# BOUNDARY REGULARITY ESTIMATES FOR NONLOCAL ELLIPTIC EQUATIONS IN $C^{1}$ AND $C^{1, \alpha}$ DOMAINS 

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Abstract. We establish sharp boundary regularity estimates in $C^{1}$ and $C^{1, \alpha}$ domains for nonlocal problems of the form $L u=f$ in $\Omega, u=0$ in $\Omega^{c}$. Here, $L$ is a nonlocal elliptic operator of order $2 s$, with $s \in(0,1)$.

First, in $C^{1, \alpha}$ domains we show that all solutions $u$ are $C^{s}$ up to the boundary and that $u / d^{s} \in C^{\alpha}(\bar{\Omega})$, where $d$ is the distance to $\partial \Omega$.

In $C^{1}$ domains, solutions are in general not comparable to $d^{s}$, and we prove a boundary Harnack principle in such domains. Namely, we show that if $u_{1}$ and $u_{2}$ are positive solutions, then $u_{1} / u_{2}$ is bounded and Hölder continuous up to the boundary.

Finally, we establish analogous results for nonlocal equations with bounded measurable coefficients in non-divergence form. All these regularity results will be essential tools in a forthcoming work on free boundary problems for nonlocal elliptic operators CRS15.

## 1. Introduction and results

In this paper we study the boundary regularity of solutions to nonlocal elliptic equations in $C^{1}$ and $C^{1, \alpha}$ domains. The operators we consider are of the form

$$
\begin{equation*}
L u(x)=\int_{\mathbb{R}^{n}}\left(\frac{u(x+y)+u(x-y)}{2}-u(x)\right) \frac{a(y /|y|)}{|y|^{n+2 s}} d y \tag{1.1}
\end{equation*}
$$

with

$$
\begin{equation*}
0<\lambda \leq a(\theta) \leq \Lambda, \quad \theta \in S^{n-1} \tag{1.2}
\end{equation*}
$$

When $a \equiv c t t$, then $L$ is a multiple of the fractional Laplacian $-(-\Delta)^{s}$.
We consider solutions $u \in L^{\infty}\left(\mathbb{R}^{n}\right)$ to

$$
\left\{\begin{align*}
L u & =f \text { in } B_{1} \cap \Omega  \tag{1.3}\\
u & =0
\end{align*} \text { in } B_{1} \backslash \Omega, ~ \$\right.
$$

with $f \in L^{\infty}\left(\Omega \cap B_{1}\right)$ and $0 \in \partial \Omega$.
When $L$ is the Laplacian $\Delta$, then the following are well known results:
(i) If $\Omega$ is $C^{1, \alpha}$, then $u \in C^{1, \alpha}\left(\bar{\Omega} \cap B_{1 / 2}\right)$.
(ii) If $\Omega$ is $C^{1}$, then solutions are in general not $C^{0,1}$.

Still, in $C^{1}$ domains one has the following boundary Harnack principle:

[^0](iii) If $\Omega$ is $C^{1}$, and $u_{1}$ and $u_{2}$ are positive in $\Omega$, with $f \equiv 0$, then $u_{1}$ and $u_{2}$ are comparable in $\bar{\Omega} \cap B_{1 / 2}$, and $u_{1} / u_{2} \in C^{0, \gamma}\left(\bar{\Omega} \cap B_{1 / 2}\right)$ for some small $\gamma>0$.
Actually, (iii) holds in general Lipschitz domains (for $\gamma$ small enough), or even in less regular domains; see [Dah77, BBB91]. Analogous results hold for more general second order operators in non-divergence form $L=\sum_{i, j} a_{i j}(x) \partial_{i j} u$ with bounded measurable coefficients $a_{i j}(x)$ [BB94].

The aim of the present paper is to establish analogous results to (i) and (iii) for nonlocal elliptic operators $L$ of the form (1.1)-(1.2), and also for non-divergence operators with bounded measurable coefficients.
1.1. $C^{1, \alpha}$ domains. When $L=\Delta$ in $(1.3)$ and $\Omega$ is $C^{k, \alpha}$, then solutions $u$ are as regular as the domain $\Omega$ provided that $f$ is regular enough. In particular, if $\Omega$ is $C^{\infty}$ and $f \in C^{\infty}$ then $u \in C^{\infty}(\bar{\Omega})$.

When $L=-(-\Delta)^{s}$, then the boundary regularity is well understood in $C^{1,1}$ and in $C^{\infty}$ domains. In both cases, the optimal Hölder regularity of solutions is $u \in C^{s}(\bar{\Omega})$, and in general one has $u \notin C^{s+\epsilon}(\bar{\Omega})$ for any $\epsilon>0$. Still, higher order estimates are given in terms of the regularity of $u / d^{s}$ : if $\Omega$ is $C^{\infty}$ and $f \in C^{\infty}$ then $u / d^{s} \in C^{\infty}(\bar{\Omega})$; see Grubb Gru15, Gru14]. Here, $d(x)=\operatorname{dist}\left(x, \mathbb{R}^{n} \backslash \Omega\right)$.

We prove here a boundary regularity estimate of order $s+\alpha$ in $C^{1, \alpha}$ domains. Namely, we show that if $\Omega$ is $C^{1, \alpha}$ then $u / d^{s} \in C^{\alpha}(\bar{\Omega})$, as stated below.

We first establish the optimal Hölder regularity up to the boundary, $u \in C^{s}(\bar{\Omega})$.
Proposition 1.1. Let $s \in(0,1), L$ be any operator of the form (1.1)-(1.2), and $\Omega$ be any bounded $C^{1, \alpha}$ domain. Let $u$ be a solution of (1.3). Then,

$$
\|u\|_{C^{s}\left(B_{1 / 2}\right)} \leq C\left(\|f\|_{L^{\infty}\left(B_{1} \cap \Omega\right)}+\|u\|_{L^{\infty}\left(\mathbb{R}^{n}\right)}\right) .
$$

The constant $C$ depends only on $n, s, \Omega$, and ellipticity constants.
Our second result gives a finer description of solutions in terms of the function $d^{s}$, as explained above.

Theorem 1.2. Let $s \in(0,1)$ and $\alpha \in(0, s)$. Let $L$ be any operator of the form (1.1) $-(1.2), \Omega$ be any $C^{1, \alpha}$ domain, and $d$ be the distance to $\partial \Omega$. Let $u$ be a solution of (1.3). Then,

$$
\left\|u / d^{s}\right\|_{C^{\alpha}\left(B_{1 / 2} \cap \bar{\Omega}\right)} \leq C\left(\|f\|_{L^{\infty}\left(B_{1} \cap \Omega\right)}+\|u\|_{L^{\infty}\left(\mathbb{R}^{n}\right)}\right) .
$$

The constant $C$ depends only on $n, s, \alpha, \Omega$, and ellipticity constants.
The previous estimate in $C^{1, \alpha}$ domains was only known for the half-Laplacian $(-\Delta)^{1 / 2}$; see De Silva and Savin DS14. For more general nonlocal operators, such estimate was only known in $C^{1,1}$ domains RS14b.

The proofs of Proposition 1.1 and Theorem 1.2 follow the ideas of RS14b, where the same estimates were established in $C^{1,1}$ domains. One of the main difficulties in the present proofs is the construction of appropriate barriers. Indeed, while any $C^{1,1}$ domain satisfies the interior and exterior ball condition, this is not true anymore
in $C^{1, \alpha}$ domains, and the construction of barriers is more delicate. We will need a careful computation to show that

$$
\left|L\left(d^{s}\right)\right| \leq C d^{\alpha-s} \quad \text { in } \Omega
$$

In fact, since $d^{s}$ is not regular enough to compute $L$, we need to define a new function $\psi$ which behaves like $d$ but it is $C^{2}$ inside $\Omega$, and will show that $\left|L\left(\psi^{s}\right)\right| \leq C d^{\alpha-s}$; see Definition 2.1.

Once we have this, and doing some extra computations we will be able to construct sub and supersolutions which are comparable to $d^{s}$, and thus we will have

$$
|u| \leq C d^{s}
$$

This, combined with interior regularity estimates, will give the $C^{s}$ estimate of Proposition 1.1 .

Then, combining these ingredients with a blow-up and compactness argument in the spirit of RS14b, RS14], we will find the expansion

$$
\left|u(x)-Q(z) d^{s}(x)\right| \leq C|x-z|^{s+\alpha}
$$

at any $z \in \partial \Omega$. And this will yield Theorem 1.2,
1.2. $C^{1}$ domains. In $C^{1}$ domains, in general one does not expect solutions to be comparable to $d^{s}$. In that case, we establish the following.
Theorem 1.3. Let $s \in(0,1)$ and $\alpha \in(0, s)$. Let $L$ be any operator of the form (1.1)-(1.2), and $\Omega$ be any $C^{1}$ domain.

Then, there exists is $\delta>0$, depending only on $\alpha, n, s, \Omega$, and ellipticity constants, such that the following statement holds.

Let $u_{1}$ and $u_{2}$, be viscosity solutions of (1.3) with right hand sides $f_{1}$ and $f_{2}$, respectively. Assume that $\left\|f_{i}\right\|_{L^{\infty}\left(B_{1} \cap \Omega\right)} \leq C_{0}$ (with $\left.C_{0} \geq \delta\right),\left\|u_{i}\right\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \leq C_{0}$,

$$
f_{i} \geq-\delta \quad \text { in } \quad B_{1} \cap \Omega
$$

and that

$$
u_{i} \geq 0 \quad \text { in } \quad \mathbb{R}^{n}, \quad \sup _{B_{1 / 2}} u_{i} \geq 1
$$

Then,

$$
\left\|u_{1} / u_{2}\right\|_{C^{\alpha}\left(\Omega \cap B_{1 / 2}\right)} \leq C C_{0}, \quad \alpha \in(0, s)
$$

where $C$ depends only on $\alpha, n, s, \Omega$, and ellipticity constants.
We expect the range of exponents $\alpha \in(0, s)$ to be optimal.
In particular, the previous result yields a boundary Harnack principle in $C^{1}$ domains.

Corollary 1.4. Let $s \in(0,1), L$ be any operator of the form $\sqrt{1.1})-(1.2)$, and $\Omega$ be any $C^{1}$ domain. Let $u_{1}$ and $u_{2}$, be viscosity solutions of

$$
\left\{\begin{aligned}
L u_{1}=L u_{2} & =0
\end{aligned} \quad \text { in } B_{1} \cap \Omega,\right.
$$

Assume that

$$
u_{1} \geq 0 \quad \text { and } \quad u_{2} \geq 0 \quad \text { in } \quad \mathbb{R}^{n}
$$

and that $\sup _{B_{1 / 2}} u_{1}=\sup _{B_{1 / 2}} u_{2}=1$. Then,

$$
0<C^{-1} \leq \frac{u_{1}}{u_{2}} \leq C \quad \text { in } \quad B_{1 / 2}
$$

where $C$ depends only on $n, s, \Omega$, and ellipticity constants.
Theorems 1.3 and 1.2 will be important tools in a forthcoming work on free boundary problems for nonlocal elliptic operators [RS15]. Namely, Theorem 1.3 (applied to the derivatives of the solution to the free boundary problem) will yield that $C^{1}$ free boundaries are in fact $C^{1, \alpha}$, and then thanks to Theorem 1.2 we will get a fine description of solutions in terms of $d^{s}$.
1.3. Equations with bounded measurable coefficients. We also obtain estimates for equations with bounded measurable coefficients,

$$
\left\{\begin{array}{rlrl}
M^{+} u & \geq-K_{0} & \text { in } B_{1} \cap \Omega  \tag{1.4}\\
M^{-} u & \leq K_{0} & & \text { in } B_{1} \cap \Omega \\
u & =0 & & \text { in } B_{1} \backslash \Omega
\end{array}\right.
$$

Here, $M^{+}$and $M^{-}$are the extremal operators associated to the class $\mathcal{L}_{*}$, consisting of all operators of the form (1.1)-(1.2), i.e.,

$$
M^{+}:=M_{\mathcal{L}_{*}}^{+} u=\sup _{L \in \mathcal{L}_{*}} L u, \quad M^{-}:=M_{\mathcal{L}_{*}}^{+} u=\inf _{L \in \mathcal{L}_{*}} L u
$$

Notice that the equation (1.4) is an equation with bounded measurable coefficients, and it is the nonlocal analogue of

$$
a_{i j}(x) \partial_{i j} u=f(x), \quad \text { with } \quad \lambda \operatorname{Id} \leq\left(a_{i j}(x)\right)_{i j} \leq \Lambda \operatorname{Id}, \quad|f(x)| \leq K_{0}
$$

For nonlocal equations with bounded measurable coefficients in $C^{1, \alpha}$ domains, we show the following.

Here, and throughout the paper, we denote $\bar{\alpha}=\bar{\alpha}(n, s, \lambda, \Lambda)>0$ the exponent in [RS14, Proposition 5.1].

Theorem 1.5. Let $s \in(0,1)$ and $\alpha \in(0, \bar{\alpha})$. Let $\Omega$ be any $C^{1, \alpha}$ domain, and $d$ be the distance to $\partial \Omega$. Let $u \in C\left(B_{1}\right)$ be any viscosity solution of (1.4). Then, we have

$$
\left\|u / d^{s}\right\|_{C^{\alpha}\left(B_{1 / 2} \cap \bar{\Omega}\right)} \leq C\left(K_{0}+\|u\|_{L^{\infty}\left(\mathbb{R}^{n}\right)}\right)
$$

The constant $C$ depends only on $n, s, \alpha, \Omega$, and ellipticity constants.
In $C^{1}$ domains we prove:
Theorem 1.6. Let $s \in(0,1)$ and $\alpha \in(0, \bar{\alpha})$. Let $\Omega$ be any $C^{1}$ domain.
Then, there exists is $\delta>0$, depending only on $\alpha, n, s, \Omega$, and ellipticity constants, such that the following statement holds.

Let $u_{1}$ and $u_{2}$, be functions satisfying

$$
\left\{\begin{aligned}
M^{+}\left(a u_{1}+b u_{2}\right) & \geq-\delta(|a|+|b|) & & \text { in } B_{1} \cap \Omega \\
u_{1}=u_{2} & =0 & & \text { in } B_{1} \backslash \Omega
\end{aligned}\right.
$$

for any $a, b \in \mathbb{R}$. Assume that

$$
\begin{aligned}
& u_{i} \geq 0 \quad \text { in } \mathbb{R}^{n} \\
&\left\|u_{i}\right\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \leq C_{0}, \text { and that } \sup _{B_{1 / 2}} u_{i} \geq 1 . \text { Then, we have } \\
&\left\|u_{1} / u_{2}\right\|_{C^{\alpha}\left(\Omega \cap B_{1 / 2}\right)} \leq C
\end{aligned}
$$

where $C$ depends only on $\alpha, n, s, \Omega$, and ellipticity constants.
The Boundary Harnack principle for nonlocal operators has been widely studied, and in some cases it is even known in general open sets; see Bogdan Bog97, Song-Wu [SW99, Bogdan-Kulczycki-Kwasnicki [BKK08], and Bogdan-KumagaiKwasnicki [BKK15]. The main differences between our Theorems 1.3 .1 .6 and previous known results are the following. On the one hand, our results allow a right hand side on the equation (1.3), and apply also to viscosity solutions of equations with bounded measurable coefficients (1.4). On the other hand, we obtain a higher order estimate, in the sense that for linear equations we prove that $u_{1} / u_{2}$ is $C^{\alpha}$ for all $\alpha \in(0, s)$. Finally, the proof we present here is perturbative, in the sense the we make a blow-up and use that after the rescaling the domain will be a half-space. This allows us to get a higher order estimate for $u_{1} / u_{2}$, but requires the domain to be at least $C^{1}$.

The paper is organized as follows. In Section 2 we construct the barriers in $C^{1, \alpha}$ domains. Then, in Section 3 we prove the regularity of solutions in $C^{1, \alpha}$ domains, that is, Proposition 1.1 and Theorems 1.2 and 1.5. In Section 4 we construct the barriers needed in the analysis on $C^{1}$ domains. Finally, in Section 5 we prove Theorems 1.3 and 1.6 .

