{n-i}$. Then we have:

Lemma 11. *Let φ be a diffeomorphism of $(\mathbb{R}/\mathbb{Z})^i \times K \times S^{2n-2i-1}(\epsilon)$ that preserves the moment map J and commuting with torus action of $(\mathbb{R}/\mathbb{Z})^n$, then φ extends to $(\mathbb{R}/\mathbb{Z})^i \times K \times B(\epsilon)^{n-i}$ as a diffeomorphism verifying the same property.*

Proof. (Theorem 10)

Only lemmas 12 and 13 are directly used in the theorem proof. To construct a diffeomorphism of M_1 to M_2 equivariant for torus action and compatible with moment maps, we cut the polytope $\Delta = \mu_i(M_i)$ in a reunion of open sets $\Omega_{i,j}$, where $0 \geq i \geq n$ and j describes the set of faces of dimension i of Δ, verifying:

- $\Omega_{i,j} \cap j$ is relatively compact in the interior of face j, and in $\Omega_{i,j}$ we have on M_1 and M_2 action-angle coordinates as in lemma 12.
- The union of $\Omega_{i,j}$ is Δ and $\Omega_{i,j} \cap \Omega_{i,k} = \emptyset$ if j different from k.
- $\Omega_{i,j} \cap (\bigcup_{a>i} \Omega_{a,b})$ contains $\mu((\mathbb{R} \times \mathbb{Z})^i \times K \times S(\epsilon)^{2n-2i-1})$ with notations of previous lemma.

Lemma 12 shows that $\mu_1^{-1}(\Omega_{i,j})$ is diffeomorphic to $\mu_2^{-1}(\Omega_{i,j})$ by an equivariant diffeomorphism transforming μ_1 in μ_2. Lemma 13 shows that giving this diffeomorphisms on the reunion , for $i > a$, of $\Omega_{i,j}$ allows to construct a diffeomorphism with same properties on the reunion of $\Omega_{i,j}$ for $i \geq a-1$: It is enough to restrict our known diffeomorphism to the complementary of the reunion of $\Omega_{a-1,j}$ and extend to each $\Omega_{a-1,j}$ using lemma 13.

□

3 Link with integrable systems

The goal of this section is to understand the link between toric manifolds and integrable systems. After defining them and looking at a few examples, we prove the main result: Arnold-Liouville theorem. Finally, a new alternative proof for the first statement is explained.

3.1 Integrable systems

Definition. *An* ***integrable system*** *on a symplectic manifold* (M^{2n}, ω) *is a set of n functions* $f_1, ..., f_n$ *generically functionally independent (i.e.* $df_1 \wedge ... \wedge df_n \neq 0$ *on a dense set) and* $\omega(X_{f_i}, X_{f_j}) = 0, \forall i, j$.

In fact, the operator $\omega(X_{f_i}, X_{f_j})$ is a particular case of a more general family of operators.

Definition. *Given a differentiable manifold* M*, an operation*

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M),$$

is a ***Poisson bracket*** *if it satisfies :*

1. *Leibniz rule:* $\{f.g, h\} = f\{g, h\} + g\{f, h\}$,
2. *is skew-symmetric:* $\{f, g\} = -\{g, f\}$,
3. *Jacobi identity:* $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$.

Let's see what are the consequences of the condition on the n functions. If we denote $F = (f_1, ..., f_n)$, lets study its level sets. A first observation that can be done is that the dimension of $F^{-1}(p)$ for any $p \in M$ is n. This is because the condition of the n functions being generically functionally independent implies that the rank of dF is n. Using the regular value theorem we obtain that $F^{-1}(p)$ is a manifold and

$$\begin{aligned} \dim\ F^{-1}(p) &= \dim\ (M) - \dim\ (dF) \\ &= 2n - n \\ &= n. \end{aligned}$$

The other condition can be written as:

$$\begin{aligned} 0 &= \{f_i, f_j\} \\ &= \omega(X_{f_i}, X_{f_j}) \\ &= i_{X_{f_i}} \omega(X_{f_j}) \\ &= df_i(X_{f_j}) \\ &= X_{f_j}(f_i)\ \ \forall i, j. \end{aligned}$$

We have that vector fields $X_{f_1}, ..., X_{f_n}$ are tangent to $F^{-1}(p)$. We can write then $T(F^{-1}(p))_p =< X_{f_1}, ..., X_{f_n} >_p$. As $\omega(X_{f_i}, X_{f_j}) = 0\ \forall i, j$ we deduce that ω vanishes in $L = F^{-1}(p)$. This leads to two interesting definitions.

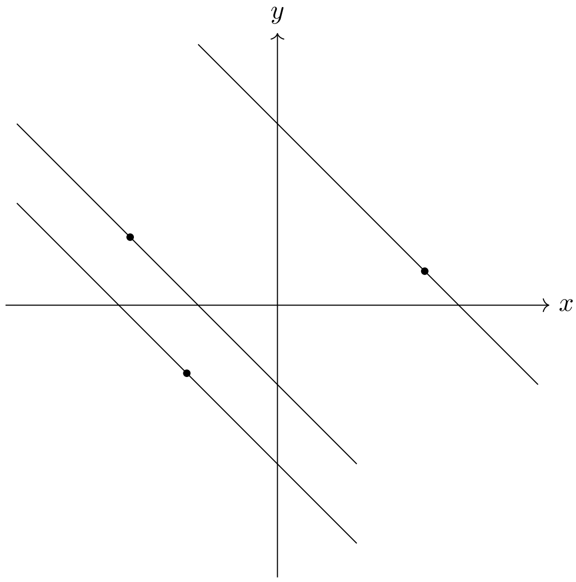
Definition. *A submanifold where the restriction of the symplectic form vanishes is called an* ***isotropic manifold****.*

