Structural Rigidity

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Abstract

This article summarizes the presently available general theory of rigidity of 3-dimensional structures. We explain how a structure, for instance a bar and joint structure, can fail to be rigid for two quite different types of reasons. First, it may not have enough bars connecting certain sets of nodes. That is, it may fail for **topological** reasons. Secondly, although it may "count" correctly, it may still fail to be rigid if it is set up with some special relative positions of its nodes and bars. This second type of failure is a question not of topology but of **projective geometry.**

Introduction

What distinguishes structural engineering from mechanical engineering is the special attention paid to the question of rigidity.* Whereas mechanisms* (linkages) are useful primarily by virtue of the relative motion of their parts, buildings must be designed to stand rigid, and to continue to stand when subjected to a variety of external forces such as gravity, loading and wind pressure.

In this article we sketch the presently available general theory of rigidity for three types of structures: (I) bar and joint structures*, typified by trusses in wood, or the bolted ironwork introduced in the last century, (2) strut and cable structures*, such as the tensegrity structures popularized by Buckminster Fuller in the fifties, which maximize reliance on the available tensile strength of wire cables, and (3) hinged panel structures*, explored recently by Janos Baracs and his students, which expand the potential for construction with prefabricated concrete or moulded plastic panels. With regard to each of these types of structures, we shall point to a number of challenging unsolved problems.

The rigidity we speak of is such that rigid structures will resist even **infinitesimal** motions. To be non-rigid in this sense, a structure need not be a true mechanism, with an easily observable motion. For mathematicians we can make the distinction clear by saying we will be using linear algebra and projective geometry rather than differential geometry.

From the standpoint of geometry, we are looking at structures whose component parts are line segments joined to one another at nodes, or else polygonal pieces of planes in space, joined to one another along edges. These joints are **articulated**, so the bars at a node are at least locally free to change in angle relative to one another, and the panels are merely hinged, rather than welded, to one another. In **Figure 1** we show (related) structures, each just rigid, of the three types.

Bar and Joint Structures

Rigidity theory for bar and joint structures in the plane was already well advanced in the latter half of the nineteenth century, thanks to the efforts of the English physicist James Clerk Maxwell and the Italian geometer Luigi Cremona (Maxwell 1864 and Cremona 1890). It was their theoretical advances which led to the development of graphical statics* as a practical discipline, at the hands of the German engineers Culmann, Henneberg and their followers (Culmann 1875 and Henneberg 1886). But during the last fifty years, the rigidity theory for bar and joint structures has been much neglected, and the best work of the last century has, for the most part, been forgotten. It is one of the tasks of our research group to correct, to extend (particularly from structures in the plane to structures in space), and to apply in new ways geometrical methods initially developed under this heading of graphical statics.

Each bar and joint structure is given topologically as a **graph** consisting of **nodes,** certain pairs of which are joined by **bars.** Each subset of the set of bars of a given structure determines a **substructure** on the same set of nodes. We will discuss rigidity of structures primarily at the level of projective geometry, where the nodes of a structure are assigned positions in projective space (usually in a plane, or in 3-space) and each bar is represented as a line segment connecting its two nodes. (The degenerate case where the two nodes of a bar are in the same position we handle by assigning a direction to the bar at that point.) To say we are working in projective space means on one hand that we are making no use of angles or distance measurement or of the concept of being parallel. It also means that we have available all the points and lines on a "plane at infinity" which are missing from affine 3-space, so any line not lying entirely in a given plane must meet the plane in a single point. Two distinct coplanar lines always meet in a single point, and two distinct planes always meet in a line.

The following introductory material we present for structures in three-dimensional space, and will make the obvious restriction to structures in the plane when it becomes necessary to do so. If a bar is

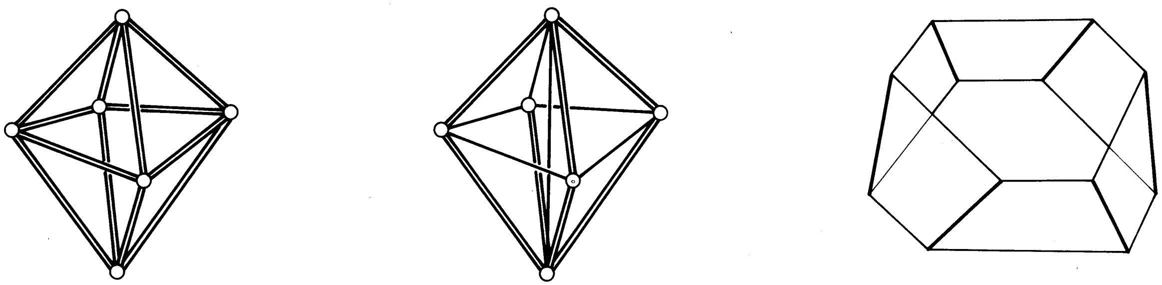


Figure 1. Just-rigid structures, of three types.

Note to Architects and Engineers

This article is written by a mathematician, and is intended primarily for readers with a mathematical background. As such, an effort has been made to give mathematical definitions of terms commonly used in statics, to place the entire discussion in a context familiar to readers with some experience in linear algebra and projective geometry, and to lay the groundwork for future research.

As a result, structures are treated here in a way which may seem unduly abstract, and unrelated to any direct and practical structural application. Even the examples given are those dictated by the theoretical development rather than those arising in architectural or engineering practice.

Part of this difficulty is in the nature of the subject, and must be patiently studied by practitioners and theoreticians alike. For instance, there is a dependence among 22 specified bars in the structure illustrated below, a dependence which causes the octahedral-tetrahedral truss to be non-rigid. This type of phenomenon must be understood in itself, and cannot be eliminated by simple algorithms and practical rules of thumb. (This example will be the subject of a short article in **Structural Topology (2).**

The other half of the difficulty in exposition can and should be corrected by the publication of further articles intended specifically for those with architectural or engineering training, perhaps with less mathematical background. In the second issue of the Bulletin, we will include a general exposition on structural rigidity, summing up the state of the art for those who wish to apply these methods in practice. The discussion and examples will be supported by intuitive and risual evidence, rather than by higher mathematics.