## 2. Barriers: $C^{1, \alpha}$ Domains

Throughout this section, $\Omega$ will be any bounded and $C^{1, \alpha}$ domain, and

$$
d(x)=\operatorname{dist}\left(x, \mathbb{R}^{n} \backslash \Omega\right)
$$

Since $d$ is only $C^{1, \alpha}$ inside $\Omega$, we need to consider the following "regularized version" of $d$.

Definition 2.1. Given a $C^{1, \alpha}$ domain $\Omega$, we consider a fixed function $\psi$ satisfying

$$
\begin{gather*}
C^{-1} d \leq \psi \leq C d  \tag{2.1}\\
\|\psi\|_{C^{1, \alpha}(\bar{\Omega})} \leq C \quad \text { and } \quad\left|D^{2} \psi\right| \leq C d^{\alpha-1} \tag{2.2}
\end{gather*}
$$

with $C$ depending only on $\Omega$.

Remark 2.2. Notice that to construct $\psi$ one may take for example the solution to $-\Delta \psi=1$ in $\Omega, \psi=0$ on $\partial \Omega$, extended by $\psi=0$ in $\mathbb{R}^{n} \backslash \Omega$.

Note also that any $C^{1, \alpha}$ domain $\Omega$ can be locally represented as the epigraph of a $C^{1, \alpha}$ function. More precisely, there is a $\rho_{0}>0$ such that for all $z \in \partial \Omega$ the set $\partial \Omega \cap B_{\rho_{0}}(z)$ is, after a rotation, the graph of a $C^{1, \alpha}$ function. Then, the constant $C$ in (2.1)-2.2) can be taken depending only on $\rho_{0}$ and on the $C^{1, \alpha}$ norms of these functions.

We want to show the following.
Proposition 2.3. Let $s \in(0,1)$ and $\alpha \in(0, s)$, L be given by (1.1)-(1.2), and $\Omega$ be any $C^{1, \alpha}$ domain. Let $\psi$ be given by Definition 2.1. Then,

$$
\begin{equation*}
\left|L\left(\psi^{s}\right)\right| \leq C d^{\alpha-s} \quad \text { in } \Omega \tag{2.3}
\end{equation*}
$$

The constant $C$ depends only on $s, n, \Omega$, and ellipticity constants.
For this, we need a couple of technical Lemmas. The first one reads as follows.
Lemma 2.4. Let $\Omega$ be any $C^{1, \alpha}$ domain, and $\psi$ be given by Definition 2.1. Then, for each $x_{0} \in \Omega$ we have

$$
\left|\psi\left(x_{0}+y\right)-\left(\psi\left(x_{0}\right)+\nabla \psi\left(x_{0}\right) \cdot y\right)_{+}\right| \leq C|y|^{1+\alpha} \quad \text { for } y \in \mathbb{R}^{n}
$$

The constant $C$ depends only on $\Omega$.
Proof. Let us consider $\tilde{\psi}$, a $C^{1, \alpha}\left(\mathbb{R}^{n}\right)$ extension of $\left.\psi\right|_{\Omega}$ satisfying $\tilde{\psi} \leq 0$ in $\mathbb{R}^{n} \backslash \Omega$. Then, since $\tilde{\psi} \in C^{1, \alpha}\left(\mathbb{R}^{n}\right)$ we clearly have

$$
\left|\tilde{\psi}(x)-\psi\left(x_{0}\right)-\nabla \psi\left(x_{0}\right) \cdot\left(x-x_{0}\right)\right| \leq C\left|x-x_{0}\right|^{1+\alpha}
$$

in all of $\mathbb{R}^{n}$. Here we used $\tilde{\psi}\left(x_{0}\right)=\psi\left(x_{0}\right)$ and $\nabla \tilde{\psi}\left(x_{0}\right)=\nabla \psi\left(x_{0}\right)$.
Now, using that $\left|a_{+}-b_{+}\right| \leq|a-b|$, combined with $(\tilde{\psi})_{+}=\psi$, we find

$$
\left|\psi(x)-\left(\psi\left(x_{0}\right)+\nabla \psi\left(x_{0}\right) \cdot\left(x-x_{0}\right)\right)_{+}\right| \leq C\left|x-x_{0}\right|^{1+\alpha}
$$

for all $x \in \mathbb{R}^{n}$. Thus, the lemma follows.
The second one reads as follows.
Lemma 2.5. Let $\Omega$ be any $C^{1, \alpha}$ domain, $p \in \Omega$, and $\rho=d(p) / 2$. Let $\gamma>-1$ and $\beta \neq \gamma$. Then,

$$
\int_{B_{1} \backslash B_{\rho / 2}} d^{\gamma}(p+y) \frac{d y}{|y|^{n+\beta}} \leq C\left(1+\rho^{\gamma-\beta}\right)
$$

The constant $C$ depends only on $\gamma, \beta$, and $\Omega$.
Proof. The proof is similar to that of [RV15, Lemma 4.2].
First, we may assume $p=0$.

Notice that, since $\Omega$ is $C^{1, \alpha}$, then there is $\kappa_{*}>0$ such that for any $t \in\left(0, \kappa_{*}\right]$ the level set $\{d=t\}$ is $C^{1, \alpha}$. Since

$$
\begin{equation*}
\int_{\left(B_{1} \backslash B_{\rho}\right) \cap\left\{d \geq \kappa_{*}\right\}} d^{\gamma}(y) \frac{d y}{|y|^{n+\beta}} \leq C \tag{2.4}
\end{equation*}
$$

then we just have to bound the same integral in the set $\left\{d<\kappa_{*}\right\}$. Here we used that $B_{r} \cap\left\{d \geq \kappa_{*}\right\}=\emptyset$ if $r \leq \kappa_{*}-2 \rho$, which follows from the fact that $d(0)=2 \rho$.

We will use the following estimate for $t \in\left(0, \kappa_{*}\right)$

$$
\mathcal{H}^{n-1}\left(\{d=t\} \cap\left(B_{2^{-k+1}} \backslash B_{2^{-k}}\right)\right) \leq C\left(2^{-k}\right)^{n-1}
$$

which follows for example from the fact that $\{d=t\}$ is $C^{1, \alpha}$ (see the Appendix in [RV15]). Note also that $\{d=t\} \cap B_{r}=\emptyset$ if $t>r+2 \rho$.

Let $M \geq 0$ be such that $2^{-M} \leq \rho \leq 2^{-M+1}$. Then, using the coarea formula,

$$
\begin{align*}
& \int_{\left(B_{1} \backslash B_{\rho}\right) \cap\left\{d<\kappa_{*}\right\}} d^{\gamma}(y) \frac{d y}{|y|^{n+\beta}} \leq \\
& \leq \sum_{k=0}^{M} \frac{1}{2^{-k(n+\beta)}} \int_{\left(B_{2^{-k+1} \backslash B_{2}-k}\right) \cap\left\{d<C 2^{-k}\right\}} d^{\gamma}(y)|\nabla d(y)| d y \\
& \leq \sum_{k=0}^{M} \frac{1}{2^{-k(n+\beta)}} \int_{0}^{C 2^{-k}} t^{\gamma} d t \int_{\left(B_{2}-k+1 \backslash B_{2-k}\right) \cap\{d=t\}} d \mathcal{H}^{n-1}(y)  \tag{2.5}\\
& \leq C \sum_{k=0}^{M} \frac{\left(2^{-k}\right)^{\gamma+1} 2^{-k(n-1)}}{2^{-k(n+\beta)}}=C \sum_{k=0}^{M} 2^{k(\beta-\gamma)}=C\left(1+\rho^{\gamma-\beta}\right) .
\end{align*}
$$

Here we used that $\gamma \neq \beta$-in case $\gamma=\beta$ we would get $C(1+|\log \rho|)$.
Combining (2.4) and (2.5), the lemma follows.
We now give the:
Proof of Proposition 2.3. Let $x_{0} \in \Omega$ and $\rho=d(x)$.
Notice that when $\rho \geq \rho_{0}>0$ then $\psi^{s}$ is smooth in a neighborhood of $x_{0}$, and thus $L\left(\psi^{s}\right)\left(x_{0}\right)$ is bounded by a constant depending only on $\rho_{0}$. Thus, we may assume that $\rho \in\left(0, \rho_{0}\right)$, for some small $\rho_{0}$ depending only on $\Omega$.

Let us denote

$$
\ell(x)=\left(\psi\left(x_{0}\right)+\nabla \psi\left(x_{0}\right) \cdot\left(x-x_{0}\right)\right)_{+},
$$

which satisfies

$$
L\left(\ell^{s}\right)=0 \quad \text { in } \quad\{\ell>0\}
$$

see [RS14, Section 2].
Now, notice that

$$
\psi\left(x_{0}\right)=\ell\left(x_{0}\right) \quad \text { and } \quad \nabla \psi\left(x_{0}\right)=\nabla \ell\left(x_{0}\right) .
$$

Moreover, by Lemma 2.4 we have

$$
\left|\psi\left(x_{0}+y\right)-\ell\left(x_{0}+y\right)\right| \leq C|y|^{1+\alpha}
$$

and using $\left|a^{s}-b^{s}\right| \leq C|a-b|\left(a^{s-1}+b^{s-1}\right)$ for $a, b \geq 0$, we find

$$
\begin{equation*}
\left|\psi^{s}\left(x_{0}+y\right)-\ell^{s}\left(x_{0}+y\right)\right| \leq C|y|^{1+\alpha}\left(d^{s-1}\left(x_{0}+y\right)+\ell^{s-1}\left(x_{0}+y\right)\right) \tag{2.6}
\end{equation*}
$$

Here, we used that $\psi \leq C d$.
On the other hand, since $\psi \in C^{1, \alpha}(\bar{\Omega})$ and $\psi \geq c d$ in $\bar{\Omega}$, then it is not difficult to check that

$$
\ell>0 \quad \text { in } \quad B_{\rho / 2}\left(x_{0}\right),
$$

provided that $\rho_{0}$ is small (depending only on $\Omega$ ). Thanks to this, one may estimate

$$
\left|D^{2}\left(\psi^{s}-\ell^{s}\right)\right| \leq C \rho^{s+\alpha-2} \quad \text { in } B_{\rho / 2},
$$

and thus

$$
\begin{equation*}
\left|\psi^{s}-\ell^{s}\right|\left(x_{0}+y\right) \leq\left\|D^{2}\left(\psi^{s}-\ell^{s}\right)\right\|_{L^{\infty}\left(B_{\rho / 2}\left(x_{0}\right)\right)}|y|^{2} \leq C \rho^{s+\alpha-2}|y|^{2} \tag{2.7}
\end{equation*}
$$

for $y \in B_{\rho / 2}$.
Therefore, it follows from (2.6) and (2.7) that

$$
\left|\psi^{s}-\ell^{s}\right|\left(x_{0}+y\right) \leq\left\{\begin{array}{lr}
C \rho^{s+\alpha-2}|y|^{2} & \text { for } y \in B_{\rho / 2} \\
C|y|^{1+\alpha}\left(d^{s-1}\left(x_{0}+y\right)+\ell^{s-1}\left(x_{0}+y\right)\right) & \text { for } y \in B_{1} \backslash B_{\rho / 2} \\
C|y|^{s} & \text { for } y \in \mathbb{R}^{n} \backslash B_{1}
\end{array}\right.
$$

Hence, recalling that $L\left(\ell^{s}\right)\left(x_{0}\right)=0$, we find

$$
\begin{aligned}
\left|L\left(\psi^{s}\right)\left(x_{0}\right)\right|= & \left|L\left(\psi^{s}-\ell^{s}\right)\left(x_{0}\right)\right| \\
= & \int_{\mathbb{R}^{n}}\left|\psi^{s}-\ell^{s}\right|\left(x_{0}+y\right) \frac{a(y /|y|)}{|y|^{n+2 s}} d y \\
\leq & \int_{B_{\rho / 2}} C \rho^{s+\alpha-2}|y|^{2} \frac{d y}{|y|^{n+2 s}}+\int_{\mathbb{R}^{n} \backslash B_{1}} C|y|^{s} \frac{d y}{|y|^{n+2 s}}+ \\
& +\int_{B_{1} \backslash B_{\rho / 2}} C|y|^{1+\alpha}\left(d^{s-1}\left(x_{0}+y\right)+\ell^{s-1}\left(x_{0}+y\right)\right) \frac{d y}{|y|^{n+2 s}} \\
\leq & C\left(\rho^{\alpha-s}+1\right)+C \int_{B_{1} \backslash B_{\rho / 2}}\left(d^{s-1}\left(x_{0}+y\right)+\ell^{s-1}\left(x_{0}+y\right)\right) \frac{d y}{|y|^{n+2 s-1-\alpha}} .
\end{aligned}
$$

Thus, using Lemma 2.5 twice, we find

$$
\left|L\left(\psi^{s}\right)\left(x_{0}\right)\right| \leq C \rho^{\alpha-s}
$$

and (2.3) follows.
When $\alpha>s$ the previous proof gives the following result, which states that for any operator (1.1)-(1.2) one has $L\left(d^{s}\right) \in L^{\infty}(\Omega)$. Here, as in Gru15, RS14b, RS14, $d$ denotes a fixed function that coincides with $\operatorname{dist}\left(x, \mathbb{R}^{n} \backslash \Omega\right)$ in a neighborhood of $\partial \Omega$, satisfies $d \equiv 0$ in $\mathbb{R}^{n} \backslash \Omega$, and it is $C^{1, \alpha}$ in $\Omega$.

Proposition 2.6. Let $s \in(0,1)$, $L$ be given by $\sqrt{1.1)}-(\sqrt{1.2})$, and $\Omega$ be any bounded $C^{1, \alpha}$ domain, with $\alpha>s$. Then,

$$
\left|L\left(d^{s}\right)\right| \leq C \quad \text { in } \Omega
$$

The constant $C$ depends only on $n, s, \Omega$, and ellipticity constants.
To our best knowledge, this result was only known in case that $L$ is the fractional Laplacian and $\Omega$ is $C^{1,1}$, or in case that $a \in C^{\infty}\left(S^{n-1}\right)$ in (1.1) and $\Omega$ is $C^{\infty}$ (in this case $L\left(d^{s}\right)$ is $C^{\infty}(\bar{\Omega})$; see Gru15).

Also, recall that for a general stable operator (1.1) (with $a \in L^{1}\left(S^{n-1}\right)$ and without the assumption (1.2)) the result is false, since we constructed in [RS14b] an operator $L$ and a $C^{\infty}$ domain $\Omega$ for which $L\left(d^{s}\right) \notin L^{\infty}(\Omega)$. Hence, the assumption (1.2) is somewhat necessary for Proposition 2.6 to be true.

Proof of Proposition 2.6. Let $x_{0} \in \Omega$, and $\rho=d(x)$.
Notice that when $\rho \geq \rho_{0}>0$ then $d^{s}$ is $C^{1+s}$ in a neighborhood of $x_{0}$, and thus $L\left(d^{s}\right)\left(x_{0}\right)$ is bounded by a constant depending only on $\rho_{0}$. Thus, we may assume that $\rho \in\left(0, \rho_{0}\right)$, for some small $\rho_{0}$ depending only on $\Omega$.

