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DERIVATION OF SHALLOW WATER MODELS FOR STEADY AND TRANSIENT PHENOMENA, REQUIRED ASSUMPTIONS AND RESULTING EQUATIONS

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Abstract. *The shallow water models describe water flow in rivers, lakes and shallow seas. They are used to study many physical phenomena of interest, such as, modeling environmental effects, commercial activities on fisheries and coastal wildlife, remediation of contaminated bays and estuaries for the purposes of improving water quality. However, without any simplifications are very difficult to solve. These assumptions, with respect to several physical aspects are the general source of uncertainty within the modeling process. In this work, we get the derivation of shallow water models for steady and transient phenomena, required assumptions and resulting equations.*

1 INTRODUCTION

In particular, the problem of the propagation of polluting agents in the coastal zones, as in ports, zones of biological or tourist interest, needs to be studied with the objective to determine which the best actions to be adopted, for the conservation of the natural patrimony. In addition, the population could be preserved of the urban and industrial polluting agents, of the natural disasters or maritime accidents.

The propagation of polluting agents is directly related with the propagation of waves. The results known in this subject had been gotten experimentally or by means of numerical simulations. The cost to make only measured experimental is considerably more raised than the cost to simulate in real time the motion of the waves, being possible to calibrate with few data. This second option provides a cost more cheap. On the other hand, to make this simulation of efficient way in real time is necessary to obtain sophisticated mathematical models that reflect the reality physical of the studied phenomena.

2 MODELS

The effects of non-linearity, such as energy dissipation by friction or breaking, are not taken into account. Here a short description of the mathematical formulations with the phenomena refraction and diffraction.

The mathematical model used in this study, in the case steady state are obtained by assuming hydrostatic pressure distribution, and in the case transient are all based on the theory of simple harmonic linear water waves.

2.1 Basic Equations

The basic equations of the hydrodynamic used to describe the motion in fluid dynamics are, mass balance or continuity equation and momentum equation, respectively:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{b} + \frac{1}{\rho}\nabla \cdot \sigma$$

where ρ is the fluid density, $\mathbf{u}^T = (u, v, w)$ is the velocity, $\mathbf{b}$ is the volume force per unit mass, σ is the stress tensor and ∇ is gradient operator.

2.2 Case Steady

Assumptions: steady state; ideal, incompressibility and homogeneous fluid; action of the pressure and gravity forces; hydrostatic approximation the momentum equation in z. Thus, integrating the equations over the depth and then taking into account the boundary conditions on the free and bottom surfaces, we obtain the two-dimensional shallow water equations, of a compact conservative form can be formulate as:

Determine $(h, \mathbf{U})$, such that:

$$\nabla \cdot (h\mathbf{U}) = 0.$$

$$\nabla \cdot (h\mathbf{U}\mathbf{U}) = -\frac{g}{2}\nabla h^2 - gh\nabla b$$

where $\mathbf{U} = (U, V)$ are the depth averaged velocity components in the x and y directions respectively, related with vertical distribution function of the velocity; g is the gravity acceleration; $h = b + \eta$ is the depth, being b is the function bottom (know) and η the surface elevation (unknow).

These equations are subject to certain conditions that can be very complicated. However, here we will only consider rather simple boundary conditions.

2.2.1 Boundary Conditions

We consider several common types of boundary conditions relevant for flow problems:

- Land boundary: the horizontal velocities are zero (non-slip condition $\mathbf{U} = 0$) and water does not pass through boundary ($\partial h/\partial \mathbf{n} = 0$).
- River boundary: Prescribed velocity ($\mathbf{U} = \hat{\mathbf{U}}$) and depth ($h = \hat{h}$).
- Open sea boundary: water is free to flow in and out (we can specify $h = \hat{h}$ and zero normal derivative of the horizontal velocity components $\partial \mathbf{U}/\partial \mathbf{n} = 0$).
- Radiation boundary: in numerical computations we often shorten the domain artificially, and in such instances we would want to impose radiation-type boundary conditions ($\partial \mathbf{U}/\partial \mathbf{n} = 0$).

2.3 Case Transient

Assumptions: ideal fluid, non-viscous and homogeneous; non-turbulent and non-laminar flow; only action of the pressure (shear stresses negligible) and gravity force.

Thus, supposing that the fluid is ideal and without energy dissipation, is assumed that the flow is irrotational and the velocity derives of a potential function.

Therefore, taking into account the boundary conditions on coordinate z, called kinematics boundary conditions, enforce that the particles on the surface/bottom do not leave it, but the boundary condition on the free surface involves the elevation (which is unknown) and consequently an extra condition is required to obtain a consistent problem. This extra condition is the so-called dynamic boundary condition, setting the pressure on the free surface to be equal to the value of the atmospheric pressure.

Thus, linearized the problem, with the hypothesis of waves of small amplitude and supposing monochrome waves (a only period), we seek for a solution harmonic in time. Therefore, expressing the space part of the potential by separation of variables, we can express the potential of the form:

$$\tilde{\Phi}(x,y,z,t) = e^{-iwt}\psi(x,y,z) = e^{-iwt}f(z)\varphi(x,y)$$

where $f(z)$ is the profile function. Assuming that the bottom is horizontal and progressive waves (transmit energy), we obtain the profile function

$$f(z) = \mu\frac{\cosh[k(z+h)]}{\cosh(kh)}$$

where k is wave number and μ constant.

Now, even if the bottom is no horizontal, assuming moderate slopes of the bottom ($|\nabla h| \ll 1$, $\Delta h \ll 1$) the profile function can also be used and, integrating along depth, a new equation is recovered

$$\nabla \cdot \left[\left(\int_{-h}^{0} f^2 dz \right) \nabla \varphi \right] + k^2 \left(\int_{-h}^{0} f^2 dz \right) \varphi \;=\; 0.$$

where

$$\int_{-h}^{0} f^2 dz \;=\; cc_g$$

and $c = w/k$ is phase velocity, $c_g = dw/dk$ is group velocity.

Thus, results the called 'mild slope' of Berkhoff equation:

$$\nabla \cdot (cc_g \nabla \varphi) + k^2 cc_g \varphi \;=\; 0 \qquad \text{at} \qquad \Omega.$$

where φ is the unknown. Thus, once the velocity determined, the free surface elevation will be obtained by means of the equation

$$\eta(x, y, z, t) \;=\; -\frac{1}{g} Real \left(\frac{\partial \tilde{\Phi}}{\partial t} \right)_{z=0}$$

2.3.1 Boundary Conditions

The equation mild slope is of elliptic type, needing a boundary condition along whole boundary. Possible conditions are:

- Partial or full reflection/absorption:($\partial \varphi / \partial \mathbf{n} - ik\alpha\varphi = 0, \alpha \in [0, 1]$).
- Periodic: ($\varphi|_{\Gamma_1} = \varphi|_{\Gamma_2}, \partial\varphi/\partial\mathbf{n}|_{\Gamma_1} = \partial\varphi/\partial\mathbf{n}|_{\Gamma_2}$)
- Natural Neumann homogeneous: ($\partial\varphi/\partial\mathbf{n} = 0$).
- Known incident wave field: ($\varphi = \varphi_0$)
- Radiation condition of Sommerfeld: ($\lim_{r\to\infty} \sqrt{r}(\partial\varphi/\partial r - ik) = 0$).

3 CONCLUSION

In this work had been carried through the definitions, assumptions and deductions necessary to formulate the model for the shallow water problem for steady and transient phenomena.

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