Realizability of Polyhedra

by Walter Whiteley

Abstract

We address ourselves to three types of combinatorial and projective problems, all of which concern the patterns of faces, edges and vertices of polyhedra. These patterns, as combinatorial structures, we call combinatorial oriented polyhedra. Which patterns can be realized in space with plane faces, bent along every edge, and how can these patterns be generated topologically? Which polyhedra are constructed in space by a series of single or double truncations on the smallest polyhedron of the type (for example from the tetrahedron for spherical polyhedra)? Which plane line drawings portraying the edge graph of a combinatorial polyhedron are actually the projection of the edges of a plane-faced polyhedron in space? Wherever possible known results and specific conjectures are given.

A great deal of our work employs surfaces which enclose sections of space (compact oriented 2-manifolds), constructed by joining plane polygons together in pairs across their edges. These finite plane-faced coverings of the closed surface are the basic cells of the work on juxtapositions or space-fillings, and they also appear as the basic visual guide in our analysis of structural dependence and rigidity in plane frameworks. These polyhedra are the essential building blocks of spatial geometry. They have been scrutinized for centuries. In this century the study has been generalized to a topological theory of combinatorial polyhedra, but many of the fundamental problems, both solved and unsolved, involve the realizability in space of the abstract patterns within particular geometric constraints.

Each generation of workers has chosen particular properties which they find desirable and then asked — which polyhedra can be realized within these constraints? The constraints which we investigate within our research group arise directly or indirectly from our prior concerns with space-fillings and rigidity. In addition, some constraints of obvious architectural or engineering significance — such as metric regularity of angles, edge-lengths or face patterns — have been widely studied elsewhere and will not be described here. Instead we will restrict our attention to a series of questions of a combinatorial, topological and projective nature which represent less traditional topics of study.

The Pattern for Convex Polyhedra

In many ways a model for the overall type of theory we are seeking is offered by the most widely studied class of polyhedra, the **convex polyhedra** . These polyhedra are formed in space by a series of plane convex polygons, the faces, formed into a rough ball so that along each edge of a polygon the face joins one other face, and the plane of each facial polygon cuts the three -dimensional space into two half-spaces, one of which contains the entire polyhedron.

Such a polyhedron will be topologically like a sphere, and for a spherical polyhedron we have Euler's formula which gives a combinatorial relationship between the number of vertices V, edges E and faces F: $V - E + F = 2$. When we have a set of vertices, edges and faces (each edge joining two vertices and separating two faces) which satisfies this formula, and thus forms a topological sphere*, the skeleton formed by the edges and vertices forms a **planar graph,** a graph* which can be drawn in the plane without any edges crosssing in their interior. (Figure 1)

Combinatorial and Topological Problems

An obvious question is: which planar graphs (with the regions formed in the plane taken as the faces)

can be topologically reproduced as the faces, edges and vertices of a convex polyhedron with plane faces and a real bend at each edge? Steinitz' theorem provides the answer — those planar graphs which are 3-connected in a vertex sense and have more than three vertices (Grunbaum 1967, p. 235). Here 3-connected means that removing any two vertices (and the adjacent edges) cannot separate the graph into several components **(Figure 1)**.

Closely related to this result, and used in some proofs, is the fact that all 3-connected planar graphs (combinatorial convex polyhedra) can be created from the graph of the tetrahedron by successively splitting faces in one of three natural ways **(Figure 2)** (Lyusternik 1963, p.75). We also know, from their definition, that any convex polyhedron can be sliced out of an underlying tetrahedron by a series of plane-cuts or truncations* — one for each face beyond the original four faces of the tetrahedron. One central theme for our further investigations will be the connections between these three approaches — the graphic characterization of a class of spatial polyhedra, the topological evolution of these graphs,

and the spatial construction by plane-truncations.

Projective Problems

We now want to study in more detail the projective construction of convex spatial polyhedron by plane slices on the tetrahedron. We have found that a useful way to observe this slicing is in a single plane projection where we record the projections of the lines where each new face plane meets all of the previous planes **(Figure 3)**. We then attempt to reverse the procedure and ask when a pattern of lines drawn in the plane represents the construction of a convex polyhedron in space. Of course we must identify, among all these lines, the 3-connected planar graph which we want to be the projected edges of the proposed polyhedron. In addition we must label each of the other lines as the projected intersection of two faces of the proposed polyhedron. We know that three planes in space 1, 2 and 3 will have a point of intersection (or, as a degenerate case, a line of intersection) so the three lines 12, 23, 31 are concurrent in space. Thus in our labeled diagram it is necessary that the three
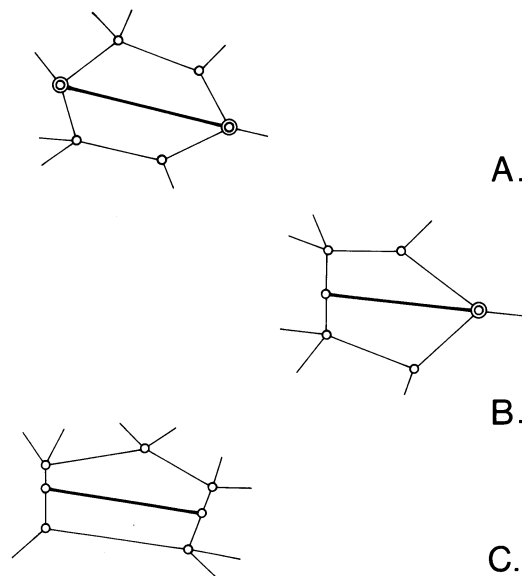
projected lines labeled 12, 23, 31 be concurrent in the plane. Finally, to ensure the convexity of the proposed polyhedron we have found that two faces which share a vertex but not an edge must have a line of intersection which intersects the face polygons only at this vertex, and all the face polygons must be convex **(Figure 4)**. Is this enough to guarantee the construction of a convex spatial polyhedron? Yes. We have proven that all such consistent diagrams of lines in the plane can be lifted back to produce convex polyhedra in space.

To fix this spatial polyhedron, we need to specify our point of projection and the position of the projection plane. For convenience we will think of orthogonal projection along the direction of the z-axis (from the point at infinity on the end of this axis) onto the xy-plane. With this frame of reference specified, if a plane diagram lifts to one convex polyhedron in space, then it lifts to a four parameter family of polyhedra — three parameters for the location of an initial face-plane, and the fourth for a factor which represents stretching along the z-axis, towards the point of projection. If one polyhedron of this four-parameter family is convex, then all of these equivalent realizations will also be convex.

If we simplify the initial information in the plane and begin only with a drawing, with straight lines, of a 3-connected planar graph, we have the obvious question — what possible complete diagrams of intersection lines for constructing a convex polyhedron can
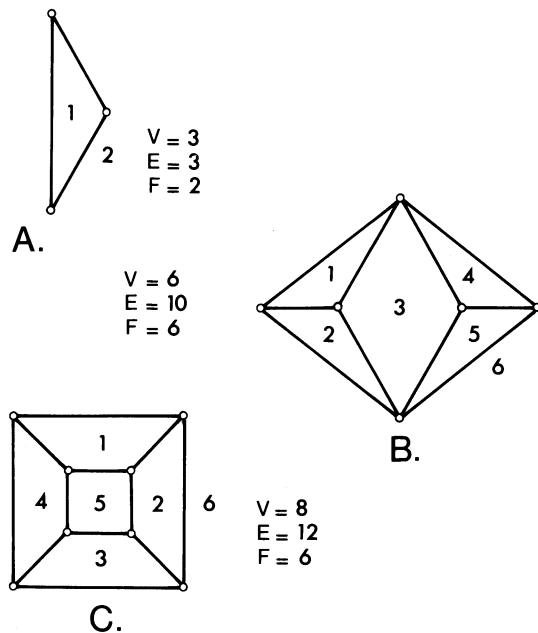

Figure 1. Both 3-connected planar graphs (C), and planar graphs which are only 2-connected, create spherical polyhedra if the regions of the plane are chosen as faces.

