Periodic points, Lie symmetries and non-integrability of planar maps

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We present a criterion for the C^0-non-integrability near elliptic fixed points of smooth planar measure preserving maps.

What kind of integrability?

A planar map F is C^0-locally integrable at an elliptic fixed point p if there exists a neighborhood U of p and a first integral V ∈ C^0(U) with m ≥ 2, i.e. V(F(x)) = V(x) such that all the level curves {V = c} are closed curves surrounding p, which is an isolated non-degenerate critical point of V in U.

Remember that a map is a measure preserving if m(F^{-1}(B)) = m(U) for any measurable set B, where m(B) = \int_B \nu(x) dx, and \nu|_{\partial U} \neq 0.

Birkhoff normal form at a \ell\resonant elliptic point

• A fixed point p of a C^0-real planar map F is elliptic when the eigenvalues of DF(p) have modulus one, but excluding \pm i1.
• When the eigenvalues are not roots of unity of order \ell < \ell \leq \ell we will say that p is not \ell\resonant.
• A C^1\resonant-map, with a not \ell\resonant fixed point p, is locally C^1\resonant-conjugate to its Birkhoff normal form:

\[
F'(z) = x_1 + \sum_{k=2}^{\frac{n-1}{\ell}} \frac{B_k(z)}{k!} + O(|z|^n),
\]

where z = x + iy, and [.] denotes the integer part.

Lie Symmetries

A vector field X is said to be a Lie symmetry of F if it satisfies

\[
X F(x) = (DF(x)) X(x)
\]

for all x ∈ U.

This implies that k – X(x) is invariant under the change of variables given by F. From a dynamic viewpoint F maps any orbit of k – X(x), into another orbit of this system. In the integrable case we have:

Theorem 1. [11], see also [2],[11]. Let F be a C^0\resonant orientation preserving map with an invariant measure with density \nu \in C^0(U) with a first integral V ∈ C^0(U). Then

(a) The vector field X = \frac{1}{\nu} (V_x V_z - V_z V_x) is a Lie Symmetry of F.
(b) If a connected component \gamma of \{V(x) = k\} without fixed points is invariant by F and \nu|_{\gamma} \geq 0, then F(x,v)_{\gamma} = 0 = V|_{\gamma}, x is conjugate to a rotation with rotation number \theta = \frac{\tau(h)}{\tau(B)}.
(c) Therefore, if F_{\gamma}\nu has rotation number \theta = \rho/p ∈ \mathbb{Q}, with p\|\rho, p = 1, then \gamma \subset U is a continuum of periodic points of F.

Main result

We present the following criterion for non-integrability of planar maps:

Theorem 2. Let F ∈ C^0\resonant be a measure preserving with a non-vanishing density \nu \in C^0\resonant, and an elliptic fixed point p, not (2n+1)-resonant, with Birkhoff constant is\ B_h = \nu - i\in \mathbb{R}. Assume that there is an unbounded sequence \{N_k\} such that F has finitely many N_k-periodic points in U if \tau \to \infty.

Proof. Suppose that F has an first integral V, then:
• F possesses a smooth Lie symmetry X = \frac{1}{\nu} (V_x V_z - V_z V_x) because it preserves an invariant measure with a smooth density.
• Since it has an elliptic fixed point, with non-zero purely imaginary Birkhoff constant, the rotation number function \theta(h) associated to each level \{V = c\} is continuous and non-constant (see Proposition 3).
• \tau \to \infty. F has no constant, there should exist closed level sets such that on them F has rational rotation numbers with all denominators bigger that some N_k.
• Since there exists a Lie symmetry X, these levels have continuous of real periods for all N_k ≥ N_0 in a given neighborhood of the elliptic point. Indeed, recall that by Theorem 1:

\[
F V = \nu = V F,
\]

\[\text{if } V \text{ is invariant by } F, \text{ without singular points of } \nu, \text{ and diffeomorphic to } S^1 \text{ then } F V = \nu \text{ is a conjugation to a rotation. Moreover if its corresponding rotation number is rational, } p\|\rho, \rho \in \mathbb{Q} \text{ then the map } F \text{ has a continuum of } p \text{-periodic points.}
\]
• But we are assuming that for an unbounded sequence of natural numbers \{N_k\}, F has finitely many N_k-periodic points in U, a contradiction.

