Title: On the Phase-lag Heat Equation with Spatial Dependent Lags

Authors:
Zhuangyi Liu
Department of Mathematics and Statistics
University of Minnesota
Duluth, MN 55812-2496, USA
E-mail:*zliu@d.umn.edu*

Ramón Quintanilla
Department of Mathematics
UPC, Colom 11
08222 Terrassa, Barcelona, Spain
E-mail:*ramon.quintanilla@upc.edu*

Yang Wang
College of Information Science and Technology
Donghua University
Shanghai 201620, P. R. China
E-mail:*wang_yang*1991@*hotmail.com*

Proposed running head: On the Time Decay for the Phase-lag Heat Equation with Spatial Dependent Lags

Mailing address:

Department of Mathematics and Statistics, University of Minnesota, Duluth, MN 55812-2496, USA
Department of Mathematics, UPC, Colom 11, 08222 Terrassa, Barcelona, Spain
College of Information Science and Technology, Donghua University, Shanghai 201620, P. R. China

Abstract: In this paper we investigate several qualitative properties of the solutions of the dual-phase-lag heat equation and the three-phase-lag heat equation. In the first case we assume that the parameter τ_T depends on the spatial position. We prove that when $2\tau_T - \tau_q$ is strict-

ly positive the solutions are exponentially stable. When this property is satisfied in a proper sub-domain, but $2\tau_T - \tau_q \geq 0$ for all the points in the case of the one-dimensional problem we also prove the exponential stability of solutions. A critical case corresponds to the situation when $2\tau_T - \tau_q = 0$ in the whole domain. In that case it is known that the solutions are not exponentially stable. We here obtain the polynomial decay of the solutions when $2\tau_T - \tau_q \geq 0$ on the whole domain. Last section of the paper is devoted to the three-phase-lag case when τ_T and τ_ν^* depend on the spatial variable. We here consider the case when $\tau_\nu^* \geq \kappa^* \tau_q$ and τ_T is a positive constant. We will obtain the analyticity of the semigroup of solutions. Exponential stability and impossibility of localization are consequences of the analyticity of the semigroup.

1. Department of Mathematics and Statistics, University of Minnesota,
Duluth, MN 55812-2496, USA
E-mail:*zliu@d.umn.edu*
2.Department of Mathematics, UPC, Colom 11,
08222 Terrassa, Barcelona, Spain
E-mail:*ramon.quintanilla@upc.edu*
3. College of Information Science and Technology, Donghua University,
Shanghai 201620, P. R. China.
E-mails: *wang_yang1991@hotmail.com*

Abstract

In this paper we investigate several qualitative properties of the solutions of the dual-phase-lag heat equation and the three-phase-lag heat equation. In the first case we assume that the parameter τ_T depends on the spatial position. We prove that when $2\tau_T - \tau_q$ is strictly positive the solutions are exponentially stable. When this property is satisfied in a proper sub-domain, but $2\tau_T - \tau_q \geq 0$ for all the points in the case of the one-dimensional problem we also prove the exponential stability of solutions. A critical case corresponds to the situation when $2\tau_T - \tau_q = 0$ in the whole domain. In that case it is known that the solutions are not exponentially stable. We here obtain the polynomial decay of the solutions when $2\tau_T - \tau_q \geq 0$ on the whole domain. Last section of the paper is devoted to the three-phase-lag case when τ_T and τ_ν^* depend on the spatial variable. We here consider the case when $\tau_\nu^* \geq \kappa^* \tau_q$ and τ_T is a positive constant. We will obtain the analyticity of the semigroup of solutions. Exponential stability and impossibility of localization are consequences of the analyticity of the semigroup.

Key words: dual-phase-lag heat equation, three-phase-lag heat equation, exponential stability, polynomial stability, analyticity of solutions.

1 Introduction

When we combine the Fourier constitutive law for the heat flux vector with the classical energy equation we obtain the well-known linear parabolic equation for the heat conduction. It is not very well accepted from the physical point of view. In fact, we have that the thermal disturbances at some point will be felt instantly anywhere for every distant. To save this drawback different theories for the heat conduction have been established in the first part of the last century. Most known theory is the Maxwell-Cattaneo law that proposes an hyperbolic damped equation for the heat conduction. This theory gives rises to two hyperbolic thermoelasticity theory which

The authors are listed in alphabetical order.

are the Lord and Shulman [15] and the Green and Lindsay [6]. Both theories are currently being studied and many authors have dedicate their attention to them. Green and Naghdi also proposed three theories [7–9] for the heat conduction based on the axioms of the continuum mechanics. They established their theories in the context of the thermoelasticity, but they also proposed some fluid theories. It is worth noting that all these theories are considered in several articles and books [10–12, 22].

In the last decade of last century Tzou [23] proposed a modification of the Fourier constitutive equation. He suggested a theory where the heat flux vector has a delay in its constitutive equation. The basic equation for this theory is

$$q(x, t + \tau_q) = -\kappa \nabla T(x, t + \tau_T), \ \ \kappa > 0. \tag{1.1}$$

In this equation T is the temperature, q is the heat flux vector and τ_T and τ_q are two delay parameters. This equation can be understood as that the temperature gradient established across a material volume at position x and at time $t+\tau_T$ results in the heat flux to flow a different instant $t + \tau_q$. This kind of proposition can be understood in terms of the microstructure of the material. In 2007 Choudhuri [3] suggested a modification of Tzou's constitutive equation and he proposed the constitutive equation

$$q(x, t + \tau_q) = -\kappa \nabla T(x, t + \tau_T) - \kappa^* \nabla \nu(x, t + \tau_\nu). \tag{1.2}$$

In this equation ν is the thermal displacement suggested by Green and Naghdi in their theories that satisfies $\dot{\nu} = T$ and τ_ν is a new delay parameter proposed by this theory. One suspects that the aim of this new theory was to establish a new model with delay in such a way that the Taylor approximations for this theory recover the theories proposed by Green and Naghdi. It is worth noting that these two last theories with delay are strongly based on an intuitive point of view, however there is not an axiomatic thermomechanical foundation of them. Furthermore, if we adjoin these equations to the classical energy equation

$$\dot{T} + \text{ div } q = 0, \tag{1.3}$$

it can be proved the existence of a sequence of solutions $T_n(x, t) = \exp(\omega_n t)\Psi_n(x)$ such that the real part of ω_n tends to infinity [4]. This result says that the associated mathematical problem is ill posed in the sense of Hadamard, which is a not suitable property for a heat conduction theory. Therefore a big interest has been developed to understand the Taylor approximations to the delay equations. These approximations propose some new and stimulating equations to study from the mathematical point of view and the most natural question is to clarify when the mathematical problems that they propose are stable and what kind of stability can be proved for them. In this paper we are going to consider two approximations. One for the Tzou model which is

$$\dot{T}+\tau_q\ddot{T}+\frac{\tau_q^2}{2}\dddot{T}=\kappa\triangle T+\kappa\tau_T\triangle\dot{T}. \tag{1.4}$$

This equation has been studied in the past. Quintanilla [19] was the first to point out the exponential stability of the solutions of this equation when $\tau_q < 2\tau_T$ and the instability when $\tau_q > 2\tau_T$. However, it was open to clarify the kind of stability we have in the limit case $\tau_q = 2\tau_T$. In the reference [20] the authors showed that the decay is not exponential. In this paper we give a polynomial decay rate for the solutions. A second problem for the equation proposed by Tzou is when we assume that the delay τ_T depends on the material point. From the known results it is natural to expect the exponential decay of solutions when $\tau_q < 2\tau_T(x)$ and we prove it. However an stimulating question rises if we assume that the inequality holds in a proper sub-domain of the solid and the equality holds in the remain. This is a nice question concerning this equation and we will prove the exponential stability in the one-dimensional case.