Figure 2. Face splitting by a new edge adds new faces to a combinatorial polyhedron.
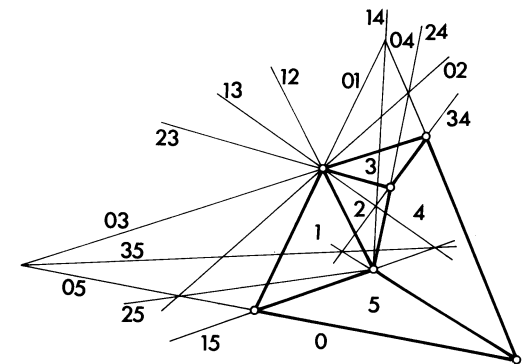

Figure 3. A convex polyhedron is shown in projection, and all lines of intersection of pairs of faces are drawn. The intersection of face 1 and face 2 is labeled 12.

be produced as extensions of this initial drawing? Equivalently what convex polyhedra exist in space which project to the initial plane drawing? We will return to this basic problem shortly, but we note that in many cases ($E \leq 2V-2$) this question can be answered by a series of direct geometric constructions from projective* (or descriptive) geometry. Consider the examples illustrated in **Figure 5. Figure 5A** shows the completion of a projected triangular prism by a sequence of steps, where points 013 and 024 are shown to confirm the consistency of the diagram. If one of these points of concurrence of three lines exists, then the other point must also exist, as a result of Desargues' theorem on perspective triangles. **Figure 5B** shows the completion of a second figure, with point 025 drawn as a check of consistency. **Figure 5C** shows an identical graph, where the point 025 has broken up, and thus no consistent completion (or spatial polyhedron) is possible.

We also recall that a thorough understanding of the statics of frameworks in the plane would provide a solution to the problem of polyhedral completions, since a convex polyhedron projects, via Maxwell's theorem, into a static stress on the plane framework which is in compression of all members which form the boundary of the plane drawing, and in tension on all other members. (See the article on rigidity in this issue of the Bulletin.) This is true because the sign of the stress (+ for compression and - for tension) is a measure of whether the interior angle between the faces at that edge is increasing (+) or decreasing (-) as we lift the plane diagram with its flat edges back to the bent edges of the spatial polyhedron. (Whiteley 1978).

General Concepts

Combinatorial Polyhedra

We want to use a similar approach to analyze other plane-faced space enclosures. A first step is to abstract a basic pattern of faces, edges and vertices which is found in the convex examples.

An **oriented combinatorial polyhedron** is a set of **vertices** $v_1,...,v_m$ and a set of faces $F_1,...,F_k$ such that

(1) each **face** is assigned a polygonal cycle of distinct vertices $u_1,...,u_n$ (at least 3).

(2) the **edges** are identified as unordered pairs of vertices which are adjacent in the polygon of some face (including the last and first vertices of the cycle).

(3) each edge (u_i,u_j) occurs in exactly two faces, once in the order u_i,u_j and once in the reverse order u_j,u_i

(4) for each vertex v_0 the set of edges which include this vertex form a single cycle without repetition $(v_0,u_1), (v_0,u_2) ... (v_0,u_n)$ such that $u_{i+1}v_0u_i$ are adjacent vertices, in the polygon of some face for each i (including $u_1v_0u_n$ when $i = n$)

(5) the structure is connected — each vertex is connected to every other vertex by a path of vertices and edges.

Among people who concentrate only on convex polyhedra (Grunbaum 1967) it is traditional to add an additional condition:

(6) two faces never have more than two vertices in common, and if they share two vertices then the edge between these vertices occurs in each of the faces.
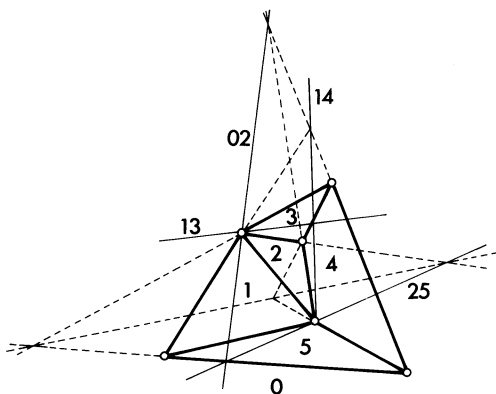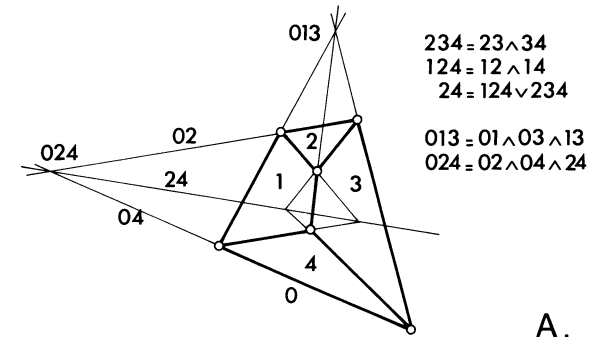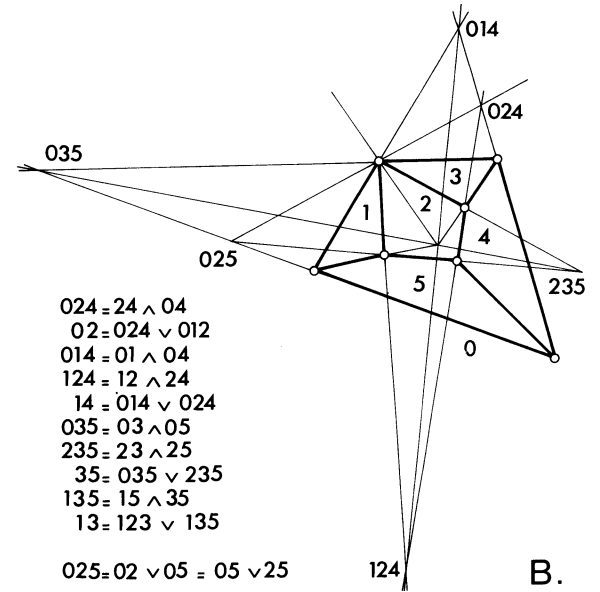
Figure 4. A non-convex polyhedron is shown in projection with bad intersection lines at vertex 0123. Line 13 splits face 3 and line 02 splits face 0. Bad intersections also occur at 1245.

$$234 = 23 \wedge 34$$
$$124 = 12 \wedge 14$$
$$24 = 124 \vee 234$$

$$013 = 01 \wedge 03 \wedge 13$$
$$024 = 02 \wedge 04 \wedge 24$$

A.

$$024 = 24 \wedge 04$$
$$02 = 024 \vee 012$$
$$014 = 01 \wedge 04$$
$$124 = 12 \wedge 24$$
$$14 = 014 \vee 024$$
$$035 = 03 \wedge 05$$
$$235 = 23 \wedge 25$$
$$35 = 035 \vee 235$$
$$135 = 15 \wedge 35$$
$$13 = 123 \vee 135$$

$$025 = 02 \vee 05 = 05 \vee 25$$

B.
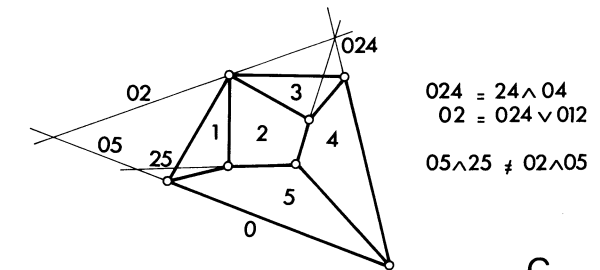
$$024 = 24 \wedge 04$$
$$02 = 024 \vee 012$$

$$05 \wedge 25 \neq 02 \wedge 05$$

C.