Proposition 3. If F ∈ C^0\resonant is measure preserving, C^0\resonant locally integrable at p, and B_h = \nu - i\in \mathbb{R} the rotation number \theta(h) associated to each curve \{V = c\} is not constant (hence there exists contin-

Proof. Suppose that \theta(h) is constant. We will prove that F is globally C^0\resonant-conjugate to the linear map L(p) = \mathcal{D} F(p).

F possesses a smooth Lie symmetry X = \frac{1}{\nu} (V_x V_z - V_z V_x) of class C^0\resonant with a non-degenerate center at p, in fact

\[
DF(p) = e^{\theta} D(V)\nu, \text{ where } \theta = \lim_{h \to 0} \theta(h)|_{F}\}

The new vector field Y(x,y) = T(x,y) X(x,y) is also a Lie Symmetry of F of class C^0\resonant\resonant, having an isochronous center at p with period function T(h) \to \infty.

\[
Y(p) = \theta(1)
\]

The isochronous center Y linearizes. Since \mathcal{D} Y(p) = e^{\theta} D(V)\nu, we prove that the "Bochner"-type map

\[
\Phi(y) = \int e^{\theta} D(V)(p)\nu = \int e^{\theta} D(V)(y)\nu dx
\]

is a C^0\resonant-bijective between F and the linear map L(p) = \mathcal{D} F(p)|_{p}

But F is also C^0\resonant-conjugate to the Birkhoff normal form, which is non linear, a contradiction.

The Cohen map case

We have applied Theorem 2 to prove the local non-integrability of a variety of maps, in particular the Cohen's one

\[
F(x,y) = \left( y, -x + \sqrt{y^2 + 1} \right)
\]

It seems that the non-integrability of the Cohen map was first conjectured by Cohen and communicated by C. de Verdieu to Moser in 1993. Ryckhil and Torgeson shown that it has not integrals given by Lie symmetries equations [5]. Inspired by Lowther [4], we have:

Theorem 4. The Cohen map is not C^0\resonant-locally integrable at its fixed point (\sqrt{3}/3, \sqrt{3}/3).

We need to prove that for an unbounded sequence \{N_k\} the Cohen map has finitely many N_k-periodic points. We will use the following result:

Theorem 5. Let F : C^N → C^N be a polynomial map. Let G denote the homogenous map corresponding to the maximum degree d of F. If y = 0 is the unique solution in C^0\resonant of the homogenous system G(y) = 0, then G(y) = 0 has finitely many solutions.

The Cohen map writes as the equation x_{n+2} = x_n + \sqrt{x_n^2 + 1}.

Therefore the N-periodic orbits satisfy the system

\[
\begin{align*}
x(0) & = 0 + 1, \\
x(1) & = 0 + 1, \\
x(2) & = 0 + 1, \\
x(3) & = 0 + 1, \\
x(4) & = 0 + 1, \\
\end{align*}
\]

Applying Theorem 5, we will prove that, for some unbounded sequence of values of N, x = 0 is the unique solution of the linear systems.

\[
\begin{align*}
x(1) & = 0 + 1, \\
x(2) & = 0 + 1, \\
x(3) & = 0 + 1, \\
x(4) & = 0 + 1, \\
x(5) & = 0 + 1, \\
\end{align*}
\]

Lemma 6. For every choice of c_j \in [-1,1], for j = 1, ..., N, and for all N \neq 3.

\[
\text{does } A_N(x_1,\ldots,x_N) = 0 \text{ mod } 2.
\]

where F_N are the Fibonacci numbers.

Recall that F_N are:

\[
\begin{align*}
1, & \ 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots \\
& \text{and modulus } 2, \ 1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots \\
& \text{Hence}
\end{align*}
\]

Corollary 7. For all N \neq 3, the Cohen map has finitely many N-periodic points.

Proof of Theorem 4. Suppose that F is integrable, then:
• F possesses a Lie symmetry because it is measure preserving.
• At the fixed point (\sqrt{3}/3, \sqrt{3}/3), F_1 = 12/\sqrt{3} ≠ 0
• Since there exists a Lie symmetry, there should exists level sets with continua periodic points for all N ≥ N_0. But we have proved that for all N \neq 3, F has finitely many N-periodic points, a contradiction.

References


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