Such "translations" will be a regular feature of the Bulletin, and will occur in both directions. Theoretical papers will be "translated" to draw out their intuitive content and practical consequences. Papers describing practical applications will be "translated" into scientifically precise language, to reveal some camouflaged but interesting and unsolved mathematical problem.

Although we may take occasion, as we have here, to point out that a certain article is intended for readers with some specific training, we do not intend to discourage readers from attempting to digest articles posed in language and expressing ideas from fields other than their own. Also, though it is unavoidable at this stage that we write some articles differently for different audiences, it is in the nature of this publication project that "translations" will be less necessary as we go along.

subjected to forces applied at its two ends, it will tend to move unless the two forces are equal in magnitude, opposite in direction, and directed down the line of the bar. In this single case the bar is in **equilibrium** under the applied load; it is either in **tension** or in **compression.** Using this idea as a starting point, we apply one of two theorems from elementary linear algebra, to arrive at a very simple explanation of the basic concepts of statics and mechanics of structures.

With any structure S having V nodes and E bars in 3-space, we associate a matrix M = M(S) called the **coordinatizing matrix** * of the structure, which has E rows and a total of 3V columns, arranged in V groups of 3 columns each. The entries in this matrix consist of six possibly non-zero entries in each row: if the row is that corresponding to a bar between nodes in positions a and b, then in the three columns for the node a we have the components of the vector a-b, and in the columns for the node b, the vector b-a. Multiplication by this matrix H is a linear transformation ·M from R^E to R^{3V}, and will convert an assignment s of scalars to the bars into an assignment sM of 3-vectors to the nodes. (A different matrix in which vectors of unit length replace the vectors a-b, gives a more useful interpretation of bars of length zero, but we shall not get into that here.)

If we think of the scalars s as assigning a compression (if positive) or a tension (if negative) to each bar, measured in force per unit length, then the result sM of multiplication by M is the resultant force on each node due to the combined effect on that node from the tension and compression in the bars incident with that node. **Figure 2** shows one such resolution on a structure which is a skew quadrilateral in space. For any scalar assignment s to the bars, the negative of the resultant -sM is an **equilibrium** system of forces on the nodes, a system of forces which will have no tendency to move the structure in any way. If the forces -sM are applied externally to the nodes, the tension-compression assignment s is one possible static response of the structure to the applied load. Any two distinct possible static responses s, t to the same external load -sM = -tM differ by a scalar assignment s-t such that (s-t)M = 0, a scalar assignment which produces no resultant force on any vertex. It is thus an internal tension compression equilibrium, which we call a **stress** * in the structure. (In engineering terminology, stress means force per unit cross-sectional area in a bar, so the scalar we assign to a bar must be multiplied by the length of the bar and divided by the cross-sectional area, to give the conventional measure of "stress".)

Let r(S) denote the rank of the matrix M, and call this the **rank** * of the structure. Let n(S), the **nullity** * of S, denote the dimension of the kernel of · M as a linear transformation from R^E to R^{3V}. Then r(S) = n(S) = E, where E is the number of bars, r = r(S) is the dimension of the space of resolvable external (equilibrium) loads, and n = n(S) is the dimension of the space of stresses (internal equilibria).

For example, the skew polygonal structure in **Figure 2** cannot normally be stressed. In order for a non-zero tension-compression assignment to resolve to zero at a node, the two bars at that node must be collinear. If the stress is non-zero in one bar, it must be non-zero in the two adjacent bars. Thus the structure normally has rank 4, nullity 0, and has rank 3, nullity 1 if and only if the structure lies entirely along a straight line in space. This is the simplest example of the phenomenon which is the main object of our study: under certain **projective geometric conditions,** a structure will have a lower rank than would be expected from purely topological consideration.

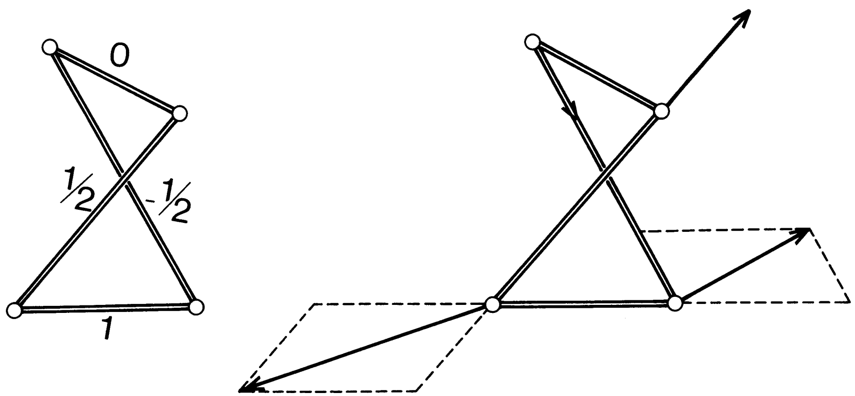
An external equilibrium load, in general, is a system of vectors acting along lines in space, whose vector sum is zero and whose total moment is zero about any axis. An arrow from a point b to a point a can be viewed as a force acting at a point b, in the specified direction and with the magnitude |a—b| . We coordinatize each such force f = ab as a 6- vector

$$(f_1,\ldots,f_6) = (a_1-b_1,\ a_2-b_2,\ a_3-b_3,\ a_2b_3-a_3b_2,\ a_3b_1-a_1b_3,\ a_1b_2-a_2b_1)$$

where (f_1,f_2,f_3) is the free-vector difference a-b, and (f_4,f_5,f_6) is the free-vector cross product axb, whose components are the moments of the force about the three coordinate axes. Thus coordinatized, the force is also known as a **line-bound vector,** because the 6 coordinates are the same for any other arrow of the same length and orientation along its own line in space, but are different for all arrows on other lines. The six coordinates of a force always satisfy the quadratic condition

$$f_1f_4 + f_2f_5 + f_3f_6 = 0$$

but this relation does not hold for the coordinates of a sum of two forces unless the forces act along coplanar (that is, intersecting) lines. The sum of two coplanar forces is the free-vector sum of those forces, acting along a line through the intersection point of their lines of action. Of course this intersection may be at infinity, in which case the sum is either another parallel and coplanar force, or is a **couple** * which when applied to a body will tend to rotate it about an axis perpendicular to the plane of the couple.



	a	b	c	d
ab	a-b	b-a	0	0
bc	0	b-c	c-b	0
cd	0	0	c-d	d-c
ad	a-d	0	0	d-a

Figure 2. A spatial structure, its coordinatizing matrix, and an equilibrium system of forces equalized by given tensions and compressions in the bars.