Figure 5. Lines of intersection of pairs of faces are constructed from the projections of the original edges in the indicated sequences (A, B). Figure C illustrates the failure of a construction when three lines fail to converge to a point.

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This restriction would exclude some traditional polyhedra, and seems unnecessary in either an architectural or geometric context (Coxeter 1973, p.4). In fact, in a general geometric context it is reasonable to modify condition (3) to drop the reference to the order of the vertices, and thus include unoriented polyhedra, polyhedra which form unoriented manifolds such as the Klein bottle (Hilbert 1952, p.309). Each combinatorial polyhedron corresponds to a unique **compact topological two-manifold,** if each face is taken as a topological disc. While unoriented surfaces with boundaries can be built in space, simply by hinging together flat panels, whenever we build an unoriented manifold without boundary (that is, an unoriented polyhedron) there will always be a self-intersection (faces crossing without an edge of the polyhedron at the intersection). For this reason unorientable polyhedra cannot be used to enclose space in a form which is useful in architecture. We will restrict our discussion to oriented polyhedra, the dissections of oriented compact two-manifolds.

Having chosen our definition of oriented polyhedra, we are aware that there are other broader possibilities — we could drop the assumption that an edge is identified by its vertices, and then permit more than one edge between a pair of vertices. There are also many possible restrictions, such as that two faces share at most one edge, or two faces share at most two vertices. The final choice of restrictions will be dictated by the requirements of the areas being studied, and we will outline below some possible
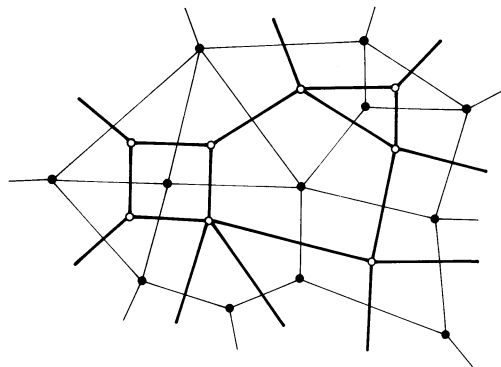
Figure 6. The topological dual (dotted lines) of a portion of a polyhedron (black lines).

choices, along with supporting reasons for such choices.

Projective Polyhedra

We begin with the idea that a projective polyhedron (or **realization**) is any assignment of points in space to the vertices, and planes in space to the faces, such that the plane of any face passes through the points of all its vertices. However this permits the singular, and absurd, examples of all faces having the same plane (with all vertices assigned points in this plane) or the dual form of all vertices assigned the same point). We will normally rule out such degenerate cases, which are of little practical and architectural interest, by insisting on "proper" realizations, this word "proper" to be defined as circumstances require. As a next step we insist that adjacent faces have different planes and the two ends of any edge are different points (all edges are **proper**). This still permits the degenerate realizations where all vertices are on a single line and all faces are assigned planes through this line. It is clear that every abstract polyhedron has such degenerate
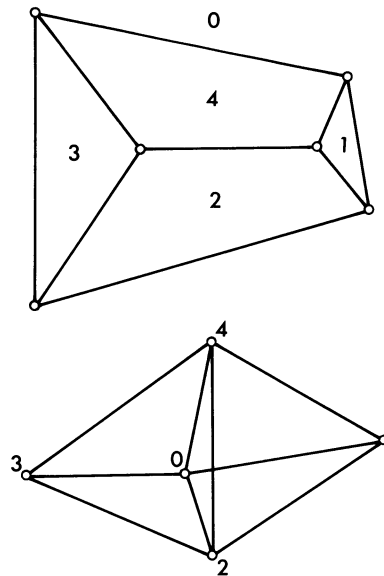
A.

B.

Figure 7. A polar pair of polyhedra.

realizations. This involves every face and every vertex being collinear, and it is unlikely that any architectural construction will have any face or vertex collinear. We will examine other possible restrictions to "proper" realizations later in the article.

We can work directly with the combinatorial polyhedra (and their generalizations or restrictions) and try to find out how they are generated combinatorially (such as by face splittings) from a few small patterns (such as the tetrahedron) and then see whether all the abstract patterns can be realized as projective polyhedra.

Alternatively we can begin by extending the type of spatial constructions we allow (generalizing the single plane cuts on the underlying tetrahedron), and then ask for a combinatorial characterization of the types of combinatorial polyhedra which are created.

Finally, given a labeled set of lines in the plane, we can ask when these lines are the projections of the corresponding edges of a projective polyhedron in space.

Duality

In our approach, a basic mathematical concept is duality — the process, either topologically or projectively, of turning vertices (points) into dual faces (planes), faces into dual vertices, and edges joining two vertices into edges separating the dual faces **(Figure 6).** In projective space this operation is achieved by a polarity — a linear transformation taking coordinates of points to coordinates of planes, which takes any four coplanar points to four concurrent planes, and thus reverses all the projective incidences, taking a plane faced polyhedron into a new plane faced polyhedron realizing the topological dual **(Figure 7).** If we polarize a convex polyhedron so that an interior point of the polyhedron becomes the new plane at infinity, we find that the dual polyhedron is also convex. This confirms that the convex polyhedra (and 3-connected planar graphs) are closed under duality. The dual of our starting tetrahedron is a tetrahedron

but the types of construction we used dualize to new constructions. The projective dual to truncation* by a plane is **stellation** * Dual to the topological face splittings are the topological vertex splittings illustrated in **Figure 8.** For the same reasons that we do not face split along an existing edge of any face (creating multiple edges) we do not attempt the dual construction, which is a vertex split creating a 2-valent vertex.

Spherical Polyhedra

Combinatorial Development

A spherical polyhedron is any combinatorial polyhedron whose underlying topological surface is a sphere, and it it is characterized by Euler's formula $V - E + F = 2$ We also know that the edges must form a 2-connected planar graph (a graph which cannot be disconnected by removing a single vertex). Every such graph belongs to a combinatorial spherical polyhedron, if we take a plane drawing and identify the regions formed in the plane as the faces **(Figure 9).** However some such polyhedra can only be realized in space with collinear faces or improper edges (for example the polyhedron formed from the graph of a triangle). However **we conjecture that any combinatorial spherical polyhedra with a planar graph which is 2-connected in a vertex sense and has every vertex at least 3-valent, will have spatial realizations with all edges proper and no face or vertex collinear.** Those graphs which are 2-

connected but not 3-connected in a vertex sense will have several different ways of assigning faces for a combinatorial spherical polyhedron (of drawing the graphs without self intersection on the unit sphere **(Figure 9)**), but any two such spherical polyhedra on the same graph are equally realizable in space. This fact follows from the converse of Maxwell's theorem in plane statics!

This class of 3-valent, 2-connected planar graphs is closed under topological vertex and face splittings. We conjecture that the entire class is generated by face splitting and hinging of pairs of pieces **(Figure 10)** , beginning with one tetrahedron.