The second model we consider is related with the Chouduri proposition. We consider the equation

$$\ddot{T}+\tau_q\dddot{T}=\kappa^*\triangle T+\tau_\nu^*\triangle\dot{T}+\kappa\tau_T\triangle\ddot{T}, \tag{1.5}$$

where $\tau_\nu^* = \kappa + \kappa^*\tau_\nu$. In the reference [21], the authors proved the exponential stability when $\tau_\nu^* > \kappa^*\tau_q$ and the instability of solutions when $\tau_\nu^* < \kappa^*\tau_q$. More recently, in [1] the authors proved the exponential stability in the case that $\tau_\nu^* = \kappa^*\tau_q$. However, several questions were still open. For instance about the regularity of solutions. In this paper we want to prove the analyticity of solutions in this last case. In fact, we are going to see this fact even in case that we assume that τ_T and τ_ν^* are functions depending on the material point. Exponential stability and impossibility of localization are consequences of this result.

2 Stability of (1.4) for spatial dependent τ_T

For equation (1.4) we recall that when $\tau_q < 2\tau_T$ it is exponential stable, whereas, when $\tau_q > 2\tau_T$, it is unstable (see [20]).

The critical case $\tau_q = 2\tau_T$ was mentioned in [1] where the authors proved that the real part of the eigenvalues of the system are all negative, but can get close to the imaginary axis arbitrarily by spectral analysis. Hence, the decay of the energy is slow. However, no specific decay rate has been obtained.

In this section, we consider the case of spatial dependent τ_T. Equation (1.4) is modified as the following.

$$\begin{cases} \dot{T} + \tau_1 \ddot{T} + \frac{\tau_1^2}{2} \dddot{T} = \kappa \triangle T + \kappa div(\tau_2(x) \nabla \dot{T}), \quad in \quad \Omega \times (0, \infty), & (2.1) \\ T(x,0) = T^0(x), \dot{T}(x,0) = \dot{T}^0(x), \ddot{T}(x,0) = \ddot{T}^0(x), \quad in \quad \Omega, & (2.2) \\ T(\cdot, t)|_{\partial\Omega} = 0, \quad for \quad t \in [0, \infty), & (2.3) \end{cases}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$, and

$$\tau_1 = \tau_q, \quad \tau_2(x) = \tau_T(x).$$

Denoting

$$a(x) = 2\tau_2(x) - \tau_1.$$

In reference to the constant coefficient equation (1.4), we assume that $a(x) \geq 0$ on Ω, and consider the stability of (2.1)-(2.3) in three cases:

(i). $a(x)$ is strictly positive, i.e., $a(x) \geq a_0 > 0$ on Ω;

(ii). the critical case $a(x) \equiv 0$ on Ω;

(iii). the partially critical case, i.e., $a(x) > 0$ only on a subdomain of positive measure $\Omega_0 \subset \Omega$.

It is important to identify a proper state space so that the "energy" of the system (2.1)-(2.3) is dissipative. For this purpose, we take the inner product of $T + \tau_1 \dot{T} + \frac{\tau_1^2}{2} \ddot{T}$ with (2.1) in $L^2(\Omega)$ to get

$$\frac{1}{2}\frac{d}{dt}\|T + \tau_1 \dot{T} + \frac{\tau_1^2}{2}\ddot{T}\|^2 = \langle \kappa \triangle T, T + \tau_1 \dot{T} + \frac{\tau_1^2}{2}\ddot{T}\rangle + \langle \kappa div(\tau_2(x)\nabla \dot{T}), T + \tau_1 \dot{T} + \frac{\tau_1^2}{2}\ddot{T}\rangle. \quad (2.4)$$

Since

$$\begin{aligned} \langle \kappa \triangle T, T + \tau_1 \dot{T} + \frac{\tau_1^2}{2}\ddot{T}\rangle &= -\langle \kappa \nabla T, \nabla(T + \tau_1 \dot{T} + \frac{\tau_1^2}{2}\ddot{T})\rangle \\ &= -\kappa\|\nabla T\|^2 - \frac{1}{2}\frac{d}{dt}\kappa\tau_1\|\nabla T\|^2 - \langle \kappa \nabla T, \frac{\tau_1^2}{2}\nabla \ddot{T}\rangle \\ &= -\kappa\|\nabla T\|^2 - \frac{1}{2}\frac{d}{dt}\kappa\tau_1\|\nabla T\|^2 - \frac{d}{dt}\langle \kappa \nabla T, \frac{\tau_1^2}{2}\nabla \dot{T}\rangle + \kappa\frac{\tau_1^2}{2}\|\nabla \dot{T}\|^2 \end{aligned}$$

and

$$\begin{aligned} \langle \kappa div(\tau_2(x)\nabla \dot{T}), T + \tau_1 \dot{T} + \frac{\tau_1^2}{2}\ddot{T}\rangle &= -\langle \kappa\tau_2(x)\nabla \dot{T}, \nabla(T + \tau_1 \dot{T} + \frac{\tau_1^2}{2}\ddot{T})\rangle \\ &= -\frac{1}{2}\frac{d}{dt}\kappa\|\tau_2^{\frac{1}{2}}(x)\nabla T\|^2 - \kappa\tau_1\|\tau_2^{\frac{1}{2}}(x)\nabla \dot{T}\|^2 - \frac{1}{4}\frac{d}{dt}\kappa\tau_1^2\|\tau_2^{\frac{1}{2}}(x)\nabla \dot{T}\|^2, \end{aligned}$$

(2.4) leads to

$$\frac{1}{2}\frac{dE(t)}{dt} = -\kappa\|\nabla T\|^2 - \kappa\tau_1\|a^{\frac{1}{2}}(x)\nabla \dot{T}\|^2. \quad (2.5)$$

where the "energy" of the system (2.1)-(2.3) is

$$\begin{aligned} E(t) &= \kappa\tau_1\|\nabla T\|^2+\kappa\|\tau_2^{\frac{1}{2}}(x)\nabla T\|^2+\frac{1}{2}\kappa\tau_1^2\|\tau_2^{\frac{1}{2}}(x)\nabla\dot{T}\|^2+\kappa\tau_1^2\langle\nabla T,\nabla\dot{T}\rangle+\|T+\tau_1\dot{T}+\frac{\tau_1^2}{2}\ddot{T}\|^2 \\ &= \kappa\tau_1(\frac{1}{2}\|\nabla T\|^2+\|\nabla T+\frac{\tau_1^2}{2}\nabla\dot{T}\|^2)+\kappa\|a^{\frac{1}{2}}(x)\nabla T\|^2+\frac{1}{2}\kappa\|a^{\frac{1}{2}}(x)\tau_1\nabla\dot{T}\|^2 \\ &\quad +\|T+\tau_1\dot{T}+\frac{\tau_1^2}{2}\ddot{T}\|^2. \end{aligned}$$

Let $H_0^1(\Omega)=\{X\in H^1(\Omega):\ X|_{\partial\Omega}=0\}$, and hence

$$\mathcal{H}:=H_0^1(\Omega)\times H_0^1(\Omega)\times L^2(\Omega).$$

We denote by $Z=(Z_1,Z_2,Z_3)$ and $W=(W_1,W_2,W_3)$, we can define the inner product

$$\begin{aligned} \langle Z,W\rangle_{\mathcal{H}} =&\kappa\tau_1\langle\nabla Z_1,\nabla W_1\rangle+\kappa\langle\tau_2^{\frac{1}{2}}(x)\nabla Z_1,\tau_2^{\frac{1}{2}}(x)\nabla W_1\rangle+\frac{1}{2}\kappa\tau_1^2\langle\tau_2^{\frac{1}{2}}(x)\nabla Z_2,\tau_2^{\frac{1}{2}}(x)\nabla W_2\rangle \\ &+\kappa\tau_1^2\langle\nabla Z_1,\nabla W_2\rangle+\langle Z_1+\tau_1 Z_2+\frac{\tau_1^2}{2}Z_3,W_1+\tau_1 W_2+\frac{\tau_1^2}{2}W_3\rangle, \end{aligned}$$

i.e.,

$$\begin{aligned} \|Z\|_{\mathcal{H}}^2 =& \kappa\tau_1(\frac{1}{2}\|\nabla Z_1\|^2+\|\nabla Z_1+\frac{\tau_1^2}{2}\nabla Z_2\|^2)+\kappa\|a^{\frac{1}{2}}(x)\nabla Z_1\|^2+\frac{1}{2}\kappa\|a^{\frac{1}{2}}(x)\tau_1\nabla Z_2\|^2 \\ &+\|Z_1+\tau_1 Z_2+\frac{\tau_1^2}{2}Z_3\|^2. \end{aligned}$$

