Regular Polylinks

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Abstract

The term **polylinks** is proposed for structures made of straight rods joined at their ends into rings of polygonal form which are linked through one another to provide rigidity. The structures are **regular** when the polygons are regular and all of the same species and size, and the corners of the structures are all alike in location and surroundings.

Thirty-one such structures have been found and wooden models have been constructed, but there is no proof that this set is complete. Since the rods in the models are of circular cross-section, each model embodies a determinate ratio of diameter to length of its rods. Accordingly the structures can be scaled down or up linearly to provide suggestive designs for jewelry or for constructivist sculpture.

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Résumé

Le terme «Poly-anneaux» est proposé pour les structures formées de tiges droites reliées à leurs extrémités en anneaux de forme polygonale et qui sont liés les uns aux autres pour obtenir un effet de rigidité. Les structures sont régulières lorsque les polygones sont réguliers et tous de mêmes espèces et taille; les coins des structures sont tous semblables quant à leur localisation et leur environnement.

Trente-trois structures de la sorte ont été découvertes et reproduites sur maquettes de bois, mais il n'est pas certain que le jeu soit complet. Puisque les tiges des maquettes sont de coupe circulaire, chaque modèle doit avoir un rapport spécifique quant au diamètre et à la longueur de ses tiges.

Conformément, les structures peuvent être réduites à l'échelle ou augmentées à l'échelle linéairement pour l'obtention de designs se prêtant à la bijouterie ou à la sculpture constructiviste.

The term **polylink** seems suitable for an assembly of plane polygonal rings linked together to form a structure that is rigid by virtue of linkage alone. The plurality denoted by the prefix «poly» applies not only to the angles in each ring, but also to the number of rings in the structure and to the number of links in which each ring participates. By analogy with the regular polyhedra, a polylink is **regular** if its rings are made of identical regular polygons, and its corners are indistinguishable. The fact that realizations of these structures can form attractive pieces of constructivist sculpture is a primary motivation for investigating them.

A method of searching out such structures is shown in **Figure 1.** Place three rings of cardboard on three adjacent faces of a cube so that each links through its

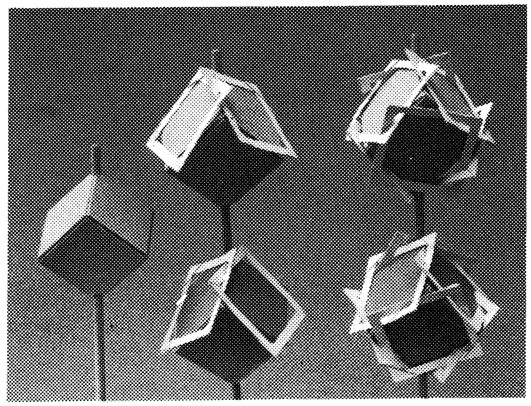

Figure 1. To make a three-dimensional threading diagram for a regular polylink, a regular polyhedron (left) is clothed with three linked cardboard rings (center), and the resulting configuration is used to direct the linkage (right) of additional rings.

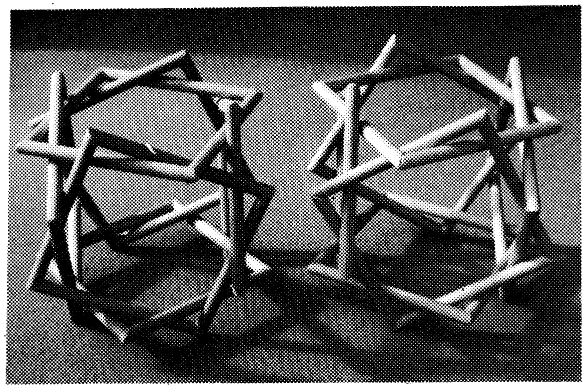

Figure 2. Two enantiomorphous regular polylinks can be constructed of dowels threaded according to the patterns of **Figure 1**.

two companions. In order to accomplish such a linkage, each ring must be turned slightly about the center of the face on which it rests. When the three rings are turned in the same sense, right- or left-handed, they establish an unambiguous pattern for linking three more rings affixed to the three remaining faces.