Definition. *The particular case where the dimension of this submanifold is 1/2 dim(M) is called a* ***Lagrangian submanifold****. All the lagrangian submanifolds (the level sets) form a Lagrangian fibration.*

Example. Let's see a first very simple example of integrable system. Consider $(\mathbb{R}^2, \omega = dx \wedge dy)$ and $F = x + y$. Let's compute its associated vector field. A general vector field is of the form $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ so:

$$\begin{aligned} i_X\omega = dF &\implies \omega(X, \cdot) = d(x + y) \\ &\iff dx \wedge dy(X, \cdot) = dx + dy \\ &\iff ady - bdx = dx + dy \\ &\iff X = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}. \end{aligned}$$

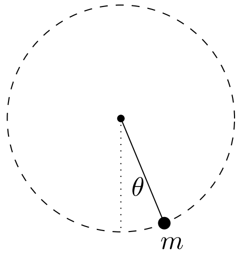
The lagrangian submanifolds are then generated by a point and the subspace $V =< (1, -1) >$. The fibration of $\mathbb{R}^2$ obtained by those lines, and a few fibres look like this.



It is clear that ω vanishes in these submanifolds: we only have the vector field X there, and we have that

$$\begin{aligned} \omega(X, X) &= dx \wedge dy(X, X) \\ &= dx(X)dy(X) - dy(X)dx(X) \\ &= -1 + 1 \\ &= 0. \end{aligned}$$

Example. An example of mechanical system which is also an integrable system is the simple pendulum. The manifold where the pendulum moves is S^1 and we can look its cotangent bundle as $T^*S^1 \cong [0, 2\pi]_\sim \times \mathbb{R}$ knowing that points at $(0, \xi)$ are identified with $(2\pi, \xi)$. We take the coordinates (θ, ξ) with θ the oriented angle between the rod and the vertical direction and ξ the velocity induced by θ.



To simplify, let's take the example where the mass and the length of the rod are 1. As we know, the hamiltonian function for this system is

$$H(\theta, \xi) = \frac{\xi^2}{2} + 1 - \cos\theta.$$

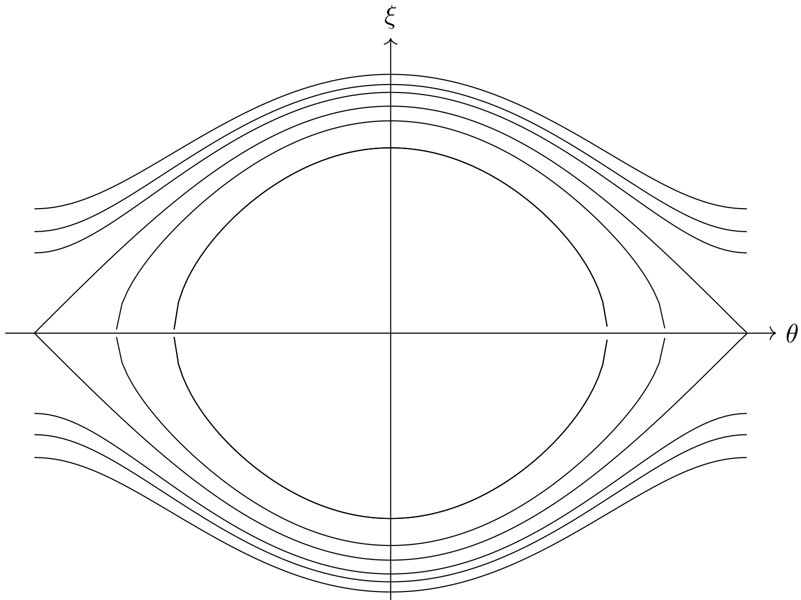
Let's compute the vector field associated to it. Since $dH = \xi d\xi + sin\theta d\theta$ we want:

$$\begin{aligned} i_X\omega = dH &\implies d\theta \wedge d\xi(a\frac{\partial}{\partial\theta} + b\frac{\partial}{\partial\xi}, \cdot) = \xi d\xi + sin\theta d\theta \\ &\implies ad\xi - bd\theta = \xi d\xi + sin\theta d\theta. \end{aligned}$$

We deduce that

$$X_H = \xi\frac{\partial}{\partial\theta} - \sin\theta\frac{\partial}{\partial\xi}.$$

Some of the lagrangian fibres in the plane (θ, ξ) look like this.



As we can see, the lagrangian submanifolds are diffeomorphic to S^1, a 1-dimension torus. This is true only for regular values of the hamiltonian, of course. If we consider the value 0, which is a singular point, the preimage is not an S^1 but a point.