Denoting $Z:=(Z_1,Z_2,Z_3)^T=(T,\dot{T},\ddot{T})^T$, we then convert system (2.1)-(2.3) to a first order evolution equation on Hilbert space $\mathcal{H}$,

$$\begin{cases} \dfrac{dZ}{dt}=\mathcal{A}Z, & (2.6) \\ Z(0)=Z_0=(T^0,\dot{T}^0,\ddot{T}^0)^T, & (2.7) \end{cases}$$

where the operator $\mathcal{A}$ is given by

$$\mathcal{A}Z=\begin{pmatrix} Z_2 \\ Z_3 \\ \dfrac{2}{\tau_1^2}\big(-Z_2-\tau_1 Z_3+\kappa\triangle Z_1+\kappa div(\tau_2(x)\nabla Z_2)\big) \end{pmatrix} \tag{2.8}$$

and

$$D(\mathcal{A})=\{Z=(Z_1,Z_2,Z_3)^T\in\mathcal{H}|Z_1,Z_2\in H^2(\Omega),Z_3\in H_0^1(\Omega)\}. \tag{2.9}$$

Theorem 2.1. *$\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on the Hilbert space $\mathcal{H}$.*

Proof. By (2.5),

$$Re\langle \mathcal{A}Z, Z\rangle_{\mathcal{H}} = \frac{1}{2}\frac{d}{dt}\|Z\|^2_{\mathcal{H}} = -\kappa\|\nabla Z_1\|^2 - \kappa\tau_1\|a^{\frac{1}{2}}(x)\nabla Z_2\|^2 \le 0. \tag{2.10}$$

Thus, $\mathcal{A}$ is dissipative. Now for $F = (f_1, f_2, f_3)^T \in \mathcal{H}$, we look for $Z = (Z_1, Z_2, Z_3)^T \in D(\mathcal{A})$ such that $(I - \mathcal{A})Z = F$. Equivalently, we consider the following system

$$\begin{cases} Z_1 - Z_2 = f_1, & (2.11)\\ Z_2 - Z_3 = f_2, & (2.12)\\ Z_3 - \frac{2}{\tau_1^2}(-Z_2 - \tau_1 Z_3 + \kappa\triangle Z_1 + \kappa div\big(\tau_2(x)\nabla Z_2\big)\big) = f_3, & (2.13)\\ Z_1|_{\partial\Omega} = Z_2|_{\partial\Omega} = Z_3|_{\partial\Omega} = 0. & (2.14) \end{cases}$$

From (2.11), (2.12) and (2.13), we have

$$(1+\frac{2}{\tau_1^2}+\frac{2}{\tau_1})Z_1-\frac{2\kappa}{\tau_1^2}\triangle Z_1-\frac{2\kappa}{\tau_1^2}div(\tau_2(x)\nabla Z_1) = f_3+(1+\frac{2}{\tau_1^2}+\frac{2}{\tau_1})f_1+(1+\frac{2}{\tau_1})f_2-\frac{2\kappa}{\tau_1^2}div(\tau_2(x)\nabla f_1). \tag{2.15}$$

Let $\phi \in H_0^1$. Multiplying (2.15) by ϕ, we get the following variational equation

$$\begin{aligned} &\langle(1 + \frac{2}{\tau_1^2} + \frac{2}{\tau_1})Z_1, \phi\rangle + \langle\frac{2\kappa}{\tau_1^2}\nabla Z_1, \nabla\phi\rangle + \langle\frac{2\kappa}{\tau_1^2}\tau_2(x)\nabla Z_1, \nabla\phi\rangle \\ &= \langle f_3, \phi\rangle + \langle(1 + \frac{2}{\tau_1^2} + \frac{2}{\tau_1})f_1, \phi\rangle + \langle(1 + \frac{2}{\tau_1})f_2, \phi\rangle + \langle\frac{2\kappa}{\tau_1^2}\tau_2(x)\nabla f_1, \nabla\phi\rangle. \end{aligned} \tag{2.16}$$

It is easy to check that the left-hand of (2.16) is a continuous and coercive bilinear form on the space $H_0^1 \times H_0^1$, and the right-hand side is a continuous linear form on the space $H_0^1 \times H_0^1$. Then, due to Lax-Milgram Lemma ([13], Theorem 2.9.1), (2.16) admits a unique solution $Z_1 \in H_0^1$. (2.16) also implies that the weak solution Z_1 of (2.15) associated with the boundary conditions (2.14) belongs to the space H^2. Therefore, $(Z_1, Z_2, Z_3)^T \in D(\mathcal{A})$ and $(I - \mathcal{A})^{-1}$ is compact in the energy space $\mathcal{H}$. Then, thanks to Lumer-Philips Theorem ([17], Theorem 1.4.3), we conclude that $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$. □

Our main results for system (2.1)-(2.3) are stated in the following two theorems.

Theorem 2.2. *Assume that $a(x) \in C^1(\Omega)$. Then the semigroup $e^{\mathcal{A}t}$ is*

(i). exponentially stable if $a(x) \ge a_0 > 0$ on Ω, i.e., there exist constants M, $\omega > 0$, such that

$$\|e^{\mathcal{A}T}Z_0\|_{\mathcal{H}} \le Me^{-\omega t}\|Z_0\|_{\mathcal{H}}, \quad \forall t > 0,\ Z_0 \in \mathcal{H};$$

(ii). polynomially stable of order $\frac{1}{2}$, if $a(x) \ge 0$ on Ω, i.e., there exists a constant $C > 0$, such that

$$\|e^{\mathcal{A}T}Z_0\|_{\mathcal{H}} \le \frac{C}{\sqrt{t}}\|Z_0\|_{D(\mathcal{A})}, \quad \forall t > 0,\ Z_0 \in D(\mathcal{A}).$$

Theorem 2.3. *Let $\Omega = [0, L]$, $\Omega_0 = (x_1, x_2)$ for $0 \le x_1 < x_2 \le L$. If $a(x) \in C^2(\Omega)$, $a(x) > 0$ on Ω_0 and $a(x) = 0$ on $\Omega \backslash \Omega_0$, then the semigroup $e^{\mathcal{A}t}$ is exponentially stable.*

Remark 2.1. *Case (i) in Theorem 2.2 extends the corresponding result for the constant coefficient case $2\tau_T > \tau_q$ considered in [20]. Case (ii) in Theorem 2.2 considers the partial critical situation which is new due to the spatial dependent τ_T. As a by-product, it improves the slow decay conclusion for the critical case $2\tau_T = \tau_q$ in [20] by a specifying polynomial decay rate. However, whether this is the best decay rate is still open.*

Remark 2.2. *The result in Theorem 2.3 is new and interesting. It reveals a transition process from exponential stability to polynomial stability as $a(x)$ changes from positive to partially positive to zero. Unfortunately, by far, we are only able to prove it for one-dimensional problem.*

Remark 2.3. *The unstable case $2\tau_T < \tau_q$ in [20] suggests that the conclusion still holds if $a(x) < 0$ on Ω. However, the picture for $a(x) < 0$ only on a subregion $\Omega_0 \subset \Omega$ is still unclear.*

The proof of Theorem 2.2 and 2.3 will be presented in next two subsections. Our main tools are the following well-known frequency domain characterization of stability for a semigroup on Hilbert space, combined with contradiction argument in [14].