Two non-coplanar forces, jointly acting on a body, will tend both to translate and to turn the body. Such sums of forces are known as **wrenches** * . They are all coordinatized by 6-vectors, but only if they are forces do their coordinates satisfy the relation $f_1f_4 + f_2f_5 + f_3f_6 = 0$ The idea of coordinatizing forces and their sums in this way is due to Plucker and Grassmann, and is part of **exterior algebra** (Plucker 1865). The line-bound vectors are the **extensors** of step 2 in this algebra. To say that a sum of extensors is zero is precisely to say their free vector sum is zero and their net moment is zero about some (or any) point as centre.

For a (non-collinear) structure with V nodes, there is a 3V-dimensional space of possible assignments of forces to its nodes. Addition of bound vectors is a linear transformation of this space R^{3V} into the space R^6. The kernel Eq of this "addition" transformation A is the 3V-6 dimensional space of **external equilibria** * , those assignments which sum to zero. Since a simple tension or compression in any one bar of the structure produces an equilibrium configuration of forces on the structure as a whole (being two equal and opposite forces acting on the same line), the same will be true (by linearity) of any assignment of tensions and compressions to the bars. Thus the image of the linear transformation $\cdot M$, right multiplication by the matrix M, is a subspace of Eq. The rank of the structure is the dimension of the range of $\cdot M$, so its rank is not greater than 3V-6, where V is the number of nodes. If r = 3V-6, the structure will resolve any external equilibrium load, and we say the structure is **rigid** * .

Mechanics of Structures

Rigidity of structures is a question not only of statics but of mechanics. Whereas right-multiplication by the coordinatizing matrix M of a structure shows how tensions and compressions in the bars resolve into forces on the nodes, left-multiplication by M shows how any movement of the nodes produces changes in the lengths of the bars: if the nodes a and b move with velocities va and vb respectively, a bar from a to b will be forced to undergo a change of length at a rate equal to

$$(v_a - v_b) \cdot (a-b)/|a-b| ,$$

the rate of **strain** in that bar. Notice that

$$(v_a - v_b) \cdot (a-b) = v_a (a-b) + v_b (b-a)$$

is the ab component of the product Mv. So the image of this linear transformation $M\cdot$ from R^{3V} back to R^E is isomorphic to the space of strains producible in the structure by moving its nodes. The kernel of the linear transformation $M\cdot$ is the space of those velocity assignments to the nodes which produce no strain in any bar of the structure, to a first order approximation. We call these the (infinitesimal) **motions*** of the structure. **(Figure 3)**

Since a matrix and its transpose have the same rank, the image of the linear transformation $M\cdot$ is also of dimension no greater than 3V-6, and its kernel must have dimension at least 6. In fact, the kernel always contains a specific 6-dimensioal subspace, the space of **isometries** * or **rigid motions** * of the (non-collinear) structure, those motions which produce no change of distance between any pair of nodes, whether or not they are connected by a bar. An isometry of 3-space is easily described as a rotation about an axis in space, combined with a translation along that axis, that is a **screw motion** * of space, or else a pure rotation or pure translation. In general the motion space of a structure has dimension 3V-r $\geqslant$ 6, so r $\leqslant$ 3V-6. The quotient space of motions modulo isometries is the space of **internal motions** * of the structure. The space of internal motions is trivial if and only of the structure is rigid.

The geometry of wrenches and screws in 3-space was extensively studied at about the turn of the century. See treatises: (Ball 1900), (Study 1903), (Jessop 1903) and (Reye 1907). See also (Klein 1939), of which the first German edition was 1908, for a nice introduction to the subject.

As is true whenever a linear transformation is represented by matrix multiplication, the subspace of R^{3V} consisting of vectors orthogonal to all rows of M is both the kernel of $M\cdot$ and the orthogonal complement of the range of $\cdot M$. Thus the space of motions of a structure is the orthogonal complement of the space of resolvable external equilibria. Similarly, the space of stresses in a structure is a structure is the orthogonal complement of the space of producible strains, both spaces being the space of vectors orthogonal to all columns of M. Any assignment f of forces to the nodes of a structure can be uniquely expressed as the sum f = m+e where m is a motion of the structure and e is a resolvable external load. Similarly, any assignment t of tensions and compressions to the bars of a structure can be uniquely expressed as the sum t = s+p where s is a stress and p is a producible strain. The relation between stress and strain is a matter of **elasticity,** which we cannot work out unless we include additional information about the elastic moduli of the bars, and use Hooke's law. The relation between force and motion is a matter of **inertia** (center of gravity and moment of inertia, for rigid structures), which we cannot work out unless we include additional information about the distribution of masses, say concentrated at the nodes, and use Newton's second law. Nevertheless it is tempting to conjecture that the orthogonal decom-

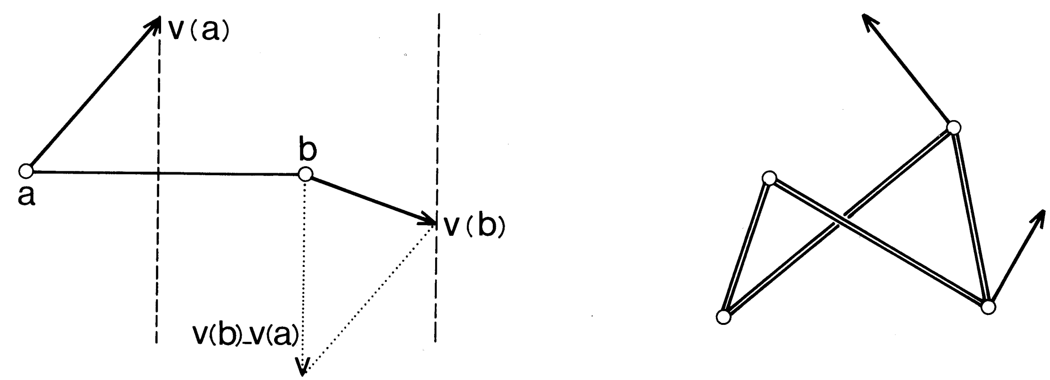


Figure 3. An internal motion of a structure.