When we try to dualize these polyhedra we notice that, since two faces may meet along more than one edge' the new dual vertices may be joined by more than one edge **(Figure 11).** Therefore the vertex-edge structure is a multi-graph — a graph with possible multiple edges between some vertices. These abstract structures are equally realizable in space, but any pair of edges on the same two vertices, which do not surround a digonal face, will produce a pinch which breaks up the space enclosed by the polyhedron in a way which is quite useless in architecture. We note that the dual of all vertices being at least 3-valent is that all faces are at least triangular, but this assumption followed from our original assumption that the vertices and edges formed a simple graph. As a final mathematical conjecture of this type we propose that all abstract

spherical polyhedra (planar multigraphs) with all faces at least triangular and all vertices at least 3-valent can be realized in space with every edge proper and no face or vertex collinear.

We define a **proper** projective realization as having no coplanar adjacent faces, no copunctual adjacent vertices, no faces collinear and no vertices collinear. We have not spoken of more subtle questions of realizability such as whether an edge should lie in the plane of a non-adjacent face. In our experience, we have found that spherical polyhedra with 2-vertex connected, 3-valent planar graphs and no faces sharing more than two vertices have realizations with every face a topological disc, and no self-intersection (i.e. an embedding of the sphere).
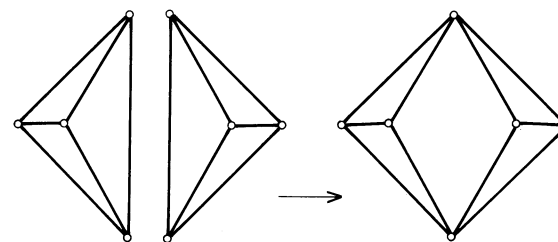
Figure 10. Topological hinging joins two polyhedra into one by identifying two pairs of vertices, two pairs of faces, and dropping two edges.
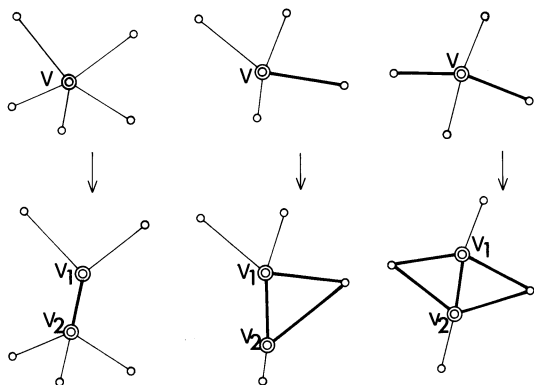
Figure 8. The combinatorial duals of face splits are three types of vertex splits: face-face vertex splits (A), face-edge vertex splits (B), and edge-edge vertex splits (C).
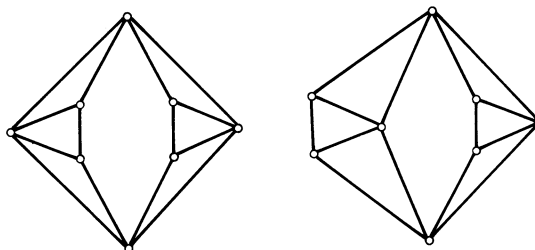
Figure 9. Two polyhedra can have the same 2-connected planar graph of edges, but different face structures (different plane regions).
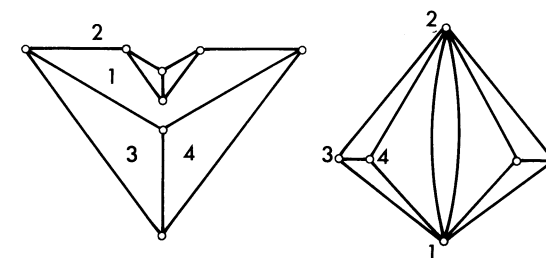
Figure 11. The dual of a polyhedron with two faces sharing several edges is a polyhedron with multiple edges between two vertices.

Constructing Projective Polyhedra.

From the point of view of spatial architectural structures it seems more useful to begin with a geometric mode of construction in space and then seek the underlying combinatorial pattern of the graphs which are created. In the context of non-convex polyhedra we allow projective cuts where the new plane may cut other planes in the exterior of the original face polygon, but this cut is then used to add a section to the face, as well as to remove sections when it meets the interior of the face polygon, **(Figure 12).** If we allow a sequence of these projective plane cuts on a tetrahedron, we will produce non-convex (and possibly self-intersecting) realizations of the same 3-connected planar graphs we studied as convex polyhedra. These are all the spherical polyhedra which satisfy the restrictive condition (6) given after our original definition.

An obvious generalization of this type of construction in space is the idea of double-plane cuts on an edge **(Figure 13).** The process of full-edge double-plane cut is self-dual, since it introduces two vertices and two faces, while removing an edge. However when a full-edge double cut is followed by one or several single-plane cuts, the sequence of cuts produces the pattern which is the dual of half-edge double-cuts (and even the dual of partial edge double-cuts, if we allow truncations to produce multi-graphs) **(Figure 14).** If we begin with the tetrahedron and close up under single cuts, and
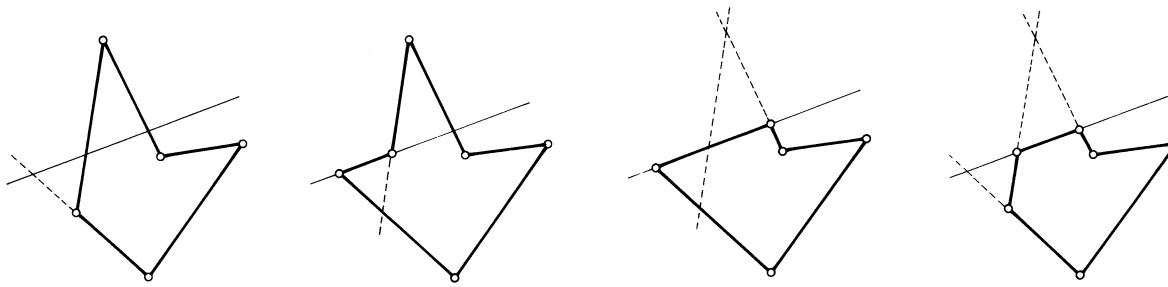
full or half-edge double-cuts, observing the restriction that the structure keeps a simple graph of edges (no multiple edges), then we create a self-dual class of generalized spherical polyhedra. We conjecture that **this class is characterized by planar graphs which are 2-connected in a vertex sense and are 3-connected in an edge sense** (removing 2 edges never disconnects the graph). Such a combinatorial class seems to be closed under face-splitting (if we forbid face splits which produce multiple edges) and is closed under topological hinging. However they are not closed under face-face vertex splits, since such a vertex split can turn a half-edge double-cut into a partial-edge double-cut (and a 3-edge-connected graph into a graph which is only 2-edge-connected). If we now close up under the partial—edge double-cuts, we create a class which is not self-dual. We conjecture that **this class consists of all spherical polyhedra which can be realized in space without self-intersection, and that it is characterized combinatorially by planar graphs which are 2-connected in a vertex sense and each vertex is at least 3-valent.** This is the same class we reached in our previous topological analysis.

Projective Diagrams

We also want to study configurations of lines in the plane to see when they describe the construction of a combinatorial spherical polyhedron in space. In general, given any combinatorial polyhedron we call a set of lines in the plane a **polyhedral completion** of the combinatorial polyhedron if there is a line for

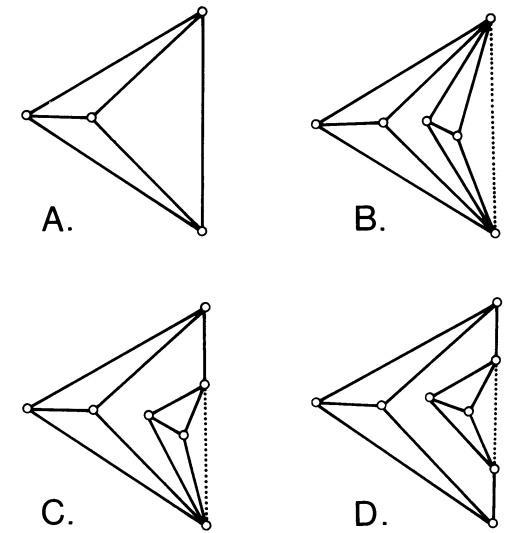Figure 13. An edge of a polyhedron (A) can be truncated by double-plane cuts in three ways: full-edge cuts (B), half-edge cuts (C), and partial-edge cuts (D).