Theorem 2.4. *[16] Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions in a Hilbert space $\mathcal{H}$. Suppose that*

$$i\mathbb{R} \subset \rho(\mathcal{A}). \tag{2.17}$$

Then, $S(t)$ is exponential stable if and only if

$$\overline{\lim}_{|\beta \to \infty|} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{H}} < \infty \tag{2.18}$$

holds.

Theorem 2.5. *[2] Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A}$ generates a bounded C_0-semigroup in $\mathcal{H}$. Assume that*

$$i\mathbb{R} \subset \rho(\mathcal{A}), \tag{2.19}$$

$$sup_{|\beta|>1} \frac{1}{\beta^k} \|(i\beta - \mathcal{A})^{-1}\| < +\infty, \quad for\ some\ k > 0. \tag{2.20}$$

Then, there exists a positive constant $C > 0$ such that

$$\|e^{t\mathcal{A}} Z_0\| \le C(\frac{1}{t})^{\frac{1}{k}} \|Z_0\|_{D(\mathcal{A})}, \quad \forall t > 0, \tag{2.21}$$

for all $Z_0 \in D(\mathcal{A})$.

2.1 Proof of Theorem 2.2

Proof. We first verify condition (2.17). Assume that it is false, i.e., there is a $\lambda = i\beta \in \sigma(\mathcal{A})$. Then there exist $\lambda_n(= i\beta_n) \to \lambda$ and normalized $Z_n = (Z_{1n}, Z_{2n}, Z_{3n})^T$ such that

$$\|(i\beta_n - \mathcal{A})Z_n\|_{\mathcal{H}} \to 0, \tag{2.22}$$

which implies

$$\begin{cases} i\beta Z_1 - Z_2 = o(1), \quad in \quad H_0^1(\Omega), & (2.23)\\ i\beta(Z_1 + \frac{\tau_1^2}{2}Z_2) - (Z_2 + \frac{\tau_1^2}{2}Z_3) = o(1), \quad in \quad H_0^1(\Omega), & (2.24)\\ i\beta(Z_1 + \tau_1^2 Z_2 + \frac{\tau_1^2}{2}Z_3) - \big(\kappa\triangle Z_1 + \kappa div(\tau_2(x)\nabla Z_2)\big) = o(1), \quad in \quad L^2(\Omega). & (2.25) \end{cases}$$

For convenience, we have omitted the subscript n here.

Thus

$$Re\langle(i\beta - \mathcal{A})Z, Z\rangle_{\mathcal{H}} = -Re\langle\mathcal{A}Z, Z\rangle_{\mathcal{H}} = \kappa\|\nabla Z_1\|^2 + \kappa\tau_1\|a^{\frac{1}{2}}(x)\nabla Z_2\|^2 = o(1). \tag{2.26}$$

Hence

$$\|\nabla Z_1\|^2 = o(1) \quad and \quad \|a^{\frac{1}{2}}(x)\nabla Z_2\|^2 = o(1). \tag{2.27}$$

Since β is finite, we get from (2.23) and (2.27) that

$$\|\nabla Z_2\| = o(1). \tag{2.28}$$

Then by (2.24), we have

$$\|\nabla Z_3\|^2 = o(1).$$

By the *Poincaré* inequality,

$$\|Z_3\|^2 = o(1). \tag{2.29}$$

We conclude that $\|Z\|_{\mathcal{H}} = o(1)$. This is a contradiction with the assumption $\|Z\|_{\mathcal{H}} = 1$. Thus, $i\mathbb{R} \subset \rho(\mathcal{A})$.

Assume that (2.18) and (2.20) are false. We can combine them in one case. Then by the uniform boundedness theorem, there exists a sequence $\beta \to \infty$ and a unit sequence $Z = (Z_1, Z_2, Z_3)^T \in D(\mathcal{A})$ such that

$$\beta^k\|(i\beta I - \mathcal{A})Z\|_{\mathcal{H}} \to 0, \tag{2.30}$$

which implies that

$$\begin{cases} \beta^k(i\beta Z_1 - Z_2) = o(1), \quad in \quad H_0^1(\Omega), & (2.31)\\ \beta^k\Big(i\beta(Z_1 + \frac{\tau_1^2}{2}Z_2) - (Z_2 + \frac{\tau_1^2}{2}Z_3)\Big) = o(1), \quad in \quad H_0^1(\Omega), & (2.32)\\ \beta^k\Big(i\beta(Z_1 + \tau_1 Z_2 + \frac{\tau_1^2}{2}Z_3) - \kappa\triangle Z_1 - \kappa div(\tau_2(x)\nabla Z_2)\Big) = o(1), \quad in \quad L^2(\Omega). & (2.33) \end{cases}$$

From dissipation, we have

$$\beta^{\frac{k}{2}}\|\nabla Z_1\|^2 = o(1), \quad and \quad \beta^{\frac{k}{2}}\|a^{\frac{1}{2}}(x)\nabla Z_2\|^2 = o(1). \tag{2.34}$$

If $a(x) \geq 0$ on Ω, from (2.31) and (2.34), we can also obtain

$$\beta^{\frac{k}{2}-1}\|\nabla Z_2\| = o(1). \tag{2.35}$$

Taking $k = 2$, we get

$$\|\nabla Z_2\| = o(1). \tag{2.36}$$

By (2.32) and (2.36), we have

$$\|\frac{\nabla Z_3}{\beta}\| = o(1). \tag{2.37}$$

Take the inner product of $\frac{Z_3}{\beta}$ with (2.33) in $L^2(\Omega)$, that is

$$\langle i\beta(Z_1 + \tau_1 Z_2 + \frac{\tau_1^2}{2}Z_3), \frac{Z_3}{\beta}\rangle - \langle \kappa\triangle Z_1, \frac{Z_3}{\beta}\rangle - \langle \kappa div(\tau_2(x)\nabla Z_2), \frac{Z_3}{\beta}\rangle = \frac{o(1)}{\beta^2}. \tag{2.38}$$

Integrating by parts, we rewrite (2.38) as

$$\langle i(Z_1 + \tau_1 Z_2 + \frac{\tau_1^2}{2}Z_3), Z_3\rangle + \langle \kappa\nabla Z_1, \frac{\nabla Z_3}{\beta}\rangle + \langle \kappa\tau_2(x)\nabla Z_2, \frac{\nabla Z_3}{\beta}\rangle = \frac{o(1)}{\beta^2}. \tag{2.39}$$

Then we can get

$$\|Z_3\|^2 = o(1). \tag{2.40}$$

Since the other terms on the left-hand of (2.39) converge to zero by (2.34), (2.36) and (2.37). Combining (2.34) and (2.40), we have $\|Z\|_{\mathcal{H}}^2 = o(1)$. This is a contradiction with the assumption $\|Z\|_{\mathcal{H}}^2 = 1$.