positions given above are in fact those occurring in nature, irrespective of elasticity and mass distribution. This is accepted as true for forces applied to rigid bodies (see for instance (Houston 1948, page 161). That is we conjecture that even though we do not know the specific response of a structure to a given applied load, we can predict the portion of the load which produces motion (thus kinetic energy). Then we expect there is a dual statement, presumably about strains, which also makes physical sense. We summarize the various relations between subspaces of R^E and R^{3V} in **Figure 4.**

Dependence

If the rank of a structure is equal to its number of bars (r = E), then it may be built up bar by bar, so that each successive substructure (each on the full set of nodes) will have a strictly smaller space of internal motions. Such a structure is **independent** *, and is characterized by the fact that it is as rigid as possible, given its number of bars. Examples of structures which are independent and rigid in 3-space are easily built up node by node, starting from a tetrahedron, as in **Figure 5.** Each successive node is attached to three nodes of the previous substructure, and is kept out of the plane of those nodes.

If the rank of a structure is less than its number of bars, the structure is **dependent** * . Since we are interested in the reasons for a structure to be dependent, we shall concentrate our attention on structures which are **minimal dependent,** which we call **circuits** * . Circuits have nullity 1, and are minimal with that property. They carry a stress which is non-zero on every bar, a stress which is unique up to a common scalar factor throughout. In **Figure 6** we draw a number of circuits, some of which are dependent for topological reasons, others for projective reasons. Dependent structures are **statically indeterminate** *, in the sense that there is some supportable external load which is in equilibrium with respect to **more than one** internal configuration of tension and compression. It is thus dependence of structures which makes it impossible to single out a specific **response** to external loading, without introducing into the theory the motion of elasticity.

Say we want an independent rigid structure: one which is rigid, and minimally so. If such a structure has V nodes, it must have exactly 3V-6 bars. Also, no proper subset of V' < V nodes can have more than 3V'-6 bars joining them, because such a substructure would be dependent, and would cause the entire structure to be dependent. But for structures in 3-space, such a count of the number of bars in various substructures is not even a correct topological estimate of rigidity. We must also take hinges into account, as the following examples show. The structures in **Figure 7** satisfy the above conditions as to the number of bars in various substructures, but are nevertheless dependent. They have been put together as follows. Take two circuits, such as a complete 5-graph, join them along a bar, then remove that bar to form a **hinge** * . If this process is repeated k times,

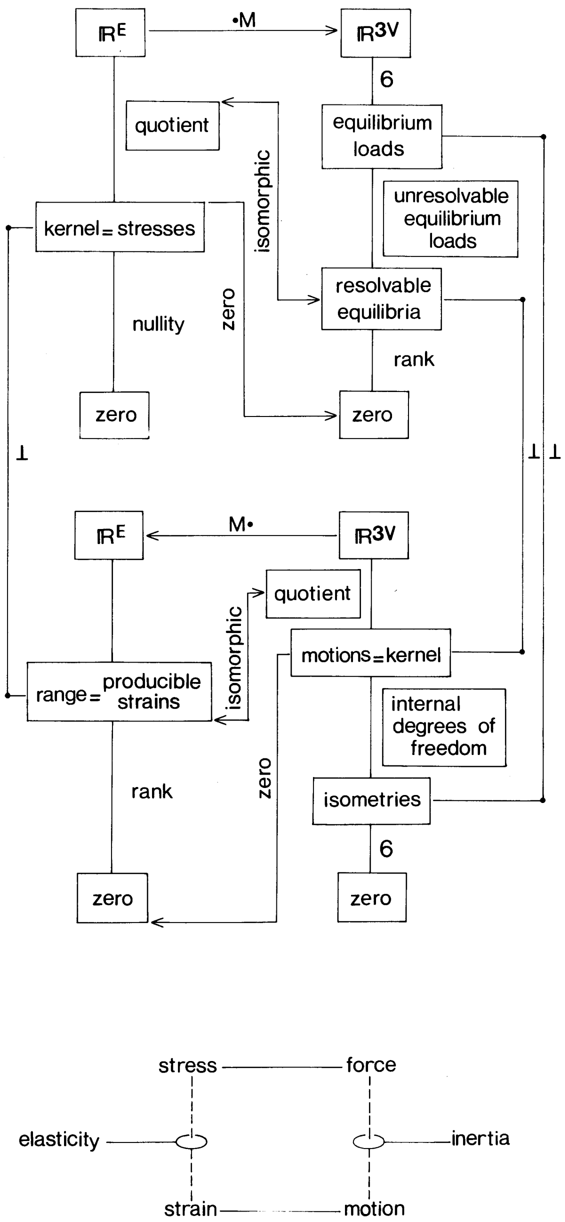


Figure 4. Summary of relations between subspaces, for statics and infinitesimal mechanics. The diagram shows containment of subspaces, difference in rank, and the role of orthogonal complementary subspaces.

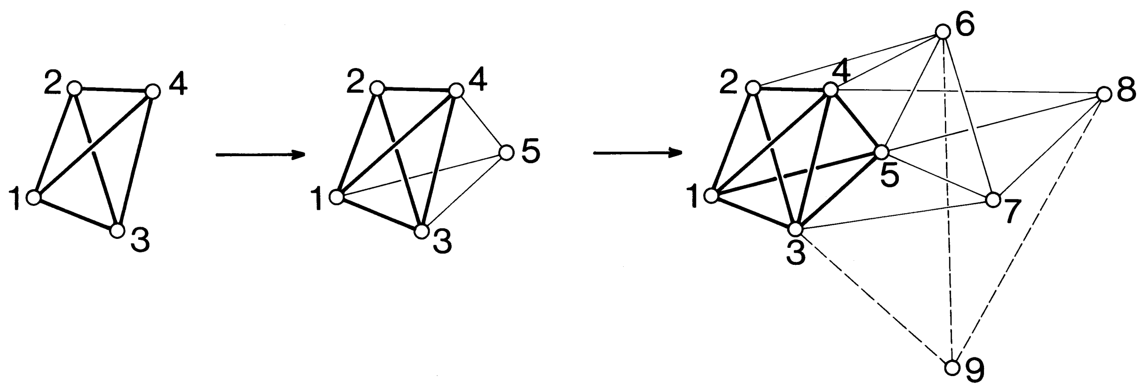


Figure 5. A sequence of simple structures which are independent and rigid in space.

we have a circuit with 3V-5-k bars, to which k-1 bars can be added to form a dependent structure with 3V-6 bars. The two added bars in the second example of **Figure 7** are drawn with broken lines, so as to reveal the underlying circuit which has 34 = 3V-8 bars on V = 14 nodes.