Let us denote

$$
\ell(x)=\left(d\left(x_{0}\right)+\nabla d\left(x_{0}\right) \cdot\left(x-x_{0}\right)\right)_{+},
$$

which satisfies

$$
L\left(\ell^{s}\right)=0 \quad \text { in } \quad\{\ell>0\} .
$$

Moreover, as in Proposition 2.3, we have

$$
\begin{equation*}
\left|d^{s}\left(x_{0}+y\right)-\ell^{s}\left(x_{0}+y\right)\right| \leq C|y|^{1+\alpha}\left(d^{s-1}\left(x_{0}+y\right)+\ell^{s-1}\left(x_{0}+y\right)\right) . \tag{2.8}
\end{equation*}
$$

In particular,

$$
\left|d^{s}\left(x_{0}+y\right)-\ell^{s}\left(x_{0}+y\right)\right| \leq C \rho^{s-1}|y|^{1+\alpha} \quad \text { for } y \in B_{\rho / 2}
$$

Hence, recalling that $L\left(\ell^{s}\right)\left(x_{0}\right)=0$, we find

$$
\begin{aligned}
\left|L\left(\psi^{s}\right)\left(x_{0}\right)\right|= & \left|L\left(\psi^{s}-\ell^{s}\right)\left(x_{0}\right)\right| \\
= & \int_{\mathbb{R}^{n}}\left|\psi^{s}-\ell^{s}\right|\left(x_{0}+y\right) \frac{a(y /|y|)}{|y|^{n+2 s}} d y \\
\leq & \int_{B_{\rho / 2}} C \rho^{s-1}|y|^{1+\alpha} \frac{d y}{|y|^{n+2 s}}+\int_{\mathbb{R}^{n} \backslash B_{1}} C|y|^{s} \frac{d y}{|y|^{n+2 s}}+ \\
& +\int_{B_{1} \backslash B_{\rho / 2}} C|y|^{1+\alpha}\left(d^{s-1}\left(x_{0}+y\right)+\ell^{s-1}\left(x_{0}+y\right)\right) \frac{d y}{|y|^{n+2 s}} \\
\leq & C\left(1+\rho^{\alpha-s}\right) .
\end{aligned}
$$

Here we used Lemma 2.5. Since $\alpha>s$, the result follows.
We next show the following.

Lemma 2.7. Let $s \in(0,1)$, $L$ be given by $\sqrt{1.1})-(1.2)$, and $\Omega$ be any $C^{1, \alpha}$ domain. Let $\psi$ be given by Definition 2.1. Then, for any $\epsilon \in(0, \alpha)$, we have

$$
\begin{equation*}
L\left(\psi^{s+\epsilon}\right) \geq c d^{\epsilon-s}-C \quad \text { in } \Omega \cap B_{1 / 2} \tag{2.9}
\end{equation*}
$$

with $c>0$. The constants $c$ and $C$ depend only on $\epsilon, s, n, \Omega$, and ellipticity constants.

Proof. Exactly as in Proposition 2.3, one finds that

$$
\begin{equation*}
\left|\psi^{s+\epsilon}\left(x_{0}+y\right)-\ell^{s+\epsilon}\left(x_{0}+y\right)\right| \leq C|y|^{1+\alpha}\left(d^{s+\epsilon-1}\left(x_{0}+y\right)+\ell^{s+\epsilon-1}\left(x_{0}+y\right)\right), \tag{2.10}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\psi^{s+\epsilon}-\ell^{s+\epsilon}\right|\left(x_{0}+y\right) \leq C \rho^{s+\epsilon+\alpha-2}|y|^{2} \tag{2.11}
\end{equation*}
$$

for $y \in B_{\rho / 2}$. Therefore, as in Proposition 2.3,

$$
\left|L\left(\psi^{s+\epsilon}-\ell^{s+\epsilon}\right)\left(x_{0}\right)\right| \leq C\left(1+\rho^{\alpha+\epsilon-s}\right) .
$$

We now use that, by homogeneity, we have

$$
L\left(\ell^{s+\epsilon}\right)\left(x_{0}\right)=\kappa \rho^{\epsilon-s},
$$

with $\kappa>0$ (see [RS14]). Thus, combining the previous two inequalities we find

$$
L\left(\psi^{s+\epsilon}\right)\left(x_{0}\right) \geq \kappa \rho^{\epsilon-s}-C\left(1+\rho^{\alpha+\epsilon-s}\right) \geq \frac{\kappa}{2} \rho^{s-\epsilon}-C
$$

as desired.
We now construct sub and supersolutions.
Lemma 2.8 (Supersolution). Let $s \in(0,1), L$ be given by (1.1)-(1.2), and $\Omega$ be any bounded $C^{1, \alpha}$ domain. Then, there exists $\rho_{0}>0$ and a function $\phi_{1}$ satisfying

$$
\left\{\begin{array}{rll}
L \phi_{1} & \leq-1 & \text { in } \Omega \cap\left\{d \leq \rho_{0}\right\} \\
C^{-1} d^{s} \leq d^{s} & \leq & \text { in } \Omega \\
\phi_{1} & \leq 0 & \text { in } \mathbb{R}^{n} \backslash \Omega .
\end{array}\right.
$$

The constants $C$ and $\rho_{0}$ depend only on $n, s, \Omega$, and ellipticity constants.
Proof. Let $\psi$ be given by Definition 2.1, and let $\epsilon=\frac{\alpha}{2}$. Then, by Proposition 2.3 we have

$$
-C_{0} d^{\alpha-s} \leq L\left(\psi^{s}\right) \leq C_{0} d^{\alpha-s}
$$

and by Lemma 2.7

$$
L\left(\psi^{s+\epsilon}\right) \geq c_{0} d^{\epsilon-s}-C_{0} .
$$

Next, we consider the function

$$
\phi_{1}=\psi^{s}-c \psi^{s+\epsilon},
$$

with $c$ small enough. Then, $\phi_{1}$ satisfies

$$
\begin{equation*}
L \phi_{1} \leq C_{0} d^{\alpha-s}+C_{0}-c c_{1} d^{\epsilon-s} \leq-1 \quad \text { in } \quad \Omega \cap\left\{d \leq \rho_{0}\right\} \tag{2.12}
\end{equation*}
$$

for some $\rho_{0}>0$. Finally, by construction we clearly have

$$
C^{-1} d^{s} \leq \phi_{1} \leq C d^{s} \quad \text { in } \quad \Omega,
$$

and thus the Lemma is proved.
Notice that the previous proof gives in fact the following.
Lemma 2.9. Let $s \in(0,1), L$ be given by $\sqrt{1.1})-(\sqrt{1.2})$, and $\Omega$ be any bounded $C^{1, \alpha}$ domain. Then, there exist $\rho_{0}>0$ and a function $\phi_{1}$ satisfying

$$
\left\{\begin{array}{rlrl}
L \phi_{1} & \leq-d^{\epsilon-s} & & \text { in } \Omega \cap\left\{d \leq \rho_{0}\right\} \\
C^{-1} d^{s} \leq \phi_{1} & \leq C d^{s} & & \text { in } \Omega \\
\phi_{1} & =0 & \text { in } \mathbb{R}^{n} \backslash \Omega
\end{array}\right.
$$

The constants $C$ and $\rho_{0}$ depend only on $n, s, \Omega$, and ellipticity constants.
Proof. The proof is the same as Lemma 2.8; see (2.12).
We finally construct a subsolution.
Lemma 2.10 (Subsolution). Let $s \in(0,1), L$ be given by (1.1)-(1.2), and $\Omega$ be any bounded $C^{1, \alpha}$ domain. Then, for each $K \subset \subset \Omega$ there exists a function $\phi_{2}$ satisfying

$$
\left\{\begin{array}{rlrl}
L \phi_{2} & \geq 1 & & \text { in } \Omega \backslash K \\
C^{-1} d^{s} \leq \phi_{2} & \leq C d^{s} & \text { in } \Omega \\
\phi_{2} & =0 & & \text { in } \mathbb{R}^{n} \backslash \Omega
\end{array}\right.
$$

The constants $c$ and $C$ depend only on $n, s, \Omega, K$, and ellipticity constants.
Proof. First, notice that if $\eta \in C_{c}^{\infty}(K)$ then $L \eta \geq c_{1}>0$ in $\Omega \backslash K$. Hence,

$$
\phi_{2}=\psi^{s}+\psi^{s+\epsilon}+C \eta
$$

satisfies

$$
L \phi_{2} \geq-C_{0} d^{\alpha-s}+c_{0} d^{\epsilon-s}-C_{0}+C c_{1} \geq 1 \quad \text { in } \quad \Omega \backslash K
$$

provided that $C$ is chosen large enough.

## 3. Regularity in $C^{1, \alpha}$ domains

The aim of this section is to prove Proposition 1.1 and Theorem 1.2 ,
3.1. Hölder regularity up to the boundary. We will prove first the following result, which is similar to Proposition 1.1 but allows $u$ to grow at infinity and $f$ to be singular near $\partial \Omega$.

Proposition 3.1. Let $s \in(0,1), L$ be any operator of the form (1.1)-(1.2), and $\Omega$ be any bounded $C^{1, \alpha}$ domain. Let $u$ be a solution to (1.3), and assume that

$$
|f| \leq C d^{\epsilon-s} \quad \text { in } \quad \Omega
$$

Then,

$$
\|u\|_{C^{s}\left(B_{1 / 2}\right)} \leq C\left(\left\|d^{s-\epsilon} f\right\|_{L^{\infty}\left(B_{1} \cap \Omega\right)}+\sup _{R \geq 1} R^{\delta-2 s}\|u\|_{L^{\infty}\left(B_{R}\right)}\right)
$$

The constant $C$ depends only on $n, s, \epsilon, \delta, \Omega$, and ellipticity constants.

Proof. Dividing by a constant, we may assume that

$$
\left\|d^{s-\epsilon} f\right\|_{L^{\infty}\left(B_{1} \cap \Omega\right)}+\sup _{R \geq 1} R^{\delta-2 s}\|u\|_{L^{\infty}\left(B_{R}\right)} \leq 1
$$

Then, the truncated function $w=u \chi_{B_{1}}$ satisfies

$$
|L w| \leq C d^{\epsilon-s} \quad \text { in } \quad \Omega \cap B_{3 / 4}
$$

$w \leq 1$ in $B_{1}$, and $w \equiv 0$ in $\mathbb{R}^{n} \backslash B_{1}$.
Let $\widetilde{\Omega}$ be a bounded $C^{1, \alpha}$ domain satisfying: $B_{1} \cap \Omega \subset \widetilde{\Omega} ; B_{1 / 2} \cap \partial \Omega \subset \partial \widetilde{\Omega}$; and $\operatorname{dist}(x, \partial \widetilde{\Omega}) \geq c>0$ in $\Omega \cap\left(B_{1} \backslash B_{3 / 4}\right)$. Let $\phi_{1}$ be the function given by Lemma 2.8. satisfying

$$
\left\{\begin{aligned}
L \phi_{1} & \leq-\tilde{d}^{\epsilon-s} & & \text { in } \widetilde{\Omega} \cap\left\{\tilde{d} \leq \rho_{0}\right\} \\
c \tilde{d}^{s} \leq \phi_{1} & \leq C \tilde{d}^{s} & & \text { in } \widetilde{\Omega} \\
\phi_{1} & =0 & & \text { in } \mathbb{R}^{n} \backslash \Omega
\end{aligned}\right.
$$

where we denoted $\tilde{d}(x)=\operatorname{dist}\left(x, \mathbb{R}^{n} \backslash \widetilde{\Omega}\right)$.
Then, the function $\varphi=C \phi_{1}$ satisfies

$$
\left\{\begin{array}{rlll}
L \varphi & \leq-C d^{\epsilon-s} & & \text { in } \Omega \cap B_{1 / 2} \cap\left\{d \leq \rho_{0}\right\} \\
\varphi & \leq C d^{s} & & \text { in } \Omega \cap B_{1 / 2} \\
\varphi & \geq 1 & & \text { in } \Omega \cap\left(B_{1} \backslash B_{3 / 4}\right) \quad \text { and in } \Omega \cap B_{1 / 2} \cap\left\{d \geq \rho_{0}\right\} \\
\varphi & \geq 0 & & \text { in } \mathbb{R}^{n} .
\end{array}\right.
$$

In particular, if $C$ is large enough then we have $L(\varphi-w) \leq 0$ in $\Omega \cap B_{1 / 2} \cap\left\{d \leq \rho_{0}\right\}$, and $\varphi-w \geq 0$ in $\mathbb{R}^{n} \backslash\left(\Omega \cap B_{1 / 2} \cap\left\{d \leq \rho_{0}\right\}\right)$.

Therefore, the maximum principle yields $w \leq \varphi$, and thus $w \leq C d^{s}$ in $B_{1 / 2}$. Replacing $w$ by $-w$, we find

$$
\begin{equation*}
|w| \leq C d^{s} \quad \text { in } \quad B_{1 / 2} \tag{3.1}
\end{equation*}
$$

Now, it follows from the interior estimates of [RS14b, Theorem 1.1] that

$$
r^{s}[w]_{C^{s}\left(B_{r}\left(x_{0}\right)\right)} \leq C\left(r^{2 s}\|L w\|_{L^{\infty}\left(B_{2 r}\left(x_{0}\right)\right)}+\sup _{R \geq 1} R^{\delta-2 s}\|w\|_{L^{\infty}\left(B_{r R}\left(x_{0}\right)\right)}\right)
$$

for any ball $B_{r}\left(x_{0}\right) \subset \Omega \cap B_{1 / 2}$ with $2 r=d\left(x_{0}\right)$. Now, taking $\delta=s$ and using (3.1), we find

$$
R^{-s}\|w\|_{L^{\infty}\left(B_{r R}\left(x_{0}\right)\right)} \leq C r^{s} \quad \text { for all } R \geq 1
$$

Thus, we have

$$
[w]_{C^{s}\left(B_{r}\left(x_{0}\right)\right)} \leq C
$$

for all balls $B_{r}\left(x_{0}\right) \subset \Omega \cap B_{1 / 2}$ with $2 r=d\left(x_{0}\right)$. This yields

$$
\|w\|_{C^{s}\left(B_{1 / 2}\right)} \leq C
$$

Indeed, take $x, y \in B_{1 / 2}$, let $r=|x-y|$ and $\rho=\min \{d(x), d(y)\}$. If $2 \rho \geq r$, then using $|u| \leq C d^{s}$

$$
|u(x)-u(y)| \leq|u(x)|+|u(y)| \leq C r^{s}+C(r+\rho)^{s} \leq \bar{C} \rho^{s} .
$$ If $2 \rho<r$ then $B_{2 \rho}(x) \subset \Omega$, and hence

$$
|u(x)-u(y)| \leq \rho^{s}[u]_{C^{s}\left(B_{\rho}(x)\right)} \leq C \rho^{s} .
$$

Thus, the proposition is proved.
The proof of Proposition is now immediate.
Proof of Proposition 1.1. The result is a particular case of Proposition 3.1.
3.2. Regularity for $u / d^{s}$. Let us now prove Theorem 1.2. For this, we first show the following.

Proposition 3.2. Let $s \in(0,1)$ and $\alpha \in(0, s)$. Let $L$ be any operator of the form (1.1)-(1.2), $\Omega$ be any $C^{1, \alpha}$ domain, and $\psi$ be given by Definition 2.1.

Assume that $0 \in \partial \Omega$, and that $\partial \Omega \cap B_{1}$ can be represented as the graph of a $C^{1, \alpha}$ function with norm less or equal than 1.

