MASTER'S THESIS

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Title: Computation of the invariant manifolds of infinity in the restricted circular planar three body problem with the parametrization method.

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Thanks to my supervisors, who not only helped me with the Master thesis. They have also recommended me to do an Erasmus programme in the Université Pierre et Marie Curie and they introduced me to my future PhD director: Marcel Guàrdia.

Abstract

This work analyses the parabolic stable manifolds of the periodic orbits at infinity for the Restricted Planar Circular Three Body Problem. Using the parametrization method we prove the existence of such manifold and we compute an approximate parametrization K with an internal dynamic Y. That leads to numerical computations that will be analysed and compared with a similar methodology called the graph transform method.

Keywords

37 Dynamical systems, 37Nxx Applications, 37N05 Dynamical systems in classical and celestial mechanics, 65 Numerical analysis, 65Pxx Numerical problems in dynamical systems, 65P10 Hamiltonian systems including symplectic integrators

Contents

Introduction

Dynamical systems have always had a strong relationship with the problems of celestial mechanics. Among them, one of the most challenging are those related to the N-body problem, which analyses the behaviour of N mass particles moving under their mutual Newtonian gravitational attraction.

The N-body problem has been studied along more than two centuries and even now it is far for being understood. Unlike the case $N = 2$, which was already solved by Newton, , when $N \geq 3$ the general solution is still a mystery. For these reason mathematicians are forced to focus on specific parts of this problem.

In this work we analyse some aspects of the restricted planar circular three body problem (RPC3BP). Here there are different points to take on account. First we claim that there are only three point bodies in the universe. Also, we suppose that one of the bodies has a so small mass that we can consider it equal to zero. Then, we also consider that the behaviour of the two massive bodies consists on circular orbits in an invariant plane. All of these conditions leads to some simplifications of the formulae involved which allows us to make studies further than the general case.

In our work we are going to study the parabolic stable manifolds associated to periodic orbits at infinity, meaning the set of points which tend to the orbit with zero velocity. Our objective is to promote a new way of proving the existence of a parabolic stable manifold for each periodic orbit at infinity. To do so we will use the parametrization method.

The parametrization method has its origin in a total of three articles, which are [CFdlL03a], [CFdlL03b] and [CFdlL05], written by X. Cabré, E. Fontich and R. de la Llave. The method is quite recent but it has been proved to be extremely useful since, by the way it works, it has extremely good synergy with numerical computations.

The main idea of the parametrization method is the following: consider a dynamical system $F : U \subset \mathbb{R}^n \to \mathbb{R}$ with a fixed point at the origin i.e. $F(0) = 0$. Given an invariant subspace E of $DF(0)$, we ask ourselves if it exists an invariant manifold tangent to this subspace. One way of finding it consist in looking for $K : U \subset \mathbb{R}^p \to \mathbb{R}^n$, such that $K(0) = 0$ and $DK(0)\mathbb{R}^p = E$, satisfying

$$F \circ K = K \circ R, \tag{1}$$

where $R : U \subset \mathbb{R}^p \to \mathbb{R}^p$ is a reparametrization.

One strategy for solving it is the following: we try to find approximate solutions of (1) i.e. $K^{\leq m}$ and $R^{\leq m}$ (polynomials of degree $\leq m$) such that

$$F^{\leq m} \circ K^{\leq m}(u) - K^{\leq m} \circ R^{\leq m}(u) = O(u^N),$$

for an N sufficiently large. If that is possible, then we can find later a true solution for (1) close to the approximated ones.

In the first chapter of this work, supported by [MHO09], we explain the general case of the N-body problem. Also, in the same chapter, we analyse with more detail the restricted circular planar three body problem. Even more, we present a suitable changes of variables, which comes from the paper [McG73], which would be of great use along all the posterior calculus.

In the second chapter we explain in more detail how the parametrization method works. We show first two cases where we can apply this methodology, even it is not strictly necessary for our work. This is done for developing the intuition of the reader. Then we move to the case we shall use for our problem.

The structure of all three cases works similar. We start by giving the necessary definitions and notation. Later we show how to compute iteratively the terms of an approximate parametrization. Finally we prove that there exist a real invariant manifold and that the approximate parametrization is close to it. This chapter is supported by the articles [BFdlLM07], [BFM], [CFdlL03a], [CFdlL03b], [CFdlL05] and the books [HCF+16] and [KK01].

In the third chapter we will move to all the computations done for proving that the new vector field of the parabolic stable manifold satisfy the hypothesis for applying the parametrization method. Once we have done that we will work with a series expansion of the differential equations for obtaining an approximated parametrization of degree four. We had chosen this degree because is the smallest one we need to know exactly the internal dynamic of the parametrization. Then, for degree greater or equal to five, it will be only necessary to compute iteratively the parametrization itself.

In the fourth and last chapter we implement numerically the parametrization method with the parametrization computed before. We use MATLAB software to do so. Later we compare the results and evolution of the error with another methodology called the graph transform method, explained in the article [MS14], which we will call simply graph method from now and beyond.

Finally we attach two appendices. The first contains a summary of the parametrization method for parabolic periodic vector fields. The other one contains the MATLAB code for the parametrization method and the graph method used in our work.

To sum up, we remark one of the most important points of this project. It works with the parametrization method which, even it is quite recent, it has been proven to be an extremely effective tool from both, the computational and theoretical point of view. There are still studies that analyse how to apply that in complete different fields. We consider that the fact of using fresh methodology is quite interesting and appealing.

Also, since we have studied the comparison of the graph method for a fixed order, this work allows people interested in that method to continue from here by computing and analysing the error of higher degrees. Even they could find a way to generalize its evolution. There is still a whole world to explore.

Chapter 1

The N-Body Problem

In this chapter we present one of the most important problems in celestial mechanics: the N-body problem.

Consider N spherical masses, which can be considered then as point masses, in $\mathbb{R}^3$. The N-body problem studies their evolution according to the classical mechanics and Newtonian gravitational forces.

First of all we introduce some notations and concepts that will be used along all the work, this is done in Section 1.1. After, in Section 1.2, we present the differential equations that the trajectories of the N-body problem follow according to their mutual gravitational attraction.

Later, in Sections 1.3 – 1.6 we show the simplest features of the N-body problem, namely first integrals, equilibrium points,... Then, in Sections 1.7 and 1.8, we restrict ourselves to the case $N = 2$ which can be solved explicitly. In Section 1.9, we consider an special case: when $N = 3$, one body has zero mass and all the bodies are moving circularly into a plane.

Finally, in Section 1.10, we present the set of orbits we will study: the parabolic orbits at infinity.

1.1 Preliminary concepts

The universal gravitation Newton's law give rise to a system of second-order differential equations in $\mathbb{R}^{3N}$. Then it can be transformed into a system of first-order equations in $\mathbb{R}^{6N}$.

This system of $6N$ first order differential equations can be written as a Hamiltonian system.

Let $H = H(t, q, p) : \mathcal{U} \subset \mathbb{R}^{2n+1} \to \mathbb{R}$, be a Hamiltonian, where t can be considered as the time, $q = (q_1, \ldots, q_n)$ as the position vector and $p = (p_1, \ldots, p_n)$ as the momentum vector. The integer n is the number of degrees of freedom of the system.

The Hamiltonian defines the following set of differential equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}(t, q, p), \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}(t, q, p), \tag{1.1}$$

for $i = 1, \ldots, n$.

These differential equations can be resumed in a simple formula if we define the $2n$-dimensional vector z, the $2n \times 2n$ skew symmetric matrix J and the gradient of H as

$$z = \begin{bmatrix} q \\ p \end{bmatrix}, \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad \nabla H = \begin{bmatrix} \frac{\partial H}{\partial z_1} \\ \ldots \\ \frac{\partial H}{\partial z_{2n}} \end{bmatrix},$$

where 0 is the $n \times n$ zero matrix and I is the $n \times n$ identity matrix. Indeed, (1.1) can be written as

$$\dot{z} = J \nabla H(z, t). \tag{1.2}$$

When H is independent of t, that is $H : \mathcal{U} \subset \mathbb{R}^{2n} \to \mathbb{R}$, we call (1.2) conservative. Note that, in this case, the differential equations (1.2) are autonomous.

A first integral for (1.2) is a smooth function $F : \mathcal{U} \to \mathbb{R}$ which is constant along the solutions of (1.2). Some examples of first integrals in systems arising from classical mechanics are the classical conserved quantities of energy, momentum, etc. Notice that if we have any constant $c \in \mathbb{R}$, then the level surface $F^{-1}(c) \subset \mathbb{R}^{2n}$ is an invariant set; i.e. if a solution starts in this set it will remain there for all time.

1.2 The general case of the N-Body Problem

Consider that we have $N \geq 2$ spherical masses in $\mathbb{R}^3$, which we are going to consider as particles i.e., all the respective masses are concentrated in single points. The only forces that act on them are their mutual Newtonian gravitational attraction. We want to study their motion along time.

For the i-th body let $q_i \in \mathbb{R}^3$ be its position and $m_i > 0$ its mass. Let $\mathcal{G}$ be the gravitational constant, which depends uniquely on the units chosen. The potential U is defined as

$$U = \sum_{1 \leq i < j \leq n} \frac{\mathcal{G} m_i m_j}{\|q_i - q_j\|}.$$

Then, from the Newton's law, we obtain the following system of equations

$$m_i \ddot{q}_i = \sum_{j=1, j \neq i}^{N} \frac{\mathcal{G} m_i m_j (q_j - q_i)}{\|q_i - q_j\|^3} = \frac{\partial U}{\partial q_i}, \quad i = 1, \ldots, N. \tag{1.3}$$

We define $p_i := m_i \dot{q}_i$ as the momentum of the i-th particle and T as the kinetic energy:

$$T = \frac{1}{2} \sum_{i=1}^{N} \frac{\|p_i\|^2}{m_i} = \frac{1}{2} \sum_{i=1}^{N} m_i \|\dot{q}_i\|^2.$$

Then the Hamiltonian H is

$$H = T - U,$$

and the equations (1.3) become

$$\dot{q}_i = \frac{p_i}{m_i} = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = \sum_{j=1, j \neq i}^{N} \frac{\mathcal{G} m_i m_j (q_j - q_i)}{\|q_i - q_j\|^3} = -\frac{\partial H}{\partial q_i}. \tag{1.4}$$

1.3 Classical first integrals

The N-body problem has ten known first integrals. We devote this section to present them.

Let

$$L = p_1 + \ldots + p_N$$

be the total momentum. From equation (1.4) we observe that $\dot{L} = 0$. Then, if we define the centre of mass C as

$$C = m_1 q_1 + \ldots + m_N q_N,$$

we obtain $\ddot{C} = 0$, since $\dot{C} = L$. Thus the centre of mass of the system moves with uniform rectilinear motion, that is, $C = at + b$, where the vectors $a, b \in \mathbb{R}^3$ are constants that depend on the initial conditions.

Each component of a and b corresponds to a first integral. So we have already found six first integrals.

Let

$$A = q_1 \times p_1 + \ldots + q_N \times p_N$$

be the total angular momentum of the system, where $\times$ denotes the cross product. We claim that it is constant along the trajectories. Indeed, we derive with respect to t and we obtain

$$\begin{aligned} \frac{dA}{dt} &= \sum_{i=1}^{N}(q_i \times \dot{p}_i + \dot{q}_i \times p_i) \\ &= \sum_{i=1}^{N} m_i q_i \times q_i + \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \frac{\mathcal{G} m_i m_j q_i \times (q_j - q_i)}{\|q_j - q_i\|^3}. \end{aligned} \tag{1.5}$$

Since $q_i \times q_i = 0$ we have that the first sum of the last equality in (1.5) is zero. The second sum is also zero because $q_i \times (q_j - q_i) = q_i \times q_j - q_i \times q_i = q_i \times q_j$ and the sum also have these terms with opposite sign.

Thus the three components of A are constant and we have obtained another three first integrals.

The last first integral is H itself, so we have found the ten known first integrals of the N-body problem.

1.4 Equilibrium points

Given a system of o.d.e., its behaviour around the hyperbolic equilibrium points is well understood, described by Hartman's theorem, at a topology level, and by the stable manifold theorem.

However, as we will prove here, there are no equilibrium points for the N-body problem. Indeed, from equation (1.3) we have that an equilibrium point has to satisfy

$$\frac{\partial U}{\partial q_i} = 0, \quad i = 1, \ldots, N.$$

Now we observe that U is an homogeneous function of degree -1 i.e. $U(tq) = t^{-1}U(q)$. Then, by the Euler's theorem on homogeneous functions, we have

$$\sum_{i=1}^{N} q_i \frac{\partial U}{\partial q_i} = -U. \tag{1.6}$$

On the one hand, if there is an equilibrium point, the left part of (1.6) should be zero and, consequently, also the right part. On the other hand, U is an strictly positive function, because is the sum of strictly positive terms.

So we have $U > 0$ and then $-U < 0$, which enters in contradiction with the equality (1.6), then we conclude that there are no equilibrium points in the N-body problem.

1.5 Central configurations

A common way to understand problems in dynamical systems is to look for the simplest possible behaviours. We have seen in the previous section that there are no equilibrium points so here we look for other types of “single solutions”.

In this section we want to see whether the N-body problem has solutions of the form $q_i(t) = \phi(t)a_i$, where a_i are constant vectors and $\phi(t)$ is the same scalar function for all i. These particular solutions are called central configurations.

Substituting these expressions into (1.3) we get

$$a_i|\phi|^3\phi^{-1}\ddot{\phi} = \sum_{j=1,j\neq i}^{N} \frac{\mathcal{G}m_j(a_j - a_i)}{\|a_j - a_i\|^3}. \tag{1.7}$$

In order to obtain solutions of this equation we notice that the right hand side of (1.7) is constant. Then, $|\phi|^3\phi^{-1}\ddot{\phi}$ must be also constant. Let $\lambda \in \mathbb{R}$ be its value. Then we have

$$a_i\lambda = \sum_{j=1,j\neq i}^{N} \frac{\mathcal{G}m_j(a_j - a_i)}{\|a_j - a_i\|^3}, \quad i = 1, \ldots, N, \tag{1.8}$$

$$\ddot{\phi} = \frac{\lambda\phi}{|\phi|^3}. \tag{1.9}$$

First of all, we notice that the equation (1.9) is a simple ordinary differential equation in $\mathbb{R}$ and one possible solution is $\phi(t) = \alpha t^{2/3}$, where $|\alpha| = -(9\lambda/2)^{1/3}$. Then (1.9) can be solved for non-positive values of λ.

However (1.8) is a non-trivial and non linear algebraic system whose solutions are only known for the case $N = 2, 3$ and some special cases with $N > 3$. We will present some of them in Section 1.6.

We will say that a configuration of N bodies is a central configuration, or c.c., if it is given by vectors $a_1, \ldots, a_N$ satisfying (1.8).

Now we observe that if $q = (q_1, \ldots, q_N)$ is a c.c., then λq is it also. For this reason we introduce the moment of inertia I, in order to measure the size of the system:

$$I = \frac{1}{2}\sum_{i=1}^{N} m_i\|q_i\|^2.$$

So we can rewrite system (1.8) as

$$\frac{\partial U}{\partial q_i}(a) + \lambda\frac{\partial I}{\partial q_i}(a) = 0, \tag{1.10}$$

with $i = 1, \ldots, N$, $a = (a_1, \ldots, a_N)$.

Notice that we can consider (1.10) as a Lagrange function with λ as the Lagrange multiplier. Because of this observation we can see a central configuration as a critical point of U restricted to $I = I_0$, where I_0 is a constant. If we set I_0 we will have already fixed the scale of the system.

We present now a method to compute λ.

Let a be the central configuration. Taking the dot product of equation (1.10) and a we obtain

$$\frac{\partial U}{\partial q_i}(a) \cdot a + \lambda\frac{\partial I}{\partial q_i}(a) \cdot a = 0.$$

Then, since U and I are homogeneous functions of degrees -1 and 2 respectively, we have from the Euler's theorem on homogeneous functions that

$$-U + 2\lambda I = 0.$$

Isolating λ we get

$$\lambda = \frac{U(a)}{2I(a)} > 0.$$

Notice that if we multiply (1.8) by m_i and we sum over i we have that

$$\sum_{i=1}^{N} m_i a_i = \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \frac{\mathcal{G} m_i m_j (a_j - a_i)}{\|a_j - a_i\|^3} = 0,$$

because of the antisymmetry.

So the centre of masses of a c.c. is the origin.

We remark that rescaling a c.c. or multiplying their vectors a_i by an orthogonal matrix give rise to other c.c.

1.6 Lagrange solutions

As we mentioned before, the central configurations are extremely hard to compute in general. However, we study here a particular case where it is possible to find it.

We will consider that we are in the planar 3-body problem i.e. the motion of the three bodies is restricted to a plane and their positions have only two degrees of freedom.

We will find c.c. of this problem by means of looking for critical points of U restricted to $I = I_0$ constant.

We define $\rho_{ij} = \|q_i - q_j\|$ as the distance between the bodies i and j. Then we can rewrite the function U as

$$U = \mathcal{G}\left(\frac{m_1 m_2}{\rho_{12}} + \frac{m_2 m_3}{\rho_{23}} + \frac{m_3 m_1}{\rho_{31}}\right).$$

Let $M = \sum_{i=1}^{3} m_i$ be the total mass of the system. We can assume that the centre of mass is at the origin. We have

$$\begin{aligned}
\sum_{i=1}^{3}\sum_{j=1}^{3} m_i m_j \rho_{ij} &= \sum_{i=1}^{3}\sum_{j=1}^{3} m_i m_j \|q_i - q_j\| \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} m_i m_j \|q_i\|^2 - 2\sum_{i=1}^{3}\sum_{j=1}^{3} m_i m_j q_i \cdot q_j + \sum_{i=1}^{3}\sum_{j=1}^{3} m_i m_j \|q_j\|^2 \\
&= 2MI - 2\sum_{i=1}^{3} m_i \left(q_i, \sum_{j=1}^{3} m_j q_j\right) + 2MI \\
&= 4MI.
\end{aligned}$$

Then we obtain

$$I = \frac{1}{4M}\sum_{i=1}^{3}\sum_{j=1}^{3} m_i m_j \rho_{ij}.$$

Now we notice that taking I as a constant is equivalent to fix

$$I^* = \frac{1}{2}(m_1 m_2 \rho_{12} + m_2 m_3 \rho_{23} + m_3 m_1 \rho_{31}),$$

since $I^* = MI$.

Thus the conditions that U has to satisfy to have a critical point on the set I^*= constant is (see (1.10)):

$$-\mathcal{G}\frac{m_i m_j}{\rho_{ij}^2} + \lambda m_i m_j \rho_{ij} = 0, \quad (i,j) = (1,2), (2,3), (3,1). \tag{1.11}$$

System (1.11) has a unique solution that is $\rho_{12} = \rho_{23} = \rho_{31} = \mathcal{G}/\lambda$.

This solution, attributed to Lagrange, corresponds to an equilateral triangle with λ as a scale parameter.

1.7 The Two-Body Problem

In the case of only two bodies, it is possible to fully solve the equations.

First we define the following constants:

$$\mu_1 = \frac{m_1}{m_1+m_2}, \quad \mu_2 = \frac{m_2}{m_1+m_2}, \quad \mu = m_1+m_2, \quad M = \frac{m_1 m_2}{m_1+m_2}.$$

Then we define the coordinates (q,u,G,v) as

$$\begin{aligned} q &= \mu_1 q_1 + \mu_2 q_2, & G &= p_1 + p_2, \\ u &= q_2 - q_1, & v &= -\mu_2 p_1 + \mu_1 p_2. \end{aligned}$$

Here we can interpret the new variables: q is the centre of mass, G is the total linear momentum, u is the position of the second body taking the first one as the origin and v is the scalar momentum.

With these variables we can define the following Hamiltonian:

$$H = \frac{\|G\|^2}{2\mu} + \frac{\|v\|^2}{2M} - \frac{m_1 m_2}{\|u\|},$$

and we obtain the corresponding equations of motion:

$$\begin{aligned} \dot{q} &= \frac{G}{\mu} = \frac{\partial H}{\partial G}, & \dot{G} &= 0 = -\frac{\partial H}{\partial q}, \\ \dot{u} &= \frac{v}{M} = \frac{\partial H}{\partial v}, & \dot{v} &= -\frac{m_1 m_2 u}{\|u\|^3} = -\frac{\partial H}{\partial u}. \end{aligned}$$

So the linear momentum G is a first integral and the centre of mass g moves following a rectilinear motion.

Since the centre of mass is in the origin we take the initial conditions $q = G = 0$. Then we can reduce the problem only to the variables u, v. That is, we need to solve:

$$\ddot{u} = -\frac{\mathcal{G}\mu u}{\|u\|^3}.$$

This is a particular case called the Kepler problem, which we will study in more detail in the following section.

1.8 The Kepler problem

1.8.1 Definition and properties

The Kepler problem is a particular case of the 2-body problem where we consider that one body is so massive that their position is fixed on the linear term. The second body is considered to have mass equal to one.

Then, if we define $q \in \mathbb{R}^3$ as the position of the second body and $k := \mathcal{G}m$, where m is the mass of the first body, we obtain that its motion is described by

$$\ddot{q} = -\frac{kq}{\|q\|^3}. \tag{1.12}$$

If we define $p = \dot{q}$ we can obtain the Hamiltonian of (1.12):

$$H = \frac{\|p\|^2}{2} - \frac{k}{\|q\|}. \tag{1.13}$$

Therefore the Kepler problem can be defined either by equation (1.12) or the Hamiltonian (1.13). Then we notice that the case we presented in Section 1.7 can be reduced to it with $m = \mu = m_1 + m_2$.

Let $A = q \times p$ the angular momentum, which we have already proved at Section 1.3 that it is constant and their components are first integrals of the system.

We will study the motion of the Kepler problem in function of A. First of all we set the following equality:

$$\frac{d}{dt}\left(\frac{q}{\|q\|}\right) = \frac{\dot{q} \cdot (q \cdot q) - (q \cdot \dot{q}) \cdot q}{\|q\|^3} = \frac{(q \times \dot{q}) \times q}{\|q\|^3} = \frac{A \times q}{\|q\|^3}. \tag{1.14}$$

Now we separate the problem in two cases, depending on whether A is zero or not.

1) If $A = 0$.

In this case we observe from (1.14) that $q = \omega\|q\|$, where ω is constant. So we can see that the motion is confined in a straight line.

Setting the line of motion as one of the coordinate axes we obtain a system with one degree of freedom. Then the computation of first integrals is straightforward.

2) If $A \neq 0$.

In this case we notice that A is orthogonal to q and p. Then the motion takes place on a plane which is orthogonal to A and invariant.

We take the last coordinate axis as the one who crosses along the vector A. Then the equations of motion are the same as (1.12) but with $q \in \mathbb{R}^2$, since the third component is zero.

We consider $A = (0, 0, c)$, with $c = \|A\| \neq 0$, and we set q in polar coordinates: $q = (r\cos\theta, r\sin\theta, 0)$. Using properly (1.14) we have

$$r^2\dot{\theta} = c,$$

which integrating we obtain the Kepler's third law: The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

As a consequence, we prove that the area described by the second body grows linearly with constant rate $c/2$. This fact is known as the Kepler's second law.

First we have that the area $a(t)$ described by the body in function of the radius r and the angle θ is equal to a $\theta/(2\pi)$ part of the total area of a circle.

We have that $a = \theta r^2/2$. Then

$$\frac{da}{d\theta} = \frac{r^2}{2} \Rightarrow \frac{da}{dt} = \frac{r^2}{2}\frac{d\theta}{dt} = \frac{\dot{\theta}r^2}{2} = \frac{c}{2}.$$

Integrating and using the fact that $a(0) = 0$ we obtain $a(t) = ct/2$.

1.8.2 Resolution of the Kepler problem

Here we present a possible strategy for solving the Kepler problem. We remark that there are others.

We multiply equation (1.14) by k and then we have

$$k\frac{d}{dt}\left(\frac{q}{\|q\|}\right) = A \times \frac{kq}{\|q\|^3} = -A \times \frac{-kq}{\|q\|^3} = -A \times \dot{p} = \dot{p} \times A.$$

Integrating we get

$$k\left(e+\frac{q}{\|q\|}\right)=p\times A, \tag{1.15}$$

where e is a constant vector, coming from the integration.