These examples illustrate some of the known topological reasons a bar and joint structure may be dependent: some substructure may have too many bars or too many hinges. But a structure may be dependent also for purely projective reasons. We give a few examples quickly at this point, and shall return to each of them in due course, to explain how we know they are dependent under the specified projective conditions. As a first example, the skeleton of the octahedron in **Figure 8** is dependent (with rank 11, nullity 1, and one internal motion) if and only if four alternate faces of the octahedron meet at a point in space. (If one set of four planes meet, the other set of four will also.) The structure in **Figure 9,** which is the cone over a polyhedral graph, is dependent (with rank 17, nullity 1, and one internal motion) if and only if the lines L and M through the node **a** lie in a plane which contains the node p, where L is the line through **a** meeting the lines bc and de, while M is the line through **a** meeting both cg and ef. The structure in **Figure 10** is dependent (with rank 17, nullity 1, and four internal motions) if and only if two conditions hold: the planes abc, a'b'c', a"b"c" meet in a line, and the planes aa'a", bb'b", cc'c" meet in a line. Finally, the 1-skeleton of the cube in **Figure 11** is dependent (with rank 11, nullity 1, and seven internal motions) in a non-coplanar configuration if and only if every one of its eight nodes has its three incident bars in a plane. This is possible when the nodes lie four each on two distinct planes, as shown.

Plane Structures

Our research group has come to understand the dependencies just described by extrapolating from the much better understood situation for bar and joint structures in the plane. We summarize below what we know about structures in the plane, and our

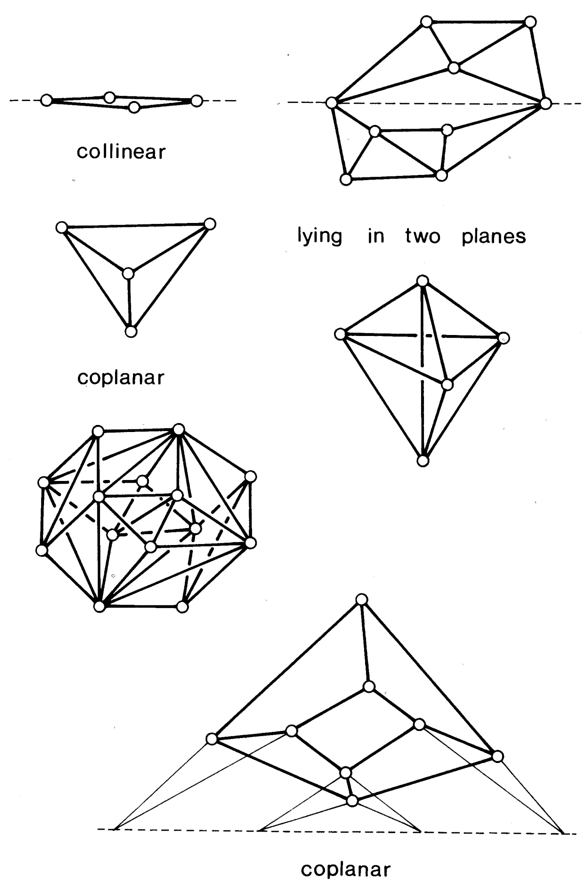


Figure 6. Some circuits (minimally dependent structures).

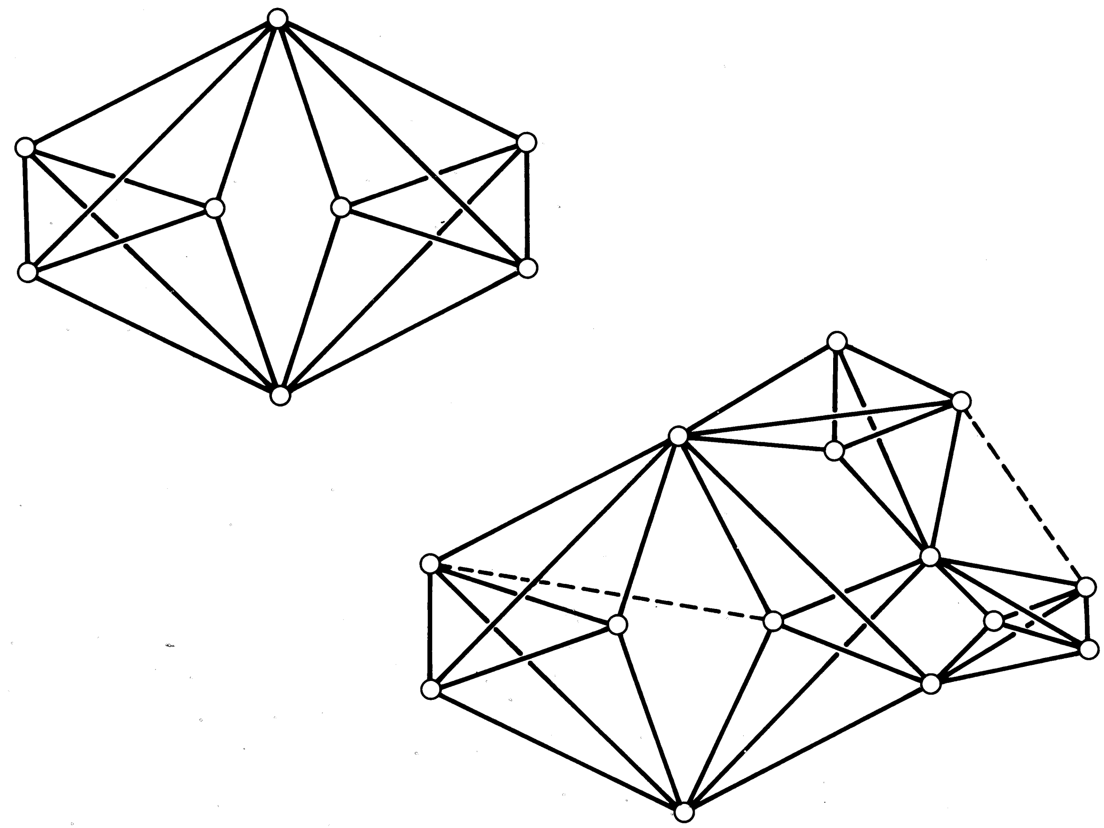


Figure 7. A generic circuit with a hinge, and a 3V-6 structure having a substructure with 3 hinges. Both have one internal degree of freedom.