Further polylinks made of regular triangular and pentagonal rings emerge from the comparable use of the other regular polyhedra. As with the example shown in **Figure 2**, all occur in enantiomorphic pairs. The use of regular polyhedra as bases for construction insures that two requirements for regularity in the resulting polylinks will be met. All rings of a polylink will stand at the same distance from its center and all its corners will be alike. These requirements will be met also by using stellations of the regular polyhedra as bases.

When an enantiomorphic pair is counted as a single polylink, the number of polylinks found by these and similar devices is thirty-one. There is no proof that these form the complete set of regular polylinks. The discovery of more, and the proof of completeness stand as challenges to interested investigators.

Construction of Models

A convenient material for making models of these structures is wooden dowelling one quarter inch in diameter, purchased from lumber yards in three foot lengths. From them pieces of a uniform trial length are sawed in a mitre box at 30° for triangles, at 45° for squares, or at 54° for pentagons. The length of the cut piece is fixed by a stop clamped in the box which has previously been sawed at the angle of the cut, so that the planes of the cuts at the ends of the piece will both be perpendicular to the plane of the polygon under construction.

After trimming the cut ends with sandpaper, the sticks are threaded past one another according to the cardboard diagram. The tentative assembly can be aided by connecting the rod ends by short lengths of cardboard which have been punched near each end by a quarter-inch punch. On complicated models it is helpful to secure the resulting polygonal rings to one another by tying them together at their crossings with rubber bands.

At this point it becomes clear whether the trial length is suitable. If it is too short, the linkages cannot be completed; if it is too long, the structure fits too loosely. When the right length has been found, the pieces can be glued together at their abutting ends into a rigid structure.

General Properties

Evidently a critical number for each structure is the ratio of the length to the diameter of the rods used. Hence the wooden models can be scaled down or up to provide patterns for jewelry or for constructivist sculpture (**Figure 3**). Whereas the structures are technically rigid, their stability as practical objects can be improved by fastening the rods together where they