Notice that if take the dot product of equation (1.15) with A and use that $q\cdot A=0$, we obtain that $e\cdot A=0$. We study the vector e as a function of the values of A.

If $A=0$, we have that $e=-q/\|q\|$ and then e lies on the straight line of motion and it has modulus equal to one.

If $A\neq 0$, the vector e lies into the invariant plane, orthogonal to A. From now on and for the rest of the section we consider only this case.

Taking dot product by q on both sides of equation (1.15) we get

$$k(e\cdot q+\|q\|)=q\cdot(p\times A)=A\cdot(q\times p)=A\cdot A=\|A\|^2=c^2.$$

Then

$$e\cdot q+\|q\|=\frac{c^2}{k}. \tag{1.16}$$

We study the cases depending on e:

If $e=0$ we obtain that $\|q\|=c^2/k$, which is constant. Then, since $\|q\|=r$ and $r^2\dot\theta=c$, we get $\dot\theta=k^2/c^3$. In this case the second body moves on a circle with constant angular velocity.

If $e\neq 0$ we take $\varepsilon=\|e\|>0$. It is convenient to consider the polar coordinates r,θ of the second body, where θ is the angle respect the first axis. We denote by g the angle of e respect the first axis. Then the angle f between e and the body is $f=\theta-g$.

Thus $e\cdot q=\varepsilon r\cos f$ and we can rewrite (1.16) as

$$r=\frac{c^2/k}{1+\varepsilon\cos f}. \tag{1.17}$$

Now we introduce the line l that is at distance $c^2/k\varepsilon$ of the origin and orthogonal to the vector e. Then, equation (1.17) can be rewritten as

$$r=\varepsilon\left(\frac{c^2}{k\varepsilon}-r\cos f\right). \tag{1.18}$$

The interpretation of (1.18) is that the distance from the origin to the second body is ε times the distance from it to the line l. With this we deduce Kepler's first law: the motion of the second body is on a conic section of eccentricity ε with one focus at the origin.

We recall that in the case $0<\varepsilon<1$ the motion is an ellipse, for $\varepsilon=1$ it is a parabola and for $\varepsilon>1$ a hyperbola.

1.9 The Restricted Planar Circular Three Body Problem

1.9.1 Definition, equations and Hamiltonian

The general case of the three body problem is unsolved and it is believed to be non-integrable. For this reason several simplifications are widely studied. One of them is known as the restricted planar three body problem.

The restricted three body problem considers that the mass of one of the bodies is so small that we can say that it is equal to zero. The other two bodies will be called primaries and, rescaling the units of mass, we will consider that the first has mass $1-\mu$ and the other one μ, where $\mu \in (0, 1/2]$.

Let $q_i \in \mathbb{R}^2$, $i = 1, 2, 3$ be the position of the i-th body. Namely, the motion lies in an invariant plane. Then the Restricted Planar Three Body Problem (RP3BP) has the following differential equation system:

$$\begin{aligned} m_1\ddot{q}_1 &= \frac{\mathcal{G}m_1m_2(q_2-q_1)}{\|q_2-q_1\|^3}, \\ m_2\ddot{q}_2 &= \frac{\mathcal{G}m_2m_1(q_1-q_2)}{\|q_1-q_2\|^3}, \\ \ddot{q}_3 &= \frac{\mathcal{G}m_1(q_1-q_3)}{\|q_1-q_3\|^3} + \frac{\mathcal{G}m_2(q_2-q_3)}{\|q_2-q_3\|^3}. \end{aligned} \tag{1.19}$$

Notice that the first two equations of (1.19) show that the motion of the first two particles (the primaries) are exactly the same as the 2-body problem, since the influence of the third body is zero. They describe Keplerian orbits.

For this reason the main point of the RP3BP is to compute the behaviour of the third body under the influence of the primaries. If we assume that the orbit of the primaries are circular, we have the Restricted Planar Circular Three Body Problem (RPC3BP).

Using a suitable change of variables, see [MHO09], we can fix the first body of mass $1-\mu$ at the position $(-\mu, 0)$ and the second one with mass μ at $(1-\mu, 0)$.

Defining $q_3 = (x, y)$ as the new variables and the function U as

$$U(x,y) = \frac{1-\mu}{((x+\mu)^2+y^2)^{1/2}} + \frac{\mu}{((x-1+\mu)^2+y^2)^{1/2}},$$

the system becomes

$$\begin{aligned} \ddot{x} &= x + 2\dot{y} + \frac{\partial U}{\partial x}, \\ \ddot{y} &= y - 2\dot{x} + \frac{\partial U}{\partial y}. \end{aligned} \tag{1.20}$$

Now we introduce the variables X, Y defined as

$$\begin{aligned} X &= \dot{x} - y, \\ Y &= \dot{y} + x, \end{aligned}$$

which makes the system Hamiltonian.

The Hamiltonian associated to the motion of the third particle in these variables is

$$H(x,y,X,Y) = \frac{1}{2}(X^2+Y^2) - xY + yX - U(x,y).$$

Indeed, the conjugate derivations match

$$\begin{aligned} \dot{x} &= X + y = \frac{\partial H}{\partial X}, \\ \dot{y} &= Y - x = \frac{\partial H}{\partial Y}, \\ \dot{X} &= Y + \frac{\partial U}{\partial x} = -\frac{\partial H}{\partial x}, \\ \dot{Y} &= -X + \frac{\partial U}{\partial y} = -\frac{\partial H}{\partial y}. \end{aligned} \tag{1.21}$$

1.9.2 Equilibrium solutions

As we mentioned before, the general case of the 3-body problem has no equilibrium points. However, once we have fixed the position of the primaries in the RPC3BP, there are five fixed points for the third body which are well known. That is due to the fact that we are using a reference system which is rotating.

From the definition of equilibrium point of systema(1.21) we obtain that an equilibrium point has to satisfy

$$\begin{aligned} X + y = 0, \qquad & Y - x = 0, \\ Y + \frac{\partial U}{\partial x} = 0, \quad & -X + \frac{\partial U}{\partial y} = 0. \end{aligned} \tag{1.22}$$

If we add the first equation of (1.22) with the last one and we subtract the third and the second we get

$$x + \frac{\partial U}{\partial x} = 0, \quad y + \frac{\partial U}{\partial y} = 0.$$

Considering now the function

$$V(x, y) = (x^2 + y^2) + 2U + \mu(1 - \mu),$$

known as the amended potential, we have that the last conditions are equivalent to have

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0.$$

Thus an equilibrium solution is a critical point of the amended potential. To compute these points we are going to consider two cases

1. If the points do not lie in the line of the primaries: $\mathcal{L}_4, \mathcal{L}_5$.

In this case we will work with the distances of the points respect to the primaries

$$d_1^2 = (x - 1 + \mu)^2 + y^2, \quad d_2^2 = (x + \mu)^2 + y^2. \tag{1.23}$$

From (1.23), we obtain

$$x^2 + y^2 = \mu d_1^2 + (1 - \mu) d_2^2 - \mu(1 - \mu),$$

and substituting in V we obtain

$$V = \mu d_1^2 + (1 - \mu) d_2^2 + \frac{2\mu}{d_1} + \frac{2(1 - \mu)}{d_2}.$$

Because we are looking for critical points of V, d_1 and d_2 have to satisfy

$$\frac{\partial V}{\partial d_1} = 2\mu d_1 - \frac{2\mu}{d_1^2} = 0, \quad \frac{\partial V}{\partial d_2} = 2(1 - \mu) d_2 - \frac{2(1 - \mu)}{d_2^2} = 0.$$

This system has a unique solution which is $d_1 = d_2 = 1$, corresponding to two symmetric points respect to $y = 0$, see (1.23). The two points $\mathcal{L}_4, \mathcal{L}_5$ are the vertexes of an equilateral triangle whose base is the segment created by the two primaries, one point for each orientation.

2. If the points lie in the line of primaries: $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$.

In this case we look for the points with the form $y = 0$. Then V will be expressed as a function with a single variable x.

$$V(x,0) = \widetilde{V}(x) = x^2 \pm \frac{2\mu}{x-1+\mu} \pm \frac{2(1-\mu)}{x+\mu} + \mu(1-\mu).$$

We want $\widetilde{V}$ to be positive in its domain, so the choice of signs will be determined by the interval where x lies

1) If $x < -\mu$, we take $-$ and $-$.

2) If $-\mu < x < 1-\mu$, we take $-$ and $+$.

3) If $x > 1-\mu$, we take $+$ and $+$.

Now, since $\widetilde{V}(x) \to \infty$ when $x \to \{\pm\infty, \mu, 1-\mu\}$ we have that there's at least one critical point at each interval.

We want to see that there exactly one for everyone.

Taking derivatives of $\widetilde{V}$ twice with respect x we get

$$\frac{\partial^2 \widetilde{V}}{\partial x^2} = 2 \pm \frac{4\mu}{(x-1+\mu)^3} \pm \frac{4(1-\mu)}{(x+\mu)^3},$$

that have the signs chosen as we set before. So the second derivative is always positive and V is a convex function, thus there is only one critical point per interval.

This leads us to prove the existence of the so called libration points $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$.

So we have found the five equilibrium points, also known as Lagrangian points, that we were looking for.

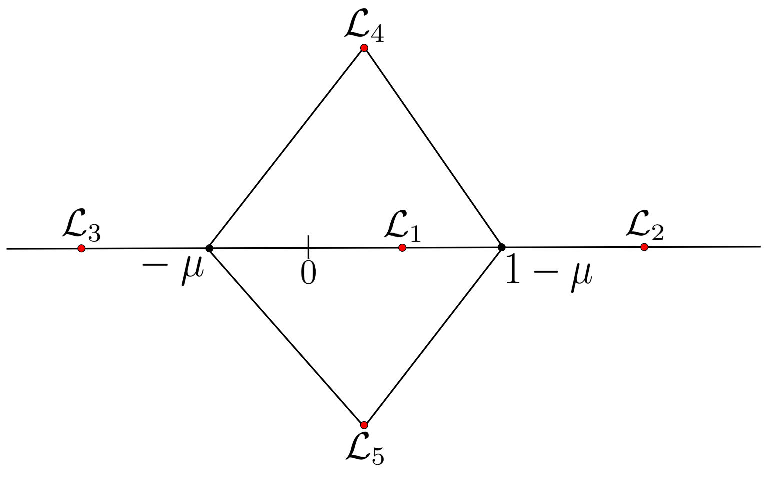

Figure 1.1: Graphical representation of the Lagrangian points (in red) respect the primaries (in black).

1.9.3 Hill's region

The Restricted Planar Circular Three Body Problem has a first integral known as the Jacobi constant. The projection of its levels sets to the position coordinates is what is called a Hill's region.

For a given value of the Jacobi constant, the motion in the RPC3BP can only take place in the corresponding Hill's region.

In order to compute the Hill's region, we introduce the following first integral for system (1.20):

$$H(x,y) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(x^2 + y^2) - U(x,y).$$

The Jacobi constant C is then defined as

$$C = -2H(x,y) + \mu(1-\mu) = (x^2 + y^2) - 2U(x,y) - (\dot{x}^2 + \dot{y}^2) + \mu(1-\mu),$$

and the amended potential V as before:

$$V(x,y) = (x^2 + y^2) - 2U(x,y) + \mu(1-\mu).$$

Notice that

$$V - (\dot{x}^2 + \dot{y}^2) = C.$$

So, from this equality, we will be able to determine where the orbits can lie.

We define the Hill's region associated to C as

$$\mathcal{R}(C) = \{(x,y) : V(x,y) \geq C\},$$

including the boundary where the velocity is zero. For a fixed value of C, the motion can only take place inside $\mathcal{R}(C)$.

1.10 The parabolic periodic orbits at infinity

We are interested in studying the parabolic stable manifold at infinity of the RPC3BP. That is, the orbits arriving to infinity with zero velocity.

The first step of our study is to use adequate coordinate: the so called McGehee coordinates.

1.10.1 The McGehee coordinates

We present here a suitable change of coordinates of the system (1.20) that will help for the study.

Let $(x,y) \in \mathbb{R}^2$ be the position of the third body on the invariant plane of the RPC3BP. Then we consider this position as a complex number $z = x + iy$ and we introduce the McGehee coordinates (q, p, θ, ω), see [McG73] and [MS14]:

$$\begin{aligned} x + iy &= \frac{2}{q^2} e^{-i\theta}, \\ \dot{x} + i\dot{y} &= e^{-i\theta}\left[p + i\left(\frac{q^2\omega}{2} - \frac{2}{q^2}\right)\right]. \end{aligned} \tag{1.24}$$

Proposition 1.10.1. *The system of o.d.e.* (1.20) *with the change of variables* (1.24) *becomes*

$$\begin{aligned} \dot{q} &= -\frac{1}{4}q^3 p, \\ \dot{p} &= -\frac{q^4}{4}\sigma_2 + \frac{q^6 w^2}{8} - \frac{q^6}{8}\mu(1-\mu)\sigma_1 \cos\theta, \\ \dot{\theta} &= 1 - \frac{1}{4}q^4\omega, \\ \dot{\omega} &= -\frac{q^4}{4}\mu(1-\mu)\sigma_1 \sin\theta, \end{aligned} \tag{1.25}$$

where

$$\sigma_1 = \frac{1}{f_\mu^3} - \frac{1}{f_{\mu-1}^3}, \quad \sigma_2 = \frac{1-\mu}{f_\mu^3} + \frac{\mu}{f_{\mu-1}^3},$$

$$f_m = \left(1 + mq^2\cos\theta + \frac{m^2}{4}q^4\right)^{1/2}.$$

Proof. We start the proof for the equations of q and θ.

We differentiate the first equation of (1.24):

$$\dot{x} + i\dot{y} = -\frac{4\dot{q}}{q^3}e^{-i\theta} - \frac{2}{q^2}i\dot{\theta}e^{-i\theta}.$$

If we equal them to the second equation of (1.24) and multiply by $e^{i\theta}$ we get

$$p + i\left(\frac{q^2\omega}{2} - \frac{2}{q^2}\right) = -\frac{4\dot{q}}{q^3} - \frac{2}{q^2}i\dot{\theta}.$$

Taking the real part we obtain

$$p = -\frac{4\dot{q}}{q^3} \Rightarrow \dot{q} = -\frac{1}{4}q^3 p. \tag{1.26}$$

And taking the imaginary part now

$$-\frac{2}{q^2}\dot{\theta} = \frac{q^2\omega}{2} - \frac{2}{q^2} \Rightarrow \dot{\theta} = 1 - \frac{1}{4}q^4\omega. \tag{1.27}$$

To compute $\dot{p}$ and $\dot{\omega}$ we differentiate the second equation of (1.24)

$$\begin{aligned}&\ddot{x} + i\ddot{y}\\ &= -i\dot{\theta}e^{-i\theta}\left[p + i\left(\frac{q^2\omega}{2} - \frac{2}{q^2}\right)\right] + e^{-i\theta}\left[\dot{p} + i\left(\frac{2\dot{q}qw + q^2\dot{w}}{2} + \frac{4\dot{q}}{q^3}\right)\right].\end{aligned} \tag{1.28}$$

Then, from system (1.20) we see that $\ddot{x} + i\ddot{y}$ has to satisfy

$$\ddot{x} + i\ddot{y} = x + iy + 2\dot{y} - 2i\dot{x} + \frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}. \tag{1.29}$$

We compute the terms of the right hand of (1.29) by parts

$$\begin{aligned} x + iy &= \frac{2}{q^2}e^{-i\theta},\\ 2\dot{y} - 2i\dot{x} &= -2i(\dot{x} + i\dot{y}) = -2ie^{-i\theta}\left[p + i\left(\frac{q^2\omega}{2} - \frac{2}{q^2}\right)\right].\end{aligned} \tag{1.30}$$

For the last term $\partial U/\partial x + i\partial U/\partial y$ we use the notation f_m, σ_1, σ_2. Then, with a few computations, we have

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = -\frac{q^4}{4}e^{-i\theta}\sigma_2 - \frac{q^6}{8}\mu(1-\mu)\sigma_1. \tag{1.31}$$

Combining (1.28), (1.29), (1.30) and (1.31) we obtain

$$\begin{aligned} &-i\dot{\theta}e^{-i\theta}\left[p + i\left(\frac{q^2\omega}{2} - \frac{2}{q^2}\right)\right] + e^{-i\theta}\left[\dot{p} + i\left(\frac{2\dot{q}qw + q^2\dot{w}}{2} + \frac{4\dot{q}}{q^3}\right)\right] =\\ &= \frac{2}{q^2}e^{-i\theta} - 2ie^{-i\theta}\left[p + i\left(\frac{q^2\omega}{2} - \frac{2}{q^2}\right)\right] - \frac{q^4}{4}e^{-i\theta}\sigma_2 - \frac{q^6}{8}\mu(1-\mu)\sigma_1.\end{aligned} \tag{1.32}$$

Multiplying (1.32) by $e^{i\theta}$ and taking the real part we get

$$\dot{\theta}\left(\frac{q^2\omega}{2}-\frac{2}{q^2}\right)+\dot{p}=\frac{2}{q^2}+q^2\omega-\frac{4}{q^2}-\frac{q^4}{4}\sigma_2-\frac{q^6}{8}\mu(1-\mu)\sigma_1\cos\theta.$$

Using (1.27) and isolating $\dot{p}$ we have

$$\dot{p}=\frac{q^6w^2}{8}-\frac{q^4}{4}\sigma_2-\frac{q^6}{8}\mu(1-\mu)\sigma_1\cos\theta, \tag{1.33}$$

and we are done for $\dot{p}$.

If we multiply (1.32) again by $e^{i\theta}$ and take its imaginary part we get

$$-\dot{\theta}p+\dot{q}q\omega+\frac{q^2\dot{\omega}}{2}+\frac{4\dot{q}}{q^3}=-2p-\frac{q^6}{8}\mu(1-\mu)\sigma_1\sin\theta.$$

Using (1.26), (1.27) and isolating $\dot{\omega}$ we have

$$\dot{\omega}=-\frac{q^4}{4}\mu(1-\mu)\sigma_1\sin\theta. \tag{1.34}$$

Summarizing (1.26), (1.27), (1.33) and (1.34) we obtain the system of differential equations for the McGehee coordinates

$$\begin{aligned}
\dot{q}&=-\frac{1}{4}q^3p,\\
\dot{p}&=-\frac{q^4}{4}\sigma_2+\frac{q^6w^2}{8}-\frac{q^6}{8}\mu(1-\mu)\sigma_1\cos\theta,\\
\dot{\theta}&=1-\frac{1}{4}q^4\omega,\\
\dot{\omega}&=-\frac{q^4}{4}\mu(1-\mu)\sigma_1\sin\theta.
\end{aligned}$$

□

The system (1.25) is the one we will use along this work. One of the great advantages it has is that the phase $q=0$ corresponds to the infinity and $p=0$ to zero velocity.

We call the parabolic infinity to the set

$$I_\infty:=\{q=p=0\},$$

which correspond to the points which have arrived at infinity with zero velocity. This set is foliated by periodic orbits, indeed, for any θ_0, $\omega_0\in\mathbb{R}$ we obtain a periodic orbit of the form $(0,0,\theta_0+t,\omega_0)$, belonging to I_∞. Any of these periodic orbits has a parabolic stable manifold associated to it, see [McG73].

Our goal is to apply the parametrization method to compute a good approximation of these parabolic manifolds.

Chapter 2

The Parametrization Method

In a dynamical system, orbits can have extremely different behaviours. The simplest ones are the equilibrium points, that is, motions that stay at a single point forever. Other simple behaviours, are either the periodic orbits or quasiperiodic motions, which lie in a torus.

The objects where these motions take place are called invariant sets. Namely, for a given initial condition in an invariant set, the motion is confined in it.

Under appropriate conditions, these simple invariant objects (equilibrium points, periodic orbits, invariant tori, etc.) have associated invariant manifolds (every point in the invariant manifold tends to the invariant set either backwards or forwards) which turn out to be also invariant sets.

These invariant manifolds provide the skeleton of the qualitative behaviour around the invariant object they are associated to.

With the arrival of the computers, the study of invariant manifolds started to be not only qualitative but also quantitative. As a consequence researchers are getting more and more interest in the development of efficient algorithms for the computation of these invariant sets.

At the same time, the problems and applications related to the computation of invariant sets had gained more complexity, motivating new research around the development of computational algorithms and software implementations.

It is an interesting fact that the synergy between software implementation and mathematical methodology had made both parts to gain force and more refinements along the last thirty years.

The parametrization method comes from this new trend. It is a novel method, introduced by X. Cabré, E. Fontich and R. de la Llave in the articles [CFdlL03a], [CFdlL03b] and [CFdlL05], which has emerged with the idea of creating new methodology in the theory of computation of invariant manifolds.

As his name suggests, the parametrization method looks for the invariant manifolds as parametrized embedded manifolds. In addition, the internal dynamics on it is also provided.

The parametrizations of the invariant manifold and the dynamics are solutions the invariance equations. Normally we will try to find the internal dynamic as simply as possible, even sometimes linear. This proceeding adapts the parametrization to the geometry of the object we are studying, and leads to simplifications that are extremely useful when we try to build software implementation. For this reason the parametrization method has good synergy between mathematical theory and software implementations.

The proofs of the parametrization method consist in computing iteratively approximations of the parametrization near the invariant manifold. There are theoretical theorems ensuring that these computable approximations converge to a parametrization of the real manifold.

With this kind of method we can see intuitively that this methodology leads to develop efficient numerical computations.

In conclusion, the parametrization method combines theoretical and practical knowledge.

Even when the method has been exploited in a big amount of situations along the last fifteen years it remains a lot of contexts to be explored.

In this section we will present how the parametrization method works for some concrete cases: maps having hyperbolic and parabolic points and periodic vector fields having parabolic equilibrium points.

For our project we will only need the case of these vector fields, since we are studying the RPC3BP. However, we have considered that it would be a good idea to present also the two cases for maps because it helps to develop the intuition about how the parametrization method works.

2.1 Preliminaries

Even when the parametrization method can deal with multidimensional invariant manifolds, we restrict our exposition to the one dimensional case. We point out that on the one hand, this will be enough for our purposes because the parabolic manifold in RPC3BP is one dimensional, when a Poincaré map is considered. On the other hand, this restriction simplifies a lot the exposition and let of the reader to gain some intuition.

We will look for the one-dimensional stable manifold of a fixed point, which we will consider it to be at the origin, of some concrete dynamical system. For the sake of notation we will define x as a one-dimensional variable, which represents the stable subspace, and y as an n-dimensional one.

The systems under consideration can be either discrete, given by a map $F(x, y) : \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$, or continuous, given by a vector field $Z(t, x, y) : \mathbb{R} \times \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$.

To finish this preliminary section we introduce some notation.

We define E_1 as the one-dimensional subspace generated by the first variable and E_2 as the n-dimensional space generated by the last n variables. We also define π_i as the projection onto E_i, for $i = 1, 2$, and $\pi_{2,l}$ as the projection onto the l-th coordinate of E_2. Then $E_{2,l} := \pi_{2,l}E_2$.

The parametrization K will be of the form $K : \mathbb{R} \to \mathbb{R}^{1+n} : u \to K(u)$ for maps and $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{1+n} : (t, u) \to K(t, u)$ for vector fields, where t represents the time. Then the invariance equations are, respectively:

$$F \circ K(u) = K(\tilde{u}), \qquad Z(t, K(t, u)) = D_u K(t, u)\dot{u} + \partial_t K(t, u).$$

We have that $\tilde{u}$ and $\dot{u}$ are the internal dynamics of the manifold for the discrete and continuous case, respectively. We define $R : \mathbb{R} \to \mathbb{R}^{1+n}$ as $R(u) := \tilde{u}$ and $Y : \mathbb{R} \to \mathbb{R}^{1+n}$ as $Y(u) := \dot{u}$. The invariance equation are rewritten as:

$$F \circ K(u) = K \circ R(u), \qquad Z(t, K(t, u)) = D_u K(t, u)Y(u) + \partial_t K(t, u).$$

Remark. *Notice that we have imposed the internal dynamics $\dot{u} = Y(u)$ to be independent of time. In fact, Y could be constructed with a time-depending form. However it is possible to get one which only depends on u.*

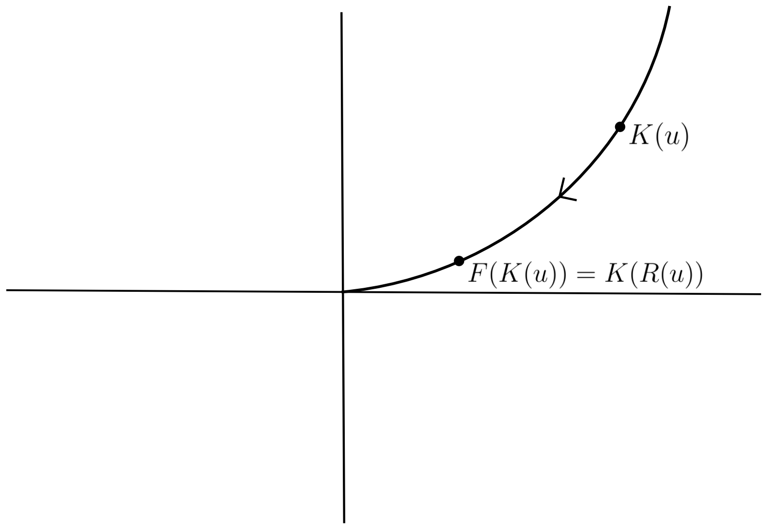

Figure 2.1: Graphical representation of the parametrization method for maps.