Example (The 2-body problem [9]). The two-body problem is the system consisting of two bodies with masses m_1, m_2 and positions $q_1, q_2 \in \mathbb{R}^3$ moving under gravitational attraction. The equations of motion are deduced from the Newton laws:

$$m_i \ddot{q}_i = G m_1 m_2 \frac{q_j - q_i}{\|q_2 - q_1\|^3}, \quad i, j = 1, 2, \quad i \neq j,$$

where G is the gravitational constant. We can introduce the negative gravitational potential

$$U := m_1 m_2 \frac{G}{\|q_2 - q_1\|}.$$

So the equations are written

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i}, \quad i, j = 1, 2, \quad i \neq j.$$

We want to describe the equations of motion using the hamiltonian formalism. The hamiltonian function corresponds to the energ of the system and is obtained as the sum of kinetic and potential energy:

$$H(q_1, q_2, p_1, p_2) := E_{kin} - U = \frac{\|p_1\|^2}{2m_1} + \frac{\|p_2\|^2}{2m_2} - U,$$

where $p_i = m_i \dot{q}_i$ are the linear momenta. The evolution of the system is given by the hamiltonian equations

$$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i}. \end{cases}$$

And in our case $-\frac{\partial H}{\partial q_i} = \mathcal{G} m_1 m_2 \frac{q_j - q_i}{\|q_2 - q_1\|^3}$ Here the underlying symplectic structure is the canonical one for the cotangent space

$$\omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2.$$

From the equations of motion we observe that that

$$\dot{p}_1 + \dot{p}_2 = 0.$$

This means the quantity $p_1 + p_2$ is preserved. The center of mass moves with constant velocity and only the relative position $q := q_2 - q_1$ of the two bodies has to be solved from the equations. Let's introduce the following change of coordinates

$$\begin{aligned} g &= \nu_1 q_1 + \nu_2 q_2, & G &= p_1 + p_2, \\ q &= q_2 - q_1, & Q &= -\nu_2 p_1 + \nu_1 p_2, \end{aligned}$$

where $\nu_i = m_i/(m_1 + m_2)$. Note that g is the center of mass and G is the total linear momentum. The coordinate q is the relative position of the second body with respect to the first one. The other "momentum" coordinate Q is chosen such that the change of coordinates preserves the symplectic form (the change is "canonical"). This coordinates are called Jacobi coordinates.

In these coordinates the hamiltonian is

$$H(g, q, G, Q) = \frac{\|G\|^2}{2\nu} + \frac{\|Q\|^2}{2M} - \mathcal{G}\frac{m_1 m_2}{\|q\|}$$

where $\nu = m_1 + m_2$ and $M = m_1 m_2/(m_1 + m_2)$.

Writing down the hamiltonian equations explicitly

$$\begin{aligned} \dot{g} &= \frac{\partial H}{\partial G} = \frac{G}{\nu}, & \dot{G} &= -\frac{\partial H}{\partial g} = 0, \\ \dot{q} &= \frac{\partial H}{\partial Q} = \frac{Q}{M}, & \dot{Q} &= -\frac{\partial H}{\partial q} = -\frac{m_1 m_2 w}{\|q\|^3}, \end{aligned}$$

we see that total linear momentum G is preserved and that the center of mass moves with constant velocity $\frac{G}{\nu}$.

Physically this means that we are viewing the system as one body with coordinates q under the influence of the central force field of a body with mass M. Now we are facing a hamiltonian system on $(\mathbb{R}^3 \backslash \{0\}) \times \mathbb{R}^3$ with hamiltonian function

$$H(q, Q) = \frac{\|Q\|^2}{2M} - \mathcal{G}\frac{m_1 m_2}{\|q\|}.$$

This is known as the **Kepler problem**.

First link with toric manifolds. The first observation that can be done relating toric manifolds to integrable systems is the following. Given a toric manifold $(M^{2n}, \omega, T^n, \mu)$, an integral system is obtained by taking as functions the n components μ_i of the moment map μ.

As a first step, since we have that the components $X_i^{\#}$ of the fundamental vector field are independent, we have that $i_{X_i^{\#}}\omega$ are independent, which are exactly the $d\mu_1, ..., d\mu_n$ because each μ_i is hamiltonian function for $X_i^{\#}$. Having $d\mu_1, ..., d\mu_n$ independent is equivalent to having the functions $\mu_1, ..., \mu_n$ funtionally independent. The fact that the moment is invariant is written, as we saw in the case where $G = T^n$, the following way (lets write X_i for $X_i^{\#}$ to simplify)

$$i_{X_j} d\mu_i = 0 \ \ \forall i, j.$$

We want to prove that $\{\mu_i, \mu_j\} = 0 \ \forall i, j$ to show $\mu_1, ..., \mu_n$ is an integrable system. But note that

$$\begin{aligned} \{\mu_i, \mu_j\} &= \omega(X_i, X_j) \\ &= i_{X_i}\omega(X_j) \\ &= i_{X_j} i_{X_i}\omega \\ &= i_{X_j} d\mu_i \\ &= 0. \end{aligned}$$

We can conclude that $\mu_1, ..., \mu_n$ determines an integrable system. A toric manifold gives us always an integrable system, and the logical question is: does an integrable system always comes from toric manifold? This is not true in general and there are example such as the spherical pendulum.

Without going into details, the spherical pendulum is a pendulum with configuration space S^2. We have two degrees of freedom and using spherical coordinates (θ, φ), where θ is the polar angle and φ the azimuthal angle, the hamiltonian H takes the form

$$H(\varphi, \theta, p_\varphi, p_\theta) = \frac{1}{2}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right) + \cos\theta.$$

where $p_\varphi = \sin^2\theta\,\dot{\varphi}$ and $p_\theta = \dot{\theta}$ are the momenta. Since H is independent of φ, the conjugate momentum p_φ is conserved and the system is integrable. But in this case it is shown if [6] that there is a singularity in the 'interior' of the image of our candidate to moment map. This is not possible since convexity theorem told us that singularities have to be in the edges of the polytope for toric manifolds. Hence, it cannot be globally a toric manifold.

