Juxtapositions

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Abstract

The paper begins with a short historical review of juxtapositions and some arguments which link morphological design in architecture to spacefillings. Three new methods for generating juxtapositions are then described in some detail: the "cube splittings", the "compound arrangements", and the "concave parallelohedra".

Introduction

Spacefilling is the least understood and most neglected among our three principal themes. Despite its apparently simple and visually intuitive nature, the problem received no attention during history and even today it hardly attracts any interest.

To our knowledge, the question was hardly raised by the classical Egyptian and Greek schools. The first valuable research on this type of problem dates from the 17th century, when Kepler studied tessellations* with regular polygons. Near the end of the 19th century the Russian crystallographer Fedorov demonstrated that there are only five convex polyhedra which fill the space by translation* (parallelohedra*). At the beginning of this century Andreini published a list of spacefillings with one, two or three kinds of regular or semiregular polyhedra, but there is no indication that his list is complete. The latest results on the problem of plane fillings are due to Grunbaum and Shephard who prepared the complete list of tesselations of the plane with convex polygons (Grunbaum 1977) and the list of 81 transitive tesselations* (Grunbaum 1976). During the writing of this article we have found references to M. Goldberg's recent work on space filling with convex polyhedra, but we have not yet seen the published report. We have also seen some information on the problem of filling the plane or space with a subset of the square or the cubic lattice (polyominos and polycubes) from M. Gardner, D. Klarner and S. Golomb.

Certainly all these make up a very meagre list of works on a very large subject, and the unanswered questions still outnumber the known results.

It is urgent, in a real and practical way, that architects come to know more about the subject of spacefilling, however little they may realize it. There is a strong social and economic pressure for a return to higher density living in our cities, but the dissatisfaction is evident with our conventional and unique form of higher density building: the skyscraping slab buildings, or rectangular prisms. Juxtapositions also have applications in industrialized space-frame construction. Thus spacefilling should no longer remain a mathematical recreation. Instead it has to develop into a conscientious effort to find more skillful subdivisions of space to create stimulating new environments.

The present approach by architects to morphological design was best described by Steve Baer (Baer 1968, p. 7): "Most of today's buildings have illegitimate designs, the exterior form appears without a history, owing nothing to any step by step process of creation. Their form as a sum of components is a forgery, worked in backwards by vertical and horizontal partitions. This is like choosing a manufacturing process for an article by deciding which byproducts it is to have, or like writing the last chapter of a novel and then arranging five thousand sentences chosen by someone else so that they lead up to it. ... If we think of the cube as a servant to the architect then its qualities are the ones which will first endear it to its master, but ultimately he will see, that although things have been done quickly and efficiently, nothing is in exactly the right place."

We do not suggest that the cube is dead and that we can solve our problems by moving tomorrow into truncated octahedra, for instance. All we are saying is that the next generation of architects should be better informed and trained in how to manipulate spacepackings beyond the cubic one. This larger inventory, along with design skill, new attitudes and new technology, will result in the correct, but perhaps unforseeable, choices of different morphologies. Thus our assignment is really to learn and not to predict.

Since any attempt to find general solutions and complete answers was beyond our means, we have concentrated our efforts since 1970 on a different approach. Because of the synthetic nature or architectural design, it proved more useful and practical to have a number of different methods to generate juxtapositions with certain prescribed properties, than to seek the complete enumeration of a given class of juxtapositions. In the following we describe three different methods to form spacepackings. This work was carried out by Janos Baracs and his students — Nabil Macarios, Michel Velly, Luong Thien Tai, Bernard Leopold, Jean Maurice and Jacques Couturier between 1971 and 1978.

First Method: "Cube Splitting"

A juxtaposition is called **regular** * if its congruent cells are regular and its regular vertices are congruent. In 3-space the only regular packing is the cubic grid. This fact gave rise to the suspicion that all the non-regular spacefillings with convex cells can be derived by some simple operation(s) from the cubic lattice. We found that, with one exception **(Figure 6)** , this is true for all known examples.

The operation we used is plane cutting of the metrically regular cube into 2n congruent parts. The convex subsets of this subdivided cubic lattice are space filling polyhedra*. **Figure 1** illustrates cuts into 2,4,6,8 and 12 congruent parts, the most important among them being a particular truncated tetrahedron shown in detail in **Figure 2** . This polyhedron has an important property: all combinatorial types of convex spacefilling polyhedra (with the same exception, which we come back to later) can be realized as finite packings of this special polyhedron (allowing translations, rotations and reflections). The pretentious name **universal brick** has been given to this polyhedron.

Figure 3 shows combinations of the basic cuts into other spacefilling polyhedra. They can also be interpreted as multiple simultaneous plane cuts of the cube.