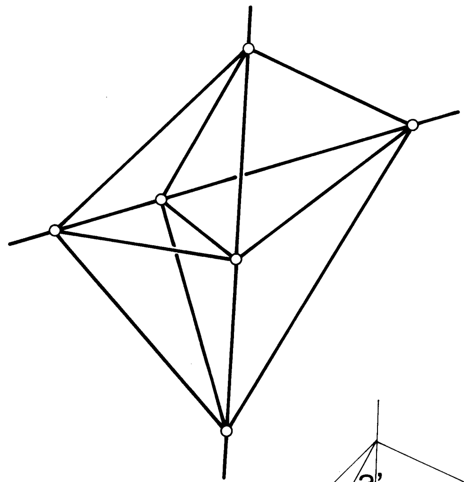
various attempts to extend results concerning plane structures to spatial structures.

If a structure on V nodes lies in a plane in 3-space, each of its nodes has an allowed motion in a direction perpendicular to the plane of the figure, all other nodes remaining fixed. These motions of single nodes span a V-dimensional space, a 3-dimensional subspace of which consists of isometries. Thus the minimum dimension of the motion space of a plane structure with V nodes in 3-space is $(V-3) + 6 = V+3$, and the maximum rank of a plane structure with V nodes is $3V - (V+3) = 2V-3$. If such a plane structure has rank 2V-3, then it has no internal motions in its own plane, and is thus **rigid in the plane.** Simple plane-rigid structures can be built up node by node, starting from a triangle, if we support each new node in succession by 2 bars to nodes in the so-far-constructed plane rigid structure, taking care only to place each new node off the straight line of the two nodes to which it is to be linked. Any such simple structure has 2V-3 bars, is independent and rigid.

The **only** topological reason for a plane structure to be dependent is that for some subset of $V' \leq V$ nodes, the substructure on that set of nodes has more than 2V'-3 bars. See (Laman 1970) and (Asimov 1978). So the mechanical properties of **generic*** plane structures (plane structures in general position) are completely determined by the count of the number of bars on each set of nodes. **Figure 12** shows generic plane structures which are circuits. Some of these were constructed by hinging two or more copies of the complete graph K_4, and the 3-connected examples were constructed from K_4 by a sequence of operations in which a **face** of the polyhedral graph is **split** by adding an edge from a vertex to the mid-point of some edge. Note that hinging and vertex-edge face-splitting both preserve the correct count of 2V-2 bars on V nodes, for topologically determined circuits.

The **projective** reasons for dependence in plane structures are also known, and to our knowledge were first available in the work of James Clerk Maxwell. Given a plane structure whose graph is three-connected and planar, it is at least topologically (by Steinitz' theorem) the projection of a spherical polyhedron. (By a **spherical polyhedron** we mean a polyhedron whose topology is that of the sphere. Steinitz states further that the polyhedron can be taken as convex, but that is not what we wish to emphasize at this point.)

What Maxwell proved is that if a plane structure is exactly **(projectively)** the projection of a spherical polyhedron with faces flat in 3-space, it is dependent. Members of our research group have recently proven the converse. Maxwell's idea was as follows. Given a spherical polyhedron in space whose projection is a plane structure S, choose an arbitrary **centre** c for the construction, and erect a line perpendicular to each face, through c. These perpendiculars may be joined in pairs to form planes perpendicular to the edges of the polyhedron. Finally, this entire cone may be intersected with a plane Q, to yield a plane structure S' whose graph is the (spherical, or spatial) dual of the 1-skeleton of the original polyhedron. If the polyhedron is also taken by perpendicular projection into the plane Q, each bar of the structure S is perpendicular to the corresponding bar of the structure S'. The structures S and S' are called **reciprocal figures***. Each determines a stress in the other, as follows. Since the oriented edges around any face of S' form a set of vectors with sum zero, they may be systematically rotated (say counter-clockwise) by a right angle and may then be used as the tension and compression forces of the bars of S. See **Figure 13,** which shows



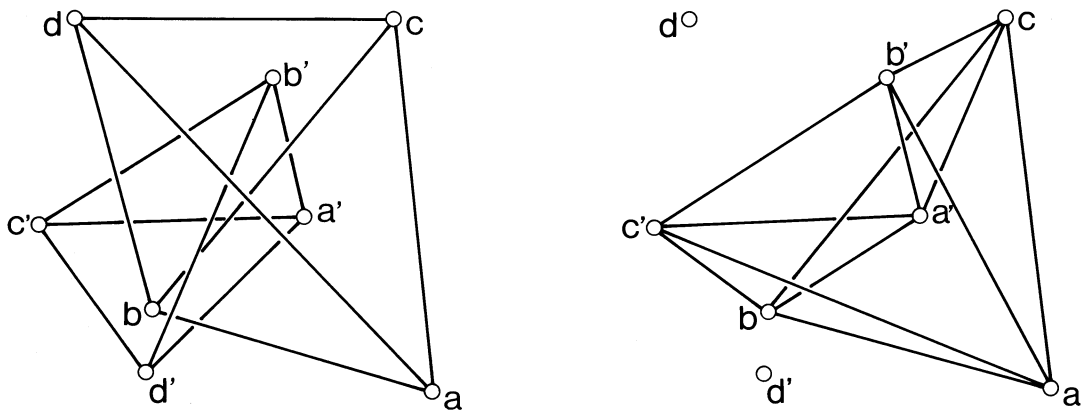


Figure 8. A dependent octahedron. The six derived points occur along the intersection of two planes in space, so they are collinear in any projection of the figure.