3.2 Arnold-Liouville theorem

Even if in general an integrable system is not a toric manifold, Arnold-Liouville theorem tells us that in the neighbourhood of a regular value an integrable system looks like a toric manifold.

Given a smooth function H on a symplectic manifold M, we define the vector of skew-symmetric gradient $sgrad(H)$ for this function using the following identity:

$$\omega(v, sgrad(H)) = v(H),$$

where v is an arbitrary tangent vector. In local coordinates $x_1, ...x_{2n}$ it has the following expression:

$$(\mathrm{sgrad}H)^i = \sum \omega_{ij}\frac{\partial H}{\partial x_j},$$

where ω_{ij} are components of the inverse matrix of Ω, the matrix of ω. The vector field sgrad H is in fact the $-X_H$, where X_H is the hamiltonian vector field.

Definition. *Given an integrable system, the decomposition of the manifold M^{2n} into connected components of common level surfaces of the integrals $f_1, ..., f_n$ is called the Liouville foliation corresponding to the integrable system $v = sgrad\ H$. By convention, $H = f_1$.*

Theorem 12. *Let (M^{2n}, ω) be a symplectic manifold and $F = (f_1, ..., f_n)$ an integrable system. Let p be a regular point (i.e. $df_1 \wedge ... \wedge df_n(p) \neq 0$). Note $F(p) = c$ and $F^{-1}(c) = L_c$ (fibre associated to c). Assuming L_c is compact and connected, then*

1. *$L_c \cong T^n$*

2. *the Liouville foliation is trivial in some neighbourhood of the Liouville torus, that is, a neighbourhood U of the torus L_c is the direct product of T^n and the disc D^n.*

3. *In a neighbourhood of L_c, $U(L_c)$, there exist coordinates of the form $(\theta_1, ..., \theta_n, p_1, ..., p_n)$ and ω is written $\omega = \sum dp_i \wedge d\theta_i$. F only depends of $p_1, ..., p_n$.*

Proof. We have that the tangent space to the manifold L_c is $T(L_c) = \langle X_{f_1}, ..., X_{f_n} \rangle$. Let's consider now for each of the vector fields X_{f_i} its associated flow $\phi^{t_i}_{X_{f_i}}$, which is defined $\forall t$ because L_c is compact. We can now consider the following action, defined with the flow of each of the vector fields:

$$\begin{aligned} \phi : \mathbb{R}^n \times M &\longrightarrow M \\ ((t_1, ..., t_n), p) &\longmapsto \phi^{t_1}_{X_{f_1}} \circ ... \circ \phi^{t_n}_{X_{f_n}}. \end{aligned}$$

This action is well defined because the condition of $\{f_i, f_j\} = 0$ implies $[X_{f_i}, X_{f_j}] = 0$ which is equivalent to the fact that the flows commute. They are all complete as well.

Lemma 13. *If the submanifold L_c is connected, then it is an orbit of the $\mathbb{R}^n$ action.*

Proof. Consider the image of $\mathbb{R}^n$ in M under the action, for a given point $p \in L_c$ this is

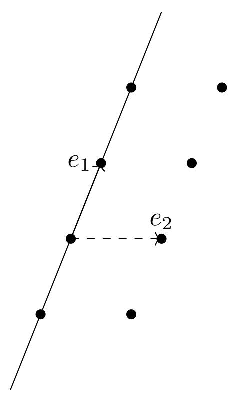
$$A_p : (t_1, ..., t_n) \longrightarrow \phi(t_1, ..., t_n)(p).$$

Since the fields are independent, this mapping is an immersion i.e. $rank(dA_p) = n$. It is a local diffeomorphism onto the image. Thus, the image or $\mathbb{R}^n$ is an open in L_c. If we assume that L_c is not a single orbit of $\mathbb{R}^n$, the it is the union of at least two. But then since each is open, L_c is disconnected which is a contradiction. □

Lemma 14. *An orbit $O(p)$ of maximal dimension of the action of $\mathbb{R}^n$ is the quotient space of $\mathbb{R}^n$ with respect with some lattice $\mathbb{Z}^k$. If $O(p)$ is compact, then $k = n$ and $O(p)$ is diffeomorphic to the n-dimensional torus.*

Proof. Note that since we assume L_c compact, $A_x : R^n \to L_c$ cannot be a injective. Hence, H_x the stationary group is non trivial. Every orbit $O(p)$ of a smooth action of $\mathbb{R}^n$ is a quotient space of $\mathbb{R}^n$ with respect to the stationary group H_p. This group is discrete since the mapping A_p is a local diffeomorphism (as it is locally injective, so points that fix p are isolated). A discrete subgroup has no accumulation points. In particular, in any bounded set there is only a finite number of elements of H_p. Lets show by induction that H_p is a lattice Z^k.

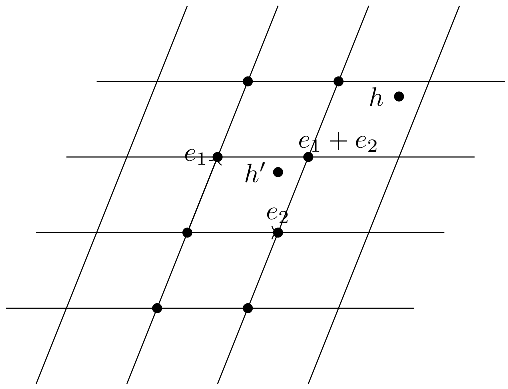
Suppose $n = 1$.