Let $u$ be any solution to (1.3), and let

$$
K_{0}=\left\|d^{s-\alpha} f\right\|_{L^{\infty}\left(B_{1} \cap \Omega\right)}+\|u\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} .
$$

Then, there exists a constant $Q$ satisfying $|Q| \leq C K_{0}$ and

$$
\left|u(x)-Q \psi^{s}(x)\right| \leq C K_{0}|x|^{s+\alpha} .
$$

The constant $C$ depends only on $n$, $s$, and ellipticity constants.
We will need the following technical lemma.
Lemma 3.3. Let $\Omega, \psi$, and $u$ be as in Proposition 3.2, and define

$$
\begin{equation*}
\phi_{r}(x):=Q_{*}(r) \psi^{s}(x), \tag{3.2}
\end{equation*}
$$

where

$$
Q_{*}(r):=\arg \min _{Q \in \mathbb{R}} \int_{B_{r}}\left(u-Q \psi^{2}\right)^{2} d x=\frac{\int_{B_{r}} u \psi^{s}}{\int_{B_{r}} \psi^{2 s} d x}
$$

Assume that for all $r \in(0,1)$ we have

$$
\begin{equation*}
\left\|u-\phi_{r}\right\|_{L^{\infty}\left(B_{r}\right)} \leq C_{0} r^{s+\alpha} \tag{3.3}
\end{equation*}
$$

Then, there is $Q \in \mathbb{R}$ satisfying $|Q| \leq C\left(C_{0}+\|u\|_{L^{\infty}\left(B_{1}\right)}\right)$ such that

$$
\left\|u-Q \psi^{s}\right\|_{L^{\infty}\left(B_{r}\right)} \leq C C_{0} r^{s+\alpha}
$$

for some constant $C$ depending only on $s$ and $\alpha$.
Proof. The proof is analogue to that of [RS14b, Lemma 5.3].
First, we may assume $C_{0}+\|u\|_{L^{\infty}\left(B_{1}\right)}=1$. Then, by (3.3), for all $x \in B_{r}$ we have

$$
\left|\phi_{2 r}(x)-\phi_{r}(x)\right| \leq\left|u(x)-\phi_{2 r}(x)\right|+\left|u(x)-\phi_{r}(x)\right| \leq C r^{s+\alpha}
$$

This, combined with $\sup _{B_{r}} \psi^{s}=c r^{s}$, gives

$$
\left|Q_{*}(2 r)-Q_{*}(r)\right| \leq C r^{\alpha}
$$

Moreover, we have $\left|Q_{*}(1)\right| \leq C$, and thus there exists the limit $Q=\lim _{r \downarrow 0} Q_{*}(r)$. Furthermore,

$$
\left|Q-Q_{*}(r)\right| \leq \sum_{k \geq 0}\left|Q_{*}\left(2^{-k} r\right)-Q_{*}\left(2^{-k-1} r\right)\right| \leq \sum_{k \geq 0} C 2^{-m \alpha} r^{\alpha} \leq C r^{\alpha}
$$

In particular, $|Q| \leq C$.
Therefore, we finally find

$$
\left\|u-Q \psi^{s}\right\|_{L^{\infty}\left(B_{r}\right)} \leq\left\|u-Q_{*}(r) \psi^{s}\right\|_{L^{\infty}\left(B_{r}\right)}+C r^{s}\left|Q_{*}(r)-Q\right| \leq C r^{s+\alpha}
$$

and the lemma is proved.
We now give the:
Proof of Proposition 3.2. The proof is by contradiction, and uses several ideas from [RS14b, Section 5].

First, dividing by a constant we may assume $K_{0}=1$. Also, after a rotation we may assume that the unit (outward) normal vector to $\partial \Omega$ at 0 is $\nu=-e_{n}$.

Assume the estimate is not true, i.e., there are sequences $\Omega_{k}, L_{k}, f_{k}, u_{k}$, for which:

- $\Omega_{k}$ is a $C^{1, \alpha}$ domain that can be represented as the graph of a $C^{1, \alpha}$ function with norm is less or equal than 1 ;
- $0 \in \partial \Omega_{k}$ and the unit normal vector to $\partial \Omega_{k}$ at 0 is $-e_{n}$;
- $L_{k}$ is of the form (1.1)-1.2);
- $\left\|d^{s-\alpha} f_{k}\right\|_{L^{\infty}\left(B_{1} \cap \Omega\right)}+\left\|u_{k}\right\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \leq 1$;
- For any constant $Q, \sup _{r>0} \sup _{B_{r}} r^{-s-\alpha}\left|u_{k}-Q \psi_{k}^{s}\right|=\infty$.

Then, by Lemma 3.3 we will have

$$
\sup _{k} \sup _{r>0}\left\|u_{k}-\phi_{k, r}\right\|_{L^{\infty}\left(B_{r}\right)}=\infty
$$

where

$$
\phi_{k, r}(x)=Q_{k}(r) \psi_{k}^{s}, \quad Q_{k}(r)=\frac{\int_{B_{r}} u_{k} \psi_{k}^{s}}{\int_{B_{r}} \psi_{k}^{2 s}}
$$

We now define the monotone quantity

$$
\theta(r):=\sup _{k} \sup _{r^{\prime}>r}\left(r^{\prime}\right)^{-s-\alpha}\left\|u_{k}-\phi_{k, r^{\prime}}\right\|_{L^{\infty}\left(B_{r^{\prime}}\right)},
$$

which satisfies $\theta(r) \rightarrow \infty$ as $r \rightarrow 0$. Hence, there are sequences $r_{m} \rightarrow 0$ and $k_{m}$, such that

$$
\begin{equation*}
\left(r_{m}\right)^{-s-\alpha}\left\|u_{k_{m}}-\phi_{k_{m}, r_{m}}\right\|_{L^{\infty}\left(B_{r_{m}}\right)} \geq \frac{1}{2} \theta\left(r_{m}\right) \tag{3.4}
\end{equation*}
$$

Let us now denote $\phi_{m}=\phi_{k_{m}, r_{m}}$ and define

$$
v_{m}(x):=\frac{u_{k_{m}}\left(r_{m} x\right)-\phi_{m}\left(r_{m} x\right)}{\left(r_{m}\right)^{s+\alpha} \theta\left(r_{m}\right)}
$$

Note that

$$
\begin{equation*}
\int_{B_{1}} v_{m}(x) \psi_{k}^{s}\left(r_{m} x\right) d x=0 \tag{3.5}
\end{equation*}
$$

and also

$$
\begin{equation*}
\left\|v_{m}\right\|_{L^{\infty}\left(B_{1}\right)} \geq \frac{1}{2} \tag{3.6}
\end{equation*}
$$

which follows from (3.4).
With the same argument as in the proof of Lemma 3.3, one finds

$$
\left|Q_{k_{m}}(2 r)-Q_{k_{m}}(r)\right| \leq C r^{\alpha} \theta(r)
$$

Then, by summing a geometric series this yields

$$
\left|Q_{k_{m}}(r R)-Q_{k_{m}}(r)\right| \leq C r^{\alpha} \theta(r) R^{\alpha}
$$

for all $R \geq 1$ (see RS14b).
The previous inequality, combined with

$$
\left\|u_{m}-Q_{k_{m}}\left(r_{m} R\right) \psi_{k_{m}}^{s}\right\|_{L^{\infty}\left(B_{r_{m} R}\right)} \leq\left(r_{m} R\right)^{s+\alpha} \theta\left(r_{m} R\right)
$$

(which follows from the definition of $\theta$ ), gives

$$
\begin{align*}
\left\|v_{m}\right\|_{L^{\infty}\left(B_{R}\right)} & =\frac{1}{\left(r_{m}\right)^{s+\alpha} \theta\left(r_{m}\right)}\left\|u_{m}-Q_{k_{m}}\left(r_{m}\right) \psi_{k_{m}}^{s}\right\|_{L^{\infty}\left(B_{r_{m} R}\right)} \\
& \leq \frac{\left(r_{m} R\right)^{s+\alpha} \theta\left(r_{m} R\right)}{\left(r_{m}\right)^{s+\alpha} \theta\left(r_{m}\right)}+\frac{C\left(r_{m} R\right)^{s}}{\left(r_{m}\right)^{s+\alpha} \theta\left(r_{m}\right)}\left|Q_{k_{m}}\left(r_{m} R\right)-Q_{k_{m}}\left(r_{m}\right)\right|  \tag{3.7}\\
& \leq C R^{s+\alpha}
\end{align*}
$$

for all $R \geq 1$. Here we used that $\theta\left(r_{m} R\right) \leq \theta\left(r_{m}\right)$ if $R \geq 1$.
Now, the functions $v_{m}$ satisfy

$$
L_{m} v_{m}(x)=\frac{\left(r_{m}\right)^{2 s}}{\left(r_{m}\right)^{s+\alpha} \theta\left(r_{m}\right)} f_{k_{m}}\left(r_{m} x\right)-\frac{\left(r_{m}\right)^{2 s}}{\left(r_{m}\right)^{s+\alpha} \theta\left(r_{m}\right)}\left(L \psi_{k_{m}}\right)\left(r_{m} x\right)
$$

in $\left(r_{m}^{-1} \Omega_{k_{m}}\right) \cap B_{r_{m}^{-1}}$. Since $\alpha<s$, and using Proposition 2.3, we find

$$
\left|L_{m} v_{m}\right| \leq \frac{C}{\theta\left(r_{m}\right)}\left(r_{m}\right)^{s-\alpha} d_{k_{m}}^{\alpha-s}\left(r_{m} x\right) \quad \text { in } \quad\left(r_{m}^{-1} \Omega_{k_{m}}\right) \cap B_{r_{m}^{-1}}
$$

Thus, denoting $\Omega_{m}=r_{m}^{-1} \Omega_{k_{m}}$ and $d_{m}(x)=\operatorname{dist}\left(x, r_{m}^{-1} \Omega_{k_{m}}\right)$, we have

$$
\begin{equation*}
\left|L_{m} v_{m}\right| \leq \frac{C}{\theta\left(r_{m}\right)} d_{m}^{\alpha-s}(x) \quad \text { in } \quad \Omega_{m} \cap B_{r_{m}^{-1}} \tag{3.8}
\end{equation*}
$$

Notice that the domains $\Omega_{m}$ converge locally uniformly to $\left\{x_{n}>0\right\}$ as $m \rightarrow \infty$.
Next, by Proposition 3.1, we find that for each fixed $M \geq 1$

$$
\left\|v_{m}\right\|_{C^{s}\left(B_{M}\right)} \leq C(M) \quad \text { for all } m \text { with } r_{m}^{-1}>2 M
$$

The constant $C(M)$ does not depend on $m$. Hence, by Arzelà-Ascoli theorem, a subsequence of $v_{m}$ converges locally uniformly to a function $v \in C\left(\mathbb{R}^{n}\right)$.

In addition, there is a subsequence of operators $L_{k_{m}}$ which converges weakly to some operator $L$ of the form (1.1)-(1.2) (see Lemma 3.1 in [RS14b]). Hence, for
any fixed $K \subset \subset\left\{x_{n}>0\right\}$, thanks to the growth condition (3.7) and since $v_{m} \rightarrow v$ locally uniformly, we can pass to the limit the equation (3.8) to get

$$
L v=0 \quad \text { in } K .
$$

Here we used that the domains $\Omega_{m}$ converge uniformly to $\left\{x_{n}>0\right\}$, so that for $m$ large enough we will have $K \subset \Omega_{m} \cap B_{r_{m}^{-1}}$. We also used that, in $K$, the right hand side in (3.8) converges uniformly to 0 .

Since this can be done for any $K \subset \subset\left\{x_{n}>0\right\}$, we find

$$
L v=0 \quad \text { in }\left\{x_{n}>0\right\}
$$

Moreover, we also have $v=0$ in $\left\{x_{n} \leq 0\right\}$, and $v \in C\left(\mathbb{R}^{n}\right)$.
Thus, by the classification result [RS14b, Theorem 4.1], we find

$$
\begin{equation*}
v(x)=\kappa\left(x_{n}\right)_{+}^{s} \tag{3.9}
\end{equation*}
$$

for some $\kappa \in \mathbb{R}$.
Now, notice that, up to a subsequence, $r_{m}^{-1} \psi_{k_{m}}\left(r_{m} x\right) \rightarrow c_{1}\left(x_{n}\right)_{+}$uniformly, with $c_{1}>0$. This follows from the fact that $\psi_{k_{m}}$ are $C^{1, \alpha}\left(\bar{\Omega}_{k_{m}}\right)$ (uniformly in $m$ ) and that $0<c_{0} d_{k_{m}} \leq \psi_{k_{m}} \leq C_{0} d_{k_{m}}$.

Then, multiplying (3.5) by $\left(r_{m}\right)^{-s}$ and passing to the limit, we find

$$
\int_{B_{1}} v(x)\left(x_{n}\right)_{+}^{s} d x=0
$$

This means that $\kappa=0$ in (3.9), and therefore $v \equiv 0$. Finally, passing to the limit (3.6) we find a contradiction, and thus the proposition is proved.

We finally give the:
Proof of Theorem 1.2. First, dividing by a constant if necessary, we may assume

$$
\|f\|_{L^{\infty}\left(B_{1} \cap \Omega\right)}+\|u\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \leq 1
$$

Second, by definition of $\psi$ we have $\psi / d \in C^{\alpha}\left(\bar{\Omega} \cap B_{1 / 2}\right)$ and

$$
\left\|\psi^{s} / d^{s}\right\|_{C^{\alpha}\left(\bar{\Omega} \cap B_{1 / 2}\right)} \leq C
$$

Thus, it suffices to show that

$$
\begin{equation*}
\left\|u / \psi^{s}\right\|_{C^{\alpha}\left(\bar{\Omega} \cap B_{1 / 2}\right)} \leq C \tag{3.10}
\end{equation*}
$$

To prove (3.10), let $x_{0} \in \Omega \cap B_{1 / 2}$ and $2 r=d\left(x_{0}\right)$. Then, by Proposition 3.2 there is $Q=Q\left(x_{0}\right)$ such that

$$
\begin{equation*}
\left\|u-Q \psi^{s}\right\|_{L^{\infty}\left(B_{r}\left(x_{0}\right)\right)} \leq C r^{s+\alpha} \tag{3.11}
\end{equation*}
$$

Moreover, by rescaling and using interior estimates, we get

$$
\begin{equation*}
\left\|u-Q \psi^{s}\right\|_{C^{\alpha}\left(B_{r}\left(x_{0}\right)\right)} \leq C r^{s} \tag{3.12}
\end{equation*}
$$

Finally, (3.11)-(3.12) yield (3.10), exactly as in the proof of Theorem 1.2 in RS14b].

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Remark 3.4. Notice that, thanks to Proposition 3.2, we have that Theorem 1.2 holds for all right hand sides satisfying $|f(x)| \leq C d^{\alpha-s}$ in $\Omega$.
3.3. Equations with bounded measurable coefficients. We prove now Theorem 1.5,

First, we show the following $C^{\alpha}$ estimate up to the boundary.
Proposition 3.5. Let $s \in(0,1)$, and $\Omega$ be any bounded $C^{1, \alpha}$ domain.
Let $u$ be a solution to

$$
\left\{\begin{align*}
M^{+} u & \geq-K_{0} d^{\epsilon-s} & & \text { in } B_{1} \cap \Omega  \tag{3.13}\\
M^{-} u & \leq K_{0} d^{\epsilon-s} & & \text { in } B_{1} \cap \Omega \\
u & =0 & & \text { in } B_{1} \backslash \Omega
\end{align*}\right.
$$

Then,

$$
\|u\|_{C^{\bar{\alpha}}\left(B_{1 / 2}\right)} \leq C\left(K_{0}+\sup _{R \geq 1} R^{\delta-2 s}\|u\|_{L^{\infty}\left(B_{R}\right)}\right)
$$

The constant $C$ depends only on $n, s, \epsilon, \delta, \Omega$, and ellipticity constants.
Proof. The proof is very similar to that of Proposition 3.5.
First, using the supersolution given by Lemma 2.8 , and by the exact same argument of Proposition 3.5, we find

$$
|w| \leq C d^{s} \quad \text { in } \quad B_{1 / 2}
$$

Now, using the interior estimates of [CS09] one finds

$$
[w]_{C^{\bar{\alpha}}\left(B_{r}\left(x_{0}\right)\right)} \leq C
$$

for all balls $B_{r}\left(x_{0}\right) \subset \Omega \cap B_{1 / 2}$ with $2 r=d\left(x_{0}\right)$, and this yields

$$
\|w\|_{C^{\bar{\alpha}}\left(B_{1 / 2}\right)} \leq C
$$

as desired.
We next show:
Proposition 3.6. Let $s \in(0,1)$ and $\alpha \in(0, \bar{\alpha})$. Let $L$ be any operator of the form (1.1)-(1.2), $\Omega$ be any $C^{1, \alpha}$ domain, and $\psi$ be given by Definition 2.1.