Figure 12. Projective truncation of a face by a new plane produces several choices of a new truncated face.
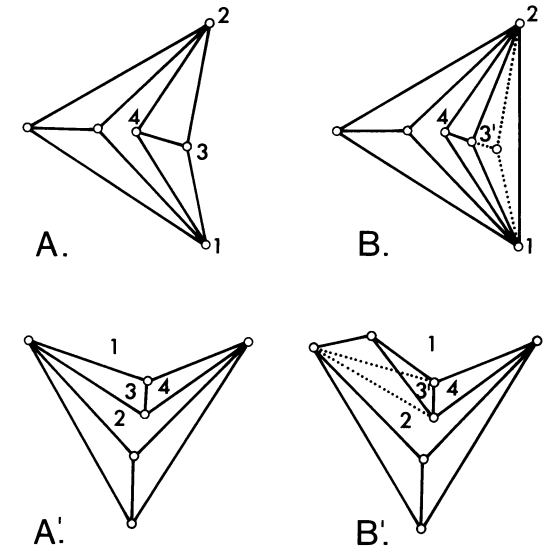
Figure 14. The dual of a double plane half-edge cut (B' to B) is also created by a full-edge cut followed by a single-plane truncation (A to B).

every pair of faces, all of the edges at a vertex form concurrent lines and for any three faces the lines 12, 23 and 31 are concurrent. For spherical polyhedra the set of original edges will at least form a 2-connected planar graph, and we have proven that every polyhedral completion of a spherical polyhedron is the description of a spherical polyhedron in space, possibly with some improper edges. For combinatorial spherical polyhedra, existence of a spatial realization over the plane diagram can be checked by a simpler diagram: the proposed edges of the polyhedron (with the appropriate incidences at the vertices) and the lines of intersection of all faces with a single plane zero (possibly one of the faces) **(Figure 15)**. The consistency condition remains the same: for every edge 12 the three lines 12, 01 and 02 must be concurrent. For a general sectioning plane 0 the lines 01 and 02 will be collinear iff the faces 1 and 2 are coplanar in the spatial polyhedron. As we noted for convex polyhedra, each configuration of the proposed edges and cross-section in the projection plane will correspond to an at least four parameter family of spatial polyhedra. In certain cases of a degenerate choice for the cross-section plane, this family will be larger.
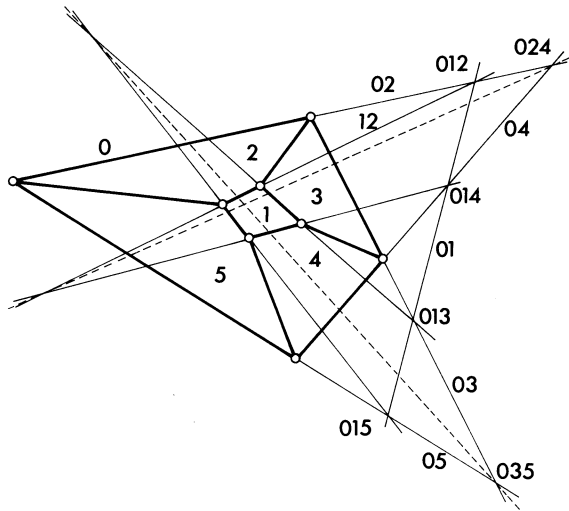

Figure 15. One cross-section of a projected polyhedron can be easily extended to show any other intersection line of two faces.

Counting the Number of Spatial Reconstructions

The key question for projective realizations is: given a diagram in the plane of the edges for a proposed projected spherical polyhedron, when can a consistent cross-section be constructed around these edges (or equivalently when is there a spatial polyhedron over these edges) and how many choices are there for this polyhedron? The problem of spatial reconstruction can be attacked in several ways. For each face F_i of the proposed polyhedron, we must choose a plane: $a_ix + b_iy + c_iz + d_i = 0$. If we assume that no faces are vertical to the projection plane placed at $z = 0$, then we can assume that $c_i = 1$, so each face introduces 3 variables. For each vertex $(u_j, v_j, 0)$ in the projection plane, we must choose a height z_j. Thus we have a total of $3F + V$ variables. The solutions for the polyhedron must satisfy a linear equation each time a vertex lies on a face: $a_iu_j + b_iv_j + z_j + d_i = 0$. A vertex with n entering edges will lie on n faces, and each edge enters 2 vertices, so the total number of equations is 2E. As we mentioned above each non-trivial solution for a spatial polyhedron belongs to a four-parameter class of equivalent solutions (3 for the first plane, a fourth for a point on the adjacent face which records the scale of the z axis, or equivalently, a dihedral angle). For convenience we will specify in advance one face which remains in the projection plane, so that the expected number of solutions becomes $N = 3F + V - 2E - 3$. If the equations are independent, and this is a positive number then this will describe the dimension of the family of possible polyhedra over the plane diagram. If the equations are independent and $N = 0$ then there is only the trivial solution and the polyhedron cannot leave the projection plane. Consider the example of a tetrahedron **(Figure 16)** $F = 4$, $V = 4$, and $E = 6$, so the expected number is 1. Every such diagram which is not all collinear has a 1-dim space of realizations with face abc fixed in the plane and vertex a at a variable height over the projection plane.

If we multiply Euler's equation by 3: $3V - 3E + 3F = 6$, and subtract this from the previous equation for N, we find that the expected space of polyhedra has dimension $N = E - (2V-3)$. This number is familiar to us from the article in this bulletin on rigidity, where it predicts the dimension of the space of stresses in

the plane, if the bars yield independent linear equations. Maxwell's theorem and its converses tell us directly that for a spherical polyhedron this space of stresses is always a record of the space of possible polyhedral realizations, with one face fixed in the projection plane. This result assumes, as above, that no face is vertical, and we usually think of the multiples of a stress as corresponding to the choice of a scale for the z axis, or equivalently the choice of one dihedral angle.

Results Drawn from Plane Statics.

If the planar graph has exactly 2V-2 edges, then the framework will contain at least one stress and at least one projected polyhedron. The known results from the statics of plane frameworks now give us additional information about possibilities for independent and dependent equations — facts which are not evident in the original analysis in terms of faces and vertices. For example, we know that if $E' \leqslant 2V'-3$ on all subgraphs with V' vertices and the diagram is in "general position" in the plane, then this stress will
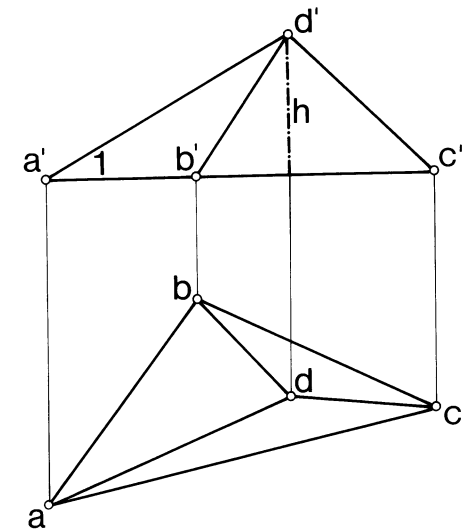

Figure 16. The projection of a tetrahedron permits reconstruction by the free choice of height h (or one angle 1) shown in a side view.

be unique, and will involve all of the edges. In this case the spatial polyhedron will have a bend at every edge of the graph. The cross-section will then be unique and every face will give a different line in a general crosssection.