Take a non-zero e_1 of H_p on the line $\mathbb{R}^1$ which is the nearest to zero. All the other elements of H_p have to be multiples of e_1. If e is not, then for some k we have

$$ke_1 < e < (k+1)e_1.$$

But then the element $e - ke_1$ would be closer to zero than e_1. Consequently, H_p is the lattice generated by e_1.



Suppose $n = 2$. Take e_1 the nearest element to zero in $\mathbb{R}^2$ and consider the straight line $l(e_1)$ generated by it. All the elements of H_p there are multiples of e_1. There are two possibilities now. If all the elements of H_p lie in $l(e_1)$ the proof is complete. Otherwise, take e_2 a non-zero vector nearest to the line $l(e_1)$. We want to see now that all the elements of H_p are linear combinations of e_1 and e_2. Assume the contrary and take $h \in H_p$ which is not decomposed into e_1 and e_2 with integer coefficients. We have then that h is in one of the parallelograms generated by e_1 and e_2 (and is not a vertex). Moving h with integer combinations of e_1 and e_2 it is clear that we can obtain an element h' closer to $l(e_1)$ than e_2 which is a contradiction.

Continuing with this argument by induction, we obtain a basis $e_1, ..., e_k$ of the subgroup H_p such that its elements are a unique linear combination of the basis with integer coefficients.

If $k < n$, then the quotient space $\mathbb{R}^n/\mathbb{Z}^k$ is a cylinder. In particular, the orbit is compact for $n = k$ only, and hence $O(p)$ is diffeomorphic to the torus T^n.

$\square$

This proves 1.

We want to prove now that in a neighbourhood of L_c, U is a direct product of T^n by a disc D^n. This follows from a more general theorem. Suppose $f : M \longrightarrow N$ a smooth map between manifolds M and N. If $y \in N$ is a regular value, then there exists a neighbourhood D of a point y such that the preimage $f^{-1}(D)$ is diffeomorphic to the direct product $D \times f^{-1}(y)$. Moreover, the direct product structure is compatible with the mapping f in the sense that $f : D \times f^{-1}(y) \to D$ is just the natural projection. It follows from this that each set $f^{-1}(z)$ with $z \in D$ is diffeomorphic to $f^{-1}(y)$. This proves 2.

Now we want to construct the action-angle variables. Consider the neighbourhood $U(L_c) = T^n \times D^n$. Choose a point x on each tori T depending smoothly on the torus. Consider T as the quotient we saw $\mathbb{R}^n/H_x$ and fix a basis $e_1, ..., e_n$in the lattice H_x. The lattice will smoothly depend on x. Indeed, the coordinates of the basis vector $e_i = (t_1, ..., t_n)$ are the solutions of the equation $\phi(t_1, ..., t_n) = x$, where x is regarded as a parameter. Using implicit function theorem, the solutions depend smoothly on x. We can use the theorem because $\frac{\partial}{\partial t_j}\phi(t)x = \text{sgrad} f_j(\phi(t)x)$, and the vector fields $\text{sgrad} f_j$ are linearly independent.

Let us now define certain angle coordinates $(\psi_1, ..., \psi_n)$ on the torus L_c in the following way. If $y = \phi(a)x$ where $a = a_1 e_1 + ... + a_n e_n \in \mathbb{R}^n$ then $\psi_1(y) = 2\pi a_1(\text{mod} 2\pi), ..., \psi_n(y) = 2\pi a_n(\text{mod} 2\pi)$. This coordinate system satisfies this property: the vector fields $\frac{\partial}{\partial \psi_1}, ..., \frac{\partial}{\partial \psi_n}$ and $\text{sgrad} f_1, ..., \text{sgrad} f_n$ are connected by a linear change with constant coefficients. That is

$$\frac{\partial}{\partial \psi_i} = \sum c_{ik} \ \text{sgrad} f_k \ \ \forall i.$$

The form ω in coordinates $(f_1, ..., f_n, \psi_1, ..., \psi_n)$ is written:

$$\omega = \sum_{i,j} \tilde{c}_{ij} df_i \wedge d\psi_{i,f} + \sum_{i,j} b_{ij} df_i \wedge df_j.$$

Notice that the terms of form $a_{ij} d\psi_i \wedge d\psi_j$ are absent since the tori are Lagrangian. In fact, the coefficients $\tilde{c}_{ij}$ coincide with the coefficients c_{ij}. Indeed,

$$\begin{aligned}\tilde{c}_{ij} &= \omega(\frac{\partial}{\partial f_i}, \frac{\partial}{\partial \psi_j}) \\ &= \omega(\frac{\partial}{\partial f_i}, \sum c_{kj} \text{sgrad} f_k) \\ &= \sum c_{kj} \frac{\partial f_k}{\partial f_i} = c_{ij}(f_1, ..., f_n).\end{aligned}$$

Note that because sgrad $f_k = -X_{f_k}$, we have that $\omega(X, \text{sgrad } f_k) = X(f_k)$, instead of the property that we had with the usual hamiltonian vector field X_{f_k} where $\omega(X, X_{f_k}) = -\omega(X_{f_k}, X) = -X(f_k)$.