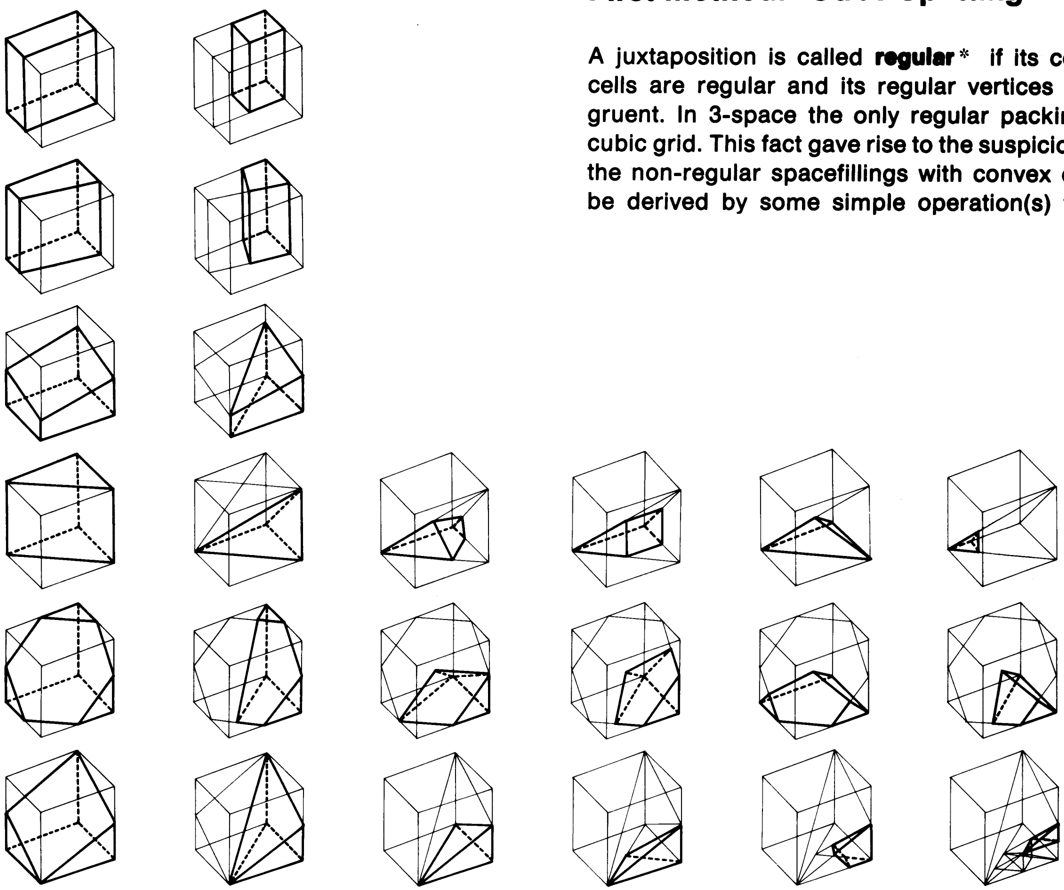


Figure 1

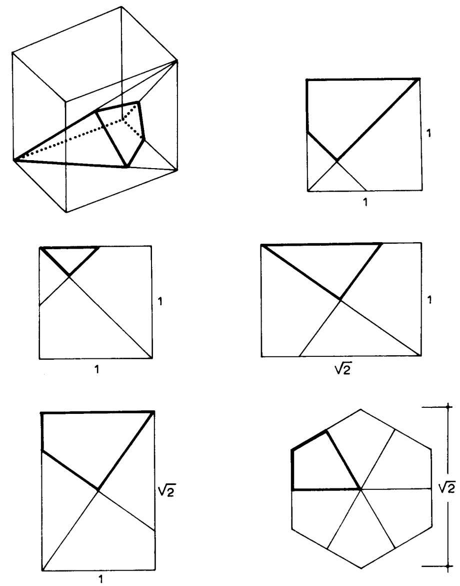


Figure 2

In **Figure 4** the polyhedra of **Figure 1** are further combined by making a new choice of subsets in the divided cubic lattice. We were surprised to notice that three well known combinatorial types of polyhedra (the rhombicdodecahedron, the elongated dodecahedron and the truncated octahedron), which fill space by translation when realized with central symmetry, are also space fillers without the central symmetry.

A list of the Schlegel diagrams* of convex spacefillers is given in **Figure 5** . We do not know if anyone has a complete list, but it seems that a finite list should exist with a surprisingly small number of entries. A reasonable question to ask is whether the 3-connected planar graph (the 1-skeleton of a convex polyhedron) contains some combinatorial properties which indicate that the polyhedron can be (or cannot be) realized as a spacefiller.

This list of Schlegel diagrams suggests that a spacefilling convex polyhedron has at most 14 faces, with the same exception mentioned above, the 16 faced produced truncated tetrahedron. This is a semiregular truncated tetrahedron with tetrahedra attached to its triangular faces as illustrated in **Figure 6.**

This first method has enumerated (hopefully) all single convex polyhedra which fill in space with copies congruent under rigid motions. It should be noted that in many instances the metric cubic lattice cannot be replaced by an affine parallelopiped lattice which permits the given space-filling. Due to the metric restrictions of this method, the probable applications lie in the fields of industrial design (packaging) and generating spaceframes.

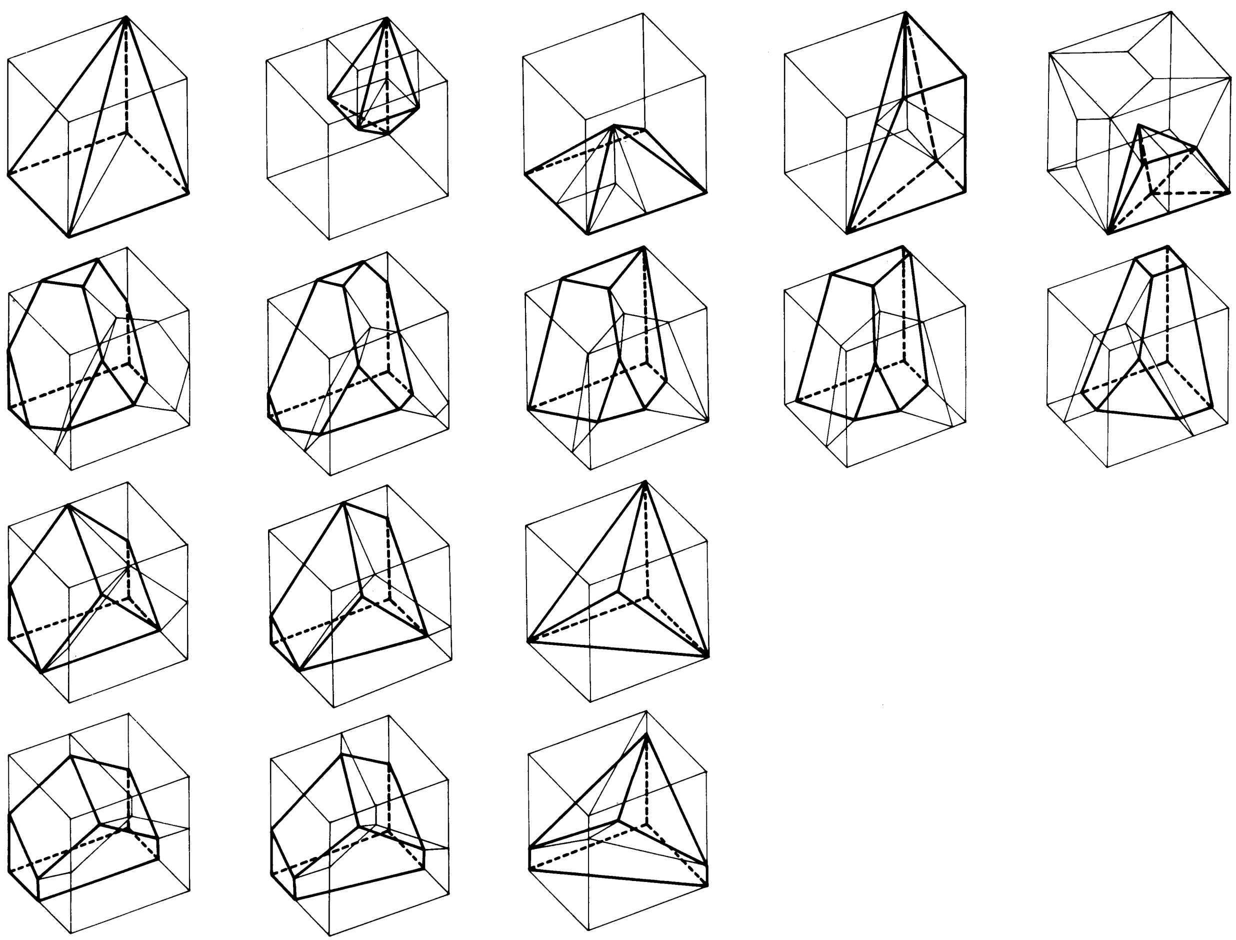


Figure 3

Second Method: Compound Arrangements

The following method will demonstrate how a given juxtaposition can lead to five other different, but closely related spacefillings by certain geometric and topological operations.

The operations are based on a generalization of the following construction in 3-space. Let P be a convex polyhedron and P' be another convex polyhedron which is its topological dual. If we can install P and P' in space so that corresponding edges of P and P' are concurrent, the arrangement is called the **compound** of P. The convex hull of all the vertices of P and P' defines the convex **union-polyhedron** P∪P', and the common solid of P and P', defines the convex **intersection-polyhedron** P∩P'. Simple considerations (easily verified on the Schlegel diagram) lead to the conclusion that P∪P' is the topological dual of P∩P'. The vertices of the union-polyhedron are the vertices of P and P', and its faces are quadrilaterals whose diagonals are the corresponding edges of P and P'. The vertices of the intersection-polyhedron are the common points of corresponding edges of P and P' and its facial planes are those of P and P'.

Note that there is a more general projective construction. If the polyhedra P and P' are realized with corresponding edge lines intersecting beyond the edge segments, then the new faces of P∪P' defined by the two corresponding edges will be concave, as will be some faces of P∩P', and the new polyhedra will be concave as well.