From now we will describe briefly how the method works only for maps. The case of vector fields is analogous.

The main idea is to look for iteratively functions $K^{\leq m}$ and $R^{\leq m}$ such that the invariance equation is satisfied up to an order error, i.e

$$F \circ K^{\leq m}(u) - K^{\leq m} \circ R^{\leq m}(u) = O(u^{m+1}), \tag{2.1}$$

for m large enough and setting $O(u^m)$ as

$$\frac{O(u^m)}{u^m} \quad \text{bounded when } h \to 0.$$

Both $K^{\leq m}$ and $R^{\leq m}$ will be computed by induction.

The case $m = 1$ will be determined from the stable manifold theorem, which says that the parametrization must contain the fixed point and must be tangent to the x-axis. Then we will have

$$K(0) = 0, \quad DK(0) = (1, 0)^T.$$

In the induction step, for $m + 1 \geq 2$, we assume that we have $K^{\leq m}, R^{\leq m}$ satisfying (2.1). Then we want to find some functions K^{m+1}, R^{m+1} such that $K^{\leq m+1} := K^{\leq m} + K^{m+1}$ and $R^{\leq m+1} := R^{\leq m} + R^{m+1}$ satisfy

$$F \circ K^{\leq m+1}(u) - K^{\leq m+1} \circ R^{\leq m+1}(u) = O(u^{m+2}).$$

When m is large enough, the error term is small enough to start a fixed point argument. That is, we look for $K^{>m}$ satisfying

$$F \circ (K^{\leq m} + K^{>m}) = (K^{\leq m} + K^{>m}) \circ R.$$

We will prove the existence and useful properties of $K^{>m}$ by using functional analysis techniques as the Banach fixed point theorem. For instance, using the appropriate Banach space, we can check that $K^{>m}(u) = O(u^{m+1})$.

For this project, which is focused in numerical approximation, we will focus on the actual computation of $K^{\leq m}$ and $R^{\leq m}$.

In Section 2.2, we will deal with maps having the origin as a hyperbolic critical point. Then we will move to Section 2.3, where the maps have a parabolic critical point. This case works remarkably different from the last one. Finally, in Section 2.4, we will deal with periodic vector fields with a parabolic critical point at the origin. There the method works in a different way as maps but the main idea is preserved.

2.2 Hyperbolic points in maps

To illustrate the parametrization method we begin by showing the simplest case, namely, invariant manifolds associated to hyperbolic critical points.

2.2.1 The maps under consideration and main result

Let us to consider $\mathcal{C}^r$ maps, with $r \geq 2$, of the form $F = (F^1, F^2) : U \in \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$:

$$\begin{aligned} F^1(x,y) &= \lambda x + \sum_{i=2}^{r} F_i^1(x,y) + O(|(x,y)|^{r+1}), \\ F^{2,l}(x,y) &= \mu_l y_l + \sum_{i=2}^{r} F_i^{2,l}(x,y) + O(|(x,y)|^{r+1}), \end{aligned} \tag{2.2}$$

for $l = 1, \ldots n$.

The variable x is one-dimensional, y is n-dimensional, $y = (y_1, \ldots, y_n)$ and the parameters λ and μ_l satisfy, respectively,

$$0 < |\lambda| < 1, \quad |\mu_l| > 1. \tag{2.3}$$

The functions $F_i^1, F_i^{2,l}$ are homogeneous polynomials of degree i. To set the notation we introduce

$$F_i^1(x,y) = \sum_{k+|\alpha|=i} a_{k,\alpha} x^k y^\alpha, \qquad F_i^{2,l}(x,y) = \sum_{k+|\alpha|=i} b_{k,\alpha}^l x^k y^\alpha,$$

where α is a multi-index vector: $y^\alpha = y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

Remark. *Since we are looking for maps with one-dimensional stable manifold, we assume that there is only one eigenvalue of the linear part that has modulus less than one. The general case can be also considered.*

The main goal in this section is to prove the following theorem:

Theorem 2.2.1. *Let $F : U \in \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$ be a $\mathcal{C}^r$ map of the form* (2.2) *satisfying* (2.3). *Then there exist $K : [-\rho, \rho] \to \mathbb{R}^{1+n}$ and $R : \mathbb{R} \to \mathbb{R}$, both $\mathcal{C}^r$, satisfying the invariance equation*

$$F \circ K = K \circ R. \tag{2.4}$$

The proof of this theorem will be divided in two parts. First we will compute the so called formal part, $K^{\leq m}$, which is an accurate approximation of K, and the exact computation of R. Then we will move to the proof of properties of the remainder $K^{>m}$.

2.2.2 Formal part

As we said in the introduction of this chapter, we search $K^{\leq m}, R^{\leq m}$ satisfying

$$F \circ K^{\leq m}(u) - K^{\leq m} \circ R^{\leq m}(u) = O(u^{m+1}). \tag{2.5}$$

We present now a proposition showing a way to compute $K^{\leq m}$, $R^{\leq m}$. As we will see along the prove there are other choices of $K^{\leq m}$ and $R^{\leq m}$.

The strategy we follow is to obtain the simplest inner dynamics R.

Proposition 2.2.1. *Let F be a $\mathcal{C}^r$ map of the form* (2.2) *satisfying* (2.3). *Take $m \in \mathbb{N}$, $1 \leq m \leq r$. Then equation* (2.5) *has a solution of the form:*

$$K^{\leq m}(u) = \left(u + \sum_{i=2}^{m} c_i^1 u^i, \sum_{i=2}^{m} c_i^{2,l} u^i \right)^T, \quad R(u) = \lambda u,$$

where λ is the only eigenvalue of the linear part of F that has modulus less than one and $c_i^1, c_i^{2,l}$ are known.

Proof. Since we are looking for invariant manifolds for the origin, the parametrization must contain it. For this reason we will ask K to satisfy $K(0) = 0$.

We have also stated that the parametrization, i.e. the stable manifold, has to be tangent to the invariant subspace associated to the eigenvalue λ. In our case it corresponds to the x-axis. So we will also ask K to satisfy $K'(0) = (1, 0)^T$, where here the second 0 is n-dimensional.

Since we want $K(0) = 0$ and $K'(0) = (1, 0)^T$, the parametrization K has to be set of the form $K(t) = (t, 0) + O(t^2)$. Then we take $K^{\leq 1} = (t, 0)^T$.

Now we want choose $R^{\leq 1}$ such that $K^{\leq 1}, R^{\leq 1}$ satisfy (2.5) with $m = 1$.

We substitute $K^{\leq 1}$ and $R^{\leq 1}$ into (2.5). The main point is that the linear terms have to be cancelled. Taking only at the first order terms of equation (2.5) we obtain that $R^{\leq 1}$ has to satisfy

$$\begin{pmatrix} \lambda u \\ 0 \end{pmatrix} - \begin{pmatrix} R^{\leq 1}(u) \\ 0 \end{pmatrix} = 0.$$

So we take $R^{\leq 1}(u) = \lambda u$.

As we commented in the introduction of this chapter, the parametrization method wants to take R the simplest possible. In this case, as we will show, it is possible to fix $R(u) = \lambda u$ and we only need to modify iteratively $K^{\leq m}$.

Now we compute $K^{\leq m}$ to eliminate the terms of orders m, for $2 \leq m \leq r$. We start with the case $m = 2$.

We want $K^{\leq 2}$ to be of the form $K^{\leq 2} = (t + c_2^1 t^2, c_2^{2,l} t^2)^T$, $l = 1, \ldots, n$. Substituting $K^{\leq 2}$ into (2.5) and taking only the coefficients of second order we get

$$\lambda c_2^1 + a_{2,0} - \lambda^2 c_2^1 = 0,$$

$$\mu_l c_2^{2,l} + b_{2,0}^l - \lambda^2 c_2^{2,l} = 0.$$

Therefore we obtain the coefficients

$$c_2^1 = \frac{a_{2,0}}{\lambda^2 - \lambda}, \quad c_2^{2,l} = \frac{b_{2,0}^l}{\lambda^2 - \mu_l}.$$

Remark. *We notice that it is always possible to compute $c_2^1, c_2^{2,l}$ because λ and μ satisfy* (2.3), *which implies that $\lambda^2 - \lambda$ and $\lambda^2 - \mu_l$ are always different from* 0.

Now we deal with the general case $K^{\leq m}$ for $2 \leq m \leq r$. For convenience, we introduce the notation $[\cdot]_j$, which means to take the coefficients of order j.

We proceed by induction. Assume that the approximated invariance equation holds true for m. That is

$$F \circ K^{\leq m}(u) - K^{\leq m} \circ R(u) = \mathcal{O}(u^{m+1}),$$

where $K^{\leq m}$ is of the form stated in the theorem. We look for coefficients $c_{m+1}^1, c_{m+1}^{2,l} \in \mathbb{R}$ such that

$$K^{\leq m+1}(u) = K^{\leq m}(u) + (c_{m+1}^1 u^{m+1}, c_{m+1}^{2,l} u^{m+1})^T. \tag{2.6}$$

The goal is then to compute $c^1_{m+1}, c^{2,l}_{m+1}$ such that $K^{\leq m+1}$ would satisfy (2.5). We introduce the following terms:

$$\gamma^1_{m+1} = \left[\sum_{i=2}^{m+1} F^1_i \circ K^{\leq m+1}\right]_{m+1} = \sum_{i=2}^{m+1} \left[F^1_i \circ K^{\leq m+1}\right]_{m+1},$$

$$\gamma^{2,l}_{m+1} = \left[\sum_{i=2}^{m+1} F^{2,l}_i \circ K^{\leq m+1}\right]_{m+1} = \sum_{i=2}^{m+1} \left[F^{2,l}_i \circ K^{\leq m+1}\right]_{m+1}.$$

Then, if we substitute $K^{\leq m+1}$ into (2.5) and we take the terms of order $m+1$ we get

$$\begin{aligned} \lambda c^1_{m+1} + \gamma^1_{m+1} - \lambda^{m+1} c^1_{m+1} = 0, \\ \mu_l c^{2,l}_{m+1} + \gamma^{2,l}_{m+1} - \lambda^{m+1} c^{2,l}_{m+1} = 0, \end{aligned} \tag{2.7}$$

and consequently we can take c^1_{m+1} and $c^{2,l}_{m+1}$ as

$$c^1_{m+1} = \frac{\gamma^1_{m+1}}{\lambda^{m+1} - \lambda}, \quad c^{2,l}_{m+1} = \frac{\gamma^{2,l}_{m+1}}{\lambda^{m+1} - \mu_l}.$$

We remark that $c^1_{m+1}, c^{2,l}_{m+1}$ are well defined for any m because of (2.3).

With this construction we have that $K^{\leq m+1}$ defined in (2.6) satisfies the approximated invariance equation

$$F \circ K^{\leq m+1} - K^{\leq m+1} \circ R = O(u^{m+2})$$

and therefore the induction proof is complete. We then note that we have found a construction of the parametrization and its internal dynamic as we wanted.

□

With the Proposition 2.2.1 we have finished the formal part. Now it only remains to deal with the reminder $K^{>m}$ that will end the proof of Theorem 2.2.1.

2.2.3 The reminder

Once we have computed $K^{\leq m}$ for m large enough it would be interesting to measure how accurate it is. We want to know if this approximate parametrization is close to the real stable manifold.

We define $K^{>m}$ as a $\mathcal{C}^r$ function having its first m derivatives at the origin equal to zero and satisfying

$$F \circ (K^{\leq m} + K^{>m}) = (K^{\leq m} + K^{>m}) \circ R. \tag{2.8}$$

The functions F, R and $K^{\leq m}$ are, respectively, the map, the internal dynamic on the invariant manifold and the approximation of the parametrization provided in Proposition 2.2.1. Then we can consider $K^{>m}$ as "the remainder" of the approximate parametrization $K^{\leq m}$.

The function $K^{>m}$ cannot be computed explicitly in the general case but there are ways to ensure its existence and uniqueness when m is large enough.

The most usual way to do this is to transform (2.8) as a fixed point equation for $K^{>m}$ and then work with the standard contraction mapping theorem:

Theorem 2.2.2. *(Banach's Contraction Mapping Principle): Let (M, d) be a complete metric space and let $\mathcal{T} : M \to M$ be a contraction mapping i.e. for all $x, y \in M$*

$$d(\mathcal{T}(x), \mathcal{T}(y)) \leq \kappa d(x, y),$$

with $0 < \kappa < 1$*. Then* $\mathcal{T}$ *has a unique fixed point* x_0*, and for each* $x \in M$

$$\lim_{n\to\infty} \mathcal{T}^n(x) = x_0.$$

Moreover,

$$d(\mathcal{T}^n(x), x_0) \le \frac{k^n}{1-k} d(x, \mathcal{T}(x)).$$

Proof. See [KK01], pages 41-43. □

We construct from (2.8) a contraction operator $\mathcal{T}$, depending on $K^{\le m}$, such that $K^{>m}$ is a fixed point i.e. $\mathcal{T}K^{>m} = K^{>m}$.

Then, if we have $K_0^{>m}$ satisfying

$$d(\mathcal{T}(K_0^{>m}), K_0^{>m}) < \delta,$$

we will have that

$$d(K^{>m}, K_0^{>m}) < \frac{\delta}{1-\kappa}.$$

With these bounds we are able to make some estimations about the accuracy of the numerical computations for $K^{\le m}$.

Since our main goal is the formal solution i.e. the computations done in Proposition 2.2.1 we will not give explicit details of the proofs involved in this part. However we will show the sketch of how the construction of the operator $\mathcal{T}$ works.

First of all we remark that the map F can be written of the form

$$F = A + N,$$

where A is the constant diagonal matrix with coefficients λ, μ_l defined in Proposition 2.2.1 and N satisfies $N(0) = 0, DN(0) = 0$.

Then, with the aim of simplifying the proofs, we will rescale the maps involved in the equations. Let H be a map and $\delta > 0$ a real number. We define H^δ as

$$H^\delta(x) = \frac{1}{\delta} H(\delta x).$$

Then we have that the invariance equation (2.4) holds in the ball of radius δ if and only if

$$F^\delta \circ K^\delta = K^\delta \circ R^\delta$$

holds in the ball of radius one. Notice also that

$$F^\delta = A + N^\delta,$$

with $N^\delta(0) = 0, DN^\delta(0) = 0$ and $\|N^\delta\|_{\mathcal{C}^r}$ is arbitrarily small in the ball of radius three centred at the origin, taking δ sufficiently small.

Remark. *For all* $\delta > 0$*, the rescaling of* K, R *will not affect the conditions* $K(0) = 0$*,* $K'(0) = (1, 0)^T$ *and* $R(u) = \lambda u$*.*

As we said before, we are looking for a $K^{>m}$, with $D^i K(0) = 0, i = 0, \ldots, m$ such that $K = K^{\le m} + K^{>m}$ satisfies the invariance equation (2.4). This is equivalent to ask for $K^{>m}$ to satisfy

$$AK^{\le m} + AK^{>m} + N \circ (K^{\le m} + K^{>m}) = K^{\le m} \circ R + K^{>m} \circ R,$$

that can be rewritten as

$$AK^{>m} - K^{>m} \circ R = -N \circ (K^{\leq m} + K^{>m}) - AK^{\leq m} + K^{\leq m} \circ R. \tag{2.9}$$

This equation leads to work with the linear operator $\mathcal{S}$ defined as

$$\mathcal{S}(H) = AH - H \circ R, \tag{2.10}$$

which acts over maps $H : B_1 \subset \mathbb{R} \to \mathbb{R}^{1+n}$, where B_1 is the unit ball centred at the origin and H belongs to the Banach Space $\Gamma_{s,l}$, defined as follows.

Let $s \in \mathbb{N} \cup \{\omega\}$ and $l \in \mathbb{N}$, with $s \geq l$. Then

$$\Gamma_{s,l} := \left\{ H : B_1 \subset \mathbb{R}; H \in \mathcal{C}^s, D^k H(0) = 0 \text{ for } 0 \leq k \leq l, \sup_{x \in B_1} \left(\frac{D^l H(x)}{|x|} \right) < \infty \right\},$$

endowed with the norm

$$\|H\|_{\Gamma_{s,l}} := \max \left\{ \|H\|_{\mathcal{C}^0(B_1)}, \ldots, \|D^s H\|_{\mathcal{C}^0(B_1)}, \sup_{x \in B_1} (D^l H(x)/|x| \right\} \quad \text{if } s \in \mathbb{N},$$
$$\|H\|_{\Gamma_{\omega,l}} := \|D^{l+1} H\|_{\mathcal{C}^0(B_1)} \quad \text{if } s = \{\omega\}$$

becomes a Banach space. As it is usual, the first problem we need to overcome to write (2.9) as a fixed point equation is to prove that the operator $\mathcal{S}$ has a left-hand side inverse. This is done in the following lemma:

Lemma 2.2.1. *Under all the assumptions made in this section, let $r \in \mathbb{N} \cup \{\omega\}$. Then $\mathcal{S} : \Gamma_{r-1,m} \to \Gamma_{r-1,m}$ is a bounded operator. Moreover $\|\mathcal{S}^{-1}\|$ can be bounded by a constant independent of the scaling parameter δ.*

Remark. *If $r = \{\omega\}$, then $\Gamma_{r-1,m} = \Gamma_{\omega,m}$.*

Proof. See [CFdlL03a], pages 19-21. □

Now, using the operator $\mathcal{S}$ defined in (2.10), we rewrite equation (2.9) as

$$\mathcal{S}(K^{>m}) = -N \circ (K^{\leq m} + K^{>m}) - AK^{\leq m} + K^{\leq m} \circ R. \tag{2.11}$$

By Proposition 2.2.1 we have that the right part of (2.11) vanishes up to order m at the origin when $K^{>m} \in \Gamma_{r-1,m}$. Since, by Lemma 2.2.2, $\mathcal{S}$ is invertible in $\Gamma_{r-1,m}$ we can rewrite the invariance equation (2.4) as the fixed point equation

$$K^{>m} = \mathcal{T}(K^{>m}),$$

where $\mathcal{T}$ is defined as

$$\mathcal{T}(K^{>m}) = \mathcal{S}^{-1}[-N \circ (K^{\leq m} + K^{>m}) - AK^{\leq m} + K^{\leq m} \circ R].$$

Then, since we are assuming that the non-linear operator

$$\mathcal{N}(H) = -N \circ (K^{\leq m} + H) - AK^{\leq m} + K^{\leq m} \circ R$$

is $\mathcal{C}^r$ small, we can prove that $\mathcal{T} = \mathcal{S}^{-1} \circ \mathcal{N}$ is a contraction in the Banach space $\Gamma_{r-1,m}$.

We present the following lemma:

Lemma 2.2.2. *Under all the assumptions made in this section, let $r \in \mathbb{N} \cup \{\omega\}$. Then $\mathcal{T}$ maps the closed unit ball $\bar{B}_1^{r-1}$ of $\Gamma_{r-1,m}$ into itself, is a contraction in $\bar{B}_1^{r-1}$ with the $\Gamma_{r-1,m}$ norm and, consequently, has a fixed point $K^{>m}$ in $\bar{B}_1^{r-1}$.*

Proof. See [CFdlL03a], page 22. □

With that lemma we have proved the existence of a $\mathcal{C}^{r-1}$ map $K^{>m}$ that satisfies (2.8). However we would like it to be $\mathcal{C}^r$ instead. In the article [CFdlL03a], at page 23, there is a proposition that proves that $K^{>m}$ is, indeed, $\mathcal{C}^r$.

Then $K = K^{\leq m} + K^{>m}$ and R computed in Proposition 2.2.1 and Lemma 2.2.2 are the functions that Theorem 2.2.1 claimed to exist.

2.3 Parabolic points in maps

In this section we will work with a type of map slightly different respect the hyperbolic case. We will see that it is not possible to obtain a parametrization with the proceeding that we presented on Section 2.2.

The reason why the procedure in the hyperbolic case does not work in the parabolic one is because it assumes that the maps have no resonances, and in the parabolic case they actually have.

However, we will be able to present an alternative way of computing the parametrization up to any order. We emphasize that in the parabolic case, for example, the internal dynamic will not be linear and the computations will be a bit harder, but not so much.

2.3.1 The maps under consideration and the main result

Let F be a $\mathcal{C}^r$ function $F = (F^1, F^2) : U \in \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$, $r \geq 2$ such that $F(0,0) = 0, DF(0,0) = \text{Id}$ that holds the following conditions:

Let be $N, M \in \mathbb{N}$ such that $2 \leq N, M \leq r$ and F satisfies

$$D^j F^1(0,0) = 0 \qquad \text{for} \quad 2 \leq j \leq N-1, \tag{2.12}$$

$$D^j F^2(0,0) = 0 \qquad \text{for} \quad 2 \leq j \leq M-1, \tag{2.13}$$

$$\frac{\partial^N F^1}{\partial x^N}(0,0) < 0, \qquad \frac{\partial^M F^2}{\partial x^M}(0,0) = 0, \tag{2.14}$$

$$\text{Spec}\frac{\partial^M F^2}{\partial x^{M-1}\partial y}(0,0) \subset \{z \in \mathbb{C} \,|\, \text{Re}\, z > 0\} \quad \text{if} \quad M \leq N. \tag{2.15}$$

We define $L = \min(N, M)$ and $\eta = 1 + N - L$ and assume that $r > 2N - 1$. We write $F = (F^1, F^2) = \text{Id} + \sum_{i=L}^{r} F_j + \tilde{F}_{r+1}$ of the form

$$\begin{aligned} F^1(x,y) &= x + \sum_{i=N}^{r} F_i^1(x,y) + O(|(x,y)|^{r+1}), \\ F^{2,l}(x,y) &= y_l + \sum_{i=M}^{r} F_i^{2,l}(x,y) + O(|(x,y)|^{r+1}), \end{aligned} \tag{2.16}$$

where $F_i^1, F_i^{2,l}$ are, as in the hyperbolic case, homogeneous polynomials of degree i and $\tilde{F}_{r+1} = O(|(x,y)|^{r+1})$. Notice that $F_i(x,y) = (F_i^1(x,y), F_i^{2,l}(x,y))$.

We will also define

$$a_{N,0} := a_{N,0,\dots,0}, \quad b_{M,0} := (b^1_{M,0,\dots,0}, \dots, b^n_{M,0,\dots,0})^\top,$$

and the matrix

$$B_{M-1,1} = \begin{pmatrix} b^1_{M-1,1,0,\dots,0} & \cdots & b^1_{M-1,0,0,\dots,1} \\ \cdots & & \cdots \\ b^n_{M-1,1,0,\dots,0} & \cdots & b^n_{M-1,0,0,\dots,1} \end{pmatrix}.$$

We notice that $a_{N,0} < 0$ and $b_{M,0} = 0$ by the condition (2.14) and $B_{M-1,1}$ has eigenvalues with real positive part by (2.15) when $M \leq N$. Otherwise we do not ask any condition over $B_{M-1,1}$.

As in the hyperbolic case, we are interested in prove the following theorem:

Theorem 2.3.1. *Let $F : U \in \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$ be a map of the form* (2.16) *satisfying the properties* (2.12)–(2.15)*. Then there exist $K : [0, \rho) \to \mathbb{R}^{1+n}$ and $R : \mathbb{R} \to \mathbb{R}$, $\mathcal{C}^r$ at $(0, \rho)$, such that satisfy the invariance equation*

$$F \circ K = K \circ R.$$

As before, the proof will be divided in two parts: the formal part and the reminder one, as we set in the previous section.