We show now that the functions b_{ij} are independent on $(\psi_1, ..., \psi_n)$. Since ω is closed(i.e. $d\omega = 0$)), we get

$$\frac{\partial b_{ij}}{\partial \psi_k} = \frac{\partial c_{kj}}{\partial f_i} - \frac{\partial c_{ki}}{\partial f_j}.$$

Functions b_{ij} are 2π-periodic (as functions on the torus) on ψ_k but as we see the derivative does not depend of ψ_k. As b_{ij} is linear with respect ψ_k and is also 2π-periodic on it, it follows that the function does not depend on ψ_k.

If we write the form the following way $\omega = (\sum c_{ij} df_j) \wedge d\psi_i + \sum b_{ij} df_i \wedge df_j = \sum \omega_i \wedge d\psi_i + \beta$, where $\omega_i = \sum c_{ij} df_j$ and $\beta = \sum b_{ij} df_i \wedge df_j$ are forms on the disc D^n (which is not dependent on $(\psi_1, ..., \psi_n)$. As ω is closed, so are ω_i and β.

Lemma 15. *In the neighbourhood $U(L_c)$, the form ω is exact, i.e. $\omega = d\alpha$.*

Proof. It is a particular case of the folllowing general statement. Let Y be a submanifold in X and there exists a mapping $f : X \longrightarrow Y$ homotopic to the identity mapping $id : X \longrightarrow X$. Then a closed differential form is exact on X if and only if its restriction to Y is exact. In our case, when X is a neighbourhood of a Liouville torus, and Y is this Liouville torus, we even have a stronger condition $\omega_{L_c} = 0$ because L_c is Lagrangian. Therefore, ω is exact.

This can be shown also with direct calculation. Since ω_i and β are closed on the disk, and a disk is contractible, by Poincar lemma they are exact and therefore there exist functions s_i and a 1-form ξ such that $ds_i = \omega_i$ and $d\xi = \beta$. Finally, let $\alpha = \sum s_i d\psi_i + \xi$ and $d\alpha = \omega$.

$\square$

Consider now the functions $s_1 = s_1(f_1, ..., f_n), ..., s_n = s_n(f_1, .., f_n)$. They are independent. Indeed, from the formula $\omega = \sum ds_i \wedge d\psi_i + \beta$ it follows that the matrix Ω of ω has the form

$$\begin{bmatrix} 0 & ... & 0 & \\ & ... & & c_{ij} \\ 0 & ... & 0 & \\ & -c_{ij} & & b_i j \end{bmatrix},$$

where $c_{ij} = \frac{\partial s_i}{\partial f_j}$. Therefore, $\det\Omega = (\det C)^2$ and $\det C \neq 0$, where C is the Jacobi matrix of the transformation $s_1, ..., s_n$ and we can consider now a new system of independent coordinates $(s_1, ..., s_n, \psi_1, ..., \psi_n)$.

Next, we represent ξ in the form $\xi = \sum g_i ds_i$ and make one more change $\varphi_i = \psi_i - g_i(s_1, ..., s_n)$. Geometrically this means that we change the initial points of reference for the angle coordinates on the Liouville tori. The level lines and even basis vectors fields are not changed.

Finally, let us show that the constructed system of action-angle variables $(s_1, ..., s_n, \varphi_1, ..., \varphi_n)$ is canonical. We have

$$\begin{aligned} \sum ds_i \wedge d\varphi_i &= \sum ds_i \wedge d(\psi_i - g_i(s_1, ..., s_n)) \\ &= \sum ds_i \wedge d\psi_i + \sum dg_i(s_1, ..., s_n) \wedge ds_i \\ &= \sum ds_i \wedge d\psi_i + d\xi \\ &= \sum ds_i \wedge d\psi_i + \beta \\ &= \omega. \end{aligned}$$

Hence, the action-angle variables have been constructed. It remains to show that the flow sgrad H straightens on Liouville tori in coordinates $(s_1, ..., s_n, \varphi_1, ..., \varphi_n)$. Indeed we have that sgrad $s_i = \frac{\partial}{\partial \varphi_i}$ and so $\frac{\partial H}{\partial \varphi_i} =$ sgrad $s_i(H) = \{s_i(f_1, ..., f_n), H\} = 0$. That means H is a function of $s_1, ..., s_n$ only. Consequently

$$v = \text{sgrad } H = \sum_i \frac{\partial H}{\partial s_i} \text{sgrad } s_i = \sum_i \frac{\partial H}{\partial s_i} \partial/\partial \varphi_i,$$

moreover, the coefficients $\frac{\partial H}{\partial s_i}$ depend only on the action variables $s_1, ..., s_n$, i.e. are constant on Liouville tori. This completes the proof of Liouville theorem. $\square$

We can now interpret this theorem with respect to Delzant theorem. Arnold-Liouville theorem is telling us that every integrable system looks locally as the pre-image of an open neighbourhood of a Delzant polytope, i.e. it looks locally like a toric manifold.

3.3 New proof of first statement in Arnold Liouville theorem

In this final subsection, we elaborate a new alternative proof of the first statement of Arnold-Liouville theorem, that is $L_c \cong T^n$. This work was done under the Severo Ochoa: Introduction to research 2017 program at ICMAT, developped and tutored by Daniel Peralta-Salas. It is based on the method used to prove Tischler theorem which we will now explain from [2] and [10]. For the proof we need the following lemma:

Lemma 16 (Ehresmann's lemma [8])**.** *A smooth mapping $f : M \longrightarrow N$ between smooth manifolds M and N such that :*

1. *f is a surjective submersion, and*
2. *f is a proper map*

is a locally trivial fibration.