If $a(x) \geq a_0 > 0$ on Ω and $k = 0$. Taking the inner product of Z_3 with (2.33) in $L^2(\Omega)$ to get

$$\langle i\beta(Z_1 + \tau_1 Z_2 + \frac{\tau_1^2}{2}Z_3), Z_3\rangle - \langle \kappa\triangle Z_1, Z_3\rangle - \langle \kappa div(\tau_2(x)\nabla Z_2), Z_3\rangle = o(1). \tag{2.41}$$

Integrating by parts, we rewrite (2.41) as

$$\langle i(Z_1 + \tau_1 Z_2 + \frac{\tau_1^2}{2}Z_3), Z_3\rangle + \langle \kappa\nabla Z_1, \frac{\nabla Z_3}{\beta}\rangle + \langle \kappa\tau_2(x)\nabla Z_2, \frac{\nabla Z_3}{\beta}\rangle = o(1). \tag{2.42}$$

By (2.32), we have $\|\frac{\nabla Z_3}{\beta}\| = O(1)$. Then by (2.34), (2.42) is

$$\begin{aligned}
& \langle i(Z_1 + \tau_1 Z_2 + \frac{\tau_1^2}{2}Z_3), Z_3\rangle + o(1) \\
= \ & \langle i(Z_1 + \tau_1 Z_2), Z_3\rangle + \langle i\frac{\tau_1^2}{2}Z_3, Z_3\rangle + o(1) \\
= \ & i\frac{\tau_1^2}{2}\|Z_3\|^2 + o(1) \quad (by\ (2.34)) \\
= \ & o(1), \qquad (2.43)
\end{aligned}$$

i.e.,

$$\|Z_3\|^2 = o(1). \tag{2.44}$$

Combining (2.34) and (2.44), we have $\|Z\|_{\mathcal{H}}^2 = o(1)$. This is a contradiction with the assumption $\|Z\|_{\mathcal{H}}^2 = 1$. □

2.2 Proof of Theorem 2.3

Proof. The proof of (2.17) is similar to the proof in Section 2.1. We will check the condition (2.18). Here we will use special multipliers introduced in [14].

Assume that (2.18) is false. Then by the uniform boundedness theorem, there exist a sequence $\beta \to \infty$ and a unit sequence $Z = (Z_1, Z_2, Z_3)^T \in D(\mathcal{A})$ such that

$$\|(i\beta I - \mathcal{A})Z\|_{\mathcal{H}} \to 0. \tag{2.45}$$

We rewrite (2.45) as

$$\begin{cases} i\beta Z_1 - Z_2 = o(1), \quad in \quad H_0^1(\Omega), & (2.46) \\ i\beta(Z_1 + \frac{\tau_1^2}{2} Z_2) - (Z_2 + \frac{\tau_1^2}{2} Z_3) = o(1), \quad in \quad H_0^1(\Omega), & (2.47) \\ i\beta(Z_1 + \tau_1 Z_2 + \frac{\tau_1^2}{2} Z_3) - \kappa Z_1'' - \kappa(\tau_2(x) Z_2')' = o(1), \quad in \quad L^2(\Omega). & (2.48) \end{cases}$$

Therefore,

$$Re\langle \mathcal{A}Z, Z\rangle_{\mathcal{H}} = \frac{1}{2}\frac{d}{dt}\|Z\|_{\mathcal{H}}^2 = -\kappa\|Z_1'\|^2 - \kappa\tau_1\|a^{\frac{1}{2}}(x) Z_2'\|^2 = o(1). \tag{2.49}$$

Taking the inner product of $\frac{a(x)Z_3}{\beta}$ with (2.48) in $L^2(\Omega)$, we get

$$\langle i\beta(Z_1 + \tau_1 Z_2), \frac{a(x)Z_3}{\beta}\rangle + \frac{\tau_1^2}{2}\langle i\beta Z_3, \frac{a(x)Z_3}{\beta}\rangle - \kappa\langle Z_1'', \frac{a(x)Z_3}{\beta}\rangle - \kappa\langle(\tau_2(x)Z_2')', \frac{a(x)Z_3}{\beta}\rangle = o(1). \tag{2.50}$$

By (2.46) and (2.47), $\|i\beta(Z_1 + \tau_1 Z_2)\| = O(1)$ and $\|\frac{a(x)Z_3}{\beta}\| = o(1)$. Integrating by parts, we rewrite (2.50) as

$$\frac{\tau_1^2}{2} i\|a^{\frac{1}{2}}(x) Z_3\|^2 + \kappa\langle Z_1', \frac{(a(x)Z_3)'}{\beta}\rangle + \kappa\langle \tau_2(x) Z_2', \frac{(a(x)Z_3)'}{\beta}\rangle = o(1). \tag{2.51}$$

As $\|Z_1'\| = o(1)$, $\|Z_3\| = O(1)$ and $\|\beta^{-1} Z_3'\| = \|Z_2'\| + o(1) = O(1)$, we have that

$$\kappa\langle Z_1', \frac{(a(x)Z_3)'}{\beta}\rangle = o(1). \tag{2.52}$$

As for the last term on the left-hand side of (2.50), by (2.47) and (2.49), we can obtain

$$\begin{aligned}\kappa\langle\tau_2(x)Z_2', \frac{(a(x)Z_3)'}{\beta}\rangle &= \kappa\langle\tau_2(x)Z_2', \frac{a'(x)Z_3}{\beta}+\frac{a(x)Z_3'}{\beta}\rangle\\ &= \kappa\langle\tau_2(x)Z_2', \frac{a(x)Z_3'}{\beta}\rangle + o(1)\\ &= \kappa\langle\tau_2(x)a(x)Z_2', Z_2'\rangle + o(1)\\ &= o(1). \end{aligned}\tag{2.53}$$

Combination of (2.51), (2.52) and (2.53) yields

$$\|a^{\frac{1}{2}}(x)Z_3\|^2 = o(1). \tag{2.54}$$

which further leads to, due to (2.47), that

$$\|a^{\frac{1}{2}}(x)\beta Z_2\|^2 = o(1). \tag{2.55}$$

Take $q(x) \in C^{1,1}([0,L];\mathbb{R})$ and $q(0) = 0$. It follows from the inner product of (2.48) with $q(x)(Z_1' + \tau_2(x)Z_2')$ in L^2 that

$$\langle i\beta(Z_1+\tau_1 Z_2+\frac{\tau_1^2}{2}Z_3), q(x)(Z_1'+\tau_2(x)Z_2')\rangle - \kappa\langle(Z_1'+\tau_2(x)Z_2')', q(x)(Z_1'+\tau_2(x)Z_2')\rangle = o(1). \tag{2.56}$$

For the terms on the left-hand side of (2.56), we have

$$\begin{aligned} &\langle i\beta(Z_1+\tau_1 Z_2+\frac{\tau_1^2}{2}Z_3), q(x)(Z_1'+\tau_2(x)Z_2')\rangle\\ =\;& \langle i\beta(Z_1+\tau_1 Z_2), q(x)Z_1'\rangle + \langle i\beta Z_1, q(x)\tau_2(x)Z_2'\rangle + \langle i\beta\tau_1 Z_2, q(x)\tau_2(x)Z_2'\rangle\\ &-\frac{\tau_1^2}{2}\langle\beta^2 Z_2, q(x)(Z_1'+\tau_2(x)Z_2')\rangle + o(1) \quad \textit{(by (2.47))}\\ =\;& \langle Z_2, q(x)\tau_2(x)Z_2'\rangle + \langle i\beta\tau_1 Z_2, q(x)\tau_2(x)Z_2'\rangle - \frac{\tau_1^2}{2}\langle\beta Z_2, q(x)\beta Z_1'\rangle - \frac{\tau_1^2}{2}\langle\beta^2 Z_2, q(x)\tau_2(x)Z_2'\rangle\\ &+o(1) \quad \textit{(by (2.46), (2.47) and dissipation)}\\ =\;& \langle i\beta\tau_1 Z_2, q(x)\tau_2(x)Z_2'\rangle + \frac{\tau_1^2}{2}\langle\beta Z_2, iq(x)Z_2'\rangle - \frac{\tau_1^2}{2}\langle\beta^2 Z_2, q(x)\tau_2(x)Z_2'\rangle\\ &+o(1) \quad \textit{(by (2.46) and } \|Z_2\| = o(1))\\ =\;& \langle i\beta\tau_1 Z_2, q(x)\tau_2(x)Z_2'\rangle - \frac{\tau_1^2}{2}\langle i\beta Z_2, q(x)Z_2'\rangle - \frac{\tau_1^2}{2}\langle\beta^2 Z_2, q(x)\tau_2(x)Z_2'\rangle + o(1). \end{aligned}\tag{2.57}$$