Assume that $0 \in \partial \Omega$, and that $\partial \Omega \cap B_{1}$ can be represented as the graph of a $C^{1, \alpha}$ function with norm less or equal than 1.

Let $u$ be any solution to (1.4), and let

$$
K_{0}=\|f\|_{L^{\infty}\left(B_{1} \cap \Omega\right)}+\|u\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} .
$$

Then, there exists a constant $Q$ satisfying $|Q| \leq C K_{0}$ and

$$
\left|u(x)-Q \psi^{s}(x)\right| \leq C K_{0}|x|^{s+\alpha}
$$

The constant $C$ depends only on $n, s$, and ellipticity constants.
Proof. The proof is very similar to that of Proposition 3.2.
Assume by contradiction that we have $\Omega_{k}$ and $u_{k}$ such that:

- $\Omega_{k}$ is a $C^{1, \alpha}$ domain that can be represented as the graph of a $C^{1, \alpha}$ function with norm is less or equal than 1 ;
- $0 \in \partial \Omega_{k}$ and the unit normal vector to $\partial \Omega_{k}$ at 0 is $-e_{n}$;
- $u_{k}$ satisfies (1.4) with $K_{0}=1$;
- For any constant $Q, \sup _{r>0} \sup _{B_{r}} r^{-s-\alpha}\left|u_{k}-Q \psi_{k}^{s}\right|=\infty$.

Then, by Lemma 3.3 we will have

$$
\sup _{k} \sup _{r>0}\left\|u_{k}-\phi_{k, r}\right\|_{L^{\infty}\left(B_{r}\right)}=\infty
$$

where

$$
\phi_{k, r}(x)=Q_{k}(r) \psi_{k}^{s}, \quad Q_{k}(r)=\frac{\int_{B_{r}} u_{k} \psi_{k}^{s}}{\int_{B_{r}} \psi_{k}^{2 s}} .
$$

We now define $\theta(r), r_{m} \rightarrow 0$, and $v_{m}$ as in the proof of Proposition 3.2. Then, we have

$$
\begin{gather*}
\int_{B_{1}} v_{m}(x) \psi_{k}^{s}\left(r_{m} x\right) d x=0  \tag{3.14}\\
\left\|v_{m}\right\|_{L^{\infty}\left(B_{1}\right)} \geq \frac{1}{2} \tag{3.15}
\end{gather*}
$$

and

$$
\begin{equation*}
\left\|v_{m}\right\|_{L^{\infty}\left(B_{R}\right)} \leq C R^{s+\alpha} \quad \text { for all } \quad R \geq 1 \tag{3.16}
\end{equation*}
$$

Moreover, the functions $v_{m}$ satisfy

$$
M^{-} v_{m}(x) \leq \frac{\left(r_{m}\right)^{2 s}}{\left(r_{m}\right)^{s+\alpha} \theta\left(r_{m}\right)}+\frac{\left(r_{m}\right)^{2 s}}{\left(r_{m}\right)^{s+\alpha} \theta\left(r_{m}\right)}\left(M^{+} \psi_{k_{m}}\right)\left(r_{m} x\right)
$$

in $\left(r_{m}^{-1} \Omega_{k_{m}}\right) \cap B_{r_{m}^{-1}}$. Using Lemma 2.3, and denoting $\Omega_{m}=r_{m}^{-1} \Omega_{k_{m}}$ and $d_{m}(x)=$ $\operatorname{dist}\left(x, r_{m}^{-1} \Omega_{k_{m}}\right)$, we find

$$
\begin{equation*}
M^{-} v_{m} \leq \frac{C}{\theta\left(r_{m}\right)} d_{m}^{\alpha-s}(x) \quad \text { in } \quad \Omega_{m} \cap B_{r_{m}^{-1}} \tag{3.17}
\end{equation*}
$$

Similarly, we find

$$
M^{+} v_{m} \geq-\frac{C}{\theta\left(r_{m}\right)} d_{m}^{\alpha-s}(x) \quad \text { in } \quad \Omega_{m} \cap B_{r_{m}^{-1}}
$$

Notice that the domains $\Omega_{m}$ converge locally uniformly to $\left\{x_{n}>0\right\}$ as $m \rightarrow \infty$.
Next, by Proposition 3.5, we find that for each fixed $M \geq 1$

$$
\left\|v_{m}\right\|_{C^{\bar{\alpha}}\left(B_{M}\right)} \leq C(M) \quad \text { for all } m \text { with } r_{m}^{-1}>2 M
$$

The constant $C(M)$ does not depend on $m$. Hence, by Arzelà-Ascoli theorem, a subsequence of $v_{m}$ converges locally uniformly to a function $v \in C\left(\mathbb{R}^{n}\right)$.

Hence, passing to the limit the equation (3.17) we get

$$
M^{-} v \leq 0 \leq M^{+} v \quad \text { in }\left\{x_{n}>0\right\} .
$$

Moreover, we also have $v=0$ in $\left\{x_{n} \leq 0\right\}$, and $v \in C\left(\mathbb{R}^{n}\right)$.

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Thus, by the classification result [RS14, Proposition 5.1], we find

$$
\begin{equation*}
v(x)=\kappa\left(x_{n}\right)_{+}^{s} \tag{3.18}
\end{equation*}
$$

for some $\kappa \in \mathbb{R}$. But passing (3.14) -multiplied by $\left(r_{m}\right)^{-s}$ - to the limit, we find

$$
\int_{B_{1}} v(x)\left(x_{n}\right)_{+}^{s} d x=0
$$

This means that $v \equiv 0$, a contradiction with (3.15).
Finally, we give the:
Proof of Theorem 1.5. The result follows from Proposition 3.6; see the proof of Theorem 1.2.

## 4. Barriers: $C^{1}$ domains

We construct now sub and supersolutions that will be needed in the proof of Theorem 1.3. Recall that in $C^{1}$ domains one does not expect solutions to be comparable to $d^{s}$, and this is why the sub and supersolutions we construct have slightly different behaviors near the boundary. Namely, they will be comparable to $d^{s+\epsilon}$ and $d^{s-\epsilon}$, respectively.

Lemma 4.1. Let $s \in(0,1)$, and $e \in S^{n-1}$. Define

$$
\Phi_{\mathrm{sub}}(x):=\left(e \cdot x-\eta|x|\left(1-\frac{(e \cdot x)^{2}}{|x|^{2}}\right)\right)_{+}^{s+\epsilon}
$$

and

$$
\Phi_{\text {super }}(x):=\left(e \cdot x+\eta|x|\left(1-\frac{(e \cdot x)^{2}}{|x|^{2}}\right)\right)_{+}^{s-\epsilon}
$$

For every $\epsilon>0$ there is $\eta>0$ such that two functions $\Phi_{\text {sub }}$ and $\Phi_{\text {super }}$ satisfy, for all $L \in \mathcal{L}_{*}$,

$$
\begin{cases}L \Phi_{\text {sub }} \geq c_{\epsilon} d^{\epsilon-s}>0 & \text { in } \mathcal{C}_{\eta} \\ \Phi_{\text {sub }}=0 & \text { in } \mathbb{R}^{n} \backslash \mathcal{C}_{\eta}\end{cases}
$$

and

$$
\begin{cases}L \Phi_{\text {super }} \leq-c_{\epsilon} d^{-\epsilon-s}<0 & \text { in } \mathcal{C}_{-\eta} \\ \Phi_{\text {super }}=0 & \text { in } \mathbb{R}^{n} \backslash \mathcal{C}_{-\eta}\end{cases}
$$

where $\mathcal{C}_{ \pm \eta}$ is the cone

$$
\mathcal{C}_{ \pm \eta}:=\left\{x \in \mathbb{R}^{n}: e \cdot \frac{x}{|x|}> \pm \eta\left(1-\left(e \cdot \frac{x}{|x|}\right)^{2}\right)\right\}
$$

The constant $\eta$ depends only on $\epsilon$, $s$, and ellipticity constants.

Proof. We prove the statement for $\Phi_{\text {sub }}$. The statement for $\Phi_{\text {super }}$ is proved similarly.
Let us denote $\Phi:=\Phi_{\text {sub }}$. By homogeneity it is enough to prove that $L \Phi \geq c_{\epsilon}>0$ on points belonging to $e+\partial \mathcal{C}_{\eta}$, since all the positive dilations of this set with respect to the origin cover the interior of $\tilde{\mathcal{C}_{\eta}}$.

Let thus $P \in \partial \mathcal{C}_{\eta}$, that is,

$$
e \cdot P-\eta\left(|P|-\frac{(e \cdot P)^{2}}{|P|}\right)=0
$$

Consider

$$
\begin{aligned}
\Phi_{P, \eta}(x) & :=\Phi(P+e+x) \\
& =\left(e \cdot(P+e+x)-\eta\left(|P+e+x|-\frac{(e \cdot(P+e+x))^{2}}{|P+e+x|}\right)\right)_{+}^{s+\epsilon} \\
& =\left(1+e \cdot x-\eta\left(|P+e+x|-|P|-\frac{(e \cdot(P+e+x))^{2}}{|P+e+x|}+\frac{(e \cdot P)^{2}}{|P|}\right)\right)_{+}^{s+\epsilon} \\
& =\left(1+e \cdot x-\eta \psi_{P}(x)\right)_{+}^{s+\epsilon}
\end{aligned}
$$

where we define

$$
\psi_{P}(x):=|P+e+x|-|P|-\frac{(e \cdot(P+e+x))^{2}}{|P+e+x|}+\frac{(e \cdot P)^{2}}{|P|}
$$

Note that the functions $\psi_{P}$ satisfy

$$
\begin{gathered}
\psi_{P}(0)=0 \\
\left|\nabla \psi_{P}(x)\right| \leq C \quad \text { in } \mathbb{R}^{n} \backslash\{-P-e\}
\end{gathered}
$$

and

$$
\left|D^{2} \psi_{P}(x)\right| \leq C \quad \text { for } x \in B_{1 / 2}
$$

where $C$ does not depend on $P$ (recall that $|e|=1$ ).
Then, the family $\Phi_{P, \eta}$ satisfies

$$
\Phi_{P, \eta} \rightarrow(1+e \cdot x)_{+}^{s+\epsilon} \quad \text { in } C^{2}\left(\overline{B_{1 / 2}}\right)
$$

as $\eta \searrow 0$, uniformly in $P$ and moreover

$$
\int_{\mathbb{R}^{n}} \frac{\left|\Phi_{P, \eta}-(1+e \cdot x)_{+}^{s+\epsilon}\right|}{1+|x|^{n+2 s}} d x \leq \int_{\mathbb{R}^{n}} \frac{C(C \eta|x|)^{s+\epsilon}}{1+|x|^{n+2 s}} d x \leq C \eta^{s+\epsilon}
$$

Thus,

$$
L \Phi_{P, \eta}(0) \rightarrow L\left((1+e \cdot)_{+}^{s+\epsilon}\right)(0) \geq c(s, \epsilon, \lambda)>0 \quad \text { as } \eta \searrow 0
$$

uniformly in $P$.
In particular one can chose $\eta=\eta(s, \epsilon, \lambda, \Lambda)$ so that $L \Phi_{P, \eta}(0) \geq c_{\epsilon}>0$ for all $P \in \partial \tilde{\mathcal{C}}_{\eta}$ and for all $L \in \mathcal{L}_{*}$, and the lemma is proved.

## 5. Regularity in $C^{1}$ domains

We prove here Theorems 1.3 and 1.6 .
Definition 5.1. Let $r_{0}>0$ and let $\rho:\left(0, r_{0}\right] \rightarrow 0$ be a nonincreasing function with $\lim _{t \downarrow 0} \rho(t)=0$. We say that a domain $\Omega$ is improving Lipschitz at 0 with inwards unit normal vector $e_{n}=(0, \ldots, 0,1)$ and modulus $\rho$ if

$$
\Omega \cap B_{r}=\left\{\left(x^{\prime}, x_{n}\right): x_{n}>g\left(x^{\prime}\right)\right\} \cap B_{r} \quad \text { for } \quad r \in\left(0, r_{0}\right]
$$

where $g: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ satisfies

$$
\|g\|_{\operatorname{Lip}\left(B_{r}\right)} \leq \rho(r) \quad \text { for } 0<r \leq r_{0}
$$

We say that $\Omega$ is improving Lipschitz at $x_{0} \in \partial \Omega$ with inwards unit normal $e \in S^{n-1}$ if the normal vector to $\partial \Omega$ at $x_{0}$ is $e$ and, after a rotation, the domain $\Omega-x_{0}$ satisfies the previous definition.

We first prove the following $C^{\alpha}$ estimate up to the boundary.
Lemma 5.2. Let $s \in(0,1)$, and let $\Omega \subset \mathbb{R}^{n}$ be a Lipschitz domain in $B_{1}$ with Lipschitz constant less than $\ell$. Namely, assume that after a rotation we have

$$
\Omega \cap B_{1}=\left\{\left(x^{\prime}, x_{n}\right): x_{n}>g\left(x^{\prime}\right)\right\} \cap B_{1},
$$

with $\|g\|_{\operatorname{Lip}\left(B_{1}\right)} \leq \ell$. Let $u \in C\left(B_{1}\right)$ be a viscosity solution of

$$
\begin{gathered}
M^{+} u \geq-K_{0} d^{\epsilon-s} \quad \text { and } \quad M^{-} u \leq K_{0} d^{\epsilon-s} \quad \text { in } \Omega \cap B_{1} \\
u=0 \quad \text { in } B_{1} \backslash \Omega .
\end{gathered}
$$

Assume that

$$
\|u\|_{L^{\infty}\left(B_{R}\right)} \leq K_{0} R^{2 s-\epsilon} \quad \text { for all } R \geq 1
$$

Then, if $\ell \leq \ell_{0}$, where $\ell_{0}=\ell_{0}(n, s, \lambda, \Lambda)$, we have

$$
\|u\|_{C^{\bar{\alpha}}\left(B_{1 / 2}\right)} \leq C K_{0} .
$$

The constants $C$ and $\bar{\alpha}$ depend only on $n, s, \epsilon$ and ellipticity constants.
Proof. By truncating $u$ in $B_{2}$ and dividing it by $C K_{0}$ we may assume that

$$
\|u\|_{L^{\infty}\left(\mathbb{R}^{n}\right)}=1
$$

and that

$$
M^{+} u \geq-d^{\epsilon-s} \quad \text { and } \quad M^{-} u \leq d^{\epsilon-s} \quad \text { in } \Omega \cap B_{1}
$$

Now, we divide the proof into two steps.
Step 1. We first prove that

$$
\begin{equation*}
|u(x)| \leq C\left|x-x_{0}\right|^{\alpha} \quad \text { in } \Omega \cap B_{3 / 4} \tag{5.1}
\end{equation*}
$$

where $x_{0} \in \partial \Omega$ is the closest point to $x$ on $\partial \Omega$. We will prove (5.1) by using a supersolution. Indeed, given $\epsilon \in(0, s)$, let $\Phi_{\text {super }}$ and $\mathcal{C}_{\eta}$ be the homogeneous supersolution and the cone from Lemma 4.1, where $e=e_{n}$. Note that $\Phi_{\text {super }}$ is a positive function satisfying $M^{-} \Phi_{\text {super }} \leq-c d^{-\epsilon-s}<0$ outside the convex cone $\mathbb{R}^{n} \backslash \mathcal{C}_{\eta}$, and it is homogeneous of degree $s-\epsilon$.

Then, we easily check that the function $\psi=C \Phi_{\text {super }}-\chi_{B_{1}\left(z_{0}\right)}$, with $C$ large and $\left|z_{0}\right| \geq 2$ such that $\Phi_{\text {super }} \geq 1$ in $B_{1}\left(z_{0}\right)$, satisfies $M^{+} \psi \leq-d^{\epsilon-s}$ in $B_{1 / 4} \cap \mathcal{C}_{\eta}$ and $\psi \geq \frac{1}{4}$ in $\mathcal{C}_{\eta} \backslash B_{1 / 4}$. Indeed, we simply use that $M^{-} \chi_{B_{1}\left(z_{0}\right)} \geq c_{0}>0$ in $B_{1 / 4}$. Note that this argument exploits the nonlocal character of the operator and a slightly more complicated one would be needed in order to obtain a result that is stable as $s \uparrow 1$.