If the compound arrangement of P and P' is formed, the the polyhedron P∪P' may also be obtained from P by a special stellation: every n-gonal face is replaced by an n-gonal pyramid with an n-valent

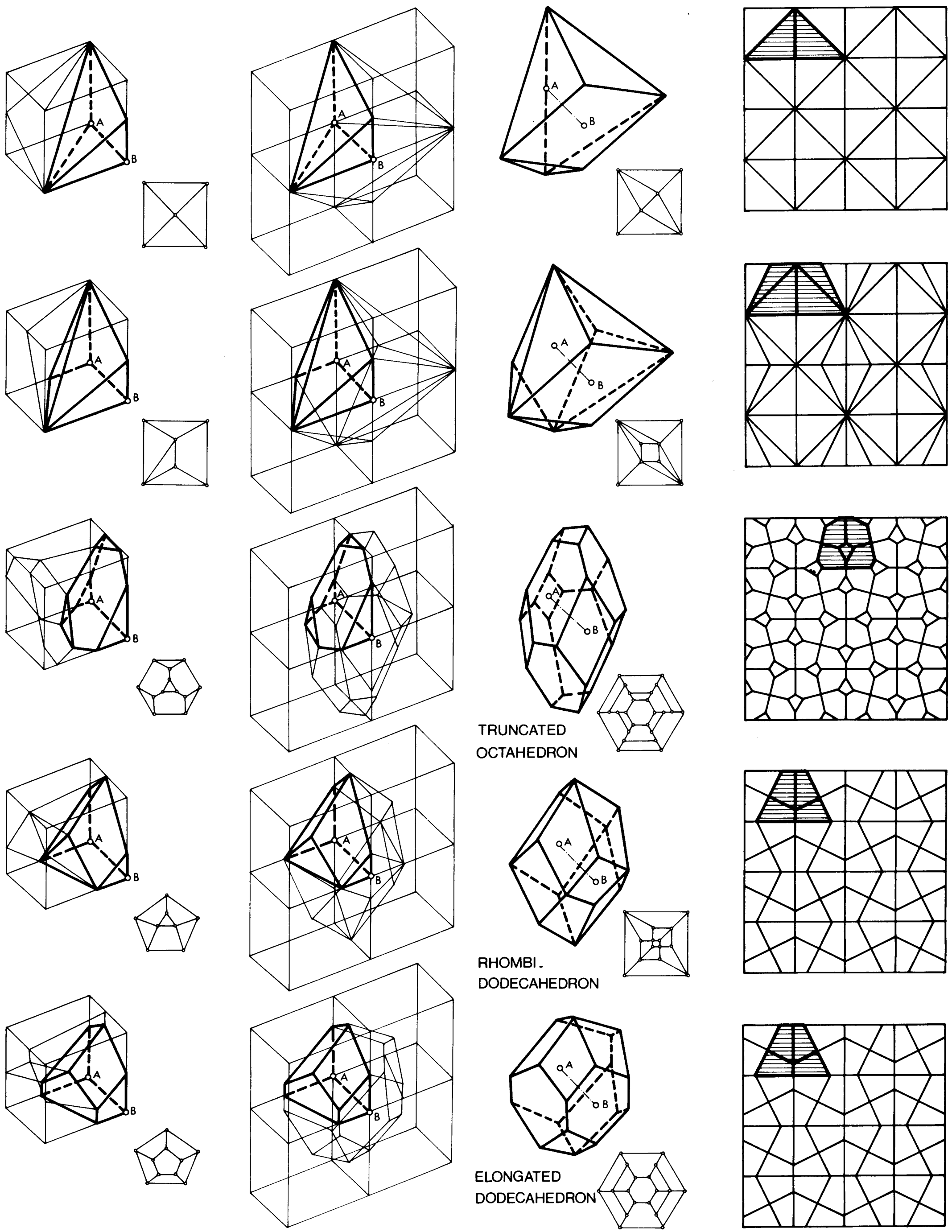


Figure 4

vertex such that the two new triangular faces adjacent to any edge of P are coplanar (thus forming a quadrilateral). Dualizing this stellation, the polyhedron P∩P' may also be obtained from P by a special truncation: every n-valent vertex is replaced by an n-gonal face such that the two new 3-valent vertices situated on any edge of P are coincident (thus forming a 4-valent vertex).

It is easy to construct this compound arrangement of P and P' in the case of regular and semi-regular polyhedra. We begin with the intersphere of the polyhedron P and for each edge we draw a second perpendicular tangent to the intersphere at the point of contact, forming an edge of P'. These edges will meet in appropriate vertices and faces to form a convex realization of P', and of the true compound. We conjecture that it is always possible to create an appropriate projective arrangement, with the faces of P and P' formed as planes and corresponding edges concurrent, for each convex polyhedron. If this is true, we ask how to construct P' to fit a given P and form the compound?

Figure 7 shows an example of a compound arrangement of the cube: P is the cube, P' is an octahedron, P∪P' is a rhombidodecahedron and P∩P' is a cuboctahedron.

Since a tesselation of the plane can be viewed as an infinite polyhedron, this method of compound arrangements leads from a given tesselation to three new tesselations. An example is given in **Figure 8.**

The relationships among the compound arrangements for plane tesselations and polyhedra are summarized in **Figure 9**, with double arrows indicating topological duality.

We will now make use of 4-dimensional space to generalize the process of compound arrangements. Just as we viewed a tesselation as an infinite polyhedron (3-polytope), we now consider spacefillings as infinite 4-polytopes.* However when we apply the idea of a compound arrangement to 4-space, we find two distinct union-polytopes and two distinct