However if this same graph is drawn in a special position, the situation can change. The cross-section may remain unique, but certain faces may become coplanar so that the polyhedron really involves a special subgraph of proper edges. If this subgraph has V' vertices and 2V'-2-k edges, the special position is forcing k edges to be flat (to separate coplanar faces) and this will appear in a general cross-section as the projective condition that 02 and 03 are collinear for each flat edge 23 or that the three points 012, 023 and 034 are collinear for each flat edge 23 **(Figure 17).** This appearance of a polyhedron with E' = 2V'-2-k edges can therefore be traced to k projective conditions in a general cross-section, each involving three points which must be, unexpectedly, collinear.
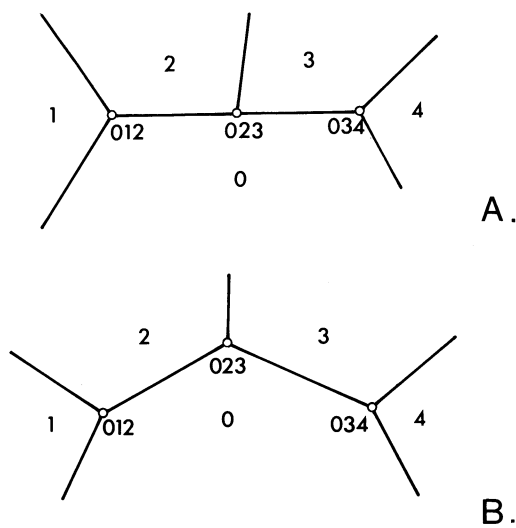
We conjecture that every polyhedron with E = 2V-2-k(k>0) edges can be constructed as a polyhedron with E = 2V-2 and E'≤2V'-2 for all subgraphs, and k flat edges. Whether this proves true or not, this analysis provides the form of the special conditions for any small (E≤2V-3) polyhedron which occurs in a graph. For example, in **Figure 18A** we see a projected polyhedron with E = 2V-2 and no flat edges. In **Figure 18B** we have added one projective condition, points 012, 034 and 013 are collinear, so faces 2 and 3 become coplanar and the polyhedron is formed with E = 2V-3 bent edges.

One type of face which always requires one projective condition to be realized is the three valent face, **(Figure 19).** For a triangular, 3-valent face we recognize that planes 1, 2 and 3 are copunctual, so lines 12, 13 and 23 must be concurrent, (or equivalently, 12 ∧ 23, 12 ∧ 13, and 23 ∧ 13 must be collinear!) For a trivalent quadrilateral face **(Figure 19B),** this calotte condition (see the article on rigidity in this bulletin) states that the points 123, 134 and 135 are collinear (along the line 13). This analysis can be generalized into an induction to show that
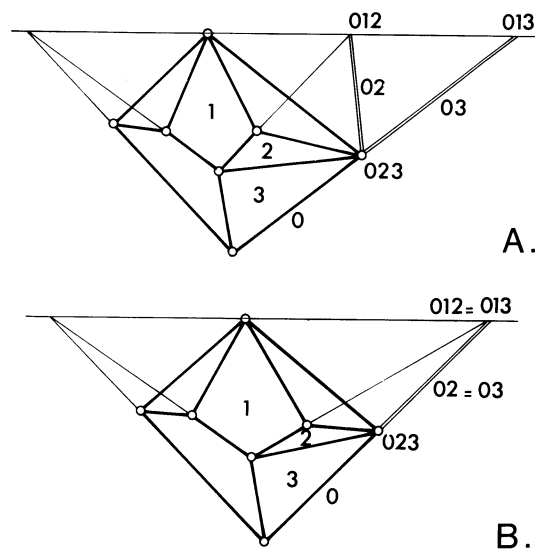
such trivalent faces always introduce one projective condition. Of course if this face is the base of a pyramid, (E = 2V0-2), all the exterior lines are concurrent at the peak, and the condition is trivially satisfied. Otherwise this result traces new projective conditions of a type which let us test any diagrams with lots of 3-valent faces, as well as letting us construct the cross-section. However, as the number of vertices grows, we obtain diagrams with E<2V-2 but no 3-valent faces, and it becomes more challenging to find the projective conditions, and the cross-sections.

If E' = 2V-2+k, k ≥ 1, then there will be a number of different stresses, and thus there will be a number of different subgraphs which each represent a projected polyhedron, and a corresponding cross-sectional completion. This family of possible polyhedra is represented by the vector space of stresses, a space of dim k+1 if the picture is in general position. We can visualize this vector space as a reading off of a set of dihedral angles in space (appropriately digested through trigonometric formulas). When the stress has a coefficient 0 on an edge, then the edge

Figure 17. An edge 23 is shown in two cross-sections: flat (A) and bent(B).

Figure 18. An entire polyhedron is shown in projection and cross-section with edge 23 bent (A) and flat (B).
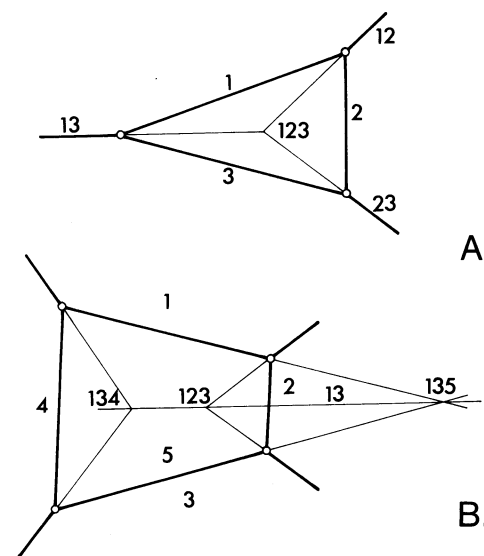
Figure 19. Trivalent polygons require one projective condition to represent faces of a polyhedron.

becomes flat. The space of all polyhedral realizations can be generated by a series of k+1 choices for dihedral angles, with all other angles determined by these choices **(Figure 20).**

If a number of appropriate projective conditions occur, then an initial edge diagram may have more polyhedral realizations than the count predicts. The space of realizations is still coordinatized by the stresses, and is algebraically recorded in the nullity of the coordinatizing matrix of the structure, as discussed in the article on rigidity.