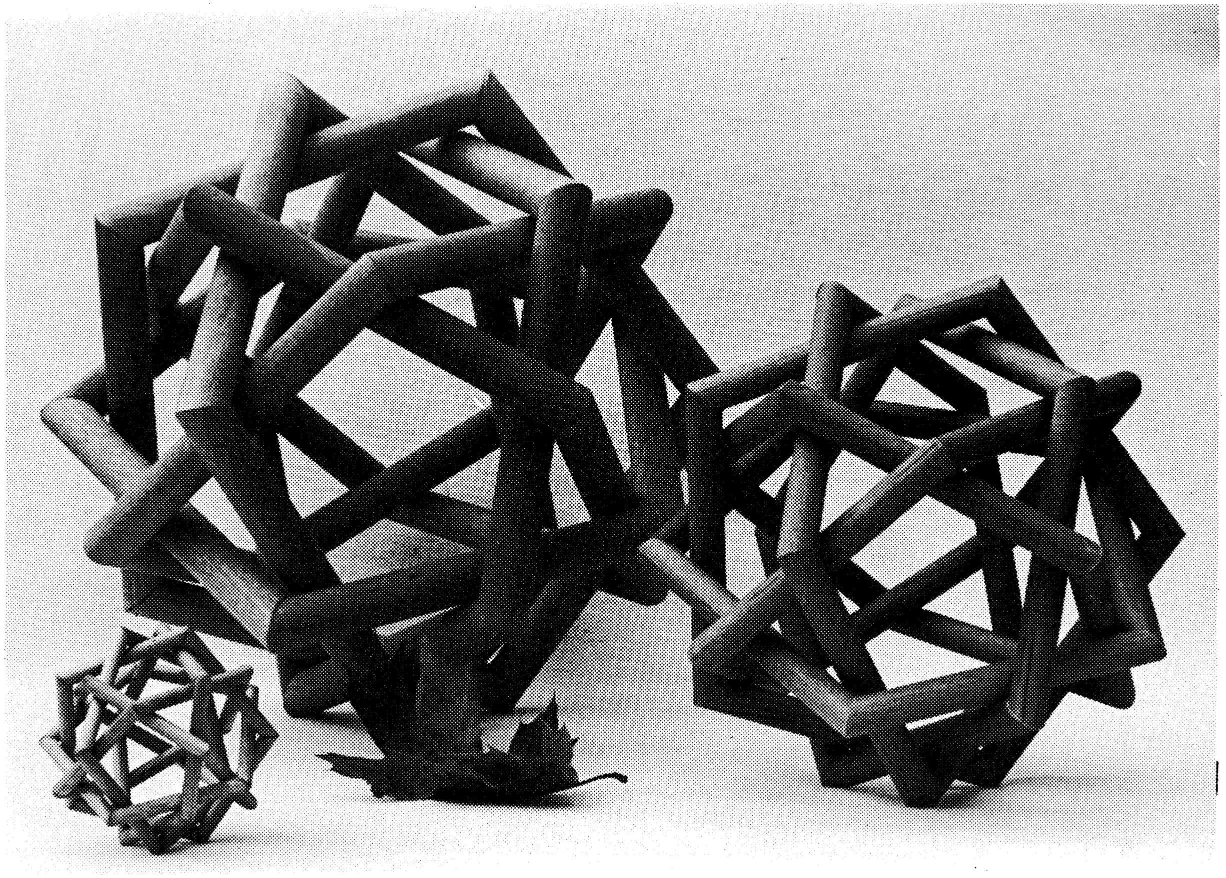

Figure 3. Scaling the size of a polylink up or down affects its appearance dramatically, even though the ratio of length to diameter of the rods stays fixed.

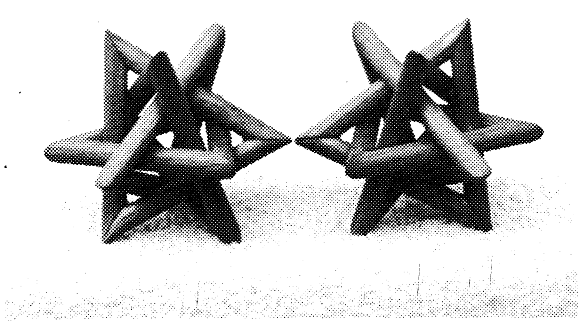

Figure 4. Two small enantiomorphous polylinks in which the ratio of length to diameter of the component rods is nine.

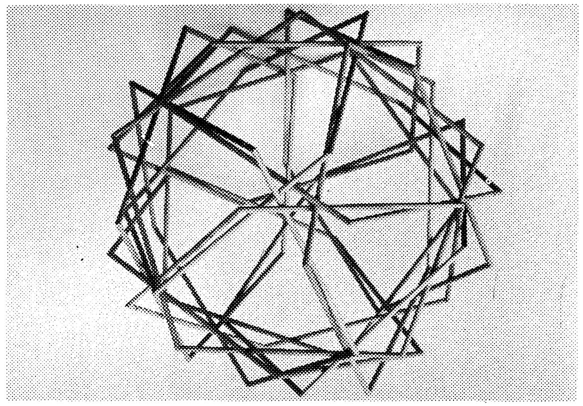

Figure 5. In this regular polylink built of pentagons on a dodecahedral base, the ratio of length to diameter of the rods is forty.

cross, in the case of jewelry by soldering, and in the sculpture by pinning.

Of the regular polylinks discovered, seventeen are made of triangles, two of squares, six of pentagons, and six of star pentagons. When constructed of quarter-inch dowelling, those so far made range in diameter from two and two-thirds inches (**Figure 4**) to nineteen inches (**Figure 5**).

The number of rings in a polylink is usually the same as the number of faces on its parental polyhedron. In three cases that number is cut in half by the fact that the centers of the rings fall at the centers of the polylinks. Thus four triangular rings (**Figure 4**) form a rigid regular assembly, as do six regular pentagonal rings and ten triangular rings. Three more regular polylinks arise by laying an additional ring alongside each ring in those polylinks.