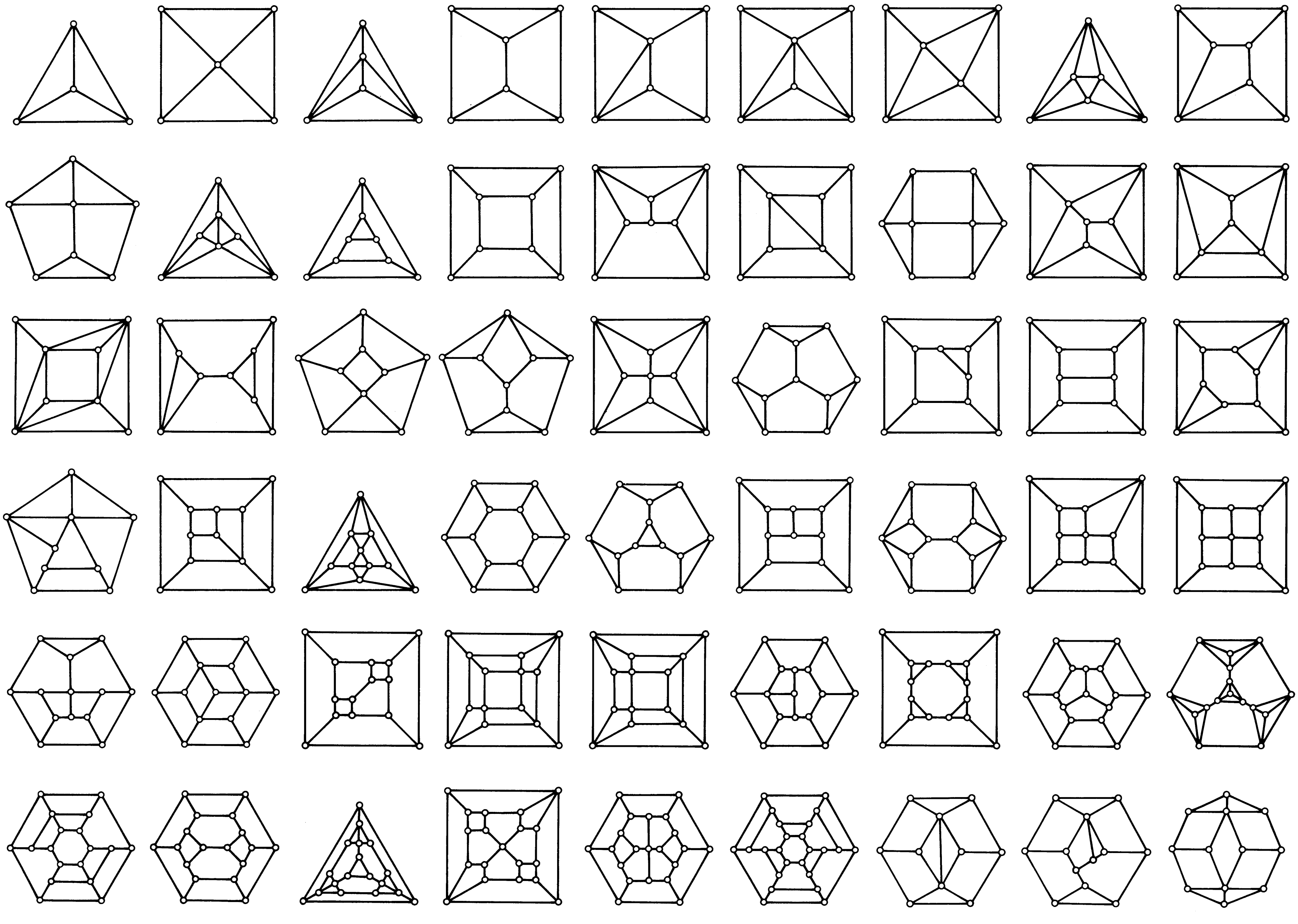


 Figure 5

intersection-polytopes. Together with P and P' these will form a family of six related juxtapositions.

We must first describe duality in 4-space: polyhedra (3-dimensional faces or cells) of P correspond to vertices of P', faces of P correspond to edges of P', and vice versa. Two vertices u and v of P' are joined by an edge of P' if the two polyhedra corresponding to u and v share a face in P. Evidently if P is a juxtaposition, then P' has the topology to be a juxtaposition, and we conjecture that it can always be realized as one.

The next step is to visualize the compound arrangement in 4-space. Edges of P' must be cospatial (lie in the same 3-space) with the corresponding faces of P, while faces of P' must be cospatial with the corresponding edges of P. To arrive at a convex compound, P must be convex and the piercing points of corresponding edges and faces must be inside the facial polygon and between the vertices of the corresponding edge.

If the two polytopes are infinite polytopes realized as spacefillings, then the conditicn about cospatial facets is trivial, but the condition for convexity remains unchanged.

Having formed a compound arrangement of the 4-polytopes and its dual, we can now look for the union-polytope and the intersection-polytope.

In 3-space a polyhedron has a unique type of union-polyhedron: the quadrilateral faces are defined by corresponding concurrent edges. In 4-space we find two distinct types of union — the **lineal union-polytope** $P_L \cup P'_F$ and the **facial union-polytope** $P_F \cup P'_L$. In both cases the 3-dimensional facets of the polytope — or the component polyhedra of the new juxtaposition — are n-gonal bipyramids. In the case of the lineal union a component polyhedron is defined by the two end vertices of an n-valent edge of P and the corresponding 2-dimensional n-gonal face of P'. (The valency of an edge in a polytope is the number of 2-faces incident in the edge.)

In the case of the facial union $P_F \cup P_L$ the component polyhedra are defined by n-gonal faces of P and the two end vertices of the corresponding edges of P'. Since the faces of P are dual to the edges of P', it is clear that the lineal union of P and P' is the facial union of P' and P:

$$P_L \cup P'_F = P'_F \cup P_L$$

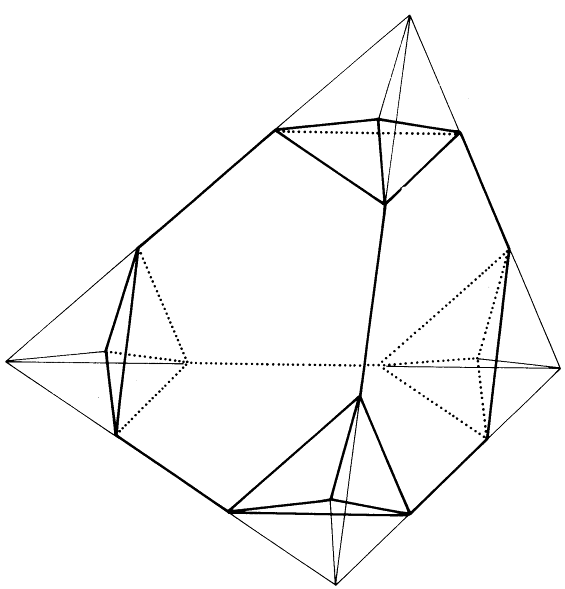


Figure 6

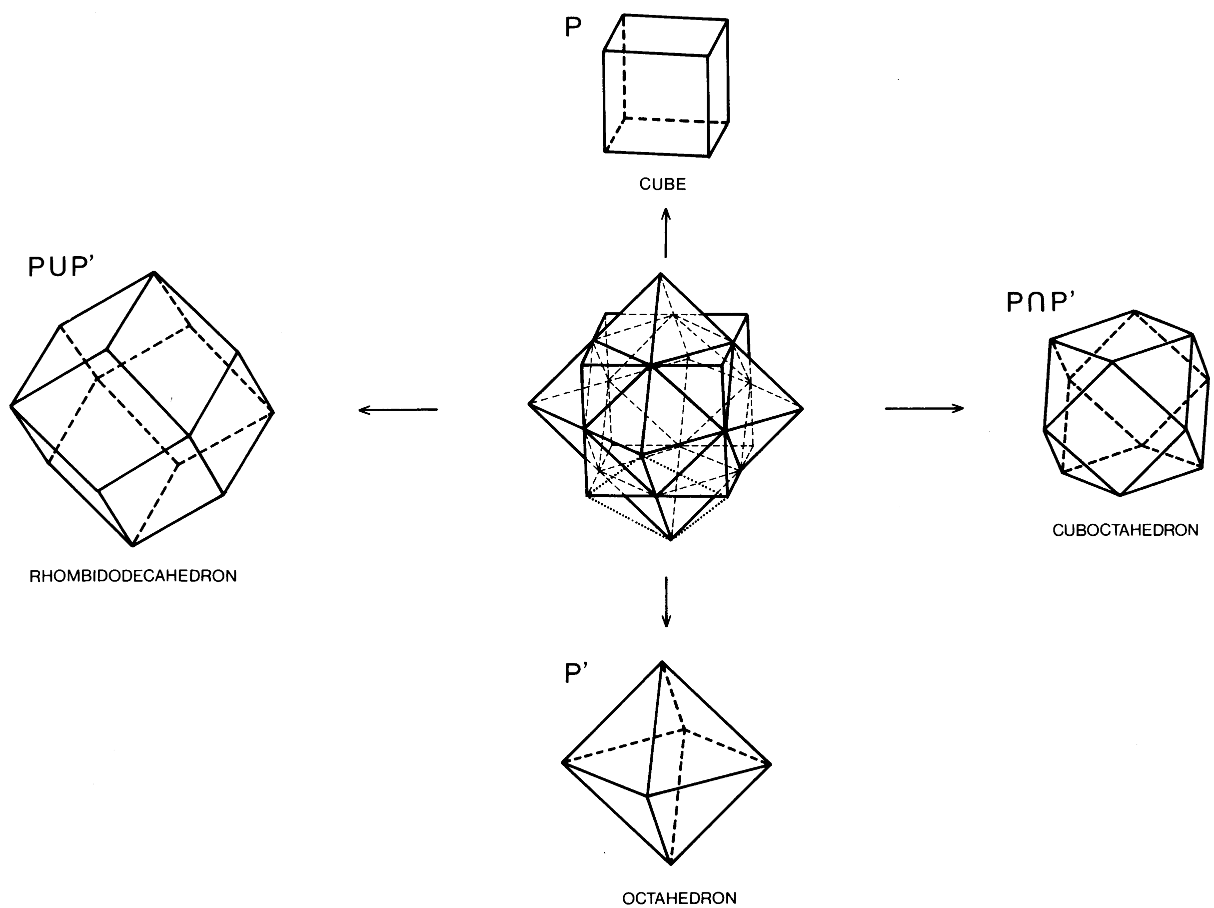


Figure 7

The **lineal intersection-polytope** can be constructed topologically by using the concept of a generalized line-graph: replace every edge of P by a vertex, and join two new vertices if the corresponding edges are incident in a vertex and share a face of a cell in P. For each 3-cell or polyhedron of P this forms a polyhedron L(C) which is the line-graph of that cell — a vertex for each edge and an edge drawn when the corresponding edges were incident in a vertex of the cell. This polyhedron is the intersection-polyhedron of the 3-dimensional cell and its 3-dual. Each vertex is replaced by a polyhedron called the vertex figure L(V): each edge to the vertex is replaced by a vertex and two new vertices are joined by a new edge if the corresponding edges shared a face in P. Since this vertex figure is a spherical polyhedron, it follows that the incidence structure of any vertex in the juxtaposition satisfies the 3-dimensional Euler formula: the number of incident edges less the number of incident faces plus the number of incident cells is equal to 2.

The **facial intersection-polytope** can be constructed topologically by using the idea of a generalized face-graph: each face of P is replaced by a new vertex and two new vertices are connected by an edge if the corresponding faces are incident in an edge and in a 3-cell (polyhedron) of P. We can describe the component polyhedra as follows: each 3-cell of P is replaced by its 3-space dual, and each vertex of P is replaced by the intersection polyhedron of its vertex figure.

If we call the 3-space dual of a cell the face-graph polyhedron of the cell, F(C), and bring in the dual polytope P', with cells C', then these constructions can be rewritten in several ways. The lineal intersection-polytope has the line-graph polyhedron for each cell of P and the face-graph polyhedron for each cell of P' (the vertex figure of the corresponding vertex of P): L(C) + F(C'). The facial intersection-polytope has the face-graph polyhedron for each cell of P and the line-graph polyhedron for each cell of P': F(C) + L(C'). This description also emphasizes the fact, implicit in the correspondence of faces in P and edges in P', that the facial intersection of P and P' is the lineal intersection of P' and P:

$$P_F \cap P'_L = P'_L \cap P_F$$

While we gave sufficient conditions for the existence of convex union-polytopes (since the bipyramids can always be constructed from cospatial edges and faces) it is an unsolved problem what the sufficient conditions are for the existence of one, or both of the intersection-polytopes.