Toroidal Polyhedra

When we move beyond the spherical polyhedra but stay with the oriented polyhedra, we are describing surfaces which can be formed, at least topologically into spheres with a number of handles H **(Figure 21)**
   Euler's formula has a modified form for these surfaces: V - E + F = 2(1-H). For each number of handles (each topological type), we can ask all the types of questions we asked for the sphere, but there are very few answers. Even when we limit ourselves to polyhedra satisfying restrictive condition (6), we do not know whether they can all be realized in space with plane faces and a bend at every edge. The combinatorial class is closed under face-splitting and vertex-splitting, but is the class of realizable toroidal polyhedra (1 handled polyhedra) closed under these operations? Can all of the realizable combinatorial toroidal polyhedra be formed by simple topological constructions from a single original example such as the complete graph on 7 points, (which forms a small toroidal polyhedron when we select 14 triangles for the faces in an appropriate way)? Can all such toroidal polyhedra be developed by a series of single-plane projective cuts and single-point stellations from the realizations of the complete graph on 7 points and its dual (a polyhedron with 7 hexagonal faces, 21 edges and 14 vertices)? Which toroidal polyhedra can be realized without self-intersection? Do we accept toroidal polyhedra for which every realization has some face cut by an edge of other faces (as happens with any combinatorial toroidal polyhedron smaller than the complete graph on seven points)?
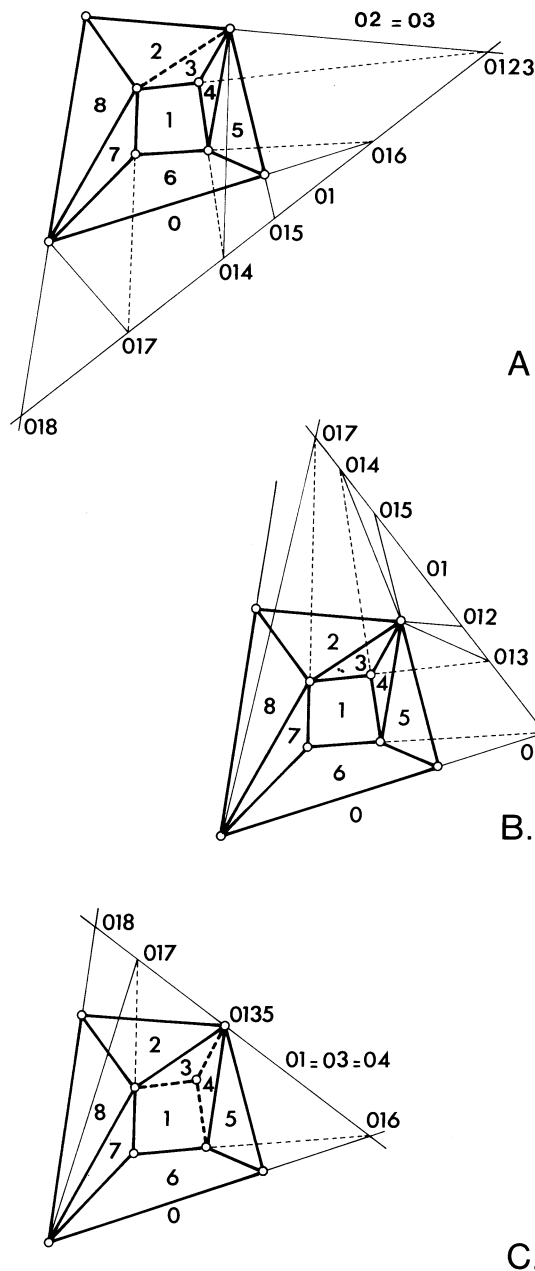

Figure 20. An ambiguous polyhedral projection has a 2-dimensional space of different realizations: (A) with edge 23 flat, (B) with bends at all edges (proper), and (C) with edges 34 and 14 flat.

At the level of projective geometry we have the corresponding problem of polyhedral completions in the plane. We conjecture that any polyhedral completion for an oriented polyhedron can be lifted up to the construction of a plane-faced polyhedron in space, but we are uncertain whether this completion can be simplified to a single cross-section, or perhaps two consistent cross-sections which would still guarantee the spatial construction.

The counting arguments for the number of spatial polyhedra to expect over the drawing in the plane of the graph of a combinatorial polyhedron with H handles can also be given. The expected number of solutions if the equations are independent, is still recorded by the number N = 3F + V-2E-3. Using the revised form of Euler's formula we obtain N=E-(2V-3)+ 6H. Thus for the torus, with 1 handle, we expect N = E - (2V+3) spatial polyhedra. A careful analysis of the plane statics provides the additional condition on a stress for there to be a corresponding spatial polyhedron: the net force across any closed path on the surface of the polyhedron is 0. For the projection of an H handled sphere, this condition gives 6H equations, as we would expect. The analysis of these conditions for non-spherical polyhedra is just beginning.

Concluding Comments

Overarching all of our analysis of realizability in general and projective drawings of the graphs in particular, is the question of algorithms — how can we decide when a diagram represents a projected polyhedron? By Maxwell's theorem and its converses we have an algebraic method based on the linear algebra used to determine the rank (and nullity) of the rigidity matrix. However, we have found that in all examples where there is a unique polyhedron present, there is a series of straight lines and derived points which can be drawn constructively from the diagram to recreate the cross-section (and even the polyhedral completion). As the number of vertices and edges has grown larger, we have employed procedures which still appear ad hoc but we wonder whether there is a practical algorithm in projective (or descriptive) geometry which will analyze the

drawing to determine whether there is any polyhedron present, and if there is, will create the cross-section of at least one of those present.

This decision problem for plane diagrams which claim to be the projection of spatial polyhedra also has direct application in computer graphics, where it is called scene analysis. In scene analysis the problem can be altered by omitting any hidden edges (edges which would lie behind some plane when projected) and permitting any form of hidden edges which will match a polyhedral image on the visible portion **(Figure 22)**. In this field the recent work of Huffman has reproduced some of the techniques which we have found in Maxwell's theory of graphical statics (Huffman, 1977, a,b,c). The reappearance of the technique of reciprocal diagrams, without any reference to statics, highlights the intrinsic geometric character of these methods which may have seemed artificial to the reader when they appeared in the plane statics. However we believe that the cross-sectional diagram and its projective constructions will emerge as a more direct approach than reciprocal figures of Maxwell and Cremona and the dual figures of Huffman for solving problems of spatial reconstructions from projected images.

In this article we have summarized several trends which have evolved within our group. Like so many of our results, the seeds come from examples and questions generated by Janos Baracs and his students over a number of years. These ideas have been further refined in our regular seminars, to the point where this presentation has become possible. We hope that a number of these problems will be solved in the near future, so that we can all proceed to the other, equally exciting problems which will follow in the wake of any solutions.

Explanation of Terms

Topological. Properties of rubber sheet geometry, unaffected by continuous deformations of the space.

Compact 2-manifold. A set of points in space (R^3, or R^n) which is topologically closed and bounded, such that the set of points in a neighbourhood of each point is a topological disc (equivalent to the interior of the unit circle in the plane). Also included are images (immersions) of such surfaces in R^3.

Topological sphere. Closed surface or manifold which is topologically equivalent to the unit sphere in R^2. Characterized as a compact surface, without boundary such that every closed path (image of the unit circle) separates the surface into at least two components, both topological discs.
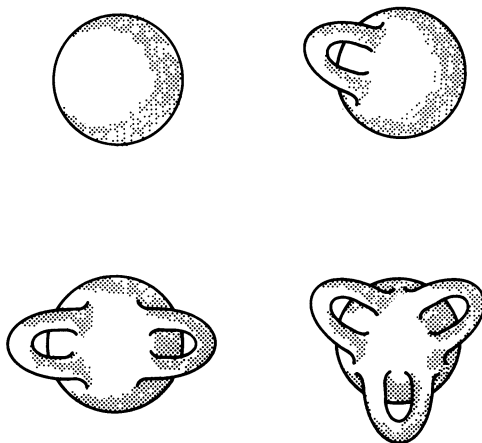
Graph. Set of vertices, and unordered pairs of vertices (edges). Technically, an undirected graph without loops.

Valence. The number of edges in a graph entering a vertex.

Drawing of a graph. A choice of points in the plane (or space) for the vertices of the graph and of arcs for the edges, such that two vertices joined by an edge are assigned different points.