Note that the supersolution $\psi$ vanishes in $B_{1 / 4} \backslash \mathcal{C}_{\eta}$. Then, if $\ell_{0}$ is small enough, for every point in $x_{0} \in \partial \Omega \cap B_{3 / 4}$ we will have

$$
x_{0}+\left(B_{1 / 4} \backslash \mathcal{C}_{\eta}\right) \subset B_{1} \backslash \Omega
$$

Then, using translates of $\psi$ (and $-\psi$ ) upper (lower) barriers we get $|u(x)| \leq$ $\psi\left(x_{0}+x\right) \leq C\left|x-x_{0}\right|^{s-\epsilon}$, as desired.

Step 2. To obtain a $C^{\alpha}$ estimate up to the boundary, we use the following interior estimate from [CS09]: Let $r \in(0,1)$,

$$
M^{+} u \geq-r^{\alpha-2 s} \text { and } M^{-} u \leq r^{\alpha-2 s} \text { in } B_{r}(x)
$$

and

$$
|u(z)| \leq r^{\alpha}\left(1+\frac{(z-x)^{\alpha}}{r^{\alpha}}\right) \quad \text { in all of } \mathbb{R}^{n}
$$

Then,

$$
[u]_{C^{\alpha}\left(B_{r / 2}(x)\right)} \leq C
$$

with $C$ and $\alpha>0$ depending only $s$, ellipticity constants and dimension.
Combining this estimate with (5.1), it follows that

$$
\|u\|_{C^{\alpha}\left(B_{1 / 2}\right)} \leq C
$$

Thus, the lemma is proved.
We will also need the following.
Lemma 5.3. Let $s \in(0,1), \alpha \in(0, \bar{\alpha})$, and $C_{0} \geq 1$. Given $\epsilon \in(0, \alpha]$, there exist $\delta>0$ depending only on $\epsilon, n, s$, and ellipticity constants, such that the following statement holds.

Assume that $\Omega \subset \mathbb{R}^{n}$ is a Lipchitz domain such that $\partial \Omega \cap B_{1 / \delta}$ is a Lipchitz graph of the form $x_{n}=g\left(x^{\prime}\right)$, in $\left|x^{\prime}\right|<1 / \delta$ with

$$
[g]_{\operatorname{Lip}\left(B_{1 / \delta}\right)} \leq \delta,
$$

and $0 \in \partial \Omega$.
Let $\varphi \in C\left(\mathbb{R}^{n}\right)$ be a viscosity solution of

$$
\begin{gathered}
M^{+} \varphi \geq-\delta d^{\epsilon-s} \quad \text { and } \quad M^{-} \varphi \leq \delta d^{\epsilon-s} \quad \text { in } \Omega \cap B_{1 / \delta} \\
\varphi=0 \quad \text { in } B_{1 / \delta} \backslash \Omega
\end{gathered}
$$

satisfying

$$
\varphi \geq 0 \quad \text { in } B_{1}
$$

Assume that $\varphi$ satisfies

$$
\sup _{B_{1}} \varphi=1 \quad \text { and } \quad\|\varphi\|_{L^{\infty}\left(B_{2} l\right)} \leq C_{0}\left(2^{l}\right)^{s+\alpha} \quad \text { for all } l \geq 0
$$

Then, we have

$$
\begin{equation*}
\int_{B_{1}} \varphi^{2} d x \geq \frac{1}{2} \int_{B_{1}}\left(x_{n}\right)_{+}^{2 s} d x \tag{5.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\frac{1}{2}\right)^{s+\epsilon} \leq \frac{\sup _{B_{2^{l-1}}} \varphi}{\sup _{B_{2^{l}}} \varphi} \leq\left(\frac{1}{2}\right)^{s-\epsilon} \quad \text { for all } l \leq 0 \tag{5.3}
\end{equation*}
$$

Proof. Step 1. We first prove that, for $\delta$ small enough, we have (5.2) and

$$
\begin{equation*}
\left(\frac{1}{2}\right)^{s+\epsilon} \leq \frac{\sup _{B_{1 / 2}} \varphi}{\sup _{B_{1}} \varphi} \leq\left(\frac{1}{2}\right)^{s-\epsilon} \tag{5.4}
\end{equation*}
$$

In a second step we will iterate (5.4) to show (5.3).
The proof of (5.4) is by compactness. Suppose that there is a sequence $\varphi_{k}$ of functions satisfying the assumptions with $\delta=\delta_{k} \downarrow 0$ for which one of the three possibilities

$$
\begin{align*}
& \left(\frac{1}{2}\right)^{s+\epsilon}>\frac{\sup _{B_{1 / 2}} \varphi_{k}}{\sup _{B_{1}} \varphi_{k}}  \tag{5.5}\\
& \frac{\sup _{B_{1 / 2}} \varphi_{k}}{\sup _{B_{1}} \varphi_{k}}>\left(\frac{1}{2}\right)^{s-\epsilon} \tag{5.6}
\end{align*}
$$

or

$$
\begin{equation*}
\int_{B_{1}} \varphi_{k}^{2} d x<\frac{1}{2} \int_{B_{1}}\left(x_{n}\right)_{+}^{2 s} d x \tag{5.7}
\end{equation*}
$$

holds for all $k \geq 1$.
Let us show that a subsequence of $\varphi_{k}$ converges locally uniformly $\mathbb{R}^{n}$ to the function $\left(x_{n}\right)_{+}^{s}$. Indeed, thanks to Lemma 5.2 and the Arzela-Ascoli theorem a subsequence of $\varphi_{k}$ converges to a function $\varphi \in C\left(\mathbb{R}^{n}\right)$, which satisfies $M^{+} \varphi \geq 0$ and $M^{-} \varphi \leq 0$ in $\mathbb{R}_{+}^{n}$, and $\varphi=0$ in $\mathbb{R}_{-}^{n}$. Here we used that $\delta_{k} \rightarrow 0$. Moreover, by the growth control $\|\varphi\|_{L^{\infty}\left(B_{R}\right)} \leq C R^{s+\alpha}$ and the classification theorem [RS14, Proposition 5.1], we find $\varphi(x)=K\left(x_{n}\right)_{+}^{s}$. But since $\sup _{B_{1}} \varphi_{k}=1$, then $K=1$.

Therefore, we have proved that a subsequence of $\varphi_{k}$ converges uniformly in $B_{1}$ to $\left(x_{n}\right)_{+}^{s}$. Passing to the limit (5.5), (5.6) or (5.7), we reach a contradiction.

Step 2. We next show that we can iterate (5.4) to obtain (5.3) by induction. Assume that for some $m \leq 0$ we have

$$
\begin{equation*}
\left(\frac{1}{2}\right)^{s+\epsilon} \leq \frac{\sup _{B_{2^{l-1}}} \varphi}{\sup _{B_{2^{l}}} \varphi} \leq\left(\frac{1}{2}\right)^{s-\epsilon} \quad \text { for } m \leq l \leq 0 \tag{5.8}
\end{equation*}
$$

We then consider the function

$$
\bar{\varphi}=\frac{\varphi\left(2^{-m} x\right)}{\sup _{B_{2} m} \varphi},
$$

and notice that

$$
2^{(s+\epsilon) l} \leq \sup _{B_{2} l} \varphi \leq 2^{(s-\epsilon) l} \quad \text { for } m \leq l \leq 0
$$

Thus,

$$
M^{+} \bar{\varphi} \geq-\frac{\delta 2^{2 s m}}{2^{(s+\epsilon) m}} \geq-\delta \quad \text { in }\left(2^{-m} \Omega\right) \cap B_{2^{-m} / \delta}
$$

and similarly

$$
M^{-} \bar{\varphi} \leq \delta \quad \text { in }\left(2^{-m} \Omega\right) \cap B_{2^{-m} / \delta}
$$

Clearly

$$
\bar{\varphi}=0 \quad \text { in }\left(2^{-m} \mathcal{C} \Omega\right) \cap B_{2^{-m} / \delta}
$$

and

$$
\varphi \geq 0 \quad \text { in } B_{2^{-m}} \supset B_{1} .
$$

Since $\partial \Omega$ is Lipchitz with constant $\delta$ in $B_{1 / \delta}$ and $2^{-m} \geq 1(m \leq 0)$ we have that the rescaled domain $\left(2^{-m} \Omega\right) \cap B_{2^{-m} / \delta}$ is also Lipchitz with the same constant $1 / \delta$ in a larger ball.

Finally, using (5.8) again we find

$$
\sup _{B_{2 l}} \bar{\varphi}=\frac{\sup _{B_{2^{l+m}}} \varphi}{\sup _{B_{2} m} \varphi} \leq 2^{(s+\epsilon) l} \leq 2^{(s+\alpha) l} \quad \text { for } l \geq 0 \text { with } l+m \leq 0
$$

For $l+m>0$ we have

$$
\sup _{B_{2^{l}}} \bar{\varphi}=\frac{\sup _{B_{2^{l+m}}} \varphi}{2^{(s+\epsilon) m} \varphi} \leq \frac{C_{0} 2^{(s+\alpha)(l+m)}}{2^{(s+\epsilon) m}}=C_{0} 2^{(s+\alpha) l} 2^{(\alpha-\epsilon) m} \leq C_{0} 2^{(s+\alpha) l}
$$

Hence, using Step 1, we obtain

$$
\left(\frac{1}{2}\right)^{s+\epsilon} \leq \frac{\sup _{B_{1 / 2}} \bar{\varphi}}{\sup _{B_{1}} \bar{\varphi}} \leq\left(\frac{1}{2}\right)^{s-\epsilon}
$$

Thus (5.8) holds for $l=m-1$, and the lemma is proved.
We next prove the following.
Proposition 5.4. Let $s \in(0,1), \alpha \in(0, \bar{\alpha})$, and $C_{0} \geq 1$.
Let $\Omega \subset \mathbb{R}^{n}$ be a domain that is improving Lipschitz at 0 with unit outward normal $e \in S^{n-1}$ and with modulus of continuity $\rho$ (see Definition 5.1). Then, there exists $\delta>0$, depending only on $\alpha, s, C_{0}$, ellipticity constants, and dimension such that the following statement holds.

Assume that $r_{0}=1 / \delta$ and $\rho(1 / \delta)<\delta$. Suppose that $u, \varphi \in C\left(\mathbb{R}^{n}\right)$ are viscosity solutions of

$$
\begin{cases}M^{+}(a u+b \varphi) \geq-\delta(|a|+|b|) d^{\alpha-s} & \text { in } B_{1 / \delta} \cap \Omega  \tag{5.9}\\ u=\varphi=0 & \text { in } B_{1 / \delta} \backslash \Omega\end{cases}
$$

for all $a, b \in \mathbb{R}$. Moreover, assume that

$$
\begin{equation*}
\|a u+b \varphi\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \leq C_{0}(|a|+|b|) R^{s+\alpha} \quad \text { for all } R \geq 1 \tag{5.10}
\end{equation*}
$$

$$
\varphi \geq 0 \quad \text { in } B_{1}, \quad \text { and } \quad \sup _{B_{1}} \varphi=1
$$

Then, there is $K \in \mathbb{R}$ with $|K| \leq C$ such that

$$
|u(x)-K \varphi(x)| \leq C|x|^{s+\alpha} \quad \text { in } B_{1}
$$

where $C$ depends only on $\rho, C_{0}, \alpha$, s, ellipticity constants, and dimension.
Proof. Step 1 (preliminary results). Fix $\epsilon \in(0, \alpha)$. Using Lemma 5.3, if $\delta$ is small enough we have

$$
\begin{equation*}
\int_{B_{1}} \varphi^{2} d x \geq \frac{1}{2} \int_{B_{1}}\left(x_{n}\right)_{+}^{2 s} d x \geq c(n, s)>0 \tag{5.11}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\frac{1}{2}\right)^{s+\epsilon} \leq \frac{\sup _{B_{2^{l-1}} \varphi}}{\sup _{B_{2} l} \varphi} \leq\left(\frac{1}{2}\right)^{s-\epsilon} \quad \text { for all } l \leq 0 \tag{5.12}
\end{equation*}
$$

In particular, since $\sup _{B_{1}} \varphi=1$ then

$$
\begin{equation*}
(r / 2)^{s+\epsilon} \leq \sup _{B_{r}} \varphi \leq(2 r)^{s-\epsilon} \quad \text { for all } r \in(0,1) \tag{5.13}
\end{equation*}
$$

Step 2. We prove now, with a blow-up argument, that

$$
\begin{equation*}
\left\|u(x)-K_{r} \varphi(x)\right\|_{L^{\infty}\left(B_{r}\right)} \leq C r^{s+\alpha} \tag{5.14}
\end{equation*}
$$

for all $r \in(0,1]$, where

$$
\begin{equation*}
K_{r}:=\frac{\int_{B_{r}} u \varphi d x}{\int_{B_{r}} \varphi^{2} d x} \tag{5.15}
\end{equation*}
$$

Notice that (5.14) implies the estimate of the proposition with $K=\lim _{r} \searrow_{0} K_{r}$. Indeed, we have $\left|K_{1}\right| \leq C$-which is immediate using 5.10 with $a=1$ and $b=0$ and (5.11- and

$$
\begin{aligned}
\left|K_{r}-K_{r / 2}\right|(r / 2)^{s+\epsilon} & \leq\left\|K_{r} \varphi-K_{r / 2} \varphi\right\|_{L^{\infty}\left(B_{r}\right)} \\
& \leq\left\|u-K_{r} \varphi\right\|_{L^{\infty}\left(B_{r}\right)}+\left\|u-K_{r / 2} \varphi\right\|_{L^{\infty}\left(B_{r}\right)} \\
& \leq C r^{s+\alpha}
\end{aligned}
$$

Thus,

$$
|K| \leq\left|K_{1}\right|+\sum_{j=0}^{\infty}\left|K_{2^{-j}}-K_{2^{-j-1}}\right| \leq C+C \sum_{j=0}^{\infty} 2^{-j(\alpha-\epsilon)} \leq C
$$

provided that $\epsilon$ is taken smaller that $\alpha$.
Let us prove 5.14 by contradiction. Assume that we have a sequences $\Omega_{j}, e_{j}, u_{j}$, $\varphi_{j}$ satisfying the assumptions of the Proposition, but not 5.14. That is,

$$
\lim _{j \rightarrow \infty} \sup _{r>0} r^{-s-\alpha}\left\|u_{j}(x)-K_{r, j} \varphi_{j}\right\|_{L^{\infty}\left(B_{r}\right)}=\infty
$$

were $K_{r, j}$ is defined as in 5.15 with $u$ replaced by $u_{j}$ and $\varphi$ replace by $\varphi_{j}$.