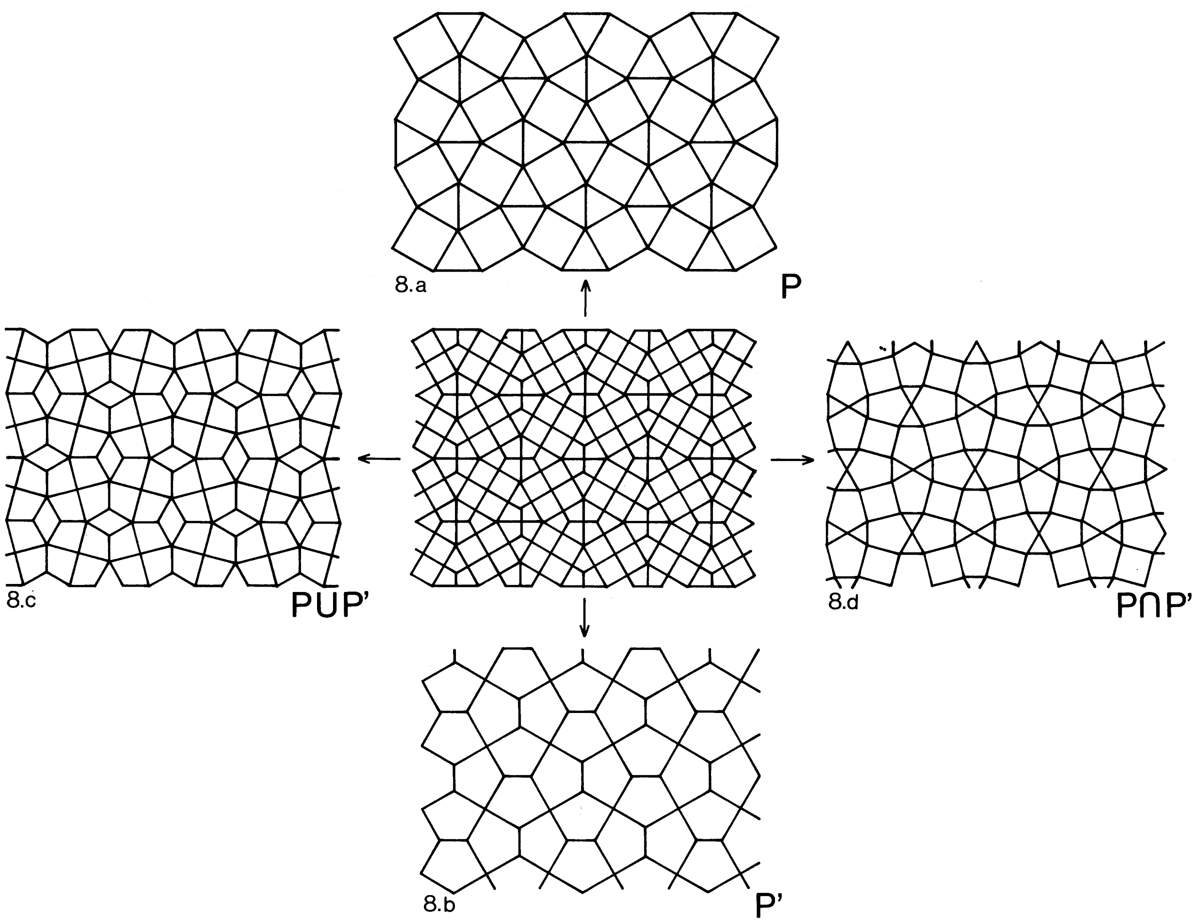


Figure 8

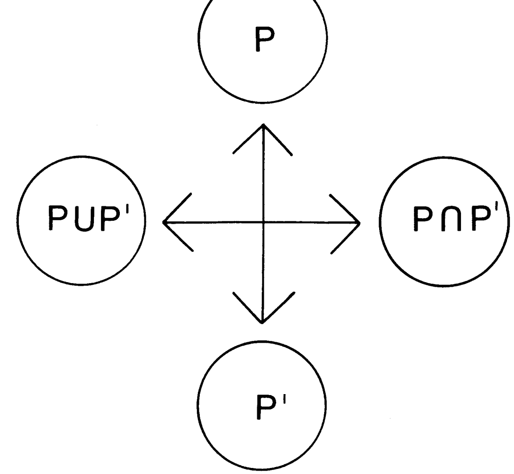
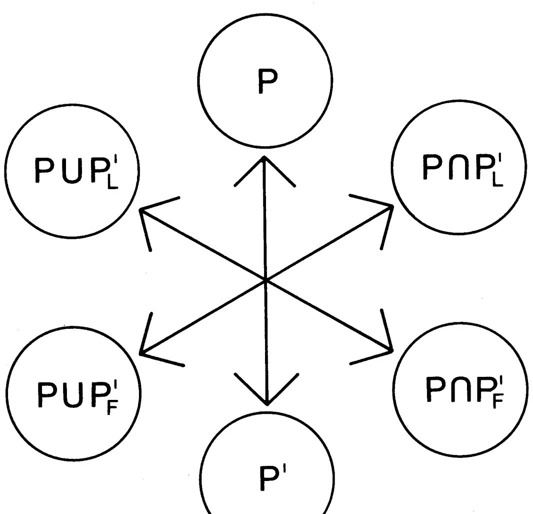


Figure 9

It is always a pleasure to observe a geometric phenomenon in a certain dimension which can be generalized to n dimensions. We mentioned earlier the duality between P∪P' and P∩P' in 3-space. This duality also exists in 4-space, taking the lineal union to the lineal intersection, and the facial union to the facial intersection. We conjecture that such a duality holds for n-space, and in n-space it appears that the number of distinct types of unions (or of intersections) is n-2.

If these constructions can all be carried out, we have in 4-space the polytope, its dual, their two intersections and dual union-polytopes, forming a family of six distinct polytopes (or six juxtapositions in 3-space if the polytope is infinite and the constructions produce juxtapositions) whose relationships are shown in **Figure 10**. In **Figure 11**, we present the six related spacefillings of the hexagonal prism.



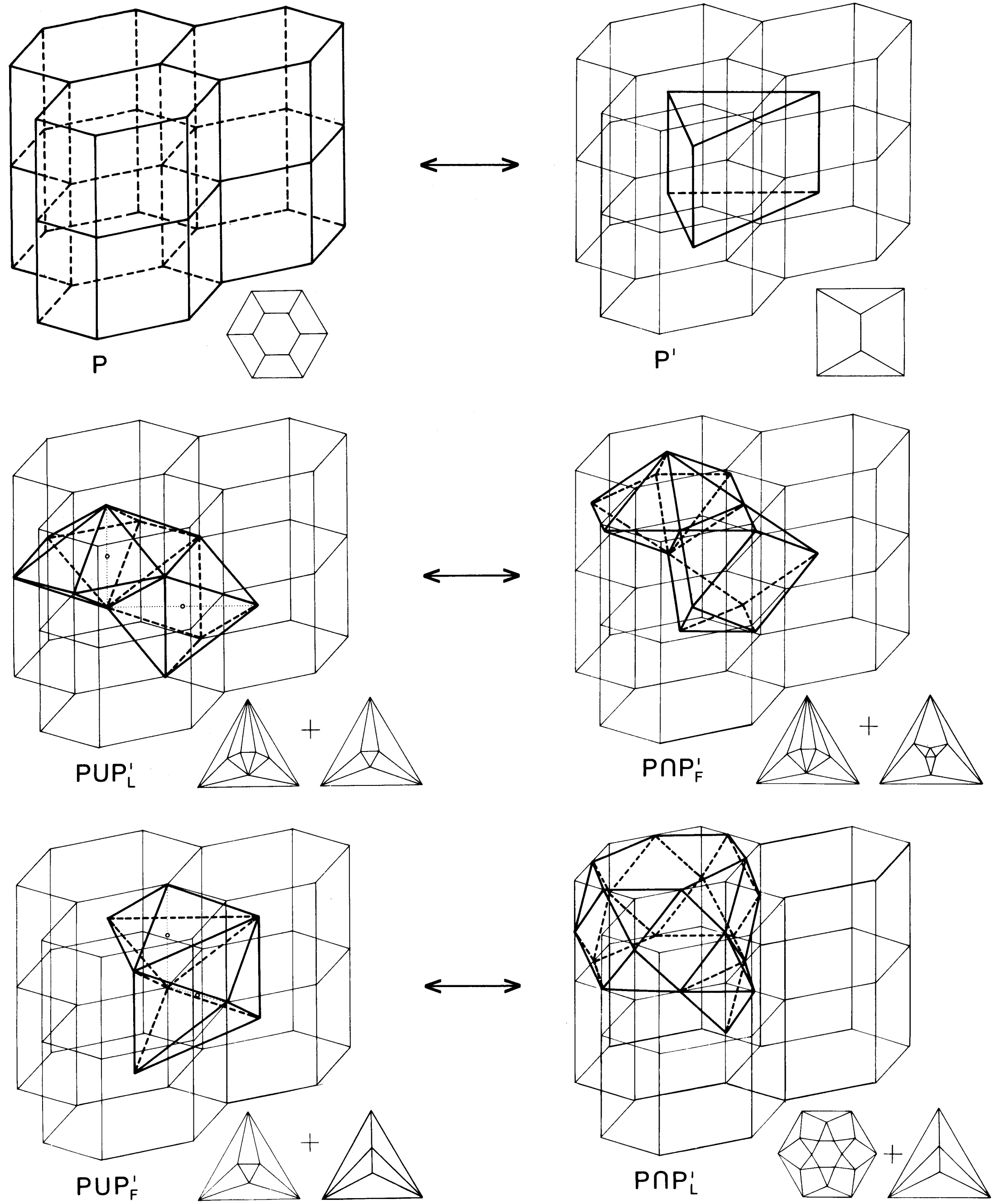
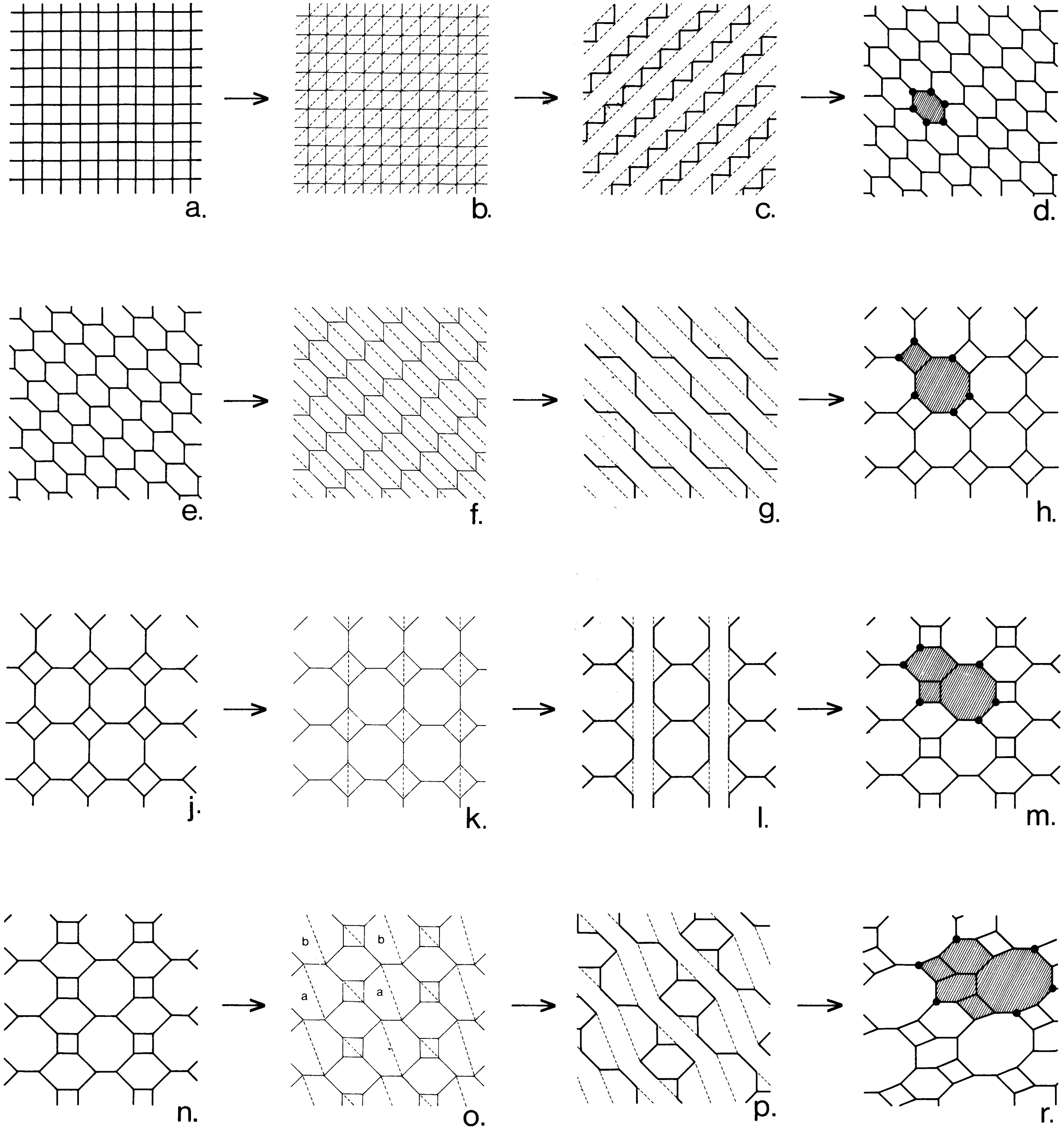