Theorem 17 (Tischler theorem)**.** *Let M be a compact manifold such that there exists a nonsingular 1-form and closed ω. Then M admits a fibration over S^1.*

Proof. Let $[\nu_1], ..., [\nu_n]$ be a basis of $H^1(M, \mathbb{Z}) \in H^1(M, \mathbb{R})$. Let p be the first Betti number, by the correspondence with the set of homotopy classes of maps $f : M \to S^1$ we write:

$$[\nu_i] = f_i^*[\theta], \ \ 1 \leq i \leq p,$$

for some smooth maps f_i and θ being the standard volume form on S^1. The forms are then $\nu_i = f_i^*(\theta)$. Our closed one-form ω is then such that $[\omega] = \sum_{i=1}^n c_i[\nu_i]$. At the form level that is

$$\omega = \sum_{i=1}^n c_i \nu_i + dF,$$

for certain function F and real numbers c_i. We can assume that $dF = 0$ since we can take $\nu_1 = (f_1 + \pi \circ F)^*(d\theta)$, where π is a projection to S^1. Since M is compact, F has a compact image and the projection to S^1 can be done. Since $\mathbb{Q}$ are dense in $\mathbb{R}$, we can choose rational number q_i such that the form

$$\omega' = \sum_{i=1}^n q_i \nu_i$$

is as close as we want to ω in the desired C^∞ topology. In particular, we can take them such that ω' is also nonsingular. For a suitable integer N, $N\omega'$ can then be written:

$$\begin{aligned} N\omega' &= \sum_{i=1}^n n_i f_i^*(d\theta) \\ &= d(\sum_{i=1}^n p_i f_i), \end{aligned}$$

with $n_i \in \mathbb{Z}\ \forall i$. The function $\Theta = \sum_{i=1}^n p_i f_i$ quotients to $\tilde{\Theta} : M \to S^1$. As we took the coefficients such that $N\omega'$ stills nonsingular we have that $\tilde{\Theta}$ is a surjective submersion. M is compact so $\tilde{\Theta}$ is proper and by Ehresmann's lemma it is a fiber map which proves the theorem. □

Back to integrable systems. Let (M^{2n}, ω) be a symplectic manifold and $f = (f_1, ..., f_n)$ an integrable system. Take p a regular point and $c = f(p)$. Since it has dimension n denote L^n any connex component of $f^{-1}(c)$(or all of it if assumed connected) and assume it compact, just as Arnold-Liouville hypotheses.

Write X_i the hamiltonian vector asociated to f_i. Recall that vector fields $X_1, ..., X_n$ are tangent to L^n for all $p \in L^n$. We can write then $T(L^n)_p =< X_{f_1}, ..., X_{f_n} >_p$. Take now in $\mathbb{R}^n$ the canonical basis of vector fields $\{\partial_i = \frac{\partial}{\partial x_i}\}_{i=1}^n$. We can consider its pullbacks by f, $S_i := f^*(\partial_i)$, which are vector fields in M. They satisfy:

$$S_i(f_j) = \delta_{ij}.$$

They are determined by this condition modulo T_pL^n.

Lemma 18. *Let $j : L^n \longrightarrow M^{2n}$ be the inclusion of the regular level set L^n into M. Define the one-forms $\alpha_i = i_{S_i}\omega$ then the one-forms $\beta_i = j^*\alpha_i$ are closed.*

Proof. By definition of S_i, we have $S_i(f_j) = \delta ij$. Then $\forall i, j,$:

$$\begin{aligned}
\alpha_i(X_j) &= \omega(S_i, X_j) \\
&= -\omega(X_j, S_i) \\
&= -i_{X_j}\omega(S_i) \\
&= -df_j(S_i) \\
&= -S_i(f_j) \\
&= -\delta_{ij}.
\end{aligned}$$

To prove that β_i is closed, we just have to check that $d\alpha_i(X, Y) = 0 \ \forall X, Y \in \Gamma(L^n)$. We know that $T_pL^n = \langle X_1, ..., X_n\rangle_p$ so we just have to check it for any pair of elements in the basis.

We'll use the invariant formula of the exterior derivative for a one form β:

$$d\beta(X, Y) = Y(\beta(X)) - X(\beta(Y)) - \beta([X, Y]).$$

Applying it in our case we have

$$\begin{aligned}
d\alpha_i(X_j, X_k) &= X_k(\alpha_i(X_j)) - X_j(\alpha_i(X_k)) - \alpha_i([X_j, Y_k]) \\
&= X_k(-\delta_{ij}) - X_j(-\delta_{ik}) - \alpha_i(0) \\
&= 0.
\end{aligned}$$

We conclude that $d(j^*\alpha_i) = d\beta_i = 0$ and so our forms β_i are closed in L^n.

□

Lemma 19. *The one-forms $\beta_1, ..., \beta_n$ are linearly independent and non degenerate in all points of L^n.*

Proof. As seen in the previous lemma, we have that $\beta_i(X_j) = -\delta_{ij}$. We deduce that $\beta_i = X_i{}^*$.