It turns out that the first two inner product terms on the right-hand side of (2.57) can be combined as

$$\frac{1}{2}\langle i\beta\tau_1 Z_2, q(x)a(x)Z_2'\rangle.$$

By dissipation it converges to zero. Thus, (2.57) is

$$\langle i\beta(Z_1+\tau_1 Z_2+\frac{\tau_1^2}{2}Z_3), q(x)(Z_1'+\tau_2(x)Z_2')\rangle = -\frac{\tau_1^2}{2}\langle\beta^2 Z_2, q(x)\tau_2(x)Z_2'\rangle + o(1). \tag{2.58}$$

Then, take the real part of right hand side (2.58)

$$\begin{aligned}
& Re\langle i\beta(Z_1+\tau_1 Z_2+\frac{\tau_1^2}{2}Z_3), q(x)(Z_1'+\tau_2(x)Z_2')\rangle \\
= & -\frac{\tau_1^2}{2}Re\langle \beta^2 Z_2, q(x)\tau_2(x)Z_2'\rangle + o(1) \\
= & \frac{\tau_1^2}{4}\langle \beta^2 Z_2, (q(x)\tau_2(x))' Z_2\rangle + o(1) \\
= & \frac{\tau_1^2}{4}\langle \beta^2 Z_2, (q(x)\tau_2(x))' Z_2\rangle + o(1) \quad (\|Z_2\|^2 = o(1)).
\end{aligned} \tag{2.59}$$

On the other hand,

$$\begin{aligned}
& -\kappa Re\langle (Z_1'+\tau_2(x)Z_2')', q(x)(Z_1'+\tau_2(x)Z_2')\rangle \\
= & \frac{1}{2}\kappa\|\big(q'(x)\big)^{\frac{1}{2}}(Z_1'+\tau_2(x)Z_2')\|^2 - \frac{1}{2}\kappa q(x)|Z_1'+\tau_2(x)Z_2'|^2\Big|_0^L \\
= & \frac{1}{2}\kappa\|\big(q'(x)\big)^{\frac{1}{2}}\tau_2(x)Z_2'\|^2 - \frac{1}{2}\kappa q(L)|Z_1'(L)+\tau_2(L)Z_2'(L)|^2 \\
& +o(1) \quad (\textit{by dissipation}).
\end{aligned} \tag{2.60}$$

Thus, (2.56) can be written as

$$\frac{\tau_1^2}{4}\langle \beta^2 Z_2, (q(x)\tau_2(x))' Z_2\rangle + \frac{1}{2}\kappa\|\big(q'(x)\big)^{\frac{1}{2}}\tau_2(x)Z_2'\|^2 - \frac{1}{2}\kappa q(L)|Z_1'(L)+\tau_2(L)Z_2'(L)|^2 = o(1). \tag{2.61}$$

Let us also take the inner product of (2.48) with $(q(x)\tau_2(x))'Z_2$ in $L^2(\Omega)$ to get

$$\begin{aligned}
& \langle i\beta(Z_1+\tau_1 Z_2), (q(x)\tau_2(x))' Z_2\rangle + \frac{\tau_1^2}{2}\langle i\beta Z_3, (q(x)\tau_2(x))' Z_2\rangle \\
- & \kappa\langle (Z_1'+\tau_2(x)Z_2')', (q(x)\tau_2(x))' Z_2\rangle = o(1).
\end{aligned} \tag{2.62}$$

For the terms on the left-hand side of (2.62), we have

$$\begin{aligned}
& \langle i\beta(Z_1+\tau_1 Z_2), (q(x)\tau_2(x))' Z_2\rangle \\
= & \langle Z_2+\tau_1 Z_3, (q(x)\tau_2(x))' Z_2\rangle + o(1) \quad (\textit{by } (2.46) \textit{ and } (2.47)) \\
= & o(1) \quad (\|Z_2\|^2 = o(1)),
\end{aligned} \tag{2.63}$$

$$\begin{aligned}
& \frac{\tau_1^2}{2}\langle i\beta Z_3, (q(x)\tau_2(x))' Z_2\rangle \\
= & -\frac{\tau_1^2}{2}\langle iZ_3, (q(x)\tau_2(x))'\beta Z_2\rangle \\
= & -\frac{\tau_1^2}{2}\langle \beta^2 Z_2, (q(x)\tau_2(x))' Z_2\rangle + o(1) \quad (\textit{by } (2.47)),
\end{aligned} \tag{2.64}$$

and

$$\begin{aligned}
& -\kappa\langle (Z_1'+\tau_2(x)Z_2')', (q(x)\tau_2(x))' Z_2\rangle \\
= & \kappa\langle \tau_2(x)Z_2', (q(x)\tau_2(x))' Z_2'\rangle + \kappa\langle \tau_2(x)Z_2', (q(x)\tau_2(x))'' Z_2\rangle + o(1) \quad (\textit{by dissipation}) \\
= & \kappa\langle \tau_2(x)Z_2', (q(x)\tau_2(x))' Z_2'\rangle + o(1) \quad (\|Z_2\|^2 = o(1) \textit{ and } \|\tau_2(x)Z_2'\| = O(1)),
\end{aligned} \tag{2.65}$$

we rewrite (2.62) as

$$-\frac{\tau_1^2}{2}\langle \beta^2 Z_2, (q(x)\tau_2(x))' Z_2\rangle + \kappa\langle \tau_2(x) Z_2', (q(x)\tau_2(x))' Z_2'\rangle = o(1). \tag{2.66}$$

Combination of (2.61) and (2.66) yields

$$\kappa \int_0^L \tau_2(x)\big((q(x)\tau_2(x))' + q'(x)\tau_2(x)\big)|Z_2'|^2 dx - \kappa q(L)|Z_1'(L) + \tau_2(L)Z_2'(L)|^2 = o(1). \tag{2.67}$$

Take $q(x) = \frac{1}{2\sqrt{\tau_2(x)}}\int_0^x \frac{1}{\sqrt{\tau_2^3(t)}} a(t)dt$, which is a solution to the first order differential equation $\tau_2(x)\big((q(x)\tau_2(x))' + q'(x)\tau_2(x)\big) = a(x)$. (2.67) then becomes

$$\kappa \int_0^L a(x)|Z_2'|^2 dx - \kappa q(L)|Z_1'(L) + \tau_2(L)Z_2'(L)|^2 = o(1). \tag{2.68}$$

The first term in (2.68) converges to zero by dissipation. Therefore,

$$\kappa q(L)|Z_1'(L) + \tau_2(L)Z_2'(L)|^2 = o(1). \tag{2.69}$$

Take $q(x) = \frac{1}{2\sqrt{\tau_2(x)}}\int_0^x \frac{1}{\sqrt{\tau_2(t)}} dt$ in (2.67), which is a solution to the first order differential equation $(q(x)\tau_2(x))' + q'(x)\tau_2(x) = 1$. Together with (2.69), we thus have

$$\kappa \int_0^L \tau_2(x)|Z_2'|^2 dx = o(1), \tag{2.70}$$

i.e.,

$$\kappa \|\tau_2^{\frac{1}{2}}(x) Z_2'\|^2 dx = o(1). \tag{2.71}$$

Take the inner product of $\frac{Z_3}{\beta}$ with (2.48) in $L^2(\Omega)$, we have

$$\langle i\beta(Z_1 + \tau_1 Z_2 + \frac{\tau_1^2}{2} Z_3), \frac{Z_3}{\beta}\rangle - \langle \kappa Z_1'', \frac{Z_3}{\beta}\rangle - \langle \kappa(\tau_2(x) Z_2')', \frac{Z_3}{\beta}\rangle = o(1). \tag{2.72}$$

Integrating by parts, we rewrite (2.72) as

$$\langle i(Z_1 + \tau_1 Z_2), Z_3\rangle + i\langle \frac{\tau_1^2}{2} Z_3, Z_3\rangle + \langle \kappa Z_1', \frac{Z_3'}{\beta}\rangle + \langle \kappa \tau_2(x) Z_2', \frac{Z_3'}{\beta}\rangle = o(1), \tag{2.73}$$