Figure 10

Figure 11



For every given spacefilling the five related infinite polytopes can always be defined topologically. In most cases of juxtaposition we do not know of affine or metric realizations of the other five polytopes as juxtapositions, but we conjecture that it is always possible to create the other five as juxtapositions.

We are just beginning the geometric exploration of these constructions and have not yet had time to question the possible applications of this method.

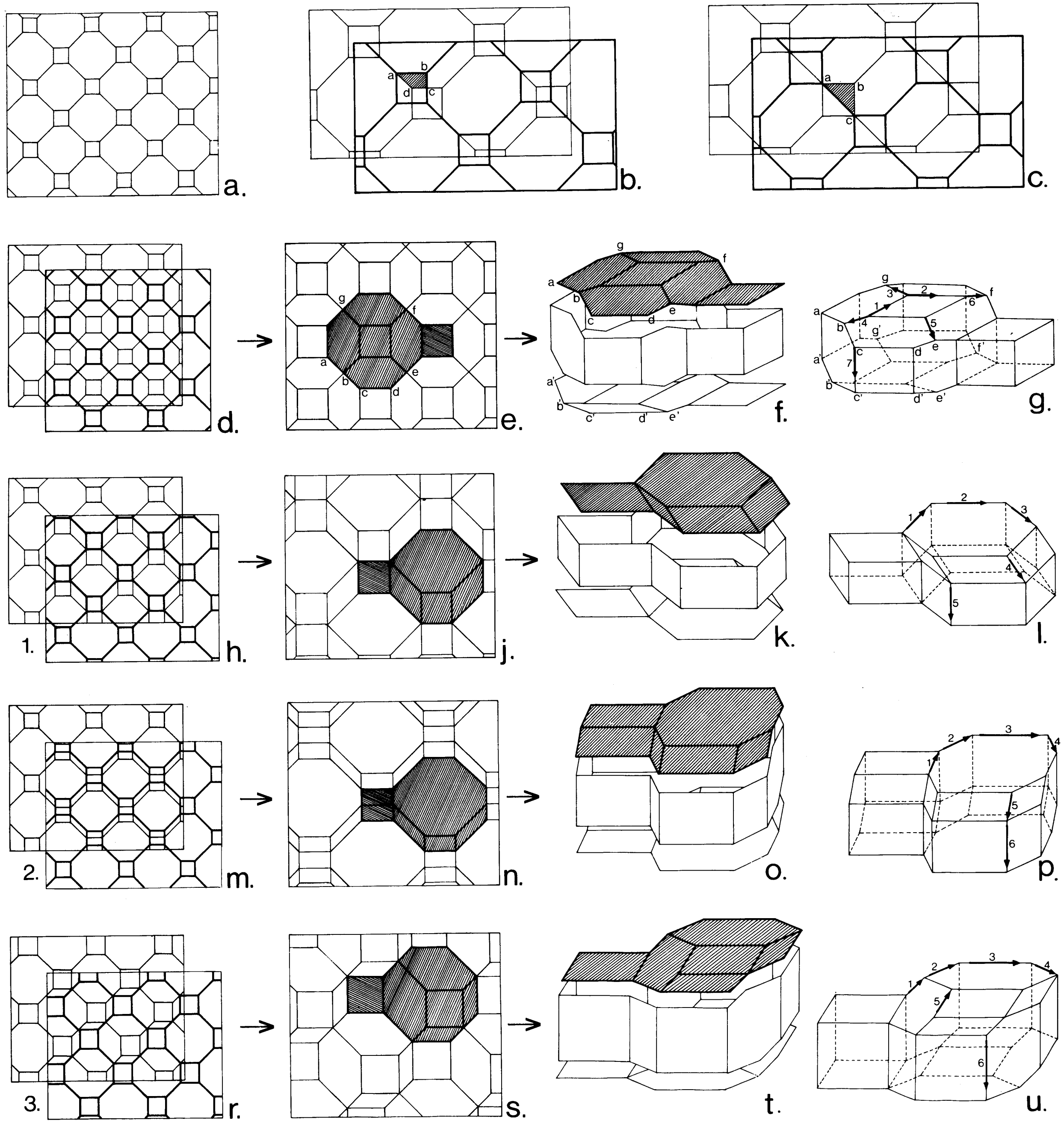
Third Method: "Concave Parallelohedra"

Among the three methods presented here, the method of concave parallelohedra has proven to be the most useful for architectural applications, and it has been tested in several student projects. It is quite unfortunate that mathematicians are preoccupied with convex domains, since most of our well functioning spaces built for any use are concave.

The method used by (Fejes-Toth, 1964) to demonstrate the existence of the five convex parallelohedra can be extended to generate an infinite number of concave parallelohedra. We demonstrate the method by an example.

As a first step **(Figure 12)** we generate concave parallelogons which cover the plane by parallel displacement (translation). This operation begins with a square (or parallelogram) tesselation **(Figure 12A),** which is then split by parallel lines in **Figure 12B** . The strips are then translated by a chosen vector **Figure 12C,** and parallel edges of equal length are inserted between corresponding vertices in **Figure 12D.** This well known hexagonal tesselation is subjected to the same operation in **Figure 12F** resulting in a semiregular tesselation **(Figure 12H).** Using the same process from here on becomes more interesting: in **Figure 12M** and **Figure 12R** we arrive at new tesselations of the plane, with the shaded concave polygon as the fundamental region* of the tesselation.