Since $X_1, ..., X_n$ are basis of the tangent space at every point in L^n, we have that $\beta_1, ..., \beta_n$ form a basis of the cotangent space at every point in L^n. In particular all β_i are independent. This implies that $\beta_1 \wedge ... \wedge \beta_n$ is a volume form and so each of the forms is non degenerate. □

Lemma 20. *As cohomology classes in $H^1_{DR}(L^n, \mathbb{R})$, $\{[\beta_i]\}_{i=1}^n$ are all different.*

Proof. Suppose we have β_i and β_j with $i \neq j$ such that $[\beta_i] = [\beta_j]$. Then there exists $f \in C^\infty(L^n)$ such that

$$\beta_i = \beta_j + df.$$

Recall now that having β_i all linearly independent is equivalent to saying that $\beta_1 \wedge ... \wedge \beta_n$ is a volume form of L^n. But since we have the last equation, then

$$\beta_i \wedge \beta_j = \beta_i \wedge (\beta_i + df) = \beta_i \wedge df.$$

By the properties of the wedge product and the fact that β_i is closed, we have that:

$$\begin{aligned} \beta_i \wedge df &= d\beta_i \wedge f - d(\beta_i \wedge f) \\ &= 0 - f d\beta_i \\ &= 0. \end{aligned}$$

It is a contradiction because then $\beta_1 \wedge ... \wedge \beta_n$ is not a volume form. $\square$

Theorem 21 (1. in Arnold-Liouville theorem)**.** *We have $L^n \cong T^n$.*

Proof. By the canonical correspondence between $H^1(L^n, \mathbb{Z}$ and the set of homotopy classes of maps $f : L^n \longrightarrow S^1$, we can choose a base $[\nu_i]$ of $H^1(L^n, \mathbb{R})$ such that $\forall i$ there exists a submersion $g_i : L^n \to S^1$ with $\nu_i = g_i^*(d\theta)$, where θ is the standard angle on S^1.

With this basis, we have that our independent β_i can be written as:

$$\beta_i = \sum_{j=1}^n a_{ij}\nu_i + dF_i, \ \ \forall i.$$

Using the argument on Tishcler theorem proof, we can choose $q_{ij} \in \mathbb{Q}$ $\forall i, j$ and obtain non degenerate forms $\tilde{\beta}_i = \sum_{j=1}^n q_{ij}\nu_i + dF_i$. Taking suitable $N_i \in \mathbb{Z}$ we obtain forms $\beta_i' = N_i\tilde{\beta}_i$ such that

$$\beta_i' = \sum k_{ij}\nu_i + dH_i,$$

where $k_{ij} = N_i q_{ij} \in \mathbb{Z}$ and $H_i = N_i F_i \in C^\infty(L^n)$.

Without loss of generality we can assume $dH_i = 0$. Indeed, the image $H_i \in C^\infty(L^n)$ is contained in a closed interval since L^n is compact. H_i quotients then into S^1 with a projection π_i, and we can redefine $g_i := g_i + \pi \circ H_i$. The basis ν_i were defined as $\nu_i = g_i^*(d\theta) = d(\tilde{g}_i)$, with $\tilde{g}_i = \theta \circ g_i$. In particular the forms β_i' can be written

$$\beta_i' = d(\sum p_i\tilde{g}_i).$$

If we define the functions $\theta_i = \sum p_i\tilde{g}_i$, then its quotients (by their period) $\tilde{\theta}_i : L^n \longrightarrow S^1$ are n submersions of L^n to S^1. Consider the function

$$\begin{aligned} \Theta : L^n &\longrightarrow S^1 \times ... \times S^1 = T^n \\ p &\longmapsto (\tilde{\theta}_1(p), ..., \tilde{\theta}_n(p)). \end{aligned}$$

We have to check that Θ is a submersion into T^n. Since we had that β'_i generate all $H^1(L^n, \mathbb{R})$, we can deduce that all $d\theta_i$ are independent as one-forms from L^n to $\mathbb{R}^n$ and so $d\tilde{\theta_i}$ are also independent. We deduce that Θ is a submersion so also an immersion since it is a function between same dimension manifolds. An immersion between compact manifolds is an embedding, and with same dimension manifolds it is in fact a diffeomorphism. The conclusion obtained is the first statement of Arnold-Liouville theorem:

$$L^n \cong T^n.$$

$\square$

4 Bibliography

References

[1] A. V. Bolsinov, A. T. Fomenko. *Integrable Hamiltonian systems*, CRC Press LL, 2004.

[2] A. Candel, L. Conlon. *Foliations I*, Providence, RI: Amer. Math. Soc., 1999.

[3] A. Cannas da Silva. *Lectures on Symplectic Geometry*, Springer-Verlag number 1764 of Lecture Notes in Mathematics series, revised January 2006.

[4] A. Cannas da Silva. *Symplectic Toric Manifolds*, July 2001.

[5] W. Chen. *Lectures on Symplectic Geometry*, UMassAmherst spring 2007.

[6] R. H. Cushman, L.M. Bates. *Global Aspects of Classical Integrable Systems*, 2nd Edit. Birkhäuser, 1st 1997.

[7] T. Delzant. *Hamiltoniens périodiques et images convexes de l'application moment* , Bulletin de la SMF tome 116 num. 3 p. 315-339, 1988.

[8] C. Ehresmann. *Les connexions infinitésimales dans un espace fibré différentiable*, Colloque de Topologie, Bruxelles 29-55, 1950.

[9] A. Kiesenhofer. *Integrable systems on b-symplectic manifolds*, PhD Thesis supervised by E. Miranda, October 2016.

[10] D. Tischler. *On fibering certaing foliated manifolds over S^1* , Topology 9 153-154, 1970.

[11] Wikipedia, The Free Encyclopedia. *"Fiber bundle"* https://en.wikipedia.org/wiki/Fiber_bundle.

[12] A. Zamorzaev. *Unofficial notes for Introduction to Symplectic Topology*, Stanford University, Autumn 2012.