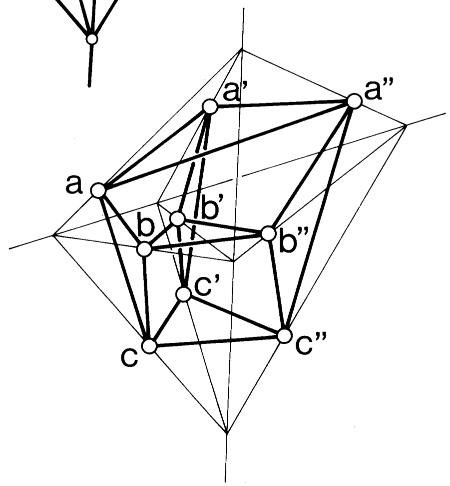


Figure 10. A dependent spatial structure with a 4-polytopial graph. The nodes may occur anywhere along the lines of the accompanying construction, the meeting of two triples of collinear planes. Thus the dependent structure need not be actually the projection of a 4-polytope.

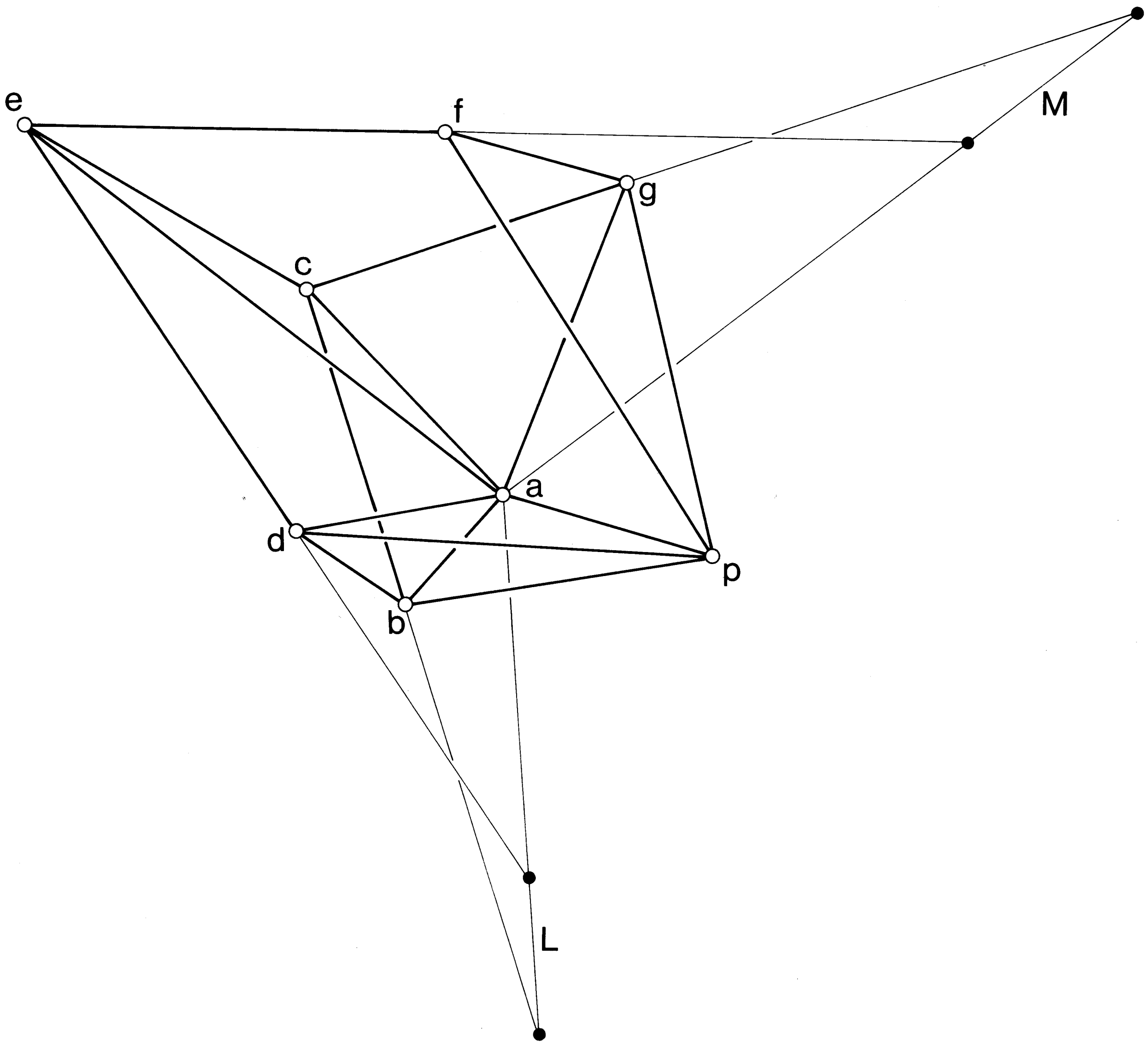


Figure 9. A dependent cone over a polyhedral graph.

pairs of reciprocal figures, some drawn parallel rather than perpendicular, all of which appeared in Maxwell's original papers, (Maxwell 1864) and (Maxwell 1870). Any such pair of reciprocal figures in the plane are obtainable by simultaneous projection of a pair of polyhedra which are images of one another under a **polarity*** of 3-space (Crapo 1978). The corresponding scalars of dependence for each of the plane structures give the rates of change of the dihedral angles of the projected polyhedra as they move to take up their "inflated" positions in space (Whiteley 1978).

Maxwell also showed how to reduce plane structures with non-planar graphs to structures with planar

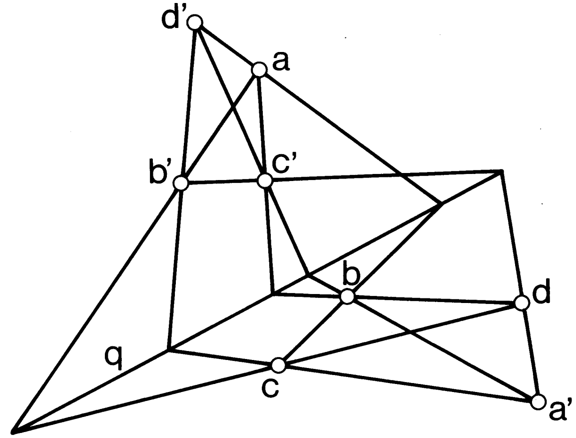


Fig. 11a.