 Figure 12



The second step is to superimpose a chosen tesselation **(Figure 13A)** on a translated version of the same tesselation **(Figure 13D).** The intersection of the two congruent grids produces a new subdivision of the fundamental region into faces which we want to all be zonagons*. Accordingly, the motions illustrated in **Figure 13B** and **Figure 13C** are not allowed, since the shaded polygons are not zonagons.

Next, the fundamental region, subdivided into zonagons, (shaded region in **Figure 13E**) is interpreted as a projected polyhedron and lifted into space in **Figures 13F and 13G** . The union of the two convex zonahedra produced here is called the concave parallelohedron, which forms the fundamental region of this spacefilling. **Figures 13H, 13M and 13R** indicate different compound grids obtained by different translations of the tesselation. These result in three combinatorially different concave parallelohedra for spacefilling **(Figure 13L, 13P and 13V).**

Figure 13

The photograph **(Figure 14)** shows a practical application of the method of concave parallelohedra in a student project on medium density housing by T.T. Luong in 1974.

The concave parallelohedron which we generate by this method is a fusion of a finite number of convex zonahedra*. We have not been explicit about the process of recognizing this fusion in the plane drawing, but by practising on examples the reader will find that the task is not difficult. A convex zonahedron is uniquely defined by its star — a sheaf of n lines, each parallel to one of the n different directions in which the edges of the zonahedron occur (Coxeter 1973). Based on visual evidence from a number of examples, we conjecture that any given star of n lines defines one, or more, concave parallelohedra. If this is true, we would also like to know, as an alternative to the method described here, how to construct the concave parallelohedron directly from a given star.

In conclusion we mention further conjectures related to this third method. First, we suspect that any zonahedron fills in space with a combination of a few other zonahedra. Second, it appears that any space-filling generated by more than one type of zonahedron has a fundamental region which can be derived, by a series of geometric operations, from the five convex parallelohedra.

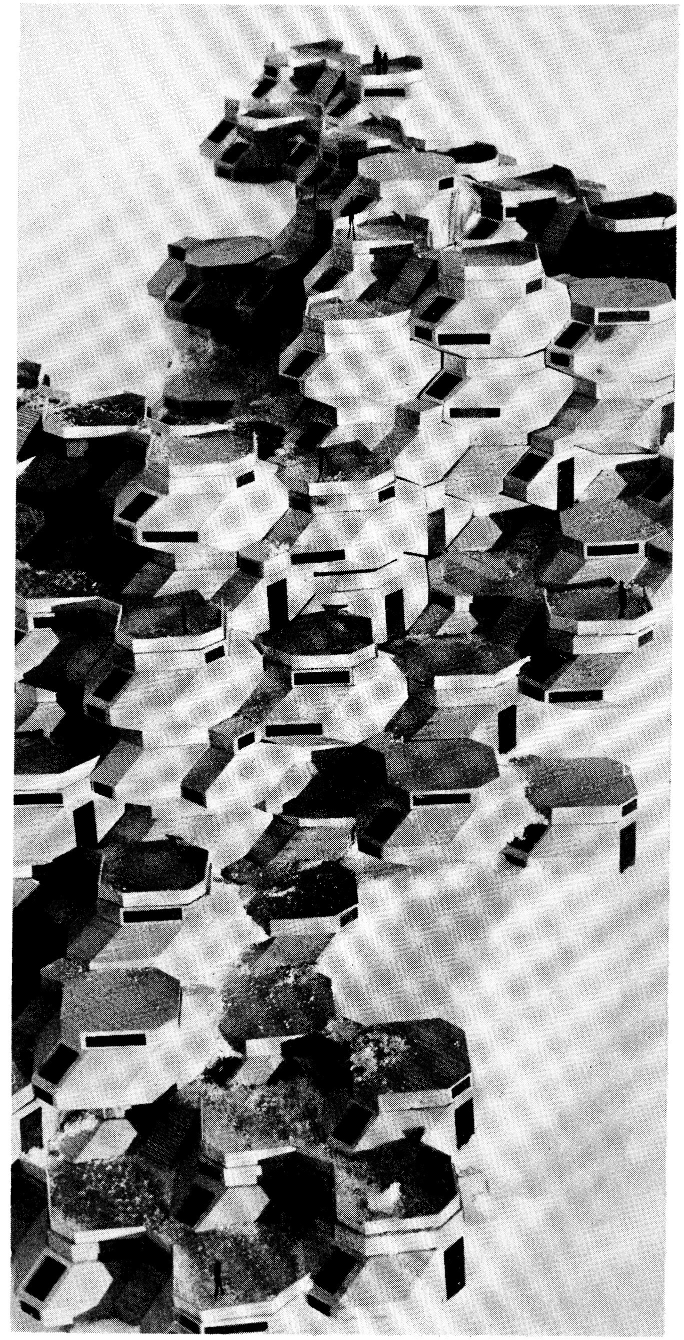


Figure 14

Terms and Definitions

Tesselation: a periodical covering of the plane with polygons, without gaps or overlaps. The mutual relation of any two polygons is one of three possibilities:

(1) they are disjoint;

(2) they have precisely one common point which is a vertex of each of the two polygons;

(3) they share a segment which is an edge of each of the two polygons.

Transitive tesselation: the group of isometries which leaves the tesselation invariant is transitive on the polygons. (All polygons of such a tesselation must be congruent).

Parallelohedron: convex polyhedron whose translated replicas juxtapose.

Convex polyhedron: a polyhedron which lies entirely in a halfspace defined by any of its facial planes.

Translation: parallel displacement.

Spherical polyhedron: a covering of the topological sphere with plane polygons, without any gaps or overlaps. The mutual relation of any two polygons is one of three possibilities:

(1) they are disjoint;

(2) they have precisely one common point which is a vertex of each of the two polygons

(3) they share a segment that is an edge of each of the two polygons.

Juxtaposition: periodical filling of space with spherical polyhedra, without gaps and overlaps. The mutual relation of any two polyhedra is one of four possibilities:

(1) they are disjoint;

(2) they have precisely one common point which is a vertex of each of the polyhedra;

(3) they share a segment which is an edge of each of the two polyhedra;

(4) they share a polygon which is a face of each of the two polyhedra.

Affine properties: geometric properties which remain invariant in parallel projections.

Intersphere: sphere tangent to all edges of a spherical polyhedron.

Schlegel diagrams: plane graph representation of a spherical polyhedron.

Plane graph: a graph of edges and vertices drawn in the plane such that the common points of edges are vertices of the graph.

Polytope: a generalization of a polyhedron to R^n, dimension n=4. Every pair of incident faces of dimensions k-l and k+l are jointly incident with exactly two faces of dimension k.

Fundamental region: subset of a tesselation or juxtaposition whose translated replicas fill the plane or space in the given arrangement.

Zonagon: convex polygon, pairs of opposite edges are parallel and equal in length.

Zonahedron: convex polyhedron, all of whose faces are zonagons.

Bibliography

The code in the first block of each bibliographic item consists of three parts, separated by dashes. The first letter indicates whether the item is a

B ook
A rticle
P reprint, or
C ourse notes.

The middle letter(s) indicates whether the piece was intended primarily for an audience of

M athematicians,
A rchitects, or
E ngineers.

The final letter(s) indicates if the piece touches on one or more of the principal themes of our work:

P olyhedra,
J uxtaposition' or
R igidity.

The key words or other annotations in the third column are intended to show the relevance of the work to research in structural topology, and do not necessarily reflect its overall contents, or the intent of the author.

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