On the regular polyhedra (except the tetrahedron) parental to the regular polylinks, all faces come in parallel pairs, and so therefore do the rings in the polylinks. Since the rings lie in planes perpendicular to the axes of symmetry, the number of links in which each ring participates must obey the symmetry requirements of the axis. Thus each triangular ring must engage in some multiple of three links, each square ring a multiple of four, and each pentagonal ring in a multiple of five.

Icosalinks and Star Pentagons

Since the icosahedron is the regular polyhedron with the largest number (20) of faces, it provides the greatest number of distinguishable polylinks. Twelve have been discovered. In one of them each triangle is linked to three others; in two to six others; in four to nine; in two to twelve; and in three to eighteen. It is noteworthy that none has been found in which each triangle links with fifteen others.

The model shown in **Figure 6** illustrates the fascinating complexity that can arise when twenty regular triangles are regularly linked on the base of a regular icosahedron. Here each triangle links with nine others. The seemingly disorderly pile of linked triangular rings

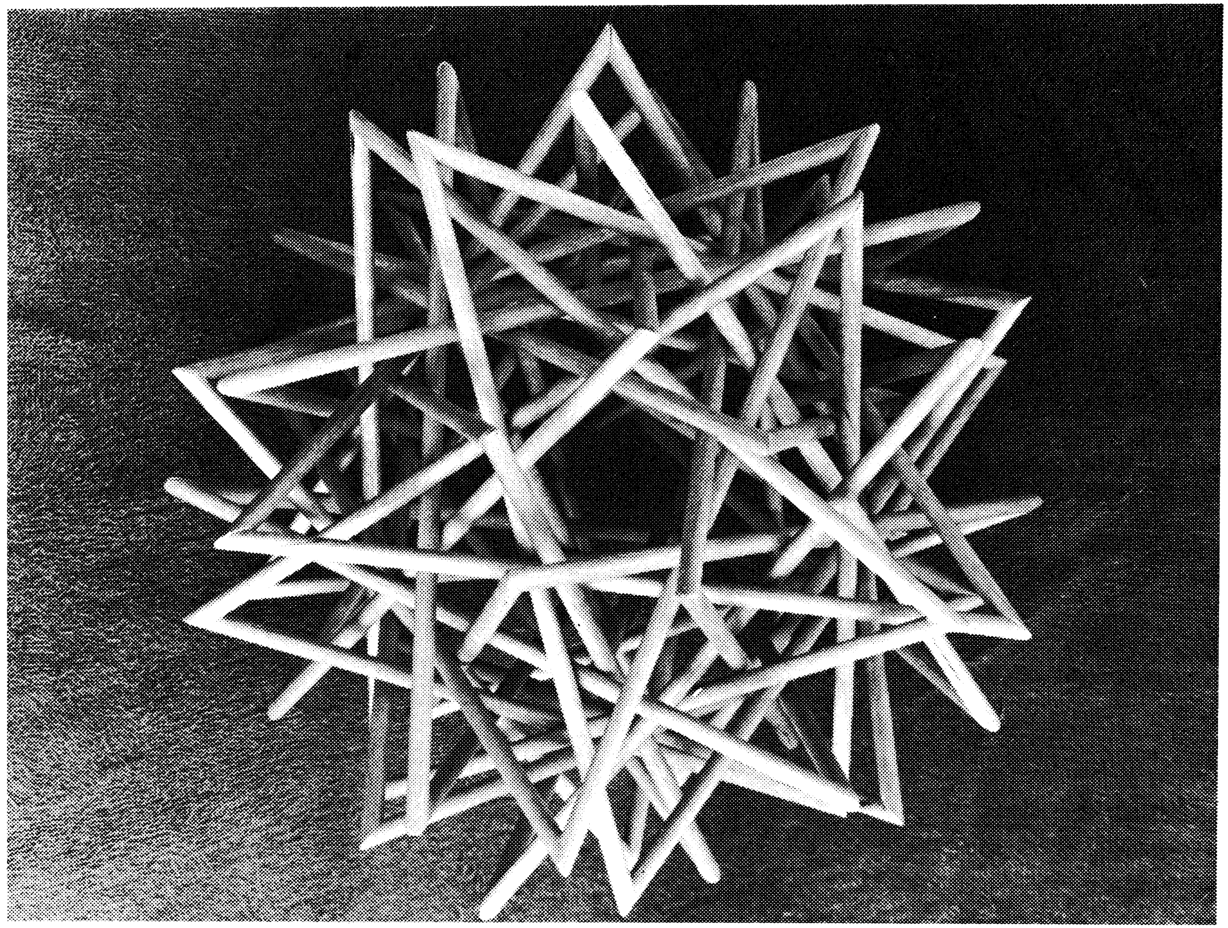

Figure 6. A regular polylink of twenty triangular rings viewed along an axis of five-fold rotational symmetry.