2.3.2 Formal part

As we commented at the beginning of this section, since $1 \in \operatorname{spec} DF(0)$, equation (2.7) can not be solved.

Therefore we have to present an alternative method for the computation of $K^{\leq m}, R^{\leq m}$. To do this we will use the following proposition, presented in [BFdlLM07]:

Proposition 2.3.1. *Let F be a $\mathcal{C}^r$ of the form* (2.16) *satisfying* (2.12)–(2.15)*. Let $m \in \mathbb{N}$ be such that $2 \leq m \leq r$. Then there exists polynomials $K^{\leq m}, R^{\leq m}$ of the form*

$$R^{\leq m} = \begin{cases} R^1 + R^N & \text{if } N \leq m < 2N-1, \\ R^1 + R^N + R^{2N-1} & \text{if } m \geq 2N-1, \end{cases}$$

$$K^{\leq m} = \begin{cases} \sum_{i=1}^{m-N+1} K^i & \text{if } N \leq M, \\ (\sum_{i=1}^{m-N+1} K_1^i, \sum_{i=1}^{m-M+1} K_{2,l}^i) & \text{if } N > M, \end{cases}$$

where K^i, R^i are homogeneous polynomials of degree i, $K^1(u) = (u,0)^T, R^1(u) = u$, such that

$$F \circ K^{\leq m}(u) - K^{\leq m} \circ R^{\leq m}(u) = O(u^{m+1}).$$

Remark. *When $N < M$ we do not need condition* (2.15) *to determine K and R.*

Proof. As we settled in the introduction, the parametrization has to contain the fixed point, which is the origin. For this reason we ask K and R to satisfy that $K(0) = 0, R(0) = 0$.

Then $K^{\leq m}$ and $R^{\leq m}$ will be of the form

$$K^{\leq m}(u) = \sum_{i=1}^{m} K^i(u), \quad R^{\leq m}(u) = \sum_{i=1}^{m} R^i(u). \tag{2.17}$$

For this proof we define $K_1^i := \pi_1 K^i, K_{2,l}^i := \pi_{2,l} K^i$. Then we use the notation

$$K_1^{\leq m}(u) = \sum_{i=1}^{m} c_i^1 u^i, \quad K_{2,l}^{\leq m}(u) = \sum_{i=1}^{m} c_i^{2,l} u^i, \quad R^{\leq m}(u) = \sum_{i=1}^{m} d_i u^i.$$

So $K^{\leq m} = (K_1^{\leq m}, K_{2,l}^{\leq m})$, $K^i(u) = (c_i^1 u^i, c_i^{2,1} u^i, \dots, c_i^{2,n} u^i)$ and $R^i(u) = d_i u^i$.

The main point of the proof is to solve iteratively the equation

$$F \circ K^{\leq m}(u) - K^{\leq m} \circ R^{\leq m}(u) = O(u^{m+1}) \tag{2.18}$$

and determine K^m and R^m for each step.

Actually what we are going to compute in each iteration are the coefficients $c_i^1, c_i^{2,l}, d_i$.

We start computing the case $m = 1$, so we want to know $K^1(u), R^1(u)$. We substitute them into the equation (2.18) and obtain

$$K^1(u) - K^1 \circ R^1(u) = O(u^2).$$

Then, as we mentioned at the introduction of the hyperbolic case, we want $K(u)$ such that $K(0) = 0$ and $K'(0) = (1,0)^T$. For this reason we take

$$K^1(u) = (K_1^1(u), K_{2,l}^1(u)) = (u, 0), \quad R^1(u) = u.$$

We remind that we have defined E_1 as the subspace generated by the one-dimensional variable x and $E_{2,l}$ as the projection of the l-component of the subspace E_2 generated by the n-dimensional variable y.

We divide the proof in two different cases, depending on the relation between N and M:

1) Case $N \leq M$.

To compute the order m we are going to divide the calculus in different cases:

a) If $2 \leq m < N$.

First of all we notice that this case is void if $N = 2$.
From (2.17) and, since $F = \mathrm{Id} + O(|(x,y)|^N)$, we write

$$K^1 + K^2 + \ldots + K^m - K^1 \circ R^{\leq m} - \ldots - K^m \circ R^{\leq m} = O(u^{m+1}). \tag{2.19}$$

Therefore all the terms of order m are obtained form the left part of (2.19). We will determine K^m and R^m such that we can eliminate precisely these terms.
We start with the second order i.e. $m = 2$:
When we look at the second order terms of (2.19) we get

$$K^2 - K^1 \circ R^2 - K^2 \circ R^1 = 0.$$

Then we can take $R^2 = 0$ and let K^2 be free.
Now we want to see that we can take $R^m = 0$ and K^m free for $m = 2, \ldots, N-1$. We will prove it by induction.
Suppose that $R^p = 0$ for $2 \leq p < l \leq N-1$, where $l = p+1$. Then we have

$$K^1 + \ldots + K^l - K^1 \circ (R^1 + R^l) - \ldots - K^l \circ (R^1 + R^l) = O(u^{l+1}).$$

From here we compare the terms of order l to obtain the equality

$$K^l - K^1 \circ R^l - K^l \circ R^1 = 0.$$

If we project into the first component we obtain directly that $R^l = 0$. Then we have that K^l is free.
So we have $R^2 = \ldots = R^{N-1} = 0$ and $K^2, \ldots, K^{N-1}$ are free.

b) If $m \geq N$.

In this case we have

$$\begin{aligned}\sum_{j=1}^{m} K^j + &\sum_{j=1}^{m} F_j \circ K^{\leq m} \\ &- \sum_{j=1}^{m} K^j \circ (R^1 + R^N + \ldots + R^m) = O(u^{m+1}).\end{aligned} \tag{2.20}$$

When $m = N$ we compare the terms of order N form (2.20) and we have

$$K^N + F_N \circ K^1 - K^1 \circ R^N - K^N \circ R^1 = 0. \tag{2.21}$$

Since $R^1(u) = u$ we have that K^N is free. Also if we project (2.21) onto E_1 we obtain that $F_N^1 \circ K^1 = R^N$. Thus $a_{N,0} = d_N$. We notice that the projection of the left-side of (2.21) onto E_2 vanishes directly.
When $m > N$ we can obtain the terms from the expression

$$\begin{aligned} &K^1 + \ldots + K^m + F_N \circ (K^1 + \ldots + K^{m-N+1}) + \ldots \\ &+ F_m \circ K^1 = K^1 \circ (R^1 + R^N + \ldots + R^m) + \ldots \\ &+ K^m \circ R^1 + O(t^{m+1}). \end{aligned} \tag{2.22}$$

We are going to compute K^{m-N+1} and R^m assuming that we already know K^p and R^q for all $p < m-N+1$, $q < m$. To do so we will work on the projections onto E_1 and $E_{2,l}$ by $\pi_1, \pi_{2,l}$ respectively.

b.1) Projection onto $E_{2,l}$.
When we project (2.22) by $\pi_{2,l}$ and we take only the terms of order m we get

$$\begin{aligned} c_m^{2,l} &+ b^l_{N-1,1,0,\dots,0} c^{2,1}_{m-N+1} + b^l_{N-1,0,1,\dots,0} c^{2,2}_{m-N+1} + \dots \\ &+ b^l_{N-1,0,0,\dots,1} c^{2,n}_{m-N+1} = (m-N+1) c^{2,l}_{m-N+1} d_N \\ &+ c_m^{2,l} + \Gamma_m^{2,l}, \end{aligned} \tag{2.23}$$

where $\Gamma_m^2 := (\Gamma_m^{2,1}, \dots, \Gamma_m^{2,n})$ depends on the coefficients of $F, K^2, \dots, K^{m-N}$, $R^N, \dots, R^{m-1}$. Then, if we write (2.23) in matrix notation we have

$$(B_{N-1,1} - (m-N+1) a_{N,0} \mathrm{Id}) c^2_{m-N+1} = \Gamma^2_m, \tag{2.24}$$

where $c^2_{m-N+1} = (c^{2,1}_{m-N+1}, \dots, c^{2,n}_{m-N+1})$.
The reason why the matrix $B_{N-1,1} - (m-N+1)a_{N,0}\mathrm{Id}$ is invertible depends on the relation between N and M.
If $N = M$ we have that $B_{N-1,1} = B_{M-1,1}$ has eigenvalues with real positive part and, since $a_{N,0} < 0$, we have that all of them are non-zero.
If $N < M$ we have that $B_{N-1,1} = 0$ and $(m-N+1)a_{N,0}\mathrm{Id}$ is invertible.
Hence in both cases we can solve (2.24) and obtain c^2_{m-N+1} that will determine K_2^{m-N+1}.

b.2) Projection onto E_1.
We proceed exactly as the case **b.1** but projecting (2.22) by π_1. Then we get

$$\begin{aligned} c_m^1 &+ N a_{N,0,\dots,0} c^1_{m-N+1} + a_{N-1,1,0,\dots,0} c^{2,1}_{m-N+1} \\ &+ a_{N-1,0,1,\dots,0} c^{2,2}_{m-N+1} + \dots \\ &= d_m + (m-N+1) c^1_{m-N+1} d_N + c^1_m + \Gamma^1_m. \end{aligned}$$

Then, isolating c_{m-N+1} and d_m, we have

$$(2N-m-1) a_{N,0} c^1_{m-N+1} - d_m = \text{“known terms”}. \tag{2.25}$$

Notice that these “known terms” depend on c^2_{m-N+1}, so its mandatory to compute them first in the step **b.1**. The way we solve equation (2.25) will depend on the value of m.
If $m \neq 2N-1$ we can take $d_m = 0$ and determine c^1_{m-N+1} from (2.25).
If $m = 2N-1$ we have fixed the value of d_N and coefficient c^1_N is free.

Then we have obtained $K^{\leq m}, R^{\leq m}$ as we claimed at the statement of the proposition in the case $N \leq M$.

2) Case $N > M$.

This case is not of our interest for the work so we will not develop its proof. However we notice that it works on a similar way as the first one. See [BFdlLM07] for further details.

□

Then we have finished the formal part and we have R and a approximation of K, that we will se now that it is accurate.

2.3.3 The reminder

Our interest is, as in the hyperbolic case, to see how accurate is the computed parametrization $K^{\leq m}$. We want to see that if m is large enough, then it will exist a real stable manifold near of $K^{\leq m}$.

Let $k \in \mathbb{N}$ be such that $2N-1 \leq k \leq r$. We can decompose $F = P + Q_k$, where P is of the form

$$P(x,y) = \begin{pmatrix} x + a_{N,0}x^N + y^T f_{N-1}(x,y) + f_{N+1}(x,y) \\ y + B_{M-1,1}x^{M-1}y + y^T g_{M-2}(x,y)y + g_{M+1}(x,y) \end{pmatrix}$$

and Q_k is of order $O(|(x,y)|^k)$.

We apply Proposition 2.3.1 to the Taylor polynomial P with $m = k-1$. We notice that sometimes we will use the term m and other times k. In any case they are used, they will always satisfy the relation

$$m + 1 = k.$$

Then we get the corresponding $K^{\leq m} : \mathbb{R} \to \mathbb{R}^{1+n}$ and $R : \mathbb{R} \to \mathbb{R}$. We define the remainder of order k:

$$\begin{aligned} T_k &:= P \circ K^{\leq k-1} - K^{\leq k-1} \circ R \\ &= P \circ K^{\leq m} - K^{\leq m} \circ R, \end{aligned} \tag{2.26}$$

where T_k is a polynomial such that $T_k(u) = O(u^k) = O(u^{m+1})$ and $D^l T_k(u) = O(u^{k-l}) = O(u^{m+1-l})$ for $1 \leq l \leq r$.

Our main goal is to prove the existence of a $\mathcal{C}^r$ function $K^{>m}$ such that

$$F \circ (K^{\leq m} + K^{>m}) - (K^{\leq m} + K^{>m}) \circ R = 0. \tag{2.27}$$

To do so we will transform the equation (2.27) into a fixed point equation for $K^{>m}$ and we will use the fixed point theorem in some appropriate Banach spaces.

First of all, considering $L = \min(M, N)$, we fix $\eta = 1 + N - L$. Given E a Banach space, $t_0 \in (0,1), r \geq 0$ and $k \in \mathbb{R}$, we introduce the Banach space

$$\mathcal{X}_r^k = \{f : (0,t_0) \to E | f \in \mathcal{C}^r, \max_{0\leq j\leq r} \sup_{t\in(0,t_0)} t^{-k+j\eta}|D^j f(t)| < \infty\},$$

with the norm

$$\|f\|_{r,k} := \max_{0\leq j\leq r} \sup_{t\in(0,t_0)} t^{-k+j\eta}|D^j f(t)|.$$

We perform some scaling in the maps, as we have done in the hyperbolic case but slightly different. For any $\delta > 0$, we define

$$E_\delta(x,y) = (x, \delta y),$$

and

$$\begin{aligned} \widetilde{F} &= E_\delta^{-1} \circ F \circ E_\delta, \\ \widetilde{P} &= E_\delta^{-1} \circ P \circ E_\delta, \\ \widetilde{Q_k} &= E_\delta^{-1} \circ Q_k \circ E_\delta, \\ \widetilde{K^{\leq m}} &= E_\delta^{-1} \circ \widetilde{K}^{\leq m}, \\ \widetilde{T}_k &= E_\delta^{-1} \circ \widetilde{T}_k. \end{aligned}$$

Scaling the equations (2.26) and (2.27) we obtain

$$\widetilde{T}_k = \widetilde{P} \circ \widetilde{K^{\leq m}} - \widetilde{K^{\leq m}} \circ \widetilde{R},$$

and

$$\widetilde{F} \circ (\widetilde{K}^{\leq m} + \widetilde{K}^{>m}) - (\widetilde{K}^{\leq m} + \widetilde{K}^{>m}) \circ \widetilde{R} = 0. \tag{2.28}$$

We have:

$$\widetilde{P}(x, y) = \begin{pmatrix} x + a_{N,0}x^N + \delta y^T f_{N-1}(x, \delta y) + f_{N+1}(x, \delta y) \\ y + B_{M-1,1}x^{M-1}y + \delta y^T g_{M-2}(x, \delta y)y + \delta^{-1}g_{M+1}(x, \delta y) \end{pmatrix}$$

To avoid cumbersome notation, we skip the tilde symbol of our notation.

We will see that if δ is sufficiently small we can ensure that there exists $K^{>m}$ such that (2.28) is satisfied.

The first step is to write equation (2.28) as a fixed point equation. Notice that (2.28) can be rewritten as

$$\begin{gathered}(DP \circ K^{\leq m})K^{>m} - K^{>m} \circ R = -T_k - Q_k \circ (K^{\leq m} + K^{>m}) \\ -P \circ (K^{\leq m} + K^{>m}) + P \circ K^{\leq m} + (DP \circ K^{\leq m})K^{>m}.\end{gathered} \tag{2.29}$$

This decomposition motivates to define the linear operator

$$\mathcal{L}^0(H) = (DP \circ K^{\leq m})H - H \circ R. \tag{2.30}$$

To deal with the derivatives of $K^{>m}$, we introduce the linear operators

$$\mathcal{L}^j(H) = (DP \circ K^{\leq m})H - H \circ R(DR)^j, \quad j \geq 1. \tag{2.31}$$

We note that, if we have $T \in \mathcal{C}^r$ such that H_* is a $\mathcal{C}^r$ solution of $\mathcal{L}^0(H_*) = T$, then, for $0 \leq j \leq r$, $H_*^j := D^j H_*$ has to be a solution of

$$\mathcal{L}^j(H_*^j) = T^j,$$

where T^j is defined by the recurrence

$$\begin{aligned} T^0 &= T, \\ T^{j+1} &= DT^j - D(DP \circ K^{\leq m})D^j H_* \circ R(DR)^{j-1}D^2R. \end{aligned} \tag{2.32}$$

We recall that $\eta = 1 + N - L$ and we define α and σ as

$$\alpha := \frac{1}{N-1}, \quad \sigma := \delta\alpha|a_{N,0}|^{-1} \sup_{t\in(0,t_0)} |f_{N-1}(t, 0)t^{-N+1}|.$$

Then we have the following lemma:

Lemma 2.3.1. *If $k > 2N - 1$ and $\sigma < \alpha(k - 2N + 1)$, then the operators $\mathcal{L}^j : \mathcal{X}_0^{k-N+1-j\eta} \to C^0, j \geq 0$ defined in (2.30) and (2.31) are one to one.*

Proof. See [BFdlLM07], page 849. □

Now we want to find the inverse of $\mathcal{L}^j$ in an appropriate Banach space.

We notice that the equation (2.31) can be rewritten as

$$H = [(DP)^{-1} \circ K^{\leq m}]H \circ R(DR)^j + [(DP)^{-1} \circ K^{\leq m}]T. \tag{2.33}$$

If we iterate (2.33) and assume that

$$\lim_{i\to\infty} \left[\prod_{n=0}^{i} (DP)^{-1} \circ K^{\leq m} \circ R^n\right] H \circ R^i(DR^i)^j = 0,$$

we can define the formal operator

$$\mathcal{S}^j(T) = \sum_{i\geq 0}\left[\prod_{n=0}^{i}(DP)^{-1}\circ K^{\leq m}\circ R^n\right] T\circ R^i(DR^i)^j. \tag{2.34}$$

The following lemma shows the relation between the operators $\mathcal{S}^j$ and $\mathcal{L}^j$ defined in (2.34) and (2.30),(2.31) respectively:

Lemma 2.3.2. *If $k > 2N-1$ and $\sigma < \alpha(k-2N+1)$, then (2.34) defines a bounded linear operator*

$$S^j : \mathcal{X}_0^{k-j\eta} \to \mathcal{X}_0^{k-N+1-j\eta},$$

satisfying

$$\mathcal{L}^j \circ \mathcal{S}^j = Id \qquad on \quad \mathcal{X}_0^{k-j\eta}.$$

Moreover, (2.34) also defines a bounded linear operator

$$S^j : \mathcal{X}_1^{k-j\eta} \to \mathcal{X}_1^{k-N+1-j\eta}$$

and, if $T \in \mathcal{X}_1^{k-j\eta}$, we have

$$D[\mathcal{S}^j(T)] = \mathcal{S}^{j+1}(\widetilde{T}),$$

where

$$\widetilde{T} = DT - D(DP\circ K^{\leq m})\mathcal{S}^j(T) + jS^j(T)\circ R(DR)^{j-1}D^2R. \tag{2.35}$$

Remark. *The conditions for k and σ of this lemma and Lemma 2.3.1 are the same. In addition, formulae (2.32) and (2.35) are very related.*

Proof. See [BFdlLM07], page 851. □

To deal with the r-derivative of $K^{>m}$, we will need to work with the operators $\mathcal{S}^0$ and $\mathcal{S}^1$ defined on the space $\mathcal{X}_s^{k-N+1}$, with $s \leq r$.

Proposition 2.3.2. *Let $r > 0$, $k > 2N-1$ and $\sigma < \alpha(k-2N+1)$. Then if $0 \leq s \leq r$*

$$\mathcal{S}^0 : \mathcal{X}_s^k \to \mathcal{X}_s^{k-N+1} \qquad and \qquad \mathcal{S}^1 : \mathcal{X}_s^{k-\eta} \to \mathcal{X}_s^{k-N+1-\eta}$$

are bounded linear operators.

Remark. *The conditions for k and σ of this lemma are the same than the ones in lemmas 2.3.1 and 2.3.2.*

Proof. See [BFdlLM07], page 853. □

Once we have found a right inverse of the linear operator $\mathcal{L}_0$, we are able to write equation (2.29) as a fixed point equation. Indeed, using the definition (2.30) of $\mathcal{L}^0$ we can rewrite equation (2.29) as

$$\mathcal{L}^0(K^{>m}) = \mathcal{F}(K^{>m}),$$

where

$$\begin{aligned}\mathcal{F}(K^{>m}) = &-T_k - Q_k\circ(K^{\leq m}+K^{>m}) - P\circ(K^{\leq m}+K^{>m})\\ &+P\circ K^{\leq m} + (DP\circ K^{\leq m})K^{>m}.\end{aligned}$$

If $\mathcal{L}^0\circ\mathcal{S}^0 = \mathrm{Id}$, in some appropriate Banach space, we have that for proving the existence of $K^{>m}$ it is enough to solve the fixed point equation

$$K^{>m} = \mathcal{S}^0 \circ \mathcal{F}(K^{>m}). \tag{2.36}$$

One of the main questions the reader could, and should, ask is which is this appropriate Banach Space. Now we are able to answer it.

We have that T_k and $Q_k \circ K^{\leq m}$ belong to $\mathcal{X}_r^k$ and, by Proposition 2.3.2, $\mathcal{S}^0 \circ \mathcal{F}(0) = \mathcal{S}^0(-T_k - Q_k \circ K^{\leq m})$ belongs to $\mathcal{X}_r^{k-N+1}$. For this reason we will look for a solution of the equation (2.36) in $\mathcal{X}_r^{k-N+1}$.

Proposition 2.3.3. *If t_0 is small enough, equation* (2.36) *has a unique fixed point $K^{>m} : [0, t_0) \to \mathbb{R}^{1+n}$ in the sphere of radius ρ, $\mathcal{B}_{r-1,\rho}^{k-N+1}$.*

Proof. See [BFdlLM07], page 856. □

Finally, combining the results of Propositions 2.3.1 and 2.3.3, we have found $K = K^{\leq m} + K^{>m}$ and R satisfying the statement of Theorem 2.3.1.

2.4 Parabolic periodic orbits in time periodic vector fields

In this section we present the parametrization method for some concrete vector fields T-periodic in time. This is the case where the computations will be used in our project, since the N-body problem is a continuous dynamical system and T-periodic with $T = 2\pi$.

2.4.1 The vector fields under consideration and main result

Definition 2.4.1. A T**-periodic vector field** is a function $Z(z, t)$, where $z \in \mathbb{R}^{1+n}, t \in \mathbb{R}$ and $Z(z, t + T) = Z(z, t)$.

We want to find a parametrization $K(u, t) : \mathbb{R} \times \mathbb{T} \to \mathbb{R}^{n+1}$, with $\mathbb{T} = \mathbb{R}/[0, T]$, of a one-dimensional invariant manifold . From the invariance condition, $K(u, t)$ has to satisfy

$$\frac{d}{dt}K(u, t) = Z(K(u, t), t),$$

that means

$$Z(K(u, t), t) = \partial_u K(u, t)\dot{u} + \partial_t K(u, t).$$

So, if we consider $Y(u) = \dot{u} : \mathbb{R} \to \mathbb{R}$ as the internal dynamic of K, we obtain the form

$$Z(K(u, t), t) = \partial_u K(u, t)Y(u) + \partial_t K(u, t). \tag{2.37}$$

We also ask K to satisfy $K(0, t) = 0$ and $\partial_u K(0, t) = \vec{v}$ for a direction vector v. In our case $\vec{v}$ will be $\vec{e}_1$.

We also assume that the vector field Z is of the form

$$Z(x, y, t) = \begin{pmatrix} -ax^N \quad + p_N(x, y) + f(x, y, t) \\ x^{N-1}By + q_N(x, y) + g(x, y, t) \end{pmatrix}, \tag{2.38}$$

where $a > 0$ and the $n \times n$ matrix B has eigenvalues with positive real part.

The functions p_N, q_N are, respectively, one-dimensional and n-dimensional sums of homogeneous polynomials of order N such that

$$p_N(x, 0) = q_N(x, 0) = 0, \quad \partial_y q_N(x, 0) = 0,$$

and f, g are $\mathcal{C}^r$ functions, with $r \geq 2$, of order $O(\|(x, y)\|^{N+1})$.

Remark. *It is also possible to solve the problem when the polynomials of* (2.38) *have different degrees, but we will not do that case.*

In this section, as before, our main goal is to prove the following theorem:

Theorem 2.4.1. *Let $Z : \mathcal{U} \subset \mathbb{R}^{1+n} \times \mathbb{R} \to \mathbb{R}^{1+n}$ be a T-periodic vector field of the form* (2.38)*. Then, for any ρ small enough, there exist $K(u,t) : [0,\rho) \times \mathbb{T} \to \mathbb{R}^{1+n}$ and $Y : \mathbb{R} \to \mathbb{R}$ such that satisfy the invariance equation* (2.37)*. Moreover $K(u,t)$ is T-periodic in t.*

As always, we divide the proof of the theorem by diving it into the formal part and the remainder.