Projective construction. A construction in the plane, or in space, which is preserved by any central projection. In the plane, this means constructions by pencil and straight edge (including points and a line at infinity), without reference to lengths, angles or parallels.

$$124 = 24 \wedge 12$$
$$134 = 13 \wedge 34$$
$$14 = 124 \vee 134$$
$$146 = 14 \wedge 16$$
$$46 = 146 \vee 346$$
$$145 = 14 \wedge 15$$
$$45 = 145 \vee 245$$

$$456 = 45 \wedge 46$$

Figure 21. Oriented topological manifolds can be visualized as spheres with 0, 1, 2, etc., handles.
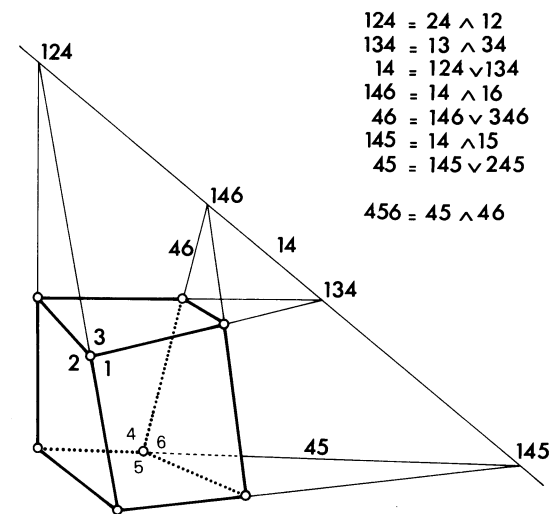
Figure 22. The hidden vertex of a projected cube is located by a direct projective construction.

Truncation. Slicing a polyhedron with a plane in projective space, to introduce a new face in the plane and remove a section of the polyhedron. As a general projective construction, this consists of choosing a plane P and a closed path of faces and edges on the polyhedron $F_1E_1F_2E_2...F_nE_nF_1$ where E_i will be vertex between F_i, F_{i+1} if the plane P passes through such a vertex and the chosen path separates the polyhedron into two pieces, one of which is a topological disc which is to be cut off. The new truncated polyhedron is then formed from the original polyhedron by

(1) splitting the edge E_i by a new vertex V_i where the line pierces the truncating plane, dropping the section of the edge which goes to the truncated disc.

(2) splitting any vertex on the original cycle, dropping all edges connected to the truncated disc.

(3) splitting the faces F_i by new edges $V_{i-1}V_i$, dropping the part of the face attached to the truncated disc.

(4) adding a new face $V_1V_2...V_n$ in the truncating plane P.

(5) dropping all vertices, faces and edges in the truncated disc.

Stellation. The projective dual of truncation and therefore the introduction of a new vertex and the removal of a section of the polyhedron. As a projective construction, this requires the choice of a point V for the new vertex, and of a cycle of vertices and edges on the polyhedron

$$V_1E_1V_2E_2...V_kE_kV_1$$

where E_i will be replaced by a face F_i if the new vertex V lies on the plane of a face between V_i and V_{i+1}, and the chosen cycle separates the polyhedron into two components, one of which is the topological disc which is to be eliminated. The stellated polyhedron is formed by:

(1) adding the new vertex V,

(2) splitting each vertex V_i of the cycle by dropping all edges connected to the eliminated topological disc and adding a new edge from V_i to the new vertex V,

(3) adding a triangular face VV_iV_{i+1} to each edge E_i of the cycle, dropping the old face at that edge which connects to the eliminated topological disc,

(4) modifying each face F_i in the cycle by adding the vertex V (and edges V_iV, VV_{i+1}) and dropping the portion attached to the eliminated disc

(5) dropping all edges, vertices and faces in the disc cut off by the cycle.

Figure 23. Some 3-valent 2-connected planar graphs will have a collinear vertex (A) or a collinear face (B) in every proper realization.

References

(Grunbaum 1967) provides an exhaustive (sometimes exhausting) exposition and bibliography on convex polytopes, while (Lyusternik 1963) provides a simpler, more readable, but sometimes erroneous survey of convex polyhedra. (Coxeter 1973) provides an exposition of regular polytopes, while (Hilbert 1952) provides the other extreme — a brief introduction to the combinatorial topology of polyhedra and surfaces. The construction of spherical polyhedra by inductive methods (truncations, stellations, etc.) is examined in recent writing for non-mathematicians (Loeb 1976). The only literature we have seen on recognizing projections of polyhedra is the very old work on graphical statics (Maxwell 1864) (Cremona 1890) and recent work on scene analysis (Duda 1973) (Huffman 1977, a,b,c). The construction of a polyhedral completion from a single projection originates with Janos Baracs, and is described in visual form in his course notes (Baracs 1978). The work of our group on the problem of projected polyhedra, Maxwell's theorem and related work on rigidity is available in preprint (Crapo 1978)(Whiteley 1978).

Added in Proof

Branko Grunbaum sent us a counterexample to the conjecture on page 50, column 1. In a simplified version (**Figure 23A**), we see a graph which will have a collinear vertex **a** in every proper realization. The dual graph (**Figure 23B**) will have a collinear face in every proper realisation. These examples also show that three constructions: topological hinging, face-splitting and **vertex-splitting** will all be required in order to construct the family of all 3-valent 2-connected planar graphs. This affects the conjecture on page 50, column 2.

We point out that the creation of convex polyhedra by truncation of a tetrahedron is, in general, a **projective** construction. While affine truncation of a tetrahedron in affine space can create a convex polyhedron projectively equivalent to any given convex polyhedron, it is impossible thus to create an affine equivalent of the regular cube. Throughout the above article, constructions described are projective unless otherwise specified.
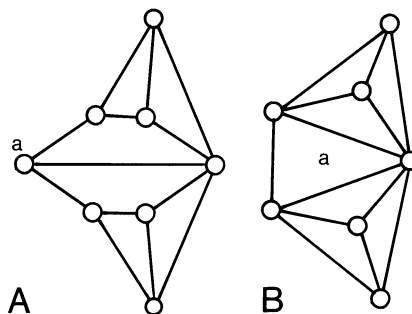
Bibliography

The code in the first block of each bibliographic item consists of three parts, separated by dashes. The first letter indicates whether the item is a

B ook
A rticle
P reprint, or
C ourse notes.

The middle letter(s) indicates whether the piece was intended primarily for an audience of

M athematicians,
A rchitects, or
E ngineers.

The final letter(s) indicates if the piece touches on one or more of the principal themes of our work:

P olyhedra,
J uxtaposition' or
R igidity.

The key words or other annotations in the third column are intended to show the relevance of the work to research in structural topology, and do not necessarily reflect its overall contents, or the intent of the author.

Baracs 1978 Janos Baracs C—AME—PJR	**Topologie Structurale.** Course Notes, U. de Montréal, Montréal, Québec, 1978.	Survey of field: polyhedra, juxtapositions, rigidity.
Crapo 1978 Henry Crapo and Walter Whiteley P—M—RP	**Stressed Frameworks and Projected Polytopes** Groupe de Recherche Topologie Structurale, U. de Montréal, Québec, 1978	Bar and joint frameworks, static stress, projected polyhedron
Coxeter 1973 H.S.M. Coxeter B—M—P	**Regular Polytopes** Dover Publ., New York, 1973	Regular solids, tesselations, polytopes, star polyhedra and star polytopes
Cremona 1890 Luigi Cremona B—ME—RP	**Graphical Statics (Translation of Le figure reciproche nelle statical grafica, 1872)** Oxford University Press, London, 1890	Projective geometry, skew polarity, reciprocal diagrams, stress diagrams
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Whiteley 1978 Walter Whiteley P—M—RP	**Motions of Panels Structures, Stresses in Frameworks and Polyhedral Projections.** Groupe de Recherche Topologie Structurale, U. de Montréal, Québec, 1978.	Articulated panel structures, motions, frameworks, static equilibria, stress, projected polyhedral, convexity, tensegrity frameworks.