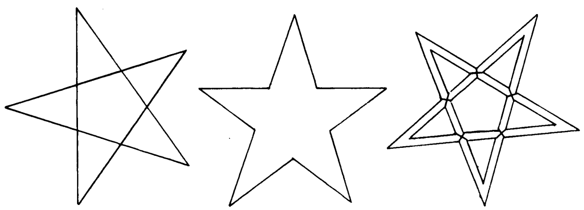

Figure 7. The mathematician's regular star-pentagon has five sides of equal length which cross one another (left). It is often simplified to the shape shown in the center. Preserving in polylinks the purism of the true star-pentagon would reqûire the construction shown at the right, and would entail a choice between two alternative forms of linkage.

acquires orderliness when it is viewed along an axis of rotational symmetry.

The design, construction and enumeration of star-pentagonal polylinks raise a special problem. A regular star-pentagon, sometimes called a «pentagram», has five sides which cross one another, as the left side of **Figure 7** shows. Often the polygon is thought about in the simplified form shown at the center of **Figure 7**. But if the polygonal rings are constructed as shown at the right, they can be linked in two distinguishable ways: through either the triangular regions or through the pentagonal region. For the construction and enumeration reported here, this purism has been ignored and only the simplified form of the star-polygon has been used (**Figure 8**).

Compounds and Archimedians

Like the regular polyhedra, the regular polylinks can be compounded to form more complicated polylinks, and the compounds are most interesting when they have the same symmetry as their ingredients. In the compound polylink shown in **Figure 9**, the three ingredient polylinks conspire to provide a structure having the same symmetry as each ingredient taken separately. Each of the compound's axes of four-fold rotational symmetry is shared with one of the comparable axes in each of its three ingredient polylinks.

Whereas the ingredients of these compounds are regular, the compounds themselves are not. Many other attractive polylinks can be constructed by departing from the requirement of regularity. For example,