2.4.2 Formal part

The following proposition provides the invariant manifolds of parabolic periodic orbits:

Proposition 2.4.1. *Let $Z(z,t)$ be a T-periodic vector field, satisfying the hypotheses of Theorem 2.4.1, with z defined as $z = (x,y)^T$, for $x \in \mathbb{R}, y \in \mathbb{R}^n$. Then for any $m \in \mathbb{N}$, $2 \le m \le r$, there exist $K^{\le m}(u,t) : \mathbb{R} \times \mathbb{T} \to \mathbb{R}^{1+n}, Y^{\le m+N-1}(u) : \mathbb{R} \to \mathbb{R}$ of the form*

$$K^{\le m}(u,t) = \sum_{j=1}^{m} K^j(u,t), \qquad Y^{\le m+N-1}(u) = \sum_{j=N}^{m+N-1} Y^j(u),$$

satisfying the invariance condition up to order $m+N$, namely

$$Z(K^{\le m}(u,t),t) - D_u K^{\le m}(u,t) Y^{\le m+N-1}(u) + \partial_t K^{\le m}(u) = O(u^{m+N}). \tag{2.39}$$

In addition, K^j and Y^j can be taken of the form

$$\begin{aligned} K^j(u,t) &= K_j u^j + \hat{K}_j(t) u^{j+N-1}, \\ Y^j(u) &= Y_j u^j, \end{aligned} \tag{2.40}$$

where $K_j, \in \mathbb{R}^{1+n}$, $Y_j \in \mathbb{R}$ are constants and $\hat{K}_j(t) : \mathbb{R} \to \mathbb{R}^{1+n}$ is a T-periodic $\mathcal{C}^r$ function with zero mean i.e.

$$\int_0^T \hat{K}_j(s) ds = 0.$$

Proof. We define the error

$$\begin{aligned} E^m(u,t) &:= \\ &Z(K^{\le m}(u,t),t) - \partial_u K^{\le m}(u,t) Y^{\le m+N-1}(u) - \partial_t K^{\le m}(u,t). \end{aligned} \tag{2.41}$$

Despite the notation, E^m is expected to be of order $m+N$ respect to u i.e. $E^m(u,t) = O(u^{m+N})$.

We will start the computations of $K^{\le m}, Y^{\le m+N-1}$ with the case $m = 1$. Since K has to contain the origin and has to be tangent to the first axis, we take

$$K^{\le 1}(u,t) = K_1 u = \begin{pmatrix} u \\ 0 \end{pmatrix}.$$

Then we want to compute $Y^{\le N}$ such that (2.39) is satisfied for m equal to one. We substitute $K^{\le 1}$ into (2.39) and we obtain

$$E^1(u,t) = \begin{pmatrix} -au^N & + & f(u,0,t) \\ 0 & + & g(u,0,t) \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} Y^{\le N} - 0. \tag{2.42}$$

Taking $Y^{\le N} = -au^N$ we have that (2.42) is of order $O(u^{N+1})$, as we wanted.

Now we will construct $K^{\leq m+1}, Y^{\leq m+N}$ by induction.

Suppose that we have already computed $K^{\leq m}, Y^{\leq m+N-1}$ satisfying $E^m(u) = O(u^{m+N})$. Then we want to find $K^{m+1}(u,t)$ and $Y^{m+N}(u)$ such that

$$K^{\leq m+1}(u,t) = K^{\leq m}(u,t) + K^{m+1}(u,t), \quad Y^{\leq m+N}(u) = Y^{\leq m+N-1}(u) + Y^{m+N}(u)$$

satisfy (2.39) for the case $m+1$.

From definition of E^{m+1} and (2.41) we have

$$\begin{aligned} E^{m+1} = Z(K^{\leq m} + K^{m+1}) - \partial_u(K^{\leq m} + K^{m+1})(Y^{\leq m+N-1} + Y^{m+N}) \\ - \partial_t(K^{\leq m} + K^{m+1}), \end{aligned}$$

that is equivalent to

$$\begin{aligned} E^{m+1} = Z(K^{\leq m}) &- \partial_u K^{\leq m} Y^{\leq m+N-1} - \partial_t K^{\leq m} \\ &+ DZ(K^{\leq m})K^{m+1} \\ &- \partial_u K^{\leq m} Y^{m+N} \\ &- \partial_u K^{m+1} Y^{\leq m+N-1} \\ &- \partial_u K^{m+1} Y^{m+N} \\ +Z(K^{\leq m} + K^{m+1}) &- Z(K^{\leq m}) - DZ(K^{\leq m})K^{m+1} \\ &- \partial_t K^{m+1}. \end{aligned} \tag{2.43}$$

We recall that we want $E^{m+1} = O(u^{m+N+1})$. We will see that all the terms in (2.43) are at least of order $\mathcal{O}(u^{m+N})$. The ones of order $m+N+1$ or higher will not be considered, since they do not represent an inconvenience.

However, the terms of order $m+N$ need to be eliminated or changed to higher order. This is the criteria that we will use in order to choose K^{m+1}, Y^{m+N}.

First we compute the order of $DZ(K^{\leq m})$:

$$DZ(K^{\leq m}) = \begin{pmatrix} -aNu^{N-1} & c^T u^{N-1} \\ 0 & Bu^{N-1} \end{pmatrix} + O(u^N) =: Au^{N-1} + O(u^N),$$

for some $c \in \mathbb{R}^n$ and A as

$$A = \begin{pmatrix} -aN & c^T \\ 0 & B \end{pmatrix}.$$

Now we estimate all the lines of (2.43).

1. The line $Z(K^{\leq m}) - \partial_u K^{\leq m} Y^{\leq m+N-1} - \partial_t K^{\leq m}$ is $O(u^{m+N})$ because it is exactly E^m and, by induction hypothesis, it has that order.

2. The line $DZ(K^{\leq m})K^{m+1}$ is $O(u^{m+N})$ because it is $O(u^{(N-1)+(m+1)})$.

3. Using previous argument, we can see that the lines $\partial_u K^{\leq m} Y^{m+N}$ and $\partial_u K^{m+1} Y^{\leq m+N-1}$ are both of order $O(u^{m+N})$ and $\partial_u K^{m+1} Y^{m+N}$ is $O(u^{2m+N}) = O(u^{m+N+1})$, since m is greater or equal to one.

4. By Taylor's theorem, we have that $Z(K^{\leq m} + K^{m+1}) - Z(K^{\leq m})$ $-DZ(K^{\leq m})K^{m+1}$ is of the same order as $D^2 Z(K^{\leq m}) \left(K^{l+1}\right)^2$, that is $O(u^{(N-2)+(2m+2)}) = O(u^{2m+N}) = O(u^{m+N+1})$, for the same reason we explained in point **3**.

5. And the last line $\partial_t K^{m+1}$ is $O(u^{m+1})$.

Recall that we are looking for K^{m+1} of the form:

$$K^{m+1}(u,t) = K_{m+1}u^{m+1} + \hat{K}_{m+1}(t)u^{m+N}.$$

Then, since we want $E^{m+1} = O(u^{m+N+1})$, we need to choose K^{m+1}, Y^{m+N} satisfying

$$\begin{aligned} &E^m + AK^{m+1}u^{N-1} - \partial_u K^{m+1} Y^{\le m+N-1} - \partial_u K^{\le m} Y^{m+N} - \partial_t K^{m+1} \\ &= O(u^{m+N+1}). \end{aligned} \tag{2.44}$$

We solve the computation of K^{m+1} and Y^{m+N} using the form (2.40).

We start by solving the equation for $K_{m+1}u^{m+1}$. In this case is the equation associated with the independent of t terms of order $m+N$.

We consider $K_{m+1} = (K^1_{m+1}, K^{2,l}_{m+1}) \in \mathbb{R} \times \mathbb{R}^n$, $l = 1, \dots, n$. As in Section 2.2, we define $[\cdot]_j$ as the terms of order j and $[\cdot]^1_j, [\cdot]^2_j$ as their respective projections to the one-dimensional variable x and n-dimensional y.

To obtain the t-independent coefficients of order $m+N$ of the equation (2.44) we take the average with respect to t in every term of (2.44) in the interval $[0,T]$. Since $Y^N(u) = -au^N$ and $\partial_t K^{m+1}(u,t)$ is of order $O(u^{m+1})$, we obtain

$$\begin{pmatrix} -aN & c^T \\ 0 & B \end{pmatrix} K_{m+1} + a(m+1)K_{m+1} - \begin{pmatrix} Y_{m+N} \\ 0 \end{pmatrix} = \left[-\overline{E^m}\right]_{m+N}, \tag{2.45}$$

where $\overline{E^m}$ is the average respect to t of E^m in $[0,T]$. We remark that it is known because it only depends on $K^{\le m-1}$ and Y^{m+N-1}.

Now we separate (2.45) in two equations, one corresponding to the variable x and another for the other n variables y:

$$-aNK^1_{m+1} + c^T K^2_{m+1} + a(m+1)K^1_{m+1} - Y_{m+N} = \left[-\overline{E^m}\right]^1_{m+N}, \tag{2.46}$$

$$(B + a(m+1)\mathrm{Id})K^2_{m+1} = \left[-\overline{E^m}\right]^2_{m+N}. \tag{2.47}$$

We start solving equation (2.47).

Since B has eigenvalues with positive real part we have that $B + a(m+1)\mathrm{Id}$ is invertible. In fact, it has eigenvalues with positive real part. So (2.47) can always be solved and we obtain a unique solution:

$$K^2_{m+1} = -(B + a(m+1)Id)^{-1} \left[\overline{E^m}\right]^2_{m+N}.$$

Once we have computed K^2_{m+1} we can solve (2.46). Notice that we have

$$(m+1-N)aK^1_{m+1} - Y_{m+N} = \left[-\overline{E^m}\right]^1_{m+N} - c^T K^2_{m+1}, \tag{2.48}$$

where the right part of (2.48) is now known.

We solve it depending on the value of m:

- If $m+1 \ne N$ we can take $Y_{m+N} = 0$ and solve the equation (2.48) for K^1_{m+1}.
- If $m+1 = N$ we have that K^1_N is a free term and Y_{2N-1} is uniquely determined by known terms.

To solve equation (2.44), the last computation which has to be done is the construction of the periodic part $\hat{K}_{m+1}(t)$ of $K^{m+1}(u,t)$. Notice that $\hat{K}_{m+1}(t)$ is the coefficient associated to u^{m+N}.

Substituting $u^{m+N}\hat{K}_{m+1}(t)$ into (2.44) and taking into account the above computations for $u^{m+1}K_{m+1}$, we obtain that we have to solve

$$\hat{E}^m - u^{m+N}\partial_t \hat{K}_{m+1}(t) = O(u^{m+N+1}), \tag{2.49}$$

where $\hat{E}^m(u,t) := E^m(u,t) - \overline{E^m}(u,t)$.

Equation (2.49) can be solved if we take $\hat{K}_{m+1}(t)$ satisfying

$$\partial_t \hat{K}_{m+1}(t) = \left[\hat{E}^m(u,t)\right]_{m+N} = \left[E^m(u,t) - \overline{E^m}(u,t)\right]_{m+N},$$

or equivalently

$$\hat{K}_{m+1}(t) = \int_0^t \left[E^m(u,s) - \overline{E^m}(u,s)\right]_{m+N} ds. \tag{2.50}$$

Remark. *The function $\hat{K}_{m+1}(t)$ is T-periodic since $\left[E^m(u,t) - \overline{E^m}(u,t)\right]_{m+N}$ has zero mean.*

Finally, from (2.45) and (2.50) we have computed $K^{m+1}(u,t)$, $Y^{m+N}(u)$ of the form (2.40) such that $E^{m+1}(u,t) = O(u^{m+N+1})$.

□

2.4.3 The reminder

In this case we will work in a different way than the case of maps. Actually we are going to prove that the proof corresponding to the reminder part for vector fields can be reduced to the corresponding one for maps.

To prove the existence of the reminder we use the following proposition:

Proposition 2.4.2. *It exists $K(u,t)$ and $Y(u)$ satisfying* (2.37) *with the vector field Z defined in* (2.38).

Proof. We start constructing two Poincaré maps $F(x,y,t) : \mathbb{R}^{1+n} \times \mathbb{R} \to \mathbb{R}^{1+n}$, $R(u) : \mathbb{R} \to \mathbb{R}$, which will come from the flows of the vector field Z and the dynamic Y.

Using Section 2.4.2 we show that the maps F and R satisfy

$$F(K^{\leq m}(u,t),t) - K^{\leq m}(R(u),t) = O(u^{m+N}).$$

Then, by Theorem 2.3.1, it will exist a parametrization $K(u,t)$ which satisfy the invariance equation for maps, meaning

$$F(K(u,t)) = K(R(u),t).$$

Remark. *Here the variable t of $K(u,t)$ is considered as a parameter.*

Finally we proof that this parametrization $K(u,t)$ also satisfies the invariance equation (2.37).

So we start by proving the reduction to the map case:

Form flows to maps

From the invariance equation (2.37) we define $\varphi(s;t,x,y)$, $\psi(s;t,u)$ as the flows of the vector fields Z and Y respectively, satisfying

$$\begin{aligned}\psi(t,t,x,y) &= (x,y),\\ \varphi(t,t,u) &= u.\end{aligned}$$

Then (2.37) is equivalent to

$$\varphi(s;t,K(u,t)) - K(\psi(s;t,u),s) = 0.$$

We have, by Proposition 2.4.1, that there exists a large $m \in \mathbb{N}$ such that the corresponding $K^{\leq m}(u,t)$ and $Y(u) := Y^{\leq m+N-1}(u)$ satisfy

$$Z(K^{\leq m}(u,t),t) - D_u K^{\leq m}(u,t)Y(u) + \partial_t K^{\leq m}(u) = O(u^{m+N}).$$

Then, for $s \in [t, t+T]$, we have

$$\varphi(s;t,K^{\leq m}(u,t)) - K^{\leq m}(\psi(s;t,u),s) = O(u^{m+N}). \tag{2.51}$$

We introduce now the Poincaré maps

$$\begin{aligned}F(x,y,t) &:= \varphi(t+T,t,x,y),\\ R(x) &:= \psi(T,0,x) = \psi(t+T,t,x).\end{aligned} \tag{2.52}$$

Applying (2.51) with $s = t+T$ we obtain

$$F(K^{\leq m}(u,t),t) - K^{\leq m}(R(u),t) = O(u^{m+N}).$$

We remark that the map F is of the form $F = P + Q_k$ as we set in Section 2.3.3. The details of this justification are in [BFM], page 50.

Then F satisfies the conditions of the Theorem 2.3.1, with $k = m$, and it will exist a parametrization $K(u,t) := K^{\leq m}(u,t) + K^{>m}(u,t)$ with the internal dynamic $R(u) = \psi(t+T,t,u)$ such that

$$F(K(u,t),t) = K(\psi(t+T,t,u),t). \tag{2.53}$$

Notice that, by the uniqueness of the solution, $K^{>m}(u,t)$ and consequently $K(u,t)$ are T-periodic respect to t.

From maps to periodic flows

We have finished the first part, which is to reduce the vector field case to the map one. Now we want to see that if K satisfies (2.53), then it will also satisfy (2.37).

Using the definitions (2.52) of F and R, equation (2.53) and the properties of general solutions of vector fields we obtain

$$K(u,s) = \varphi(s;s+T,K(R(u),s)), \qquad R(\psi(s;t,u)) = \psi(s;t,R(u)).$$

We introduce

$$\mathcal{K}_s(u,t) = \varphi(t;s,K(\psi(s;t,u),s)),$$

remarking that $\mathcal{K}_t(u,t) = K(u,t)$.

We want to check that $\mathcal{K}_s(u,t)$ satisfies the invariant equation (2.53) for all $s \in \mathbb{R}$, namely

$$F(\mathcal{K}_s(u,t),t) = \mathcal{K}_s(\psi(t+T,t,u),t).$$

Computing both parts of the equality we have

$$\begin{aligned} F(\mathcal{K}_s(u,t),t) &= \varphi(t+T,s,K(\psi(s;t,u),s)) = \varphi(t+T,s+T,K(\psi(s;t,R(u)),s)) \\ &= \varphi(t,s,K(\psi(s;t-T,u),s)) \end{aligned}$$

and

$$\mathcal{K}_s(R(u),t)) = \varphi(t;s,K(\psi(s;t,R(u)),s)) = \varphi(t,s,K(\psi(s;t-T,u),s)).$$

So $\mathcal{K}_s(u,t)$ satisfies the invariant condition (2.53) for all s, as we wanted.

Once we know that we want to check that

$$\mathcal{K}_s(u,t) - K^{\leq m}(u,t) = O(u^{m+1}), \tag{2.54}$$

since it will be enough to proof that $K(u,t)$ satisfies (2.37).

Indeed, applying Taylor's theorem we obtain

$$\begin{aligned} \mathcal{K}_s(u,t) = \varphi(t;s,K(\psi(s;t,u),s)) = \varphi(t;s,K^{\leq m}(\psi(s;t,u),s)) \\ + \int_0^1 D\varphi(t;s,K^{\leq m}(\psi(s;t,u),s) + wK^{>m}(\psi(s;t,u),s))K^{>m}(\psi(s;t,u),s)dw. \end{aligned}$$

Using now equality (2.51) we have

$$\begin{aligned} \mathcal{K}_s(u,t) - K^{\leq m}(u,t) = O(u^{m+N}) + \int_0^1 DK^{\leq m}(u+w(\varphi(s;t,u)-u),t)[\psi(s;t,u)-u]dw \\ + \int_0^1 D\varphi(t;s,K^{\leq m}(\psi(s;t,u),s) + wK^{>m}(\psi(s;t,u),s))K^{>m}(\psi(s;t,u),s)dw. \end{aligned}$$

Since $\psi(s;t,0) = 0$ and $\psi(s;t,u) = u + O(u^N)$ we obtain that (2.54) is satisfied.

From the uniqueness statement of Proposition 2.3.3 we have that

$$\mathcal{K}_s(u,t) = K(u,t).$$

Then we get

$$K(\psi(s;t,u),s) = \varphi(s;t,\mathcal{K}_s(u,t)) = \varphi(s;t,K(u,t)).$$

□

With this statement we have finished the remaining part, since K satisfies the invariant equation for vector fields and all the properties stated and Theorem 2.4.1 is proved.

Chapter 3

Parabolic manifolds in the RPC3BP

In this Chapter we show how the parametrization method can be used to compute an approximation of the parabolic stable manifold of each periodic orbits at infinity in the RPC3BP. Recall that, as we presented in Section 1.10, the parabolic infinity can be seen as the set of orbits $(0, 0, \theta_0 + t, \omega_0)$ inside $I_\infty := \{q = p = 0\}$.

To do so, we recall here system (1.25):

$$\begin{aligned} \dot{q} &= -\frac{1}{4}q^3 p, \\ \dot{p} &= -\frac{q^4}{4}\sigma_2 + \frac{q^6 w^2}{8} - \frac{q^6}{8}\mu(1-\mu)\sigma_1 \cos\theta, \\ \dot{\theta} &= 1 - \frac{1}{4}q^4\omega, \\ \dot{\omega} &= -\frac{q^4}{4}\mu(1-\mu)\sigma_1 \sin\theta, \end{aligned} \tag{3.1}$$

with

$$\sigma_1 = \frac{1}{f_\mu^3} - \frac{1}{f_{\mu-1}^3}, \quad \sigma_2 = \frac{1-\mu}{f_\mu^3} + \frac{\mu}{f_{\mu-1}^3}, \tag{3.2}$$

$$f_m = \left(1 + mq^2\cos\theta + \frac{m^2}{4}q^4\right)^{1/2}. \tag{3.3}$$

The strategy we will use is the following:

1) We emphasize that system (3.1) is not in the form stated in Proposition 2.4.1. Therefore, the first thing we need to do is to prove that our result can be applied to system (3.1). This is done by a change of variables and considering θ as the new independent variable, in Section 3.1.

   Once we know that system (3.1) has a parabolic manifold, it does not matter how we find it. As a consequence we can work, if necessary, with the initial variables (q, p, θ, ω) but with the variable θ as the new independent variable.

   We consider the new system:

$$\begin{aligned} \frac{dq}{d\theta} &= -\frac{1}{4}q^3 p\,\frac{1}{1-\frac{1}{4}q^4\omega}, \\ \frac{dp}{d\theta} &= \left[-\frac{q^4}{4}\sigma_2 + \frac{q^6 w^2}{8} - \frac{q^6}{8}\mu(1-\mu)\sigma_1\cos\theta\right]\frac{1}{1-\frac{1}{4}q^4\omega}, \\ \frac{d\omega}{d\theta} &= -\frac{q^4}{4}\mu(1-\mu)\sigma_1\sin\theta\,\frac{1}{1-\frac{1}{4}q^4\omega}. \end{aligned} \tag{3.4}$$

2) Since the Taylor expansions of the functions involved plays a crucial role in the parametrization method, we explicitly compute the Taylor expansion of the vector field in (3.4). This is done in Sections 3.2 and 3.3.

3) Finally, in Section 3.4, by using the algorithm presented in Section 2.4, we compute the formal expansion of the parabolic manifold up to order four.

3.1 The parabolic invariant manifold

As we said at the beginning of this chapter, system (3.1) does not fit the setting in Theorem 2.4.1. However, there is a change of coordinates, preserving the parabolic character of a given periodic orbit ($q = p = 0, \omega = \omega_0$) in the parabolic infinity I_∞, such that in these new coordinates, the new systems satisfies the hypotheses of Theorem 2.4.1.

Indeed, it is clear that, in a neighbourhood of $q = 0$,

$$\frac{1}{1 - \frac{1}{4}q^4 w} = 1 + O(q^4)$$

and from expressions (3.2) of σ_1, σ_2,

$$\sigma_1 = O(q^2), \qquad \sigma_2 = O(1).$$

Then, system (3.4) is

$$\begin{aligned} \frac{dq}{d\theta} &= -\frac{1}{4}q^3 p + O(q^7), \\ \frac{dp}{d\theta} &= -\frac{1}{4}q^4 + O(q^6), \\ \frac{d\omega}{d\theta} &= O(q^6), \end{aligned}$$

which, as we commented, does not satisfy the hypotheses of the Theorem 2.4.1.

We introduce the change of variables

$$\begin{aligned} q &= \frac{\alpha + \beta}{2}, \\ p &= \frac{\alpha - \beta}{2}, \\ \omega &= \omega_0 + \gamma q, \end{aligned} \tag{3.5}$$

with $\omega_0 \in \mathbb{R}$ a the constant associated to the orbit of $I_\infty = (0, 0, \theta(t), \omega_0)$ and α, β and γ as the new variables. Note that conversely, we have that

$$\begin{aligned} \alpha &= q + p, \\ \beta &= q - p, \\ \gamma &= \frac{\omega - \omega_0}{q}. \end{aligned} \tag{3.6}$$

We present the following proposition.

Proposition 3.1.1. *The vector field generated by the variables α, β and γ is of the form*

$$\begin{aligned} \frac{d\alpha}{d\theta} &= -\frac{1}{32}(\alpha^4 + 3\alpha^3\beta + 3\alpha^2\beta^2 + \alpha\beta^3) + O(|(\alpha, \beta)|^6), \\ \frac{d\beta}{d\theta} &= \frac{1}{32}(\alpha^3\beta + 3\alpha^2\beta^2 + 3\alpha\beta^3 + \beta^4) + O(|(\alpha, \beta)|^6), \\ \frac{d\gamma}{d\theta} &= \frac{1}{32}(\alpha^3 + \alpha^2\beta - \alpha\beta^2 - \beta^3)\gamma + O(|(\alpha, \beta)|^5), \end{aligned} \tag{3.7}$$

and it satisfies the hypotheses of Theorem 2.4.1 with period $T = 2\pi$, $N = 4$, $a = 1/32$ *and* B *as the matrix*

$$B = \begin{pmatrix} \frac{1}{32} & 0 \\ 0 & \frac{1}{32} \end{pmatrix}.$$

Proof. The 2π-periodicity comes directly form the 2π-periodicity of θ in (3.1). Also the value of N, a and B which (3.7) satisfies Theorem 2.4.1's hypothesis.