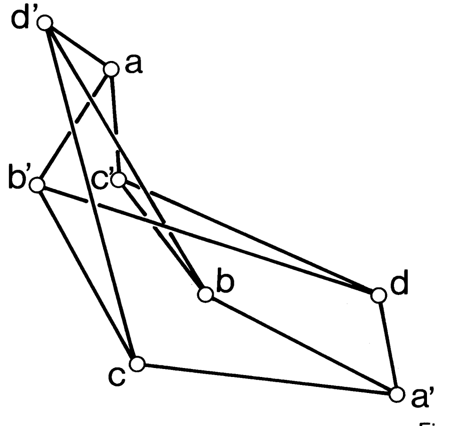


Fig. 11b.

Figure 11. A dependent cube. At each node the three bars are coplanar, but the entire structure is not coplanar.

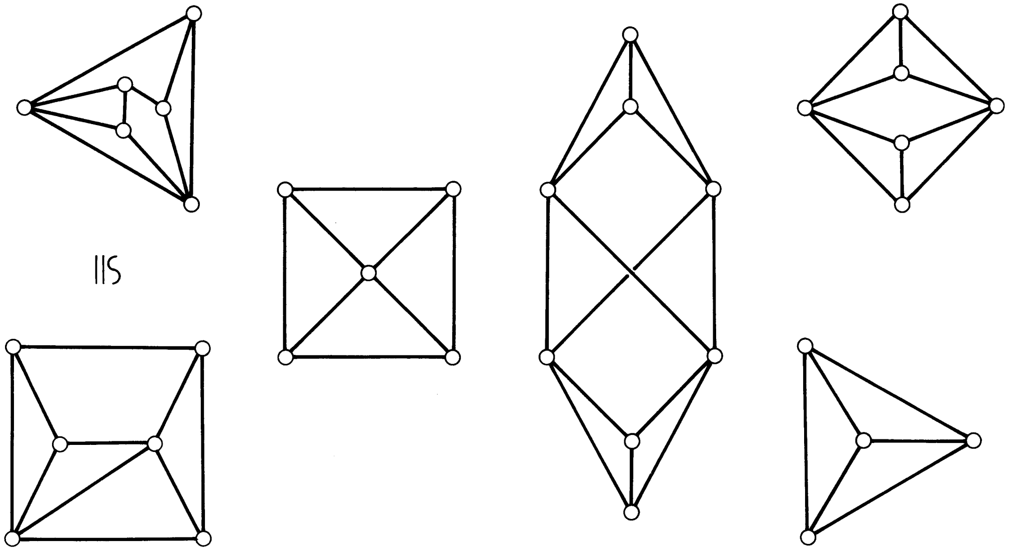


Figure 12. Generic circuits in the plane.

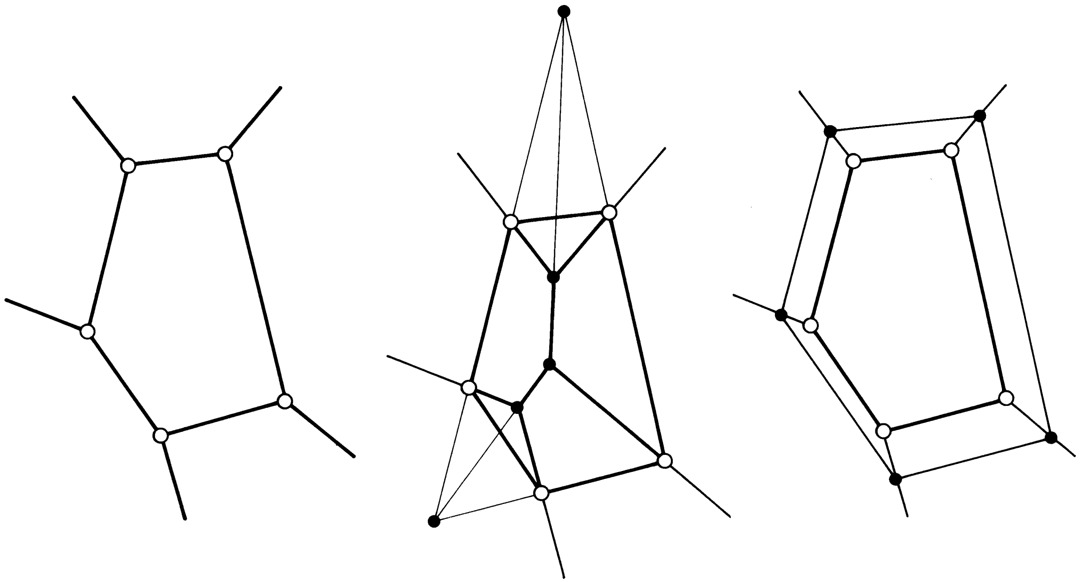


Figure 15. A funicular polygon, with its associated calotte and parallel redrawing.

graphs, by introducing new nodes at the crossing points of any pair of edges, and showing how such alterations do not affect the statics of the structure **(Figure 14).** This is the use of **Bow's Notation** * in (Maxwell 1876). Maxwell's theorem has also been generalized to oriented (not necessarily spherical) polyhedra. Polyhedral projections have reciprocal figures relative to such polyhedra, and structures with reciprocal figures are dependent, but **neither converse holds** (Crapo 1978).

In the article on polyhedra in this issue of the bulletin, we look into the matter of **projective conditions*** and **projective choices** * for the construction of a polyhedron over a given plane figure. Generally speaking, if a spherical polyhedron has V vertices and 2V-2 edges, it also has V faces, and any drawing of it in the plane has a polyhedral preimage under projection which is unique up to the choice of one dihedral angle and the position of one initial face. If it has more edges, say (2V-2) + k in number, then there are V+k faces, and k additional free choices of dihedral angles can be made in constructing the polyhedron over the given plane figure. If there are fewer than 2V-2 edges, say (2V-2) - k in number, then k independent projective conditions are required to be satisfied in order for a plane structure with that graph to be a polyhedral projection, and thus dependent.