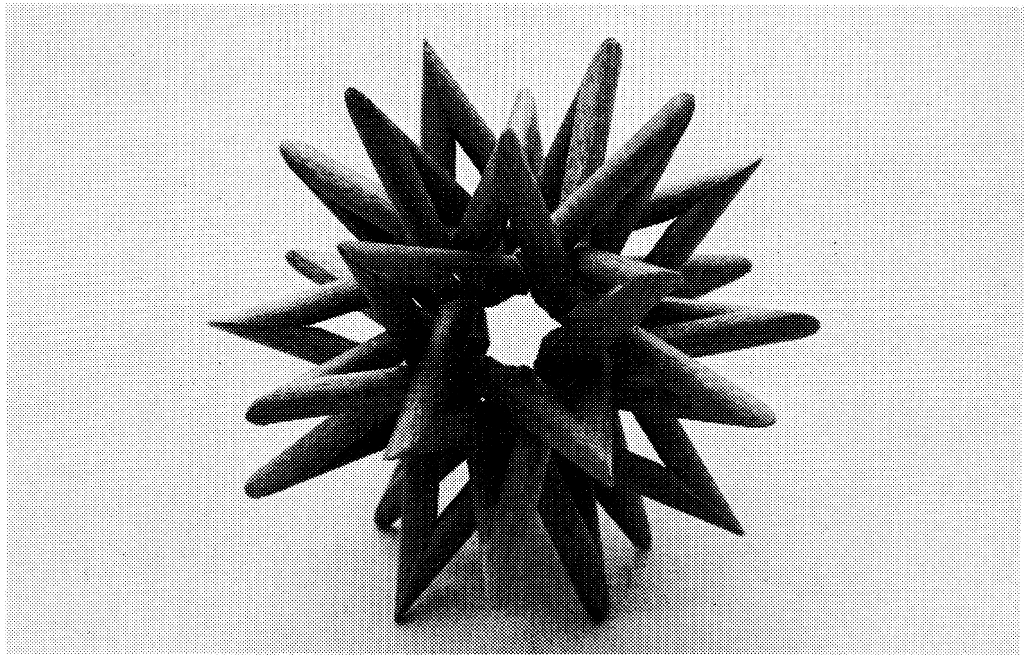

Figure 8. The regular polylink composed of six simplified star-pentagonal rings.

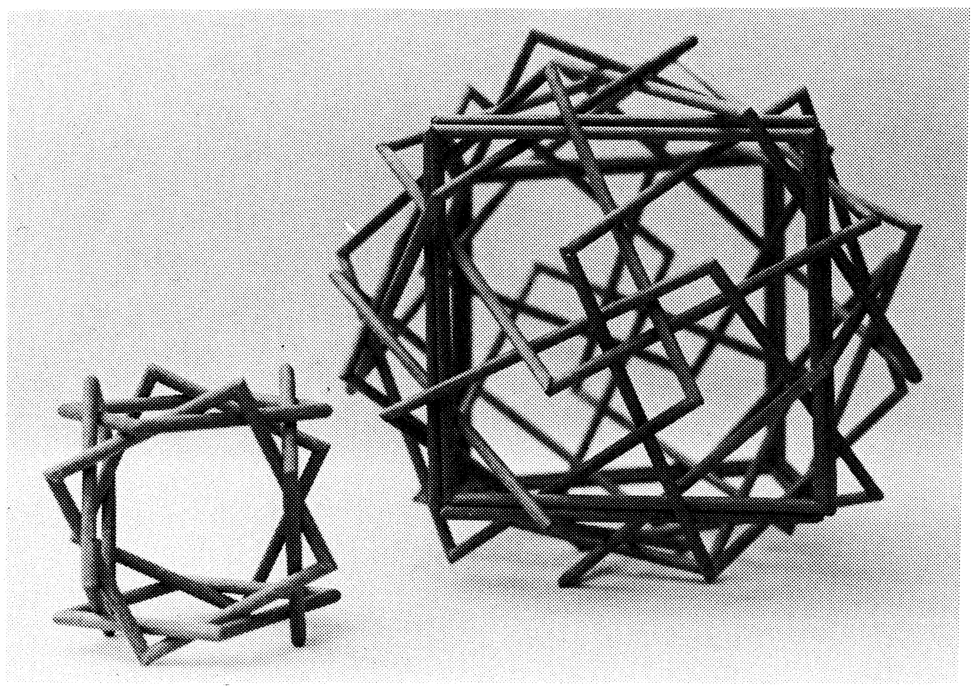

Figure 9. In this compound (right) of three ingredient regular polylinks (left), one of the axes of four-fold rotational symmetry of each ingredient polylink falls along one or another of the three axes of four-fold rotational symmetry of the compound.

the Archimedian semi-regular polyhedra form bases for polylinks whose rings are shaped as regular polygons of more than one species.

The snub polyhedra furnish especially interesting examples. The snub cube comes in two enantiomorphous forms, and each fathers two enantiomorphous polylinks (**Figure 10**). Each of these four polylinks has six square and thirty-two triangular rings. When made of quarter-inch dowelling each has the diameter eleven inches.

Some of the polyhedra dual to the Archimedians can also form parents of interesting polylinks. But these are not closely related to the regular polylinks (**Figure 11**).