Performing the change of variables (3.6): comes directly from its form.

$$\begin{aligned}
\frac{d\alpha}{d\theta} &= \frac{dq}{d\theta} + \frac{dp}{d\theta} = -\frac{1}{4}q^3(q+p) + O(q^6) = -\frac{1}{4}\left(\frac{\alpha+\beta}{2}\right)^3 \alpha + O(|(\alpha,\beta)|^6), \\
\frac{d\beta}{d\theta} &= \frac{dq}{d\theta} + \frac{dp}{d\theta} = \frac{1}{4}q^3(q-p) + O(q^6) = \frac{1}{4}\left(\frac{\alpha+\beta}{2}\right)^3 \beta + O(|(\alpha,\beta)|^6), \\
\frac{d\gamma}{d\theta} &= \frac{1}{q}\left(\frac{d\omega}{d\theta} - \gamma\frac{dq}{d\theta}\right) = \frac{1}{4}\gamma q^2 p + O(q^5) = \frac{1}{4}\gamma\left(\frac{\alpha+\beta}{2}\right)^2 \frac{\alpha-\beta}{2} + O(|(\alpha,\beta)|^5).
\end{aligned}$$

Developing the binomials we get the form of the statement. □

Then, we have that the original system (3.1) has an invariant manifold and a parametrization $K^{\leq m}$ near it can be computed by using the parametrization method.

3.2 The Taylor expansion of the vector field

Once we had proven the existence of the real stable manifold, we want to compute an approximate parametrization. To do so we will work with the Taylor expansion respect to q of system (3.1). Namely, (p, ω) are acting as parameters. This is done in the following proposition:

Proposition 3.2.1. *System* (3.1) *can be written as:*

$$\begin{aligned}
\frac{dq}{d\theta} &= \sum_{n\geq 0} q^{4n+3}\left(-\frac{\omega^n p}{4^{n+1}}\right), \\
\frac{dp}{d\theta} &= \sum_{n\geq 0} q^{2n+4}\Psi_n(\theta,\omega), \\
\frac{d\omega}{d\theta} &= \sum_{n\geq 0} q^{2n+4}\Phi_n(\theta,\omega),
\end{aligned} \tag{3.8}$$

where $\Psi_n(\theta,\omega)$ *and* $\Phi_n(\theta,\omega)$ *can be computed recursively, see* (3.18) *and* (3.19).

Proof. Since q is small enough, $\frac{d\theta}{dt} \neq 0$ and hence from (3.1) we have that

$$\begin{aligned}
\frac{dq}{d\theta} &= \frac{dq}{dt}\frac{dt}{d\theta} = -\frac{1}{4}q^3 p \,\frac{1}{1-\frac{1}{4}q^4\omega}, \\
\frac{dp}{d\theta} &= \frac{dp}{dt}\frac{dt}{d\theta} = \left[-\frac{q^4}{4}\sigma_2 + \frac{q^6 w^2}{8} - \frac{q^6}{8}\mu(1-\mu)\sigma_1\cos\theta\right]\frac{1}{1-\frac{1}{4}q^4\omega}, \\
\frac{d\omega}{d\theta} &= \frac{d\omega}{dt}\frac{dt}{d\theta} = -\frac{q^4}{4}\mu(1-\mu)\sigma_1\sin\theta\,\frac{1}{1-\frac{1}{4}q^4\omega},
\end{aligned}$$

which is exactly (3.4).

Then, again since q is small enough and ω is bounded we have that

$$\frac{1}{1-\frac{1}{4}q^4\omega} = \sum_{n\geq 0}\left(\frac{1}{4}q^4\omega\right)^n,$$

and (3.4) is rewritten as:

$$\begin{aligned}
\frac{dq}{d\theta} &= -\frac{1}{4}q^3 p \sum_{n\geq 0}\left(\frac{1}{4}q^4\omega\right)^n, \\
\frac{dp}{d\theta} &= \left[-\frac{q^4}{4}\sigma_2 + \frac{q^6 w^2}{8} - \frac{q^6}{8}\mu(1-\mu)\sigma_1\cos\theta\right]\sum_{n\geq 0}\left(\frac{1}{4}q^4\omega\right)^n, \\
\frac{d\omega}{d\theta} &= -\frac{q^4}{4}\mu(1-\mu)\sigma_1\sin\theta\sum_{n\geq 0}\left(\frac{1}{4}q^4\omega\right)^n.
\end{aligned} \tag{3.9}$$

Now, we compute the series expansion of σ_1 and σ_2 in (3.2). To do so we will compute the series expansion of $1/f_m^3$ defined in (3.3).

Since

$$1 + mq^2\cos(\theta) + \frac{m^2}{4}q^4 = \left(1+\frac{m}{2}q^2e^{-i\theta}\right)\left(1+\frac{m}{2}q^2e^{i\theta}\right)$$

we obtain

$$\frac{1}{f_m^3} = \frac{1}{\left(1+\frac{m}{2}q^2e^{-i\theta}\right)^{\frac{3}{2}}\left(1+\frac{m}{2}q^2e^{i\theta}\right)^{\frac{3}{2}}}. \tag{3.10}$$

We emphasize that, if $z\in\mathbb{C}$, $|z|<1$, then

$$h(z) = \frac{1}{(1+z)^{\frac{3}{2}}} = \sum_{n\geq 0} c_n z^n, \quad c_n = \binom{-\frac{3}{2}}{n},$$

where c_n are the binomial coefficients With the aid of this expansion we can rewrite (3.10) as

$$\begin{aligned}
\frac{1}{f_m^3} &= \left[\sum_{n_1\geq 0} c_{n_1}\left(\frac{m}{2}q^2e^{-i\theta}\right)^{n_1}\right]\left[\sum_{n_2\geq 0} c_{n_2}\left(\frac{m}{2}q^2e^{i\theta}\right)^{n_2}\right] \\
&= \sum_{n_1,n_2\geq 0} c_{n_1}c_{n_2}\left(\frac{m}{2}\right)^{n_1+n_2} q^{2(n_1+n_2)}e^{-i\theta n_1}e^{i\theta n_2} \\
&= \sum_{n\geq 0} q^{2n}\left(\frac{m}{2}\right)^n \sum_{n_1=0}^{n} c_{n_1}c_{n-n_1}e^{-i\theta n_1}e^{i\theta(n-n_1)}.
\end{aligned}$$

We define $\zeta_n(\theta)$ as

$$\zeta_n(\theta) = \sum_{n_1=0}^{n} c_{n_1}c_{n-n_1}e^{-i\theta n_1}e^{i\theta(n-n_1)}, \tag{3.11}$$

that is

$$\zeta_n(\theta) = \begin{cases} 1 & n=0, \\ 2\displaystyle\sum_{k=0}^{\frac{n-1}{2}} c_k c_{n-k}\cos((n-2k)\theta) & n \text{ odd}, \\ c_{n/2}^2 + 2\displaystyle\sum_{k=0}^{\frac{n}{2}-1} c_k c_{n-k}\cos((n-2k)\theta) & \text{otherwise}. \end{cases}$$

Then $1/f_m^3$ will have the series expansion

$$\frac{1}{f_m^3} = \sum_{n\geq 0} q^{2n}\left(\frac{m}{2}\right)^n \zeta_n(\theta). \tag{3.12}$$

By definition (3.2) of σ_1, σ_2 and using expression (3.12) of $1/f_m^3$, one gets

$$\sigma_1 = \sum_{n\geq 0} q^{2n}\left[\left(\frac{\mu}{2}\right)^n - \left(\frac{\mu-1}{2}\right)^n\right]\zeta_n(\theta),$$
$$\sigma_2 = \sum_{n\geq 0} q^{2n}\left[(1-\mu)\left(\frac{\mu}{2}\right)^n + \mu\left(\frac{\mu-1}{2}\right)^n\right]\zeta_n(\theta).$$

To simplify the notation we introduce

$$a_n = \left(\frac{\mu}{2}\right)^n - \left(\frac{\mu-1}{2}\right)^n, \quad b_n = (1-\mu)\left(\frac{\mu}{2}\right)^n + \mu\left(\frac{\mu-1}{2}\right)^n,$$

and we obtain

$$\begin{aligned}\sigma_1 &= \sum_{n\geq 0} q^{2n} a_n \zeta_n(\theta),\\ \sigma_2 &= \sum_{n\geq 0} q^{2n} b_n \zeta_n(\theta),\end{aligned} \tag{3.13}$$

where $\zeta_n(\theta)$ were defined in (3.11). After these preliminaries computations, we prove the equalities of the statement one by one.

- **Equation for q:**

  From expression (3.9) of our system we obtain that

$$\begin{aligned}\frac{dq}{d\theta} &= -\frac{1}{4}q^3 p \sum_{n\geq 0}\left(\frac{1}{4}q^4\omega\right)^n\\ &= -\sum_{n\geq 0} q^{4n+3}\frac{\omega^n p}{4^{n+1}}\\ &= \sum_{n\geq 0} q^{4n+3}\left(-\frac{\omega^n p}{4^{n+1}}\right),\end{aligned}$$

  as we wanted.

- **Equation for p:**

  Again from (3.9) we have

$$\begin{aligned}\frac{dp}{d\theta} &= \left[-\frac{q^4}{4}\sigma_2 + \frac{q^6 w^2}{8} - \frac{q^6}{8}\mu(1-\mu)\sigma_1\cos\theta\right]\sum_{n\geq 0}\left(\frac{1}{4}q^4\omega\right)^n\\ &= -\frac{q^4}{4}\sigma_2\sum_{n\geq 0}\left(\frac{1}{4}q^4\omega\right)^n\\ &+ \frac{q^6 w^2}{8}\sum_{n\geq 0}\left(\frac{1}{4}q^4\omega\right)^n\\ &- \left[\frac{q^6}{8}\mu(1-\mu)\sigma_1\cos\theta\right]\sum_{n\geq 0}\left(\frac{1}{4}q^4\omega\right)^n\end{aligned} \tag{3.14}$$

  We compute the three last lines in (3.14) separately:

  1) It follows straightforwardly that:

$$\frac{q^6 w^2}{8}\sum_{n\geq 0}\left(\frac{1}{4}q^4\omega\right)^n = \sum_{n\geq 0} q^{4n+6}\frac{\omega^{n+2}}{2^{2n+3}}. \tag{3.15}$$

2) From expression (3.13) of σ_2 we have

$$\begin{aligned}\frac{q^4}{4}\sigma_2\sum_{l\geq 0}\left(\frac{1}{4}q^4\omega\right)^l &= \frac{q^4}{4}\left(\sum_{k\geq 0}q^{2k}b_k\zeta_k(\theta)\right)\left(\sum_{l\geq 0}\left(\frac{1}{4}q^4\omega\right)^l\right)\\ &= \sum_{k,\,l\geq 0}q^{2k+4l+4}b_k\zeta_k(\theta)\frac{\omega^l}{4^{l+1}}\\ &= \sum_{n\geq 0}q^{2n+4}\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\frac{b_{n-2m}\zeta_{n-2m}(\theta)}{4^{m+1}}\omega^m.\end{aligned}$$

Defining now $P^2_{n,m}(\theta)$, with notation associated to σ_2, as

$$P^2_{n,m}(\theta) = \frac{b_{n-2m}\zeta_{n-2m}(\theta)}{4^{m+1}},$$

we obtain

$$\frac{q^4}{4}\sigma_2\sum_{l\geq 0}\left(\frac{1}{4}q^4\omega\right)^l = \sum_{n\geq 0}q^{2n+4}\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}P^2_{n,m}(\theta)\omega^m. \tag{3.16}$$

3) Using again (3.13) for σ_1, we have

$$\begin{aligned}&\left[\frac{q^6}{8}\mu(1-\mu)\sigma_1\cos\theta\right]\sum_{n\geq 0}\left(\frac{1}{4}q^4\omega\right)^n\\ &= \frac{q^6}{8}\mu(1-\mu)\left[\frac{e^{i\theta}+e^{-i\theta}}{2}\right]\left(\sum_{k\geq 0}q^{2k}a_k\zeta_k(\theta)\right)\left(\sum_{l\geq 0}\left(\frac{1}{4}q^4\omega\right)^l\right)\\ &= \sum_{n\geq 0}q^{2n+6}\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}(e^{i\theta}+e^{-i\theta})\frac{\mu(1-\mu)}{4^{m+2}}a_{n-2m}\zeta_{n-2m}(\theta)w^m.\end{aligned}$$

As in the previous case we define $P^1_{n,m}(\theta)$ as

$$P^1_{n,m}(\theta) = (e^{i\theta}+e^{-i\theta})\frac{\mu(1-\mu)}{4^{m+2}}a_{n-2m}\zeta_{n-2m}(\theta),$$

obtaining

$$\begin{aligned}&\left[\frac{q^6}{8}\mu(1-\mu)\sigma_1\cos\theta\right]\sum_{n\geq 0}\left(\frac{1}{4}q^4\omega\right)^n\\ &\qquad = \sum_{n\geq 0}q^{2n+6}\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}P^1_{n,m}(\theta)w^m.\end{aligned} \tag{3.17}$$

Then, substituting (3.15), (3.16) and (3.17) into (3.14) we obtain

$$\frac{dp}{d\theta} = \sum_{n\geq 0}q^{2n+4}\Psi_n(\theta,\omega),$$

with

$$\Psi_n(\theta,\omega) = \begin{cases} -\dfrac{1}{4} & n=0,\\[2ex] \dfrac{\omega^{\frac{n+3}{2}}}{2^{n+2}} - \displaystyle\sum_{m=0}^{\frac{n-1}{2}}(P^1_{n-1,m}(\theta)+P^2_{n,m}(\theta))\omega^m & n \text{ odd},\\[2ex] -P^2_{n,\frac{n}{2}}(\theta)w^{\frac{n}{2}} - \displaystyle\sum_{m=0}^{\frac{n}{2}-1}(P^1_{n-1,m}(\theta)+P^2_{n,m}(\theta))\omega^m & \text{otherwise.}\end{cases} \tag{3.18}$$

This conclude the proof for the p component of our system.

- **Equation for ω:**

Recall that $\dot{\omega} = -\frac{q^4}{4}\mu(1-\mu)\sigma_1 \sin\theta$. Therefore, using expression (3.13) of σ_1, we have that

$$\begin{aligned}\dot{\omega} &= -\frac{q^4}{4}\mu(1-\mu)\left[\frac{e^{i\theta}-e^{-i\theta}}{2i}\right]\sum_{n\geq 0} q^{2n} a_n \zeta_n(\theta) \\ &= \sum_{n\geq 0} q^{2n+4}\left(-\frac{\mu(1-\mu)}{4} a_n \zeta_n(\theta)\frac{e^{i\theta}-e^{-i\theta}}{2i}\right).\end{aligned}$$

Defining

$$\Omega_n(\theta) = -\frac{\mu(1-\mu)}{4} a_n \zeta_n(\theta)\frac{e^{i\theta}-e^{-i\theta}}{2i},$$

we obtain

$$\dot{\omega} = \sum_{n\geq 0} q^{2n+4}\Omega_n(\theta).$$

Therefore, using expression (3.9) of $\frac{d\omega}{d\theta}$, we have

$$\begin{aligned}\frac{d\omega}{d\theta} &= \left(\sum_{k\geq 0} q^{2k+4}\Omega_k(\theta)\right)\left(\sum_{l\geq 0} q^{4l}\frac{\omega^l}{4^l}\right) \\ &= \sum_{k,l\geq 0} q^{2k+4l+4}\Omega_k(\theta)\frac{\omega^l}{4^l} \\ &= \sum_{n\geq 0} q^{2n+4}\sum_{m=0}^{\lfloor \frac{n}{2}\rfloor}\Omega_{n-2m}(\theta)\frac{\omega^m}{4^m}.\end{aligned}$$

Taking $\Phi_n(\theta,\omega)$ as

$$\Phi_n(\theta,\omega) = \sum_{m=0}^{\lfloor \frac{n}{2}\rfloor}\Omega_{n-2m}(\theta)\frac{\omega^m}{4^m} \tag{3.19}$$

we have

$$\frac{d\omega}{d\theta} = \sum_{n\geq 0} q^{2n+4}\Phi_n(\theta,\omega),$$

as we wanted in our statement.

□

3.3 The dominant terms of our system

In this section we will compute the coefficients $\Psi_n(\theta,\omega)$ and $\Phi_n(\theta,\omega)$, with $n = 0, 1, 2$, appearing in Proposition 3.2.1.

Proposition 3.3.1. *The coefficients* $\Psi_n(\theta,\omega)$ *and* $\Phi_n(\theta,\omega)$ *introduced in* (3.8) *and defined in* (3.18) *and* (3.19), *satisfy, for* $n=0,1,2$*:*

$$\begin{aligned}
\Psi_0(\theta,\omega) &= -\frac{1}{4},\\
\Psi_1(\theta,\omega) &= \frac{\omega^2}{8},\\
\Psi_2(\theta,\omega) &= -\frac{\omega}{16}+\frac{3}{16}\mu(1-\mu)\cos^2(\theta)-\frac{\mu(1-\mu)}{64}(15\cos(2\theta)+9),\\
\Phi_0(\theta,\omega) &= 0,\\
\Phi_1(\theta,\omega) &= \frac{3}{16}\mu(1-\mu)\sin(2\theta),\\
\Phi_2(\theta,\omega) &= -\frac{\mu(1-\mu)(2\mu-1)}{64}(15\cos(2\theta)+9)\sin(\theta).
\end{aligned}$$

Remark. *All the notation and formulae we will use, along the proof of this result, comes from Proposition 3.2.1. We will use it without mention.*

Proof. First we compute c_n, a_n, b_n.

n	c_n	a_n	b_n
0	1	0	1
1	-3/2	1/2	0
2	15/8	(2μ-1)/4	$\mu(1-\mu)/4$

We compute now $\zeta_n(\theta)$:

$$\begin{aligned}
\zeta_0(\theta) &= 1,\\
\zeta_1(\theta) &= 2c_0c_1\cos(\theta) = -3\cos(\theta),\\
\zeta_2(\theta) &= c_1^2+2c_0c_2\cos(2\theta) = \frac{15}{4}\cos(2\theta)+\frac{9}{4}.
\end{aligned}$$

We start by proving the result for $\Psi_n(\theta,\omega)$, $n=0,1,2$. For that we need to compute $P^1_{n,m}(\theta)$ and $P^2_{n,m}(\theta)$. It is straightforwardly checked that:

$$\begin{aligned}
P^1_{0,0}(\theta) &= 0,\\
P^1_{1,0}(\theta) &= -\frac{3}{16}\mu(1-\mu)\cos^2(\theta),\\
P^1_{2,0}(\theta) &= \frac{\mu(1-\mu)(2\mu-1)}{128}\cos(\theta)(15\cos(2\theta)+9),\\
P^1_{2,1}(\theta) &= 0.\\
P^2_{0,0}(\theta) &= \frac{1}{4},\\
P^2_{1,0}(\theta) &= 0,\\
P^2_{2,0}(\theta) &= \frac{\mu(1-\mu)}{64}(15\cos(2\theta)+9),\\
P^2_{2,1}(\theta) &= \frac{1}{16}.
\end{aligned}$$

Therefore, we obtain $\Psi_n(\theta,\omega)$, $n=0,1,2$ from (3.18):

$$\begin{aligned}
\Psi_0(\theta,\omega) &= -\frac{1}{4},\\
\Psi_1(\theta,\omega) &= \frac{\omega^2}{8} - P^1_{0,0}(\theta) - P^2_{1,0}(\theta) = \frac{\omega^2}{8},\\
\Psi_2(\theta,\omega) &= -P^2_{2,1}(\theta)\omega - P^1_{1,0}(\theta) - P^2_{2,0}(\theta)\\
&= -\frac{\omega}{16} + \frac{3}{16}\mu(1-\mu)\cos^2(\theta) - \frac{\mu(1-\mu)}{64}(15\cos(2\theta)+9),
\end{aligned}$$

getting the result of the statement.

Now we move on the computation of $\Phi_n(\theta,\omega)$, $n=0,1,2$. As for $\Psi_n(\theta,\omega)$, we compute first $\Omega_n(\theta)$:

$$\begin{aligned}
\Omega_0(\theta) &= 0,\\
\Omega_1(\theta) &= -\frac{\mu(1-\mu)}{4}a_1\zeta_1(\theta)\sin(\theta) = \frac{3}{16}\mu(1-\mu)\sin(2\theta),\\
\Omega_2(\theta) &= -\frac{\mu(1-\mu)}{4}a_2\zeta_2(\theta)\sin(\theta) = -\frac{\mu(1-\mu)(2\mu-1)}{64}(15\cos(2\theta)+9)\sin(\theta).
\end{aligned}$$

And then, we can compute $\Phi_n(\theta,\omega)$:

$$\begin{aligned}
\Phi_0(\theta,\omega) &= 0,\\
\Phi_1(\theta,\omega) &= \frac{3}{16}\mu(1-\mu)\sin(2\theta),\\
\Phi_2(\theta,\omega) &= -\frac{\mu(1-\mu)(2\mu-1)}{64}(15\cos(2\theta)+9)\sin(\theta).
\end{aligned}$$

□

3.4 Computation of the approximate parametrization

As we mentioned before, the system (3.8), or equivalently (3.1), is not on the form that we need to resolve it. For this reason we will use the change of variables (3.6) but only for the variables α and β. We can work in this way since we had already proved the existence of a true stable invariant manifold.

The new differential system is presented on the following proposition

Proposition 3.4.1. *The system* (3.8) *can be rewritten as*

$$\begin{aligned}
\frac{d\alpha}{d\theta} &= \sum_{n\geq 0}\left(\frac{\alpha+\beta}{2}\right)^{4n+3}\left(-\frac{\omega^n\left(\frac{\alpha-\beta}{2}\right)}{4^{n+1}}\right) + \sum_{n\geq 0}\left(\frac{\alpha+\beta}{2}\right)^{2n+4}\Psi_n(\theta,\omega),\\
\frac{d\beta}{d\theta} &= \sum_{n\geq 0}\left(\frac{\alpha+\beta}{2}\right)^{4n+3}\left(-\frac{\omega^n\left(\frac{\alpha-\beta}{2}\right)}{4^{n+1}}\right) - \sum_{n\geq 0}\left(\frac{\alpha+\beta}{2}\right)^{2n+4}\Psi_n(\theta,\omega),\\
\frac{d\omega}{d\theta} &= \sum_{n\geq 1}\left(\frac{\alpha+\beta}{2}\right)^{2n+4}\Phi_n(\theta,\omega).
\end{aligned} \tag{3.20}$$

Where

$$\Psi_n(\theta,\omega) = \begin{cases} -\dfrac{1}{4} & n=0,\\[2ex] \dfrac{\omega^{\frac{n+3}{2}}}{2^{n+2}} - \displaystyle\sum_{m=0}^{\frac{n-1}{2}} (P^1_{n-1,m}(\theta) + P^2_{n,m}(\theta))\omega^m & n \ \textit{odd},\\[2ex] -P^2_{n,\frac{n}{2}}(\theta) w^{\frac{n}{2}} - \displaystyle\sum_{m=0}^{\frac{n}{2}-1} (P^1_{n-1,m}(\theta) + P^2_{n,m}(\theta))\omega^m & \textit{otherwise}. \end{cases}$$

$$\Phi_n(\theta,\omega) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \Omega_{n-2m}(\theta)\frac{\omega^m}{4^m}$$

$$P^1_{n,m}(\theta) = (e^{i\theta} + e^{-i\theta})\frac{\mu(1-\mu)}{4^{m+2}} a_{n-2m}\zeta_{n-2m}(\theta),$$

$$P^2_{n,m}(\theta) = \frac{b_{n-2m}\zeta_{n-2m}(\theta)}{4^{m+1}},$$

$$\zeta_n(\theta) = \begin{cases} 1 & n=0,\\[2ex] 2\displaystyle\sum_{k=0}^{\frac{n-1}{2}} c_k c_{n-k}\cos((n-2k)\theta) & n \ \textit{odd},\\[2ex] c^2_{n/2} + 2\displaystyle\sum_{k=0}^{\frac{n}{2}-1} c_k c_{n-k}\cos((n-2k)\theta) & \textit{otherwise}. \end{cases}$$

$$\Omega_n(\theta) = -\frac{\mu(1-\mu)}{4} a_n\zeta_n(\theta)\sin(\theta),$$

$$a_n = \left(\frac{\mu}{2}\right)^n - \left(\frac{\mu-1}{2}\right)^n$$

$$b_n = (1-\mu)\left(\frac{\mu}{2}\right)^n + \mu\left(\frac{\mu-1}{2}\right)^n,$$

Proof. It comes directly from Proposition 3.2.1 and the change of variable (3.6) for α and β. □

Now we will compute, using the differential system (3.20), the parametrization $K^{\leq 4}(u,\theta)$ close to the real manifold and its dynamic $Y(u)$. This parametrization is expected to have an error order of $O(u^8)$. Then, from this approximate parametrization $K^{\leq 4}(u,\theta)$, we are going to return to the original variables.