Define, for $r \in(0,1]$ the nonincreasing quantity

$$
\theta(r)=\sup _{r^{\prime} \in(r, 1)}\left(r^{\prime}\right)^{-s-\alpha}\left\|u_{j}(x)-K_{r^{\prime}, j} \varphi_{j}\right\|_{L^{\infty}\left(B_{r^{\prime}}\right)}
$$

Note that $\theta(r)<\infty$ for $r>0$ since $\left\|u_{j}\right\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \leq 1$ and that $\lim _{r \rrbracket_{0}} \theta(r)=\infty$.
For every $m \in \mathbb{N}$, by definition of $\theta$ there exist $r_{m}^{\prime} \geq 1 / m, j_{m}, \Omega_{m}=\Omega_{j_{m}}$, and $e_{m}=e_{j_{m}}$ such that

$$
\left(r_{m}^{\prime}\right)^{-s-\alpha}\left\|u_{j_{m}}(x)-K_{r_{m}^{\prime}, j_{m}} \varphi_{j_{m}}\right\|_{L^{\infty}\left(B_{r_{m}^{\prime}}\right)} \geq \frac{1}{2} \theta(1 / m) \geq \frac{1}{2} \theta\left(r_{m}^{\prime}\right)
$$

Note that $r_{m}^{\prime} \rightarrow 0$. Taking a subsequence we may assume that $e_{m} \rightarrow e \in S^{n-1}$. Denote

$$
u_{m}=u_{j_{m}}, \quad K_{m}=K_{r_{m}^{\prime}, j_{m}} \quad \text { and } \quad \varphi_{m}=\varphi_{j_{m}}
$$

We now consider the blow-up sequence

$$
v_{m}(x)=\frac{u_{m}\left(r_{m}^{\prime} x\right)-K_{m} \varphi_{m}\left(r_{m}^{\prime} x\right)}{\left(r_{m}^{\prime}\right)^{s+\alpha} \theta\left(r_{m}^{\prime}\right)}
$$

By definition of $\theta$ and $r_{m}^{\prime}$ we will have

$$
\begin{equation*}
\left\|v_{m}\right\|_{L^{\infty}\left(B_{1}\right)} \geq \frac{1}{2} \tag{5.16}
\end{equation*}
$$

In addition, by definition of $K_{m}=K_{r_{m}^{\prime}, j_{m}}$ we have

$$
\begin{equation*}
\int_{B_{1}} v_{m}(x) \varphi_{m}\left(r_{m}^{\prime} x\right) d x=0 \tag{5.17}
\end{equation*}
$$

for all $m \geq 1$.
Let us prove that

$$
\begin{equation*}
\left\|v_{m}\right\|_{L^{\infty}\left(B_{R}\right)} \leq C R^{s+\alpha} \quad \text { for all } R \geq 1 \tag{5.18}
\end{equation*}
$$

Indeed, first, by definition of $\theta(2 r)$ and $\theta(r)$,

$$
\begin{aligned}
\frac{\left\|K_{2 r, j} \varphi_{j}-K_{r, j} \varphi_{j}\right\|_{L^{\infty}\left(B_{r}\right)}}{r^{s+\alpha} \theta(r)} & \leq \frac{2^{s+\alpha} \theta(2 r)}{\theta(r)} \frac{\left\|u_{j}-K_{r, j} \varphi_{j}\right\|_{L^{\infty}\left(B_{2 r}\right)}}{(2 r)^{s+\alpha} \theta(2 r)}+\frac{\left\|u_{j}-K_{r / 2, j} \varphi_{j}\right\|_{L^{\infty}\left(B_{r}\right)}}{r^{s+\alpha} \theta(r)} \\
& \leq 2^{s+\alpha}+1 \leq 5
\end{aligned}
$$

On the one hand, using Step 1 we have

$$
\begin{aligned}
\frac{\left|K_{2 r, j}-K_{r, j}\right|(r / 2)^{s+\epsilon}}{r^{s+\alpha} \theta(r)} & \leq \frac{\left|K_{2 r, j}-K_{r, j}\right|\left\|\varphi_{j}\right\|_{L^{\infty}\left(B_{r}\right)}}{r^{s+\alpha} \theta(r)} \\
& =\frac{\left\|K_{2 r, j} \varphi_{j}-K_{r, j} \varphi_{j}\right\|_{L^{\infty}\left(B_{r}\right)}}{r^{s+\alpha} \theta(r)} \\
& \leq 5,
\end{aligned}
$$

and therefore

$$
\begin{equation*}
\left|K_{2 r, j}-K_{r, j}\right| \leq 10 r^{\alpha-\epsilon} \theta(r), \tag{5.19}
\end{equation*}
$$

which we will use later on in this proof.

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On the other hand, by (5.12) in Step 1 we have, whenever $0<2^{l} r \leq 2^{N} r \leq 1$,

$$
\left\|\varphi_{j}\right\|_{L^{\infty}\left(B_{2^{N_{r}}}\right)} \leq 2^{(s+\epsilon)(N-l)}\left\|\varphi_{j}\right\|_{L^{\infty}\left(B_{\left.2^{l_{r}}\right)}\right)}
$$

and therefore

$$
\begin{aligned}
\frac{\left\|K_{2^{l+1} r, j} \varphi_{j}-K_{2^{l} r, j} \varphi_{j}\right\|_{L^{\infty}\left(B_{\left.2 N_{r}\right)}\right)}}{r^{s+\alpha} \theta(r)} & =\frac{\left|K_{2^{l+1} r, j}-K_{2^{l} r, j}\right|\left\|\varphi_{j}\right\|_{L^{\infty}\left(B_{2^{N_{r}}}\right)}}{r^{s+\alpha} \theta(r)} \\
& \leq \frac{\left|K_{2^{l+1} r, j}-K_{2^{l} r, j}\right| 2^{(s+\epsilon)(N-l)}\left\|\varphi_{j}\right\|_{L^{\infty}\left(B_{\left.2^{l}\right)}\right)}}{r^{s+\alpha} \theta(r)} \\
& =\frac{2^{(s+\epsilon)(N-l)}\left\|K_{2^{l+1} r, j} \varphi_{j}-K_{2^{l} r, j} \varphi_{j}\right\|_{L^{\infty}\left(B_{2} l_{r}\right)}}{r^{s+\alpha} \theta(r)} \\
& =\frac{2^{l(s+\alpha)} \theta\left(2^{l} r\right)}{\theta(r)} \frac{2^{(s+\epsilon)(N-l)}\left\|K_{2^{l+1} r, j} \varphi_{j}-K_{2^{l} r, j} \varphi_{j}\right\|_{L^{\infty}\left(B_{2} l_{r}\right)}}{\left(2^{l} r\right)^{s+\alpha} \theta\left(2^{l} r\right)} \\
& \leq 102^{(s+\epsilon) N} 2^{l(\alpha-\epsilon)} .
\end{aligned}
$$

Thus,

$$
\frac{\left\|K_{2^{N} r, j} \varphi_{j}-K_{r, j} \varphi_{j}\right\|_{L^{\infty}\left(B_{2 N_{r}}\right)}}{r^{s+\alpha} \theta(r)} \leq 2^{(s+\epsilon) N} \sum_{l=0}^{N-1} 2^{l(\alpha-\epsilon)} \leq C 2^{(s+\alpha) N}
$$

where we have used that $\epsilon \in(0, \alpha)$.
Form the previous equation we deduce

$$
\frac{\left\|K_{R r, j} \varphi_{j}-K_{r, j} \varphi_{j}\right\|_{L^{\infty}\left(B_{R r}\right)}}{r^{s+\alpha} \theta(r)} \leq C R^{s+\alpha}
$$

whenever $0<r \leq R r \leq 1$.
Hence,

$$
\begin{aligned}
\left\|v_{m}\right\|_{L^{\infty}\left(B_{R}\right)} & =\frac{1}{\theta\left(r_{m}^{\prime}\right)\left(r_{m}^{\prime}\right)^{s+\alpha}}\left\|u_{m}-K_{m} \varphi_{m}\right\|_{L^{\infty}\left(B_{R r^{\prime} m}\right)} \\
& \leq \frac{R^{s+\alpha}\left\|u_{j_{m}}-K_{R r_{m}^{\prime}, j_{m}} \varphi_{j_{m}}\right\|_{L^{\infty}\left(B_{R r^{\prime} m}\right)}}{\theta\left(r_{m}^{\prime}\right)\left(R r_{m}^{\prime}\right)^{s+\alpha}}+\frac{\left\|K_{R r_{m}^{\prime}, j_{m}} \varphi_{j_{m}}-K_{r_{m}^{\prime}, j_{m}} \varphi_{j_{m}}\right\|_{L^{\infty}\left(B_{\left.R r_{m}^{\prime}\right)}\right.}}{\left(r_{m}^{\prime}\right)^{s+\alpha} \theta\left(r_{m}^{\prime}\right)} \\
& \leq \frac{R^{s+\alpha} \theta\left(R r_{m}^{\prime}\right)}{\theta\left(r_{m}^{\prime}\right)}+C R^{s+\alpha} \\
& \leq C R^{s+\alpha}
\end{aligned}
$$

whenever $R r_{m}^{\prime} \leq 1$.
When $R r_{m}^{\prime} \geq 1$ we simply use the assumption (5.10), namely,

$$
\left\|a u_{m}+b \varphi_{m}\right\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \leq C_{0}(|a|+|b|) R^{s+\alpha} \quad \text { for all } R \geq 1
$$

twice, with $a=1, b=-K_{1, j_{m}}$ and with $a=0, b=1$ to estimate

$$
\left.\begin{array}{rl}
\left\|v_{m}\right\|_{L^{\infty}\left(B_{R}\right)} & =\frac{1}{\theta\left(r_{m}^{\prime}\right)\left(r_{m}^{\prime}\right)^{s+\alpha}}\left\|u_{m}-K_{m} \varphi_{m}\right\|_{L^{\infty}\left(B_{R r^{\prime} m}\right)} \\
& \leq \frac{R^{s+\alpha}\left\|u_{j_{m}}-K_{1, j_{m}} \varphi_{j_{m}}\right\|_{L^{\infty}\left(B_{R r^{\prime} m}\right)}}{\theta\left(r_{m}^{\prime}\right)\left(R r_{m}^{\prime}\right)^{s+\alpha}}+\frac{\left\|K_{1, j_{m}} \varphi_{j_{m}}-K_{r_{m}^{\prime}, j_{m}} \varphi_{j_{m}}\right\|_{L^{\infty}\left(B_{R r_{m}^{\prime}}\right)}}{\left(r_{m}^{\prime}\right)^{s+\alpha} \theta\left(r_{m}^{\prime}\right)} \\
& \left.\leq C_{0}\left(1+\left|K_{1, j_{m}}\right|\right) R^{s+\alpha}+\frac{\left\|K_{1, j_{m}} \varphi_{j_{m}}-K_{r_{m}^{\prime}, j_{m}} \varphi_{j_{m}}\right\|_{L^{\infty}\left(B_{1}\right)}}{\left\|\varphi_{j_{m}}\right\|_{L^{\infty}\left(B_{R r_{m}^{\prime}}\right)}} \| r_{m}^{\prime}\right)^{s+\alpha} \theta\left(r_{m}^{\prime}\right)
\end{array}\left\|\varphi_{j_{m}}\right\|_{L^{\infty}\left(B_{1}\right)}\right)
$$

where we have used $\left|K_{1, j_{m}}\right| \leq C$ (that we will prove in detail in Step 3).
Step 3. We prove that a subsequence of $v_{m}$ converges locally uniformly to a entire solution $v_{\infty}$ of the problem

$$
\begin{cases}M^{+} v_{\infty} \geq 0 \geq M^{-} v_{\infty} & \text { in }\{e \cdot x>0\}  \tag{5.20}\\ v_{\infty}=0 & \text { in }\{e \cdot x<0\}\end{cases}
$$

By assumption, the function $w=a u_{m}+b \varphi_{m}$ satisfies

$$
\begin{cases}M^{+}\left(a u_{m}+b \varphi_{m}\right) \geq-\delta(|a|+|b|) d^{\alpha-s} & \text { in } B_{1} \cap \Omega_{m}  \tag{5.21}\\ u_{m}=\varphi_{m}=0 & \text { in } B_{1} \backslash \Omega_{m}\end{cases}
$$

for all $a, b \in \mathbb{R}$.
Now, using (5.19) we obtain

$$
\begin{aligned}
\frac{\left|K_{1, j}-K_{2^{-N}, j}\right|}{\theta\left(2^{-N}\right)} & \leq \sum_{l=0}^{N-1} \frac{\left|K_{2^{-N+l+1, j}}-K_{2^{-N+l, j}}\right|}{\theta\left(2^{-N}\right)} \\
& =\sum_{l=0}^{N-1} 10 \frac{\theta\left(2^{-N+l}\right)}{\theta\left(2^{-N}\right)} 2^{(-N+l)(\alpha-\epsilon)} \\
& \leq 10 \sum_{l=0}^{N-1} 2^{(-N+l)(\alpha-\epsilon)} \leq C
\end{aligned}
$$

since $\alpha-\epsilon>0$.
Next, using (5.11) - that holds with $\varphi$ replaced by $\varphi_{j}$, the definition $K_{r, j}$, and that $\left\|\varphi_{j}\right\|_{L^{\infty}\left(B_{1}\right)}=1$ while $\left\|u_{j}\right\|_{L^{\infty}\left(B_{1}\right)} \leq C_{0}$, we obtain

$$
\begin{equation*}
\left|K_{1, j}\right|=\left|\frac{\int_{B_{1}} u_{j} \varphi_{j} d x}{\int_{B_{1}} \varphi_{j}^{2} d x}\right| \leq C \tag{5.22}
\end{equation*}
$$

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Thus

$$
\frac{\left|K_{2^{-N}, j}\right|}{\theta\left(2^{-N}\right)} \leq \frac{\left|K_{1, j}\right|}{\theta\left(2^{-N}\right)}+\frac{\left|K_{1, j}-K_{2^{-N}, j}\right|}{\theta\left(2^{-N}\right)} \leq C
$$

Using this control for $K_{r, j}$ and setting in (5.21)

$$
a=\frac{1}{\theta\left(r_{m}^{\prime}\right)} \quad \text { and } \quad b=\frac{-K_{r_{m}^{\prime}, j_{m}}}{\theta\left(r_{m}^{\prime}\right)}
$$

we obtain

$$
\begin{aligned}
M^{+} v_{m} & =\frac{\left(r_{m}^{\prime}\right)^{2 s}}{\left(r_{m}^{\prime}\right)^{s+\alpha} \theta\left(r_{m}^{\prime}\right)} M^{+}\left(\frac{1}{\theta\left(r_{m}^{\prime}\right)} u_{m}-\frac{K_{r_{m}^{\prime}, j_{m}}}{\theta\left(r_{m}^{\prime}\right)} \varphi_{m}\right)\left(r_{m}^{\prime} \cdot\right) \\
& \geq-C \delta \frac{d_{m}^{\alpha-s}}{\theta\left(r_{m}^{\prime}\right)} \quad \text { in } B_{\left(r_{m}^{\prime}\right)^{-1}} \cap\left(r_{m}^{\prime}\right)^{-1} \Omega_{m}
\end{aligned}
$$

where $d_{m}(x)=\operatorname{dist}\left(x, r_{m}^{-1} \Omega_{k_{m}}\right)$. Similarly, changing sign in the previous choices of $a$ and $b$ we obtain

$$
-M^{-}\left(v_{m}\right)=M^{+}\left(-v_{m}\right) \geq-C \delta \frac{d_{m}^{\alpha-s}}{\theta\left(r_{m}^{\prime}\right)} \quad \text { in } B_{\left(r_{m}^{\prime}\right)^{-1}} \cap\left(r_{m}^{\prime}\right)^{-1} \Omega_{m}
$$

As complement datum we clearly have

$$
v_{m}=0 \quad \text { in } B_{\left(r_{m}^{\prime}\right)^{-1}} \backslash\left(r_{m}^{\prime}\right)^{-1} \Omega_{m}
$$

Then, by Lemma 5.2 we have

$$
\left\|v_{m}\right\|_{C^{\gamma}\left(B_{R}\right)} \leq C(R) \quad \text { for all } m \text { large enough. }
$$

The constant $C(R)$ depends on $R$, but not on $m$.
Then, by Arzelà-Ascoli and the stability lemma in [CS11b, Lemma 4.3] we obtain that

$$
v_{m} \rightarrow v_{\infty} \in C\left(\mathbb{R}^{n}\right)
$$

locally uniformly, where $v_{\infty}$ satisfies the growth control

$$
\left\|v_{\infty}\right\|_{L^{\infty}\left(B_{R}\right)} \leq C R^{s+\alpha} \quad \text { for all } R \geq 1
$$

and solves (5.20) in the viscosity sense. Thus, by the Liouville-type result RS14, Proposition 5.1], we find $v_{\infty}(x)=K(x \cdot e)_{+}^{s}$ for some $K \in \mathbb{R}$.