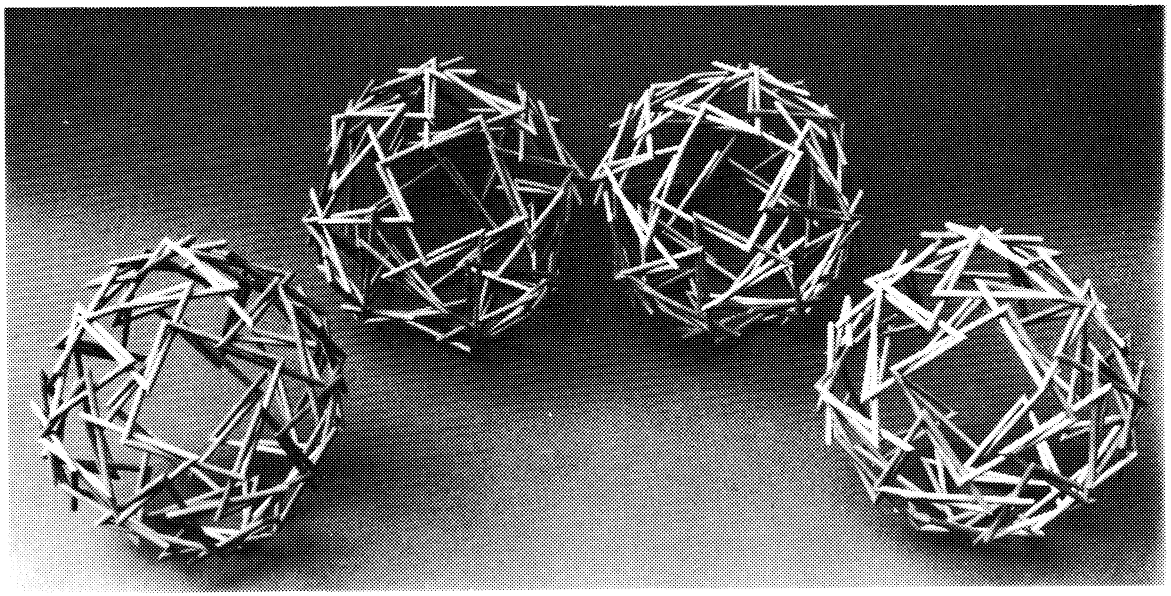

Figure 10. The two enantiomorphous Archimedian snub cubes father four polylinks.

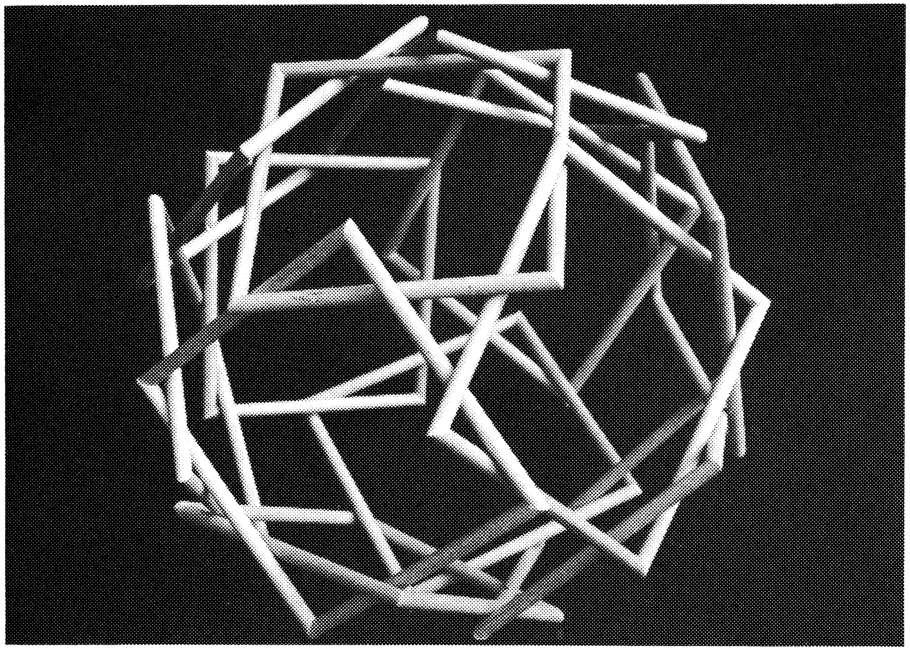

Figure 11. Twelve square rings parallel to the faces of a rhombic dodecahedron, each linking with four others, viewed along an axis of three-fold rotational symmetry.

 .... Photographs by Doug Kendall