3.4.1 Computation of the first coefficients

In this section we will compute the parametrization up to a order 4 for the system (3.1) by using (3.20). We have chosen the fourth order because it is there when the internal dynamic $Y(u)$ is computed exactly.

We remark that in this particular case the periodic variable of K will be θ instead of the usual t, which is the notation used in Chapter 2 and the Appendices.

To compute the approximated parametrization, we first compute this approximation in the variables α, β and ω, that is the system (3.20). Then we will return to the original variables corresponding to system (3.1).

The computations are resumed in the following theorem:

Theorem 3.4.1. *The parametrization $K(u,\theta)$ of the stable invariant manifold and its dynamic $Y(u)$ for the differential system* (3.20) *can be of the form*

$$K(u,\theta) = \begin{pmatrix} u + \dfrac{\omega_0^2}{2^6}u^3 + 2\Lambda u^4 \\ \dfrac{\omega_0^2}{2^6}u^3 \\ \omega_0 - \dfrac{3}{2^{11}}\mu(1-\mu)[\cos(2\theta)-1]u^6 \end{pmatrix} + O(u^5),$$
$$Y(u) = -\frac{1}{32}u^4,$$

with ω_0, $\Lambda \in \mathbb{R}$.

Remark. *We said that* K *and* Y *can be of that form rather, as we have already commented in Chapter 2, the are other possible choices.*

Proof. To start with the proof we will remind the formulae involved in this resolution. Later we will compute one by one the terms $K^m(u,\theta)$, $Y^{m+N-1}(u)$, with $m \geq 1$, satisfying

$$K^{\leq m}(u,\theta) = \sum_{j=1}^{m} K^j(u,\theta), \qquad Y^{\leq m+N-1}(u) = \sum_{j=N}^{m+N-1} Y^j(u). \tag{3.21}$$

The functions $K^j(u,\theta)$ and $Y^j(u)$ are asked to be of the form

$$K^j(u,\theta) = K_j u^j + \hat{K}_j(\theta)u^{j+N-1},$$
$$Y^j(u) = Y_j u^j,$$

where $K_j, \in \mathbb{R}^{1+n}$, $Y_j \in \mathbb{R}$ are constants and $\hat{K}_j(\theta) : \mathbb{R} \to \mathbb{R}^{1+n}$ is a T-periodic function with zero mean.

The algorithm and notation we will used are the one defined in Section 2.4, see also Appendix A. The formulas that we will use in this proof are (2.41), (2.45), (2.47), (2.48), (2.50), which are respectively

$$E^m(u,\theta) := Z(K^{\leq m}(u,\theta),\theta) - \partial_u K^{\leq m}(u,\theta)Y^{\leq m+N-1}(u) - \partial_\theta K^{\leq m}(u,\theta). \tag{3.22}$$

$$\begin{pmatrix} -aN & c^T \\ 0 & B \end{pmatrix} K_{m+1} + a(m+1)K_{m+1} - \begin{pmatrix} Y_{m+N} \\ 0 \end{pmatrix} = \left[-\overline{E^m}(u)\right]_{m+N}, \tag{3.23}$$

$$(B + a(m+1)\mathrm{Id})K^2_{m+1} = \left[-\overline{E^m}(u)\right]^2_{m+N}, \tag{3.24}$$

$$(m+1-N)aK^1_{m+1} - Y_{m+N} = \left[-\overline{E^m}(u)\right]^1_{m+N} - c^T K^2_{m+1}, \tag{3.25}$$

$$\hat{K}_{m+1}(\theta) = \int_0^\theta \left[E^m(u,s) - \overline{E^m}(u)\right]_{m+N} ds. \tag{3.26}$$

We also remember that for this case we have $a = 1/32$, $N = 4$ and the matrix, which we call it A, of (3.23) is

$$A = \begin{pmatrix} -aN & c^T \\ 0 & B \end{pmatrix} = \begin{pmatrix} -\frac{1}{8} & -\frac{3}{32} & 0 \\ 0 & \frac{1}{32} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can check that A is of this form by using the first two equations of (3.7) and the fact that the sum of the differential equation for ω starts with degree six.

Remark. *The application of this case is slightly different from the algorithm presented in the Appendices:*

First, the fixed point of the system (3.20) *is* $(0, 0, \omega_0)$, *for any* $\omega_0 \in \mathbb{R}$, *instead of the origin.*

This will not be a major problem, we only have to take that in consideration when we construct $K^{\leq 1}(u, \theta)$, *since the parametrization must contain that fixed point.*

Second, the matrix B *has a positive eigenvalue, which is* $1/32$, *and a null one. That does not satisfy the initial hypothesis, where it was demanded to have both eigenvalues with positives values.*

We can see that it will not be a problem when solving (3.24). *This is due to the fact that we are adding to* B *a diagonal matrix, which is* $a(m+1)Id$. *Then, all the eigenvalues of the matrix of* (3.24) *are positive. So the matrix* $B + a(m+1)Id$ *will be always invertible. This is enough to solve the cohomological equations.*

We compute now $K^{\leq m}(u, \theta)$ and $Y^{\leq m+N-1}(u)$ iteratively, starting with $K^1(u, \theta)$ and $Y^1(u)$:

1. $K^1(u, \theta)$, $Y^4(u)$

   This particular case has been already determined in Section 2.4. Since it has to contain the fixed point and to be tangent to the first variable axis we have

$$K^1(u, \theta) = \begin{pmatrix} u \\ 0 \\ \omega_0 \end{pmatrix}.$$

   Then, if we want to obtain an error of $O(u^5)$, we have to choose $Y^4(u)$ of the form

$$Y^4(u) = -\frac{1}{32} u^4.$$

   Using (3.22), we obtain the error associated:

$$E^1(u, \theta) = \begin{pmatrix} \displaystyle\sum_{n \geq 1} \left(\frac{u}{2}\right)^{4n+3} \left(-\frac{\omega_0^n \left(\frac{u}{2}\right)}{4^{n+1}} \right) + \sum_{n \geq 1} \left(\frac{u}{2}\right)^{2n+4} \Psi_n(\theta, \omega_0) \\ \displaystyle\sum_{n \geq 1} \left(\frac{u}{2}\right)^{4n+3} \left(-\frac{\omega_0^n \left(\frac{u}{2}\right)}{4^{n+1}} \right) - \sum_{n \geq 1} \left(\frac{u}{2}\right)^{2n+4} \Psi_n(\theta, \omega_0) \\ \displaystyle\sum_{n \geq 1} \left(\frac{\alpha + \beta}{2}\right)^{2n+4} \Phi_n(\theta, \omega_0) \end{pmatrix}.$$

Now we move to the iterative computation of $K^{m+1}(u, \theta)$ and Y^{m+N}, considering the coefficients of order $\nu := m + N$:

2. $K^2(u, \theta)$, $Y^5(u)$

   Here we use formulas (3.22)–(3.26) with $m = 1$ and $\nu = 5$.

   We have that $E^1(u, \theta)$ does not have terms of fifth order, so we get

$$\left[E^1(u, \theta)\right]_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \left[\overline{E^1}(u)\right]_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.27}$$

We start by solving equation (3.24):

$$\begin{pmatrix} \frac{1}{32}+\frac{2}{32} & 0 \\ 0 & \frac{2}{32} \end{pmatrix} K_2^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

obtaining

$$K_2^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Now we can solve (3.25):

$$-2\frac{1}{32}K_2^1 - Y_5 = 0,$$

and we take

$$K_2^1 = 0, \quad Y_5 = 0.$$

From (3.27) and (3.26) we have directly

$$\hat{K}_2(\theta) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then we finally obtain

$$K^2(u,\theta) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad Y^5(u) = 0.$$

Therefore the error is

$$E^2(u,\theta) = E^1(u,\theta).$$

3. $K^3(u,\theta)$, $Y^6(u)$

We proceed as the previous case with $m = 2$ and $\nu = 6$.

First of all we check that

$$\left[E^2(u,\theta)\right]_6 = \begin{pmatrix} \frac{\omega_0^2}{2^9} \\ -\frac{\omega_0^2}{2^9} \\ \frac{3}{2^{10}}\mu(1-\mu)\sin(2\theta)u^6 \end{pmatrix}, \quad \left[\overline{E^2}(u)\right]_6 = \begin{pmatrix} \frac{\omega_0^2}{2^9} \\ -\frac{\omega_0^2}{2^9} \\ 0 \end{pmatrix}.$$

In this case, the equation (3.24) is:

$$\begin{pmatrix} \frac{1}{32}+\frac{3}{32} & 0 \\ 0 & \frac{3}{32} \end{pmatrix} K_3^2 = \begin{pmatrix} \frac{\omega_0^2}{2^9} \\ 0 \end{pmatrix}.$$

Then we have

$$K_3^2 = \begin{pmatrix} \frac{\omega_0^2}{2^6} \\ 0 \end{pmatrix}.$$

Now we are able to solve equation (3.25):

$$-\frac{1}{32}K_3^1 - Y_6 = \left(-\frac{\omega_0^2}{2^9} + \frac{3}{32}\frac{\omega_0^2}{2^6}\right),$$

by taking (for instance)

$$K_3^1 = \frac{\omega_0^2}{2^6}, \quad Y_6 = 0.$$

To obtain $\hat{K}_3(\theta)$ we use the equation (3.26):

$$\hat{K}_3(\theta) = \int_0^\theta [E^2(u,s) - \overline{E^2}(u)]_6 \, ds = \begin{pmatrix} 0 \\ 0 \\ -\frac{3}{2^{11}}\mu(1-\mu)[\cos(2\theta)-1] \end{pmatrix}.$$

Summarizing the above computations, we have obtained

$$K^3(u,\theta) = \begin{pmatrix} \frac{\omega_0^2}{2^6}u^3 \\ \frac{\omega_0^2}{2^6}u^3 \\ -\frac{3}{2^{11}}\mu(1-\mu)[\cos(2\theta)-1]u^6 \end{pmatrix}, \quad Y^6(u) = 0.$$

Consequently, we have that the error is

$$E^3(u,\theta) = \begin{pmatrix} \sum_{n\geq 1}\left(\frac{u}{2}+\frac{\omega_0^2}{2^6}u^3\right)^{4n+3}\left(-\frac{(\omega_0+K_3^3(u,\theta))^n\left(\frac{u}{2}\right)}{4^{n+1}}\right) \\ \sum_{n\geq 1}\left(\frac{u}{2}+\frac{\omega_0^2}{2^6}u^3\right)^{4n+3}\left(-\frac{(\omega_0+K_3^3(u,\theta))^n\left(\frac{u}{2}\right)}{4^{n+1}}\right) \\ \sum_{n\geq 1}\left(\frac{u}{2}+\frac{\omega_0^2}{2^6}u^3\right)^{2n+4}\Phi_n(\theta,\omega_0+K_3^3(u,\theta)) \end{pmatrix}.$$

$$+\begin{pmatrix} \sum_{n\geq 1}\left(\frac{u}{2}+\frac{\omega_0^2}{2^6}u^3\right)^{2n+4}\Psi_n(\theta,\omega_0+K_3^3(u,\theta)) \\ -\sum_{n\geq 1}\left(\frac{u}{2}+\frac{\omega_0^2}{2^6}u^3\right)^{2n+4}\Psi_n(\theta,\omega_0+K_3^3(u,\theta)) \\ -\frac{9}{2^{15}}\mu(1-\mu)[\cos(2\theta)-1]u^9 \end{pmatrix},$$

with $K_3^3(u,\theta) = -\frac{3}{2^{11}}\mu(1-\mu)[\cos(2\theta)-1]u^6$.

4. $K^4(u,\theta)$, $Y^7(u)$

   This case is special since we are taking $m = 3$ and then we have that $m+1 = N = 4$. Then we will have resonance in the equation (3.25) and the resolution of this equation will work in a different way than the previous computations.

   First we check that $E^3(u,\theta)$ has no terms of seventh order, then

$$\left[E^3(u,\theta)\right]_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \left[\overline{E^3}(u)\right]_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.28}$$

   From (3.24) we compute K_4^2:

$$\begin{pmatrix} \frac{1}{32}+\frac{4}{32} & 0 \\ 0 & \frac{4}{32} \end{pmatrix} K_4^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We have then

$$K_4^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Moving now to equation (3.25) we check that

$$Y_7 = 0$$

and K_4^1 is a free term. Let it call $2\Lambda \in \mathbb{R}$.

Finally, we have from (3.28) and (3.26) that

$$\hat{K}_2(\theta) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then we obtain

$$K^4(u,\theta) = \begin{pmatrix} 2\Lambda u^4 \\ 0 \\ 0 \end{pmatrix}, \quad Y^7(u) = 0,$$

and the error term is

$$E^4(u,\theta) = \begin{pmatrix} \sum_{n\geq 1} \left(\frac{u}{2} + \frac{\omega_0^2}{2^6}u^3 + \Lambda u^4\right)^{4n+3} \left(-\frac{(\omega_0 + K_3^3)^n \left(\frac{u}{2} + \Lambda u^4\right)}{4^{n+1}}\right) \\ \sum_{n\geq 1} \left(\frac{u}{2} + \frac{\omega_0^2}{2^6}u^3 + \Lambda u^4\right)^{4n+3} \left(-\frac{(\omega_0 + K_3^3)^n \left(\frac{u}{2} + \Lambda u^4\right)}{4^{n+1}}\right) \\ \sum_{n\geq 1} \left(\frac{u}{2} + \frac{\omega_0^2}{2^6}u^3 + \Lambda u^4\right)^{2n+4} \Phi_n(\theta, \omega_0 + K_3^3) \end{pmatrix} \tag{3.29}$$
$$+ \begin{pmatrix} \sum_{n\geq 1} \left(\frac{u}{2} + \frac{\omega_0^2}{2^6}u^3 + \Lambda u^4\right)^{2n+4} \Psi_n(\theta, \omega_0 + K_3^3) \\ -\sum_{n\geq 1} \left(\frac{u}{2} + \frac{\omega_0^2}{2^6}u^3 + \Lambda u^4\right)^{2n+4} \Psi_n(\theta, \omega_0 + K_3^3) \\ -\frac{9}{2^{15}}\mu(1-\mu)[\cos(2\theta) - 1]u^9 \end{pmatrix}.$$

Finally, we get the form of our statement:

$$K^{\leq 4}(u,\theta) = \sum_{m=1}^{4} K^m(u,\theta) = \begin{pmatrix} u + \frac{\omega_0^2}{2^6}u^3 + 2\Lambda u^4 \\ \frac{\omega_0^2}{2^6}u^3 \\ \omega_0 - \frac{3}{2^{11}}\mu(1-\mu)[\cos(2\theta) - 1]u^6 \end{pmatrix}.$$

We notice that, following the algorithm described in Appendix A, we have

$$Y^m(u) = 0, \qquad m \geq 8.$$

Consequently we already know that the dynamics on the stable manifold is described by

$$\frac{du}{d\theta} = Y(u) = -\frac{1}{32}u^4.$$

That means that, in each new step, the only function that will be actualized iteratively will be the parametrization $K(u,\theta)$. □

Once we have computed the parametrization $K^{\leq 4}(u,\theta)$ of the system (3.20) in variables α, β and ω, we perform the change of variables (3.5) and we obtain that the parametrization $K^{\leq 4}(u,\theta)$ of the stable manifold for the system (3.1) is:

$$K^{\leq 4}(u,\theta) = \begin{pmatrix} \frac{u}{2} + \frac{\omega_0^2}{2^6}u^3 + \Lambda u^4 \\ \frac{u}{2} + \Lambda u^4 \\ \theta \\ \omega_0 - \frac{3}{2^{11}}\mu(1-\mu)[\cos(2\theta) - 1]u^6 \end{pmatrix}, \tag{3.30}$$

with $\Lambda \in \mathbb{R}$ as a free coefficient.

Chapter 4

Numerical Computations

In this chapter we implement two numerical methods to compute an approximation of the parabolic stable manifold of the T-periodic vector field defined in (3.1) and we compare them.

We start by presenting in Section 4.1 the formulae involved with the graph method. Then we will be able to present how the numerical implementations are implemented.

The first method we are going to deal with, in Section 4.2, is the parametrization method, implementing the computations developed in Chapter 3. Then, in Section 4.3, we will compare the results with the classical graph transformation method. For that we will implement the formulae in [MS14].

We choose an error tolerance and we compare both methods in different cases to see which one computes the stable manifold of the vector field (3.1) with better accuracy. The results can be encountered in Section 4.4.

In Appendix A we summarize the algorithm developed in Chapter 2, focusing in the computational aspects. In Appendix B, we present the codes in MATLAB we have written to do this numerical study.

4.1 The Graph Transformation Method

In order to make fair the comparison with the graph method we also set system of [MS14] to have θ as the time variable, like in the parametrization method. That means that we are going to integrate the vector field

$$\begin{aligned}\frac{dq}{d\theta} &= -\frac{1}{4}q^3p\left(-1+\frac{1}{8}q^4C+\frac{1}{4}q^4W\right)^{-1},\\ \frac{dp}{d\theta} &= \left(-\frac{1}{4}q^4\sigma_3+\frac{\mu(1-\mu)}{8}q^6\sigma_2\cos\theta+\frac{q^6}{8}\left(\frac{C}{2}+W\right)^2\right)\left(-1+\frac{1}{8}q^4C+\frac{1}{4}q^4W\right)^{-1},\end{aligned} \tag{4.1}$$

where W corresponds to the change of variables $W=\omega-C/2$, which is equivalent to

$$W(q,p,\theta)=\frac{C^2q^4+16p^2-16q^2\sigma_1}{2\left(8-Cq^4+4\sqrt{-Cq^4-p^2q^4+q^6\sigma_1+4}\right)}, \tag{4.2}$$

with

$$\sigma_1=\frac{1-\mu}{f_\mu}+\frac{\mu}{f_{\mu-1}},\quad \sigma_2=\frac{1}{f_\mu^3}-\frac{1}{f_{\mu-1}^3},\quad \sigma_3=\frac{1-\mu}{f_\mu^3}+\frac{\mu}{f_{\mu-1}^3}, \tag{4.3}$$

$$f_m=\left(1-mq^2\cos\theta+\frac{m^2}{4}q^4\right)^{1/2}. \tag{4.4}$$

The Jacobian constant C is defined by

$$C = q^2 - p^2 + 2\omega - \frac{1}{4}q^4\omega^2 + q^2(\sigma_1 - 1).$$

We recall that there is a direct relation between C and the constant ω_0, corresponding to the set of parabolic orbits

$$I_\infty = (0, 0, \theta_0 + t, \omega_0).$$

The relation satisfied by C and ω_0 is

$$C = 2\omega_0,$$

which means that, for the same initial angle position θ_0, the orbits with a fixed constant ω_0 in the parametrization method correspond to the ones in the graph method with Jacobi constant $C = 2\omega_0$.

Remark. *In te graph method we will work with the variables* (q, p, θ, W)*. Since* W *is only a translation of the variable* ω *of* (3.1) *we claim that the comparison with the parametrization method will be accurate.*

Remark. *The differential equation system in variables* (q, p, θ, ω) *of [MS14] is slightly different than* (3.1)*. This is due the fact that in [MS14] the primaries are setted in different positions than in our work.*

Remark. *The values for the notation* $\sigma_1, \sigma_2, \sigma_3, f_m$ *in* (4.3)*,* (4.4) *are different than the ones for* (3.2)*,* (3.3) *respectively.*

Then, for a given $q_0 > 0$, $\theta_0 \in [0, 2\pi]$, we can parametrize an initial point P_0 of the form

$$P_0 = (q_0, p^{\leq 4}(q_0, \theta_0), \theta_0, W^{\leq 4}(q_0, \theta_0)), \tag{4.5}$$

where

$$p^{\leq 4}(q, \theta) = q - \frac{C^2}{32}q^3, \quad W^{\leq 4}(q, \theta) = 0.$$

When we integrate along system (4.1) for an initial point $P_0 = (q_0, p_0, \theta_0, W_0)$ we will obtain a two-variable point $P_1 = (q_1, p_1)$. Then, we take $\theta_1 = \theta_0 + 2\pi$ and W_1 will be determined by

$$W_1 = W(q_1, p_1, \theta_1),$$

with W defined in (4.2).

In Section 4.4 we will measure the accuracy of parametrization (4.5) respect the integration along the vector field (4.1).

4.2 Numerics on the Parametrization Method

We present the strategy we have chosen to check the accuracy of the approximated parametrization $K^{\leq m}(u, t)$ for any m. After all we apply it when $m = 4$.

Let $K^{\leq m}(u, t)$ and $Y^{\leq m}(u)$ be the polynomials in u defined in the expression (3.21). In order the check the accuracy of this approximation we proceed as follows:

1. Choose a tolerance $T_0 > 0$ (for instance, 10^{-12} or 10^{-14}).

2. Fix t_0 (its value does not matter). Given $u_{t_0} > 0$, compute $x_{t_0} = K^{\leq m}(u_{t_0}, t_0)$. Compute, by integrating numerically the differential equations, the solutions of the initial value problems
$$\begin{cases} \dot{x} = X(x,t), \\ x(t_0) = x_0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{u} = Y(u), \\ u(t_0) = u_{t_0} \end{cases}$$
at time $t = t_0 + T$, denote them by x_{t_0+T} and u_{t_0+T}, respectively. The quantity that measures how invariant is the image of $K^{\leq m}$ is
$$E(u_0) = \|x_{t_0+T} - K^{\leq m}(u_{t_0+T}, t_0 + T)\|.$$

3. Find, plotting the function $E(u_0)$, the largest $u_0 > 0$ for which $E(u_0) < T_0$.

Remark. *In the computation of $K^{\leq m}$, some constants can be chosen arbitrarily. Their value will affect the goodness of the approximation. It is advisable to repeat the procedure above with several choices of these constants.*

4.3 Numerics on the Graph Transformation Method

The invariant manifold can be described as the graph over one variable. Let us write $x = (x_1, \tilde{x})^\top$ and assume that the manifold can be written as
$$\tilde{x} = \phi(x_1, t)$$
in a neighbourhood of $x_1 = 0$ with $x_1 > 0$. Let us assume that a expansion of the function ϕ is known, $\sum_{j\geq 1} \phi_j(t) x_1^j$. We can repeat the procedure of the previous section to check if the graph of a partial sum $\phi^{\leq m}(x_1, t) = \sum_{j=1}^{m} \phi_j(t) x_1^j$ describes a close to invariant object in the following way:

1. Take the tolerance $T_0 > 0$ of the previous section.

2. Fix t_0 (its value does not matter). Given $x_{1,t_0} > 0$, compute $\tilde{x}_{t_0} = \phi^{\leq m}(x_{1,t_0}, t_0)$. Compute, by integrating numerically the differential equations, the solution of the initial value problem
$$\begin{cases} \dot{x} = X(x,t), \\ x(t_0) = (x_{1,t_0}, \tilde{x}_{t_0})^\top \end{cases}$$
at time $t = t_0 + T$, $\hat{x}_{t_0+T} = (x_{1,t_0+T}, \tilde{x}_{t_0+T})^\top$. The quantity that measures how invariant is the image of $\phi^{\leq m}$ is
$$\tilde{E}(x_{1,t_0}) = \|\hat{x}_{t_0+T} - \phi^{\leq m}(x_{1,t_0+T}, t_0 + T)\|.$$

3. Find, plotting the function $\tilde{E}(x_{1,t_0})$, the largest $x_{1,t_0} > 0$ for which $\tilde{E}(x_{1,t_0}) < T_0$. Compare with the result of the previous section.

4.4 Main results and conclusions

We have implemented the algorithmic methods of Sections 4.2 and 4.3 in the respectively programs *Par_Method.m* and *Graph_Method.m* attached in Appendix B. We have used the MATLAB software to program both of them.

Once we have tried several different cases on both programmes, we have arrived at the following conclusions:

1) Both programs obtain the same error results for different values of the initial θ_0. This coincides with our assumptions since the variable θ is 2π-periodic.