The first term on the left-side of (2.73) converges to zero by (2.49) and (2.71). Since $\|\frac{Z_3'}{\beta}\| = O(1)$, which together with dissipation and (2.71), we rewrite (2.73) as

$$\|Z_3\|^2 = o(1). \tag{2.74}$$

Then by (2.49), (2.71) and (2.74), we have $\|Z\|_{\mathcal{H}}^2 = o(1)$. This is a contradiction with the assumption $\|Z\|_{\mathcal{H}}^2 = 1$. □

3 Analyticity of (1.5) for spatial dependent τ_ν^* and τ_T

For (1.5) we recall that when $\tau_\nu^* > \kappa^*\tau_q$ the problem is exponential stable, where τ_ν^* is defined $\kappa^*\tau_\nu + \kappa$ (see [21]). On the other hand, depending on the domain, it can be unstable if $\tau_\nu^* < \kappa\tau_q$. In [1], it was also proved that it is exponential stable by energy method when $\tau_\nu^* = \kappa^*\tau_q$. Here, we will consider the spatial dependent $\tau_\nu^*(x)$ and $\tau_T(x)$.

Note that (1.5) can be written as the following system when $\tau_\nu^*(x) \geq \kappa^*\tau_q$ and $\tau_T = \tau_T(x)$:

$$\begin{cases} \ddot{T} + \tau_1\dddot{T} = \kappa^*\triangle T + div(\tau_3(x)\nabla\dot{T}) + \kappa div(\tau_2(x)\nabla\ddot{T}), \quad in \quad \Omega\times(0,\infty), & (3.1)\\ T(x,0) = T^0(x), \dot{T}(x,0) = \dot{T}^0(x), \ddot{T}(x,0) = \ddot{T}^0(x), \quad in \quad \Omega, & (3.2)\\ T(\cdot,t)|_{\partial\Omega} = 0, \quad for \quad t\in[0,\infty), & (3.3)\end{cases}$$

where

$$\tau_1 = \tau_q, \quad \tau_2(x) = \tau_T(x) \quad and \quad \tau_3(x) = \tau_\nu^*(x).$$

Denoting

$$a(x) = \tau_3(x) - \kappa^*\tau_q$$

and assuming that $a(x) \geq 0$ on Ω. We would like to identify a proper state space for the system (3.1)-(3.3) with dissipative "energy". Taking the inner product of $\dot{T} + \tau_1\ddot{T}$ with (3.1) in $L^2(\Omega)$, we have

$$\frac{1}{2}\frac{d}{dt}\|\dot{T}+\tau_1\ddot{T}\|^2 = \langle\kappa^*\triangle T, \dot{T}+\tau_1\ddot{T}\rangle + \langle div(\tau_3(x)\nabla\dot{T}), \dot{T}+\tau_1\ddot{T}\rangle + \langle\kappa div(\tau_2(x)\nabla\ddot{T}), \dot{T}+\tau_1\ddot{T}\rangle. \quad (3.4)$$

Since

$$\begin{aligned}\langle\kappa^*\triangle T, \dot{T}+\tau_1\ddot{T}\rangle &= -\langle\kappa^*\nabla T, \nabla(\dot{T}+\tau_1\ddot{T})\rangle\\ &= -\frac{1}{2}\frac{d}{dt}\kappa^*\|\nabla T\|^2 - \langle\kappa^*\nabla T, \tau_1\nabla\ddot{T}\rangle\\ &= -\frac{1}{2}\frac{d}{dt}\kappa^*\|\nabla T\|^2 - \frac{d}{dt}\langle\kappa^*\nabla T, \tau_1\nabla\dot{T}\rangle + \kappa^*\tau_1\|\nabla\dot{T}\|^2,\end{aligned}$$

$$\begin{aligned}\langle\kappa div(\tau_3(x)\nabla\dot{T}), \dot{T}+\tau_1\ddot{T}\rangle &= -\langle\tau_3(x)\nabla\dot{T}, \nabla(\dot{T}+\tau_1\ddot{T})\rangle\\ &= -\|\tau_3^{\frac{1}{2}}(x)\nabla\dot{T}\|^2 - \frac{1}{2}\frac{d}{dt}\tau_1\|\tau_3^{\frac{1}{2}}(x)\nabla\dot{T}\|^2,\end{aligned}$$

and

$$\begin{aligned}\langle\kappa div(\tau_2(x)\nabla\ddot{T}), \dot{T}+\tau_1\ddot{T}\rangle &= -\langle\kappa\tau_2(x)\nabla\ddot{T}, \nabla(\dot{T}+\tau_1\ddot{T})\rangle\\ &= -\frac{1}{2}\frac{d}{dt}\kappa\|\tau_2^{\frac{1}{2}}(x)\nabla\dot{T}\|^2 - \kappa\tau_1\|\tau_2^{\frac{1}{2}}(x)\nabla\ddot{T}\|^2,\end{aligned}$$

then, (3.4) can be written as

$$\frac{1}{2}\frac{dE(t)}{dt} = -\kappa\tau_1\|\nabla\ddot{T}\|^2 - \|a^{\frac{1}{2}}(x)\nabla\dot{T}\|^2,$$

where

$$E(t) = \kappa^*\|\nabla T\|^2 + \tau_1\|\tau_3^{\frac{1}{2}}(x)\nabla\dot{T}\|^2 + \kappa\|\tau_2^{\frac{1}{2}}(x)\nabla\dot{T}\|^2 + 2\kappa^*\tau_1\langle\nabla T, \nabla\dot{T}\rangle + \|\dot{T} + \tau_1\ddot{T}\|^2$$
$$= \kappa^*\|\nabla T + \tau_1\nabla\dot{T}\|^2 + \tau_1\|a^{\frac{1}{2}}(x)\nabla\dot{T}\|^2 + \kappa\|\tau_2^{\frac{1}{2}}(x)\nabla\dot{T}\|^2 + \|\dot{T} + \tau_1\ddot{T}\|^2.$$

Therefore, we define

$$\mathcal{H}_1 = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega),$$

with inner product

$$\langle Z, W\rangle_{\mathcal{H}_1} :=\kappa^*\langle\nabla Z_1, \nabla W_1\rangle + \tau_1\langle\tau_3^{\frac{1}{2}}(x)\nabla Z_2, \tau_3^{\frac{1}{2}}(x)\nabla W_2\rangle + \kappa\langle\tau_2^{\frac{1}{2}}(x)\nabla Z_2, \tau_2^{\frac{1}{2}}(x)\nabla W_2\rangle$$
$$+ 2\kappa^*\tau_1\langle\nabla Z_1, \nabla Z_2\rangle + \langle Z_2 + \tau_1 Z_3, W_2 + \tau_1 W_3\rangle,$$

i. e.,

$$\|Z\|^2_{\mathcal{H}_1} = \kappa^*\|\nabla Z_1 + \tau_1\nabla Z_2\|^2 + \tau_1\|a^{\frac{1}{2}}(x)\nabla Z_2\|^2 + \kappa\|\tau_2^{\frac{1}{2}}(x)\nabla Z_2\|^2 + \|Z_2 + \tau_1 Z_3\|^2. \tag{3.5}$$

Let

$$Z := (Z_1, Z_2, Z_3)^T = (T, \dot{T}, \ddot{T})^T.$$

We rewrite (3.1)-(3.3) as a first order evolution equation on Hilbert space $\mathcal{H}_1$

$$\begin{cases} \dfrac{dZ}{dt} = \mathcal{A}_1 Z, \\ Z(0) = Z_0, \end{cases}$$

where the operator $\mathcal{A}_1$ is given by

$$\mathcal{A}_1 Z = \begin{pmatrix} Z_2 \\ Z_3 \\ \dfrac{1}{\tau_1}\Big(\kappa^*\triangle Z_1 + div(\tau_3(x)\nabla Z_2) + \kappa div\big(\tau_2(x)\nabla Z_3\big) - Z_3\Big) \end{pmatrix}$$

and

$$D(\mathcal{A}_1) = \{Z = (Z_1, Z_2, Z_3)^T \in \mathcal{H} | Z_1 \in H^2(\Omega), Z_2, Z_3 \in H_0^1(\Omega)\}. \tag{3.6}$$

From (3.4),

$$Re\langle\mathcal{A}_1 Z, Z\rangle_{\mathcal{H}_1} = \frac{1}{2}\frac{d}{dt}\|Z\|^2_{\mathcal{H}_1} = -\|a^{\frac{1}{2}}(x)\nabla Z_2\|^2 - \kappa\tau_1\langle\tau_2(x)\nabla Z_3, \nabla Z_3\rangle \le 0. \tag{3.7}$$

It was proved in [1] that exponential stability of the system (3.1)-(3.3). Similar to Section 2, we can prove that $e^{\mathcal{A}_1 t}$ is a C_0-semigroup of contractions in $\mathcal{H}_1$.