Step 4. We prove that as subsequence of $\tilde{\varphi}_{m}$, where

$$
\tilde{\varphi}_{m}(x)=\frac{\varphi_{m}\left(r_{m}^{\prime} x\right)}{\sup _{B_{r_{m}^{\prime}}} \varphi_{m}}
$$

converges locally uniformly to $(x \cdot e)_{+}^{s}$.
This is similar to Step 3 and we only need to use the estimates in Step 1, and the growth control 5.10, to obtain a uniform control of the type

$$
\left\|\tilde{\varphi}_{m}\right\|_{L^{\infty}\left(B_{R}\right)} \leq C_{0} R^{s+\alpha} \quad \text { for all } R \geq 1
$$

Using the estimates in Step 1 we easily show that

$$
\frac{\left(r_{m}^{\prime}\right)^{2 s}}{\sup _{B_{r_{m}^{\prime}}} \varphi_{m}} \downarrow 0
$$

Thus, we use (5.21) with $a=0$ and $b=\left(\sup _{B_{r_{m}^{\prime}}} \varphi_{m}\right)^{-1}$ to prove that $\tilde{\varphi}_{m}$ converges locally uniformly to a solution $\tilde{\varphi}_{\infty}$ of

$$
\begin{cases}M^{+} \tilde{\varphi}_{\infty} \geq 0 \geq M^{-} \tilde{\varphi}_{\infty} & \text { in }\{e \cdot x>0\} \\ \tilde{\varphi}_{\infty}=0 & \text { in }\{e \cdot x<0\}\end{cases}
$$

Then, using the Liouville-type result [RS14, Proposition 5.1] and since

$$
\left\|\tilde{\varphi}_{\infty}\right\|_{L^{\infty}\left(B_{1}\right)}=\lim _{m \rightarrow \infty}\left\|\tilde{\varphi}_{m}\right\|_{L^{\infty}\left(B_{1}\right)}=\lim _{m \rightarrow \infty} 1=1
$$

we get

$$
\tilde{\varphi}_{\infty} \equiv(x \cdot e)_{+}^{s} .
$$

Hence, $\tilde{\varphi}_{m}(x) \rightarrow(x \cdot e)_{+}^{s}$ locally uniformly in $\mathbb{R}^{n}$.
Step 5. We have $v_{m} \rightarrow K(x \cdot e)_{+}^{s}$ and $\tilde{\varphi}_{m} \rightarrow(x \cdot e)_{+}^{s}$ locally uniformly. Now, by (5.17),

$$
\int_{B_{1}} v_{m}(x) \tilde{\varphi}_{m}(x) d x=0
$$

Thus, passing this equation to the limits,

$$
\int_{B_{1}} v_{\infty}(x)(x \cdot e)_{+}^{s} d x=0
$$

This implies $K=0$ and $v_{\infty} \equiv 0$.
But then passing to the limit (5.16) we get

$$
\left\|v_{\infty}\right\|_{L^{\infty}\left(B_{1}\right)} \geq \frac{1}{2}
$$

a contradiction.
We next prove Theorems 1.3 and 1.6 .
Proof of Theorem 1.6. Step 1. We first show, by a barrier argument, that for any given $\epsilon>0$ we have

$$
c d^{s+\epsilon} \leq u_{i} \leq C d^{s-\epsilon} \quad \text { in } B_{1 / 2}
$$

where $d=\operatorname{dist}\left(\cdot, B_{1} \backslash \Omega\right)$, and $c>0$ is a constant depending only on $\Omega, n, s$, ellipticity constants.

First, notice that by assumption we have $M^{-} u_{i}=-M^{+}\left(-u_{i}\right) \leq \delta$ and $M^{+} u_{i} \geq$ $-\delta$ in $B_{1} \cap \Omega$. Therefore, since $\sup _{B_{1 / 2}} u_{i} \geq 1$, for any small $\rho>0$ by the interior Harnack inequality we find

$$
\inf _{B_{3 / 4} \cap\{d \geq \rho\}} u_{i} \geq C^{-1}-C \delta \geq c>0
$$

provided that $\delta$ is small enough (depending on $\rho$ ).

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Now, let $x_{0} \in B_{1 / 2} \cap \partial \Omega$, and let $e \in S^{n-1}$ be the normal vector to $\partial \Omega$ at $x_{0}$. By the previous inequality,

$$
\inf _{B_{\rho}\left(x_{0}+2 \rho e\right)} u_{i} \geq c .
$$

Since $\Omega$ is $C^{1}$, then for any $\eta>0$ there is $\rho>0$ for which

$$
\left(x_{0}+\mathcal{C}_{\eta}\right) \cap B_{4 \rho} \subset \Omega
$$

where $\mathcal{C}_{\eta}$ is the cone in Lemma 4.1.
Therefore, using the function $\Phi_{\text {sub }}$ given by Lemma 4.1, we may build the subsolution

$$
\psi=\Phi_{\text {sub }} \chi_{B_{4 \rho}\left(x_{0}\right)}+C_{1} \chi_{B_{\rho / 2}\left(x_{0}+2 \rho e\right)}
$$

Indeed, if $C_{1}$ is large enough then $\psi$ satisfies

$$
M^{-} \psi \geq 1 \quad \text { in } \quad\left(x_{0}+\mathcal{C}_{\eta}\right) \cap\left(B_{3 \rho}\left(x_{0}\right) \backslash B_{\rho}\left(x_{0}+2 \rho e\right)\right)
$$

and $\psi \equiv 0$ outside $x_{0}+\mathcal{C}_{\eta}$.
Hence, we may use $c_{2} \psi$ as a barrier, with $c_{2}$ small enough so that $u_{i} \geq c_{2} \psi$ in $B_{\rho}\left(x_{0}+2 \rho e\right)$. Then, by the comparison principle we find

$$
u_{i} \geq c_{2} \psi
$$

and in particular

$$
u_{i}\left(x_{0}+t e\right) \geq c_{3} t^{s+\epsilon}
$$

for $t \in(0, \rho)$. Since this can be done for all $x_{0} \in B_{1 / 2} \cap \partial \Omega$, we find

$$
\begin{equation*}
u_{i} \geq c d^{s+\epsilon} \quad \text { in } B_{1 / 2} \tag{5.23}
\end{equation*}
$$

Similarly, using the supersolution $\Phi_{\text {sup }}$ from Lemma 4.1, we find

$$
\begin{equation*}
u_{i} \leq C d^{s-\epsilon} \quad \text { in } B_{1 / 2} \tag{5.24}
\end{equation*}
$$

for $i=1,2$.
Step 2. Let us prove now that

$$
\begin{equation*}
u_{1} \leq C u_{2} \quad \text { in } \quad B_{1 / 2} \tag{5.25}
\end{equation*}
$$

To prove 5.25, we rescale the functions $u_{1}$ and $u_{2}$ and use Proposition 5.4.
Let $x_{0} \in B_{1 / 2} \cap \partial \Omega$, and let

$$
\theta(r)=\sup _{r^{\prime}>r} \frac{\left\|u_{1}\right\|_{L^{\infty}\left(B_{r^{\prime}}\left(x_{0}\right)\right)}+\left\|u_{2}\right\|_{L^{\infty}\left(B_{r^{\prime}}\left(x_{0}\right)\right)}}{\left(r^{\prime}\right)^{s+\epsilon}}
$$

Notice that $\theta(r)$ is monotone nonincreasing and that $\theta(r) \rightarrow \infty$ by 5.23. Let $r_{k} \rightarrow 0$ be such that

$$
\left\|u_{1}\right\|_{L^{\infty}\left(B_{r_{k}}\left(x_{0}\right)\right)}+\left\|u_{2}\right\|_{L^{\infty}\left(B_{r_{k}}\left(x_{0}\right)\right)} \geq \frac{1}{2}\left(r_{k}\right)^{s+\epsilon} \theta\left(r_{k}\right)
$$

with $c_{0}>0$, and define

$$
v_{k}(x)=\frac{u_{1}\left(x_{0}+r_{k} x\right)}{\left(r_{k}\right)^{s+\epsilon} \theta\left(r_{k}\right)}, \quad w_{k}(x)=\frac{u_{2}\left(x_{0}+r_{k} x\right)}{\left(r_{k}\right)^{s+\epsilon} \theta\left(r_{k}\right)}
$$

Note that

$$
\left\|v_{k}\right\|_{L^{\infty}\left(B_{1}\right)}+\left\|w_{k}\right\|_{L^{\infty}\left(B_{1}\right)} \geq \frac{1}{2} .
$$

Moreover,

$$
\left\|v_{k}\right\|_{L^{\infty}\left(B_{R}\right)}=\frac{\left\|u_{1}\right\|_{L^{\infty}\left(B_{r_{k}} R\right)}}{\left(r_{k}\right)^{s+\epsilon} \theta\left(r_{k}\right)} \leq \frac{\theta\left(r_{k} R\right)\left(r_{k} R\right)^{s+\epsilon}}{\left(r_{k}\right)^{s+\epsilon} \theta\left(r_{k}\right)} \leq R^{s+\epsilon},
$$

for all $R \geq 1$, and analogously

$$
\left\|w_{k}\right\|_{L^{\infty}\left(B_{R}\right)} \leq R^{s+\epsilon}
$$

for all $R \geq 1$.
Now, the functions $v_{k}, w_{k}$ satisfy the equation
$M^{+}\left(a v_{k}+b w_{k}\right)(x)=\frac{\left(r_{k}\right)^{2 s}}{\left(r_{k}\right)^{s+\epsilon} \theta\left(r_{k}\right)} M^{+}\left(a u_{1}+b u_{2}\right)\left(x_{0}+r_{k} x\right) \geq-C_{0}\left(r_{k}\right)^{s-\epsilon} \delta(|a|+|b|)$
in $\Omega_{k} \cap B_{r_{k}^{-1}}$, where $\Omega_{k}=r_{k}^{-1}\left(\Omega-x_{0}\right)$.
Taking $k$ large enough, we will have that $\Omega_{k}$ satisfies the hypotheses of Proposition 5.4 in $B_{1 / \delta}$, and

$$
M^{+}\left(a v_{k}+b w_{k}\right) \geq-\delta(|a|+|b|) \quad \text { in } \quad \Omega_{k} \cap B_{1 / \delta}
$$

Moreover, since $\sup _{B_{1}} v_{k}+\sup _{B_{1}} w_{k} \geq 1 / 2$, then either $\sup _{B_{1}} v_{k} \geq 1 / 4$ or $\sup _{B_{1}} w_{k} \geq$ $1 / 4$. Therefore, by Proposition 5.4 we find that either

$$
\left|v_{k}(x)-K_{1} w_{k}(x)\right| \leq C|x|^{s+\alpha}
$$

or

$$
\left|w_{k}(x)-K_{2} v_{k}(x)\right| \leq C|x|^{s+\alpha}
$$

for some $|K| \leq C$. This yields that either

$$
\begin{equation*}
\left|u_{1}(x)-K_{1} u_{2}(x)\right| \leq C\left|x-x_{0}\right|^{s+\alpha} \tag{5.26}
\end{equation*}
$$

or

$$
\begin{equation*}
\left|u_{2}(x)-K_{2} u_{1}(x)\right| \leq C\left|x-x_{0}\right|^{s+\alpha}, \tag{5.27}
\end{equation*}
$$

with a bigger constant $C$.
Now, we may choose $\epsilon>0$ so that $\epsilon<\alpha / 2$, and then (5.27) combined with (5.23)(5.24) gives $K_{2} \geq c>0$, which in turn implies (5.26) for $K_{1}=K_{2}^{-1},\left|K_{1}\right| \leq C$. Thus, in any case (5.26) is proved.

In particular, for all $x_{0} \in B_{1 / 2} \cap \partial \Omega$ and all $x \in B_{1 / 2} \cap \Omega$ we have

$$
u_{1}(x) / u_{2}(x) \leq K_{1}+\left|\frac{u_{1}(x)}{u_{2}(x)}-K_{1}\right| \leq K_{1}+C\left|x-x_{0}\right|^{s+\alpha} / u_{2}(x)
$$

Choosing $x_{0}$ such that $\left|x-x_{0}\right| \leq C d(x)$ and using (5.24), we deduce

$$
u_{1}(x) / u_{2}(x) \leq K_{1}+C d^{s+\alpha} / d^{s-\epsilon} \leq C
$$

and thus 5.25 is proved.

Step 3. We finally show that $u_{1} / u_{2} \in C^{\alpha}\left(\bar{\Omega} \cap B_{1 / 2}\right)$ for all $\alpha \in(0, \bar{\alpha})$. Since this last step is somewhat similar to the proof of Theorem 1.2 in [RS14b], we will omit some details.

We use that, for all $\alpha \in(0, \bar{\alpha})$ and all $x \in B_{1 / 2} \cap \Omega$, we have

$$
\begin{equation*}
\left|\frac{u_{1}(x)}{u_{2}(x)}-K\left(x_{0}\right)\right| \leq C\left|x-x_{0}\right|^{\alpha-\epsilon} \tag{5.28}
\end{equation*}
$$

where $x_{0} \in B_{1 / 2} \cap \partial \Omega$ is now the closest point to $x$ on $B_{1 / 2} \cap \partial \Omega$. This follows from (5.26), as shown in Step 2.

We also need interior estimates for $u_{1} / u_{2}$. Indeed, for any ball $B_{2 r}(x) \subset \Omega \cap B_{1 / 2}$, with $2 r=d(x)$, there is a constant $K$ such that $\left\|u_{1}-K u_{2}\right\|_{L^{\infty}\left(B_{r}(x)\right)} \leq C r^{s+\alpha}$. Thus, by interior estimates we find that $\left[u_{1}-K u_{2}\right]_{C^{\alpha-\epsilon}\left(B_{r}(x)\right)} \leq C r^{s+\epsilon}$. This, combined with (5.23)-(5.24) yields

$$
\begin{equation*}
\left[u_{1} / u_{2}\right]_{C^{\alpha-\epsilon}\left(B_{r}(x)\right)} \leq C \tag{5.29}
\end{equation*}
$$

Let now $x, y \in B_{1 / 2} \cap \Omega$, and let us show that

$$
\begin{equation*}
\left|\frac{u_{1}(x)}{u_{2}(x)}-\frac{u_{1}(y)}{u_{2}(y)}\right| \leq C|x-y|^{\alpha-\epsilon} . \tag{5.30}
\end{equation*}
$$

If $y \in B_{r}(x), 2 r=d(x)$, or if $x \in B_{r}(y), 2 r=d(y)$, then this follows from 5.29). Otherwise, we have $|x-y| \geq \frac{1}{2} \max \{d(x), d(y)\}$, and then 5.30 follows from 5.28.

In any case, 5.30) is proved, and therefore we have

$$
\left\|u_{1} / u_{2}\right\|_{C^{\alpha-\epsilon}\left(\bar{\Omega} \cap B_{1 / 2}\right)} \leq C
$$

Since this can be done for any $\alpha \in(0, \bar{\alpha})$ and any $\epsilon>0$, the result follows.
Proof of Theorem 1.3. The proof is the same as Theorem 1.6, replacing the Liouvilletype result [RS14, Proposition 5.1] by [RS14b, Theorem 4.1], and replacing $\bar{\alpha}$ by $s$.

Remark 5.5. Notice that in Proposition 5.4 we only require the right hand side of the equation to be bounded by $d^{\alpha-s}$. Thanks to this, Theorem 1.3 holds as well for

$$
\begin{equation*}
-\delta d^{\alpha-s} \leq f_{i}(x) \leq C_{0} d^{\alpha-s}, \quad \alpha \in(0, s) \tag{5.31}
\end{equation*}
$$

In that case, we get

$$
\left\|u_{1} / u_{2}\right\|_{C^{\alpha}\left(\Omega \cap B_{1 / 2}\right)} \leq C C_{0}
$$

with the exponent $\alpha$ in (5.31).
Proof of Corollary 1.4. The result follows from Theorem 1.3.

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