2) The program *Par_Method.m* obtains the same result independent of the value of the coefficient Λ of (3.30). That coincides with the expected result, since the error $E^4(u, \theta)$ defined in (3.29) does not depend on Λ. Even if in the expression (3.29) it appears explicitly Λ, it goes cancelled.

We remark that in higher order parametrizations it is possible that the error could depend on Λ.

3 The estimation of the error varies depending on the initial value ω_0 for *Par_Method.m* or $C = 2\omega_0$ for *Graph_Method.m*.

According to that conclusions, we fix $t_0 = 0$, $\theta_0 = 0$ and $\Lambda = 0$ in either both programs. Then we study different cases depending on $\omega_0 = C/2$. A deeper study show that the graphics present no difference between values of ω_0 with opposite sign. For this reason we show only cases of non-negative ω_0.

In the plots we choose the distance of the correspondent parametrizations to the origin as the x-variable, and the error as the y-variable. Also the plots have a red horizontal line corresponding to the tolerance, equal to 10^{-12}.

We have the following plots:

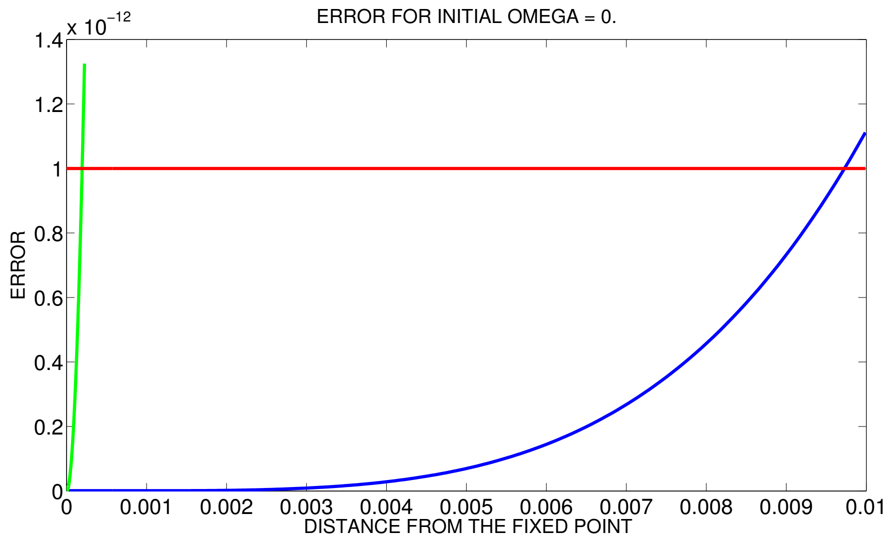

Figure 4.1: Error for $\omega_0 = 0$. Green line representing graph method and blue line the parametrization method.

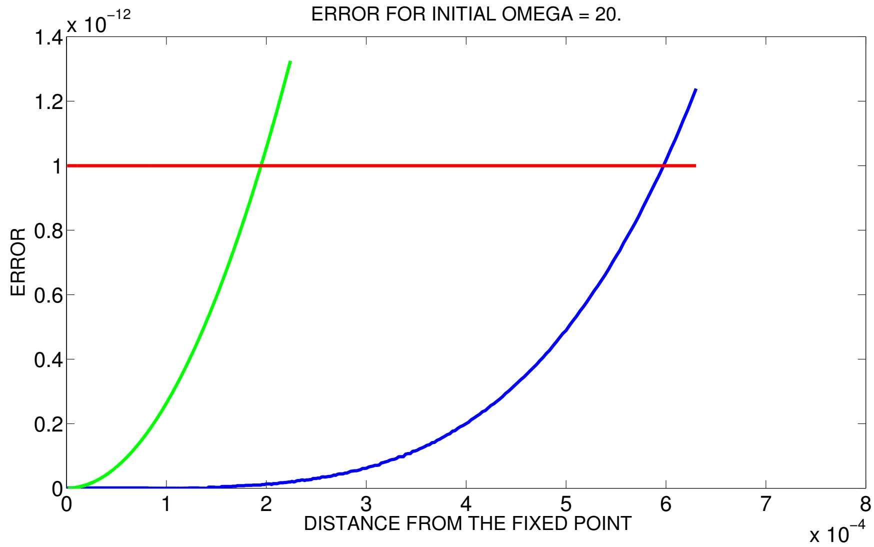

Figure 4.2: Error for $\omega_0 = 20$. Green line representing graph method and blue line the parametrization method.

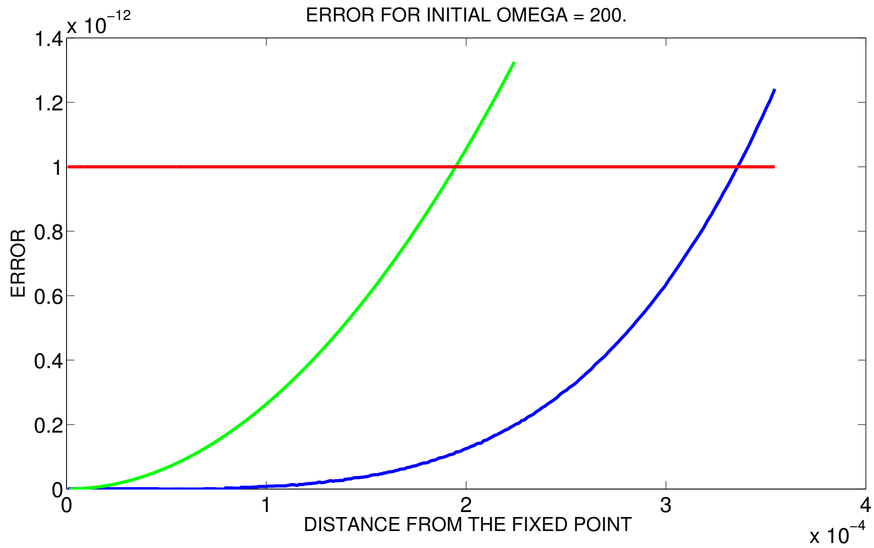

Figure 4.3: Error for $\omega_0 = 200$. Green line representing graph method and blue line the parametrization method.

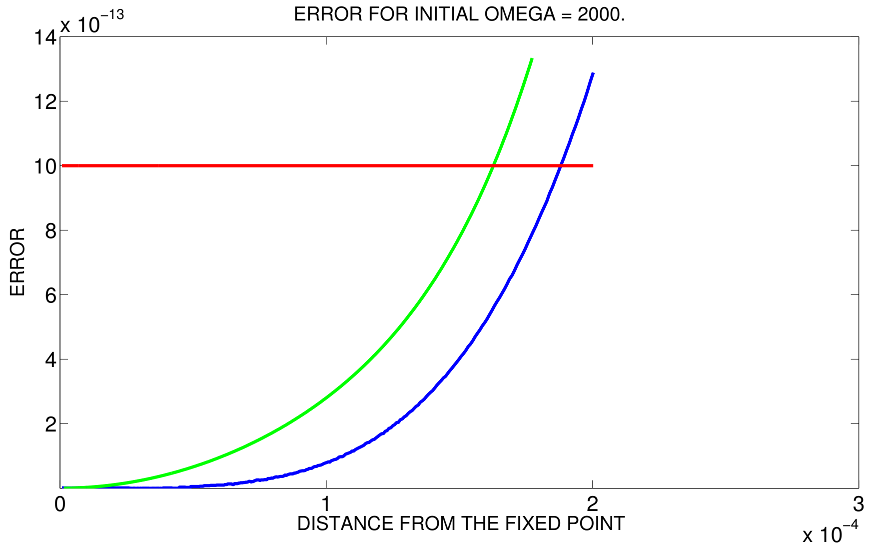

Figure 4.4: Error for $\omega_0 = 2000$. Green line representing graph method and blue line the parametrization method.

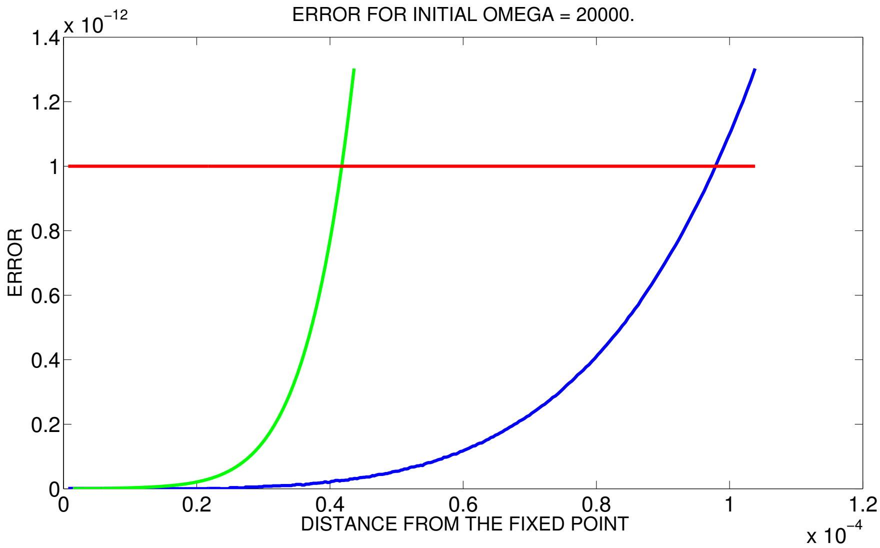

Figure 4.5: Error for $\omega_0 = 20000$. Green line representing graph method and blue line the parametrization method.

We can observe the following results:

1) The error using the parametrization method for degree four is more efficient that the graph method with the same approximation degree.

2) Once we increment the absolute value of ω_0, the distance to the fixed point which arrives to the tolerance limit becomes smaller.

3) In many of the results, the proportion between the distance to the fixed point in the parametrization method and the graph method is around two.

This fact is quite important due to the internal dynamic of the parameter u:

$$\dot{u} = Y(u) = -\frac{1}{32}u^4,$$

and the following problem:

We start in an initial small point q_0, which corresponds to be close to infinity. Then we want to estimate the required time to this point to achieve values of the distance of order $O(1)$, corresponding to closeness to the primaries.

By simply integration we have that this estimated time will be of order $O(1/q_0^3)$. This estimation means that, if we want to duplicate the distance we are interested in, we will to integrate eight times respect the original one.

So the parametrization method provides also more efficiency in the time of computation.

From the results commented before we obtain the final conclusions:

1) When the Jacobi constant $C = 2\omega_0$ is big, in absolute value, it is recommended to work with higher degree of parametrizations.

2) Under the tolerance settled, the parametrization method works more efficiently than the graph method.

Appendix A

Parametrization Method's algorithm

We start remembering all the notation and definitions that will be used along the computations.

- $Z(x, y, t) : \mathbb{R}^{1+n} \times \mathbb{T}$ is a T-periodic vector field respect t with $x \in \mathbb{R}$, $y \in \mathbb{R}^n$ and $\mathbb{T} = \mathbb{R}/[0, T]$.

  We also assume that the vector field Z has a fixed point on the origin and is of the form
$$Z(x, y, t) = \begin{pmatrix} -ax^N \quad + p_N(x, y) + f(x, y, t) \\ x^{N-1}By + q_N(x, y) + g(x, y, t) \end{pmatrix},$$
  where $a > 0$ and the $n \times n$ matrix B has eigenvalues with positive real part.

  The functions p_N, q_N are, respectively, one-dimensional and n-dimensional sums of homogeneous polynomials of order N such that
$$p_N(x, 0) = q_N(x, 0) = 0, \quad \partial_y q_N(x, 0) = 0,$$
  and f, g are $\mathcal{C}^r$ functions, with $r \geq 2$, of order $O(\|(x, y)\|^{N+1})$.

- $K(u, t) : \mathbb{R} \times \mathbb{T} \to \mathbb{R}^{1+n}$ and $Y(u) : \mathbb{R} \to \mathbb{R}$ are respectively the parametrization and the internal dynamic of $Z(x, y, t)$.

- The invariance equation we ask Z to satisfy is
$$Z(K(u, t), t) = \partial_u K(u, t) Y(u) + \partial_t K(u, t).$$

- $K^{\leq m}(u, t) : \mathbb{R} \times \mathbb{T} \to \mathbb{R}^{1+n}$ and $Y^{\leq m+N-1}(u) : \mathbb{R} \to \mathbb{R}$, with $m \geq 1$, are the approximation of $K(u, t)$ and $Y(u)$ respectively.

  We ask them to be of the form
$$K^{\leq m}(u, t) = \sum_{j=1}^{m} K^j(u, t), \qquad Y^{\leq m+N-1}(u) = \sum_{j=N}^{m+N-1} Y^j(u),$$
  satisfying the invariance condition up to order $m + N$, namely
$$Z(K^{\leq m}(u, t), t) - D_u K^{\leq m}(u, t) Y^{\leq m+N-1}(u) + \partial_t K^{\leq m}(u) = O(u^{m+N}).$$
  In addition, K^j and Y^j can be taken of the form
$$K^j(u, t) = K_j u^j + \hat{K}_j(t) u^{j+N-1},$$
$$Y^j(u) = Y_j u^j,$$
  where $K_j, \in \mathbb{R}^{1+n}$, $Y_j \in \mathbb{R}$ are constants and $\hat{K}_j(t) : \mathbb{R} \to \mathbb{R}^{1+n}$ is a T-periodic $\mathcal{C}^r$ function with zero mean i.e.
$$\int_0^T \hat{K}_j(s) ds = 0.$$

- $K_m^1 \in \mathbb{R}$ and $K_m^2 \in \mathbb{R}^n$ are respectively the first and n last components of K_m.
- A is the matrix defined as

$$DZ(K^{\leq m})(u,t) = Au^{N-1} + O(u^N),$$

for all $m \geq 1$. Also it is of the form

$$A = \begin{pmatrix} -aN & c^T \\ 0 & B \end{pmatrix},$$

where $a \in \mathbb{R}$ and $B \in \mathbb{R}^{n\times n}$ have been already defined and $c \in \mathbb{R}^n$.

- The error $E^m(u,t)$ is defined as

$$E^m(u,t) := \\ Z(K^{\leq m}(u,t),t) - \partial_u K^{\leq m}(u,t) Y^{\leq m+N-1}(u) - \partial_t K^{\leq m}(u,t) = O(u^{m+N}).$$

- $\overline{E^m}(u)$ is the average respect to t of $E^m(u,t)$.
- The operator $[\cdot]_j \in \mathbb{R}^{1+n}$ takes the coefficients of exact order j respect to u .
  $[\cdot]_j^1$ and $[\cdot]_j^2$ are respectively the first and n last components of $[\cdot]_j$.
  The operator $[\cdot]_j$ and its components $[\cdot]_j^1$, $[\cdot]_j^2$ are going to be used with E^m and $\overline{E^m}$.

Once we have already set all the elements necessary for developing the algorithm, we proceed to the running. We divide it in two cases:

1. $K^{\leq 1}(u,t)$ and $Y^N(u)$:

As we commented in Section 2.4, $K^{\leq 1}$ and Y^N are of the form

$$K^1(u,t) = \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix}, \quad Y^N(u) = -au^N.$$

Then the error will be

$$E^1(u,t) := \\ Z(K^{\leq 1}(u,t),t) - \partial_u K^{\leq 1}(u,t) Y^{\leq N}(u) = O(u^{N+1}).$$

2. $K^{m+1}(u,t)$ and $Y^{m+N}(u)$, with $m \geq 1$:

In each step we will have already computed $K^m(u,t)$, $Y^{m+N-1}(u)$ and the corresponding $E^m(u,t)$. Using the formulae of Section 2.4 we compute iteratively $K^{m+1}(u,t)$ and $Y^{m+N}(u)$.

We start by obtaining K_{m+1}^2:

$$K_{m+1}^2 = -(B + a(m+1)\mathrm{Id})^{-1} \left[\overline{E^m}\right]_{m+N}^2.$$

Now we move to the computation of K_{m+1}^1, that will depend on the value of $m+1$:

a) If $m+1 \neq N$:

$$K_{m+1}^1 = \frac{1}{(m+1-N)a} \left(\left[-\overline{E^m}\right]_{m+N}^1 - c^T K_{m+1}^2 \right), \quad Y_{m+N} = 0.$$

b) If $m+1=N$:

Then $K^1_{m+1} = K^1_N$ is a free term, which we will call it $\Lambda \in \mathbb{R}$. Also, $Y_{m+N} = Y_{2N-1}$ will be uniquely determined as

$$Y_{m+N} = \left[\overline{E^m}\right]^1_{m+N} + c^T K^2_{m+1}.$$

And then we move to the computation of $\hat{K}_{m+1}(t)$ and $Y_{m+N}(u)$ that will be

$$\hat{K}_{m+1}(t) = \int_0^t \left[E^m(u,s) - \overline{E^m}(u)\right]_{m+N} ds.$$

So finally we will have the following values for $K^{m+1}(u,t)$ and $Y^{m+N}(u)$:

$$K^{m+1}(u,t) = \begin{pmatrix} K^1_{m+1} \\ K^2_{m+1} \end{pmatrix} u^{m+1} + \hat{K}_{m+1}(t) u^{m+N}, \quad Y^{m+N}(u) = Y_{m+N} u^{m+N}.$$

And the error $E^{m+1}(u,t)$ will be

$$E^{m+1}(u,t) := \\ Z(K^{\leq m+1}(u,t),t) - \partial_u K^{\leq m+1}(u,t) Y^{\leq m+N}(u) - \partial_t K^{\leq m+1}(u,t),$$

of order $O(u^{m+N+1})$.

Appendix B

MATLAB implementation

We present here the codes in MATLAB language for the parametrization method and the graph method.

We start with the code of the parametrization method:

Par_Method.m

```
% This code is the implementation of the Parametrization Method.
% Notation is according to the memory.

clear all
% INITIAL VALUES

    % Initial w_0 constant:
    w0 = 20;          % C = 2*w0
    % Initial value for log10(u):
    ui = -6;
    % Final value for log10(u):
    uf = -3;
    % Number of values of log10(u):
    Nu = 1000;
    % Mu, mass of the first planet:
    mu = 0.001;
    % Value of theta:
    Theta = 2;
    % Free coeficient Lambda
    L = 0;
    % Tolerance
    Tol = 10^(-12);
% VARIABLES

    % Parametrizarion K up to order 4:
    K = @(U,theta) [U/2+((w0^2)/2^6)*U^3 + L*U^4, U/2+ L*U^4, theta,
        w0-(3/(2^11))*mu*(1-mu)*(cos(2*theta)-1)*U^6];

    % Field
    f = @(q,o,m) sqrt(1+m*q^2*cos(o)+0.25*m^2+q^4);
    sigma1 = @(q,o,mu) 1/(f(q,o,mu))^3 - 1/(f(q,o,mu-1))^3;
    sigma2 = @(q,o,mu) (1-mu)/(f(q,o,mu))^3 + mu/(f(q,o,mu-1))^3;
```

```
    % q = x(1), p = x(2), theta = x(3)
    Z= @(t,x) [-0.25*x(1)^3*x(2); 0.125*x(1)^6*x(4)^2 - 0.25*x(1)^4*sigma2(x(1),x(3),mu) - 0.125*x(1)^6*mu*(1-mu)*sigma1(x(1),x(3),mu)*cos(x(3)); 1-0.25*x(1)^4*x(4); -0.25*x(1)^4*mu*(1-mu)*sigma1(x(1),x(3),mu)*sin(x(3))];
    Y = @(t,x) -(1/32)*x^4;

    % Values of u
    ValU = linspace(ui,uf,Nu);

    % Matrix of errors depending on u:
    M = zeros(Nu,2);

% IMPLEMENTATION
for i=1:Nu
    u = 10^(ValU(i));

        % Initial point
        P0 = K(u,Theta);
        [~, U1] = ode45(@(t,x) Z(t,x), [0,2*pi], P0');

        FK = U1(end,:)';
        [~, U2] = ode45(@(t,x) Y(t,x), [0,2*pi], u);
        Yu = U2(end);
        KR = K(Yu,Theta)';
        P = FK - KR; % Error between the integration of the parametrization
        P(3) = P(3) - 2*pi; % and the param. under its internal dynamic.
        NormP = norm(P);
        Dist = P0 - [0,0,Theta,w0]; % Distance between the parametrized point
        NormD = norm(Dist); % and the fixed one
        M(i,1) = NormD;
        M(i,2) = NormP;
        disp(i);
end
figure
plot(M(:,1),M(:,2),M(:,1),Tol*ones(Nu,1),'r');

    title(['GRAPHIC OF THE PARAMETRIZATION ERROR FOR VALUES BETWEEN u = e' num2str(ui) ' TO u = e' num2str(uf) ', WITH INITIAL OMEGA = ' num2str(w0) ':'])
    xlabel('DISTANCE FROM THE FIXED POINT')
    ylabel('ERROR')
```

Moving on the graph method now we have:

Graph_Method.m

```
% This code is the implementation of the Graph Method, from Carles Simo and Regina Martinez.
% Notation is according to the original paper.

clear all
% INITIAL VALUES

    % C jacobi constant:
    C = 40;
    % Initial value for log10(q):
    qi = -6;
    % Final value for log10(q):
    qf = -3.7;
    % Number of q:
    Nq = 1000;
    % Mu, mass of the first planet:
    mu = 0.001;
    % Initial value theta:
    Theta = 2;
    % Tolerance
    Tol = 10^(-12);

% VARIABLES

    % Field
    f = @(q,o,m) sqrt(1-m*q^2*cos(o)+0.25*m^2+q^4);
    sigma1 = @(q,o,mu) (1-mu)/(f(q,o,mu)) + mu/(f(q,o,mu-1));
    sigma2 = @(q,o,mu) 1/(f(q,o,mu))^3 - 1/(f(q,o,mu-1))^3;
    sigma3 = @(q,o,mu) (1-mu)/(f(q,o,mu))^3 + mu/(f(q,o,mu-1))^3;

    % Variables p and W up to coefficient 4:
    p = @(q, theta) q -((C^2)/32)*q^3;  % a1=1, a2=0, a3=-C^2/32, a4=0
    W1 = @(q, theta) 0;                  % w1=0, w2=0, w3=0,      , w4=0
    W = @(q,pe,o) (C^2*q^4 + 16*pe^2 - 16*q^2*sigma1(q,0,mu))/(2*(8-C*q^4+4*sqrt(-C*(q^4)-(pe^2)*(q^4)+(q^6)*sigma1(q,0,mu)+4)));
    % q = x(1), p = x(2), theta = x(3), W(q,p,theta)
    Z= @(t,x) [(-0.25*x(1)^3*x(2))/(-1+0.125*(x(1)^4)*C+0.25*(x(1)^4)*W(x(1),x(2),t)); (-0.25*(x(1)^4)*sigma3(x(1),t,mu)+0.125*mu*(1-mu)*(x(1)^6)*sigma2(x(1),t,mu)*cos(t)+0.125*(x(1)^6)*((C/2)+W(x(1),x(2),t))^2)/(-1+0.125*(x(1)^4)*C+0.25*(x(1)^4)*W(x(1),x(2),t))];

    % Values of u
    ValQ = linspace(qi,qf,Nq);

    % Matrix of errors depending on q:
    M = zeros(Nq,2);

```

```
% IMPLEMENTATION

for i=1:Nq
    disp(i);
    q = 10^(ValQ(i));

    % Initial point
    P0 = [q, p(q,Theta), Theta, W1(q,Theta)]';
    [~, U1] = ode45(@(t,x) Z(t,x), [0,2*pi], [P0(1),P0(2)]');
    P1 = U1(end,:);
    P1 = [P1(1),P1(2), Theta + 2*pi, W(P1(1),P1(2),Theta+ 2*pi)]';
    P2 = [P1(1), p(P1(1),P1(3)),P1(3),W1(P1(1),P1(3))]';
    P = P1-P2;
    NormP = norm(P); % Error of the integration
    O = [0,0,Theta,0]'; %Fourth variable is zero since W = w - C/2 = w - w0
    Dist = P0 - O; % Distance between the fixed point and the initial one
    NormD = norm(Dist);
    M(i,1) = NormD;
    M(i,2) = NormP;
end

figure
plot(M(:,1),M(:,2),M(:,1),Tol*ones(Nq,1),'r');

    title(['GRAPHIC OF THE PARAMETRIZATION ERROR FOR VALUES BETWEEN q = e' num2str(qi) ' TO q = e' num2str(qf) ', WITH INITIAL C = ' num2str(C) ':'])
    xlabel('DISTANCE FROM THE FIXED POINT')
    ylabel('ERROR')
```

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