Theorem 3.1. *If $\tau_2(x),\ \tau_3(x) \in C^1(\Omega)$ are strictly positive on $\bar{\Omega}$, and $a(x) \ge 0$ on Ω. The semigroup $e^{\mathcal{A}_1 t}$ is analytic and exponentially stable.*

Remark 3.1. *It is worth noting that the analyticity of solutions implies the exponential stability of solutions and the impossibility of localization of solutions.*

We will use the following theorem to prove Theorem 3.1.

Theorem 3.2. *[5, 18] Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions in a Hilbert space $\mathcal{H}_1$. Suppose that*

$$i\mathbb{R} \subset \rho(\mathcal{A}). \tag{3.8}$$

Then, $S(t)$ is analytic if and only if

$$\overline{\lim}_{|\beta\to\infty|} \|\beta(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{H}_1} < \infty \tag{3.9}$$

holds.

Proof. 1. We first check the condition (3.8).

Assume that (3.8) is false. Suppose that $i\beta \in \sigma(\mathcal{A}_1)$, then there exists a normalized $Z_n = (Z_{1n}, Z_{2n}, Z_{3n})^T$ such that,

$$\|(i\beta I - \mathcal{A}_1)Z_n\|_{\mathcal{H}_1} \to 0. \tag{3.10}$$

For convenience, we will write Z_n as Z.

Thus

$$Re\langle \mathcal{A}_1 Z, Z\rangle_{\mathcal{H}_1} = -\|a^{\frac{1}{2}}(x)\nabla Z_2\|^2 - \kappa\tau_1\|\tau_2^{\frac{1}{2}}(x)\nabla Z_3\|^2 = o(1). \tag{3.11}$$

Then we can rewrite (3.10) as

$$\begin{cases} i\beta(Z_1 + \tau_1 Z_2) - (Z_2 + \tau_1 Z_3) = o(1), \quad in \quad H_0^1(\Omega), & (3.12)\\ i\beta\big(\tau_2^{\frac{1}{2}}(x)\nabla Z_2\big) - \big(\tau_2^{\frac{1}{2}}(x)\nabla Z_3\big) = o(1), \quad in \quad L^2(\Omega), & (3.13)\\ i\beta(Z_2 + \tau_1 Z_3) - \kappa^*\triangle Z_1 - div(\tau_3(x)\nabla Z_2) - \kappa div\big(\tau_2(x)\nabla Z_3\big) = o(1), \quad in \quad L^2(\Omega). & (3.14) \end{cases}$$

From (3.11), since $a(x) \geq 0$, we have

$$\|\tau_2^{\frac{1}{2}}(x)\nabla Z_3\| = o(1). \tag{3.15}$$

By (3.13), (3.15) and the fact β is finite,

$$\|\tau_2^{\frac{1}{2}}(x)\nabla Z_2\| = o(1). \tag{3.16}$$

Moreover, it follows from $\tau_2(x) \geq C > 0$ on Ω, that

$$\|\nabla(Z_2 + \tau_1 Z_3)\| = o(1). \tag{3.17}$$

Together with (3.12), we obtain

$$\|\nabla(Z_1 + \tau_1 Z_2)\| = o(1). \tag{3.18}$$

Therefore, we obtain $\|Z\|_{\mathcal{H}_1} = o(1)$. This is a contradiction with the assumption that $\|Z\|_{\mathcal{H}_1} = 1$. Then we have proved $i\mathbb{R} \subset \rho(\mathcal{A}_1)$.

2. We now check the condition (3.9).

Assume that (3.9) is false. Then by the uniform boundedness theorem, there exist a sequence $\beta \to \infty$ and a unit sequence $Z = (Z_1, Z_2, Z_3)^T \in D(\mathcal{A}_1)$ such that

$$\|(iI - \frac{1}{\beta}\mathcal{A}_1)Z\|_{\mathcal{H}_1} \to 0. \tag{3.19}$$

We can write (3.19) as

$$\begin{cases} i(Z_1 + \tau_1 Z_2) - \frac{1}{\beta}(Z_2 + \tau_1 Z_3) = o(1), \quad in \quad H_0^1(\Omega), & (3.20) \\ i\big(\tau_2^{\frac{1}{2}}(x)\nabla Z_2\big) - \frac{1}{\beta}\big(\tau_2^{\frac{1}{2}}(x)\nabla Z_3\big) = o(1), \quad in \quad L^2(\Omega), & (3.21) \\ i(Z_2 + \tau_1 Z_3) - \frac{1}{\beta}\kappa^*\triangle Z_1 - \frac{1}{\beta}div(\tau_3(x)\nabla Z_2) - \frac{1}{\beta}\kappa div\big(\tau_2(x)\nabla Z_3\big) = o(1), \; in \; L^2(\Omega) & (3.22) \end{cases}$$

From (3.19) and (3.11),

$$Re\langle(iI - \frac{1}{\beta}\mathcal{A}_1)Z, Z\rangle_{\mathcal{H}_1} = -\frac{1}{\beta}Re\langle\mathcal{A}_1 Z, Z\rangle_{\mathcal{H}_1} = \frac{1}{\beta}\|a^{\frac{1}{2}}(x)\nabla Z_2\|^2 + \frac{1}{\beta}\|\tau_2^{\frac{1}{2}}(x)\nabla Z_3\|^2 = o(1), \tag{3.23}$$

which implies that

$$\|\tau_2^{\frac{1}{2}}(x)\nabla Z_2\| = o(1) \tag{3.24}$$

due to (3.21).

Since $\tau_2(x) > C > 0$ on Ω, then we have

$$\frac{1}{\beta}\|\nabla(Z_2 + \tau_1 Z_3)\| = o(1), \tag{3.25}$$

with (3.20), we can get

$$\|\nabla(Z_1 + \tau_1 Z_2)\| = o(1). \tag{3.26}$$

Taking inner product of (3.22) with Z_3, we can obtain

$$i\langle Z_2, Z_3\rangle + i\tau_1\|Z_3\|^2 + \frac{1}{\beta}\kappa^*\langle\nabla Z_1, \nabla Z_3\rangle + \frac{1}{\beta}\langle\tau_3(x)\nabla Z_2, \nabla Z_3\rangle + \frac{1}{\beta}\kappa\langle\tau_2(x)\nabla Z_3, \nabla Z_3\rangle = o(1). \tag{3.27}$$

From the above estimates, all four other inner product terms in (3.27) converge to zero. Hence

$$\|Z_3\|^2 = o(1). \tag{3.28}$$

Then we can obtain

$$\|Z_2 + \tau_1 Z_3\| = o(1). \tag{3.29}$$

Therefore, we obtain $\|Z\|_{\mathcal{H}_1} = o(1)$ again. This is a contradiction with the assumption that $\|Z\|_{\mathcal{H}_1} = 1$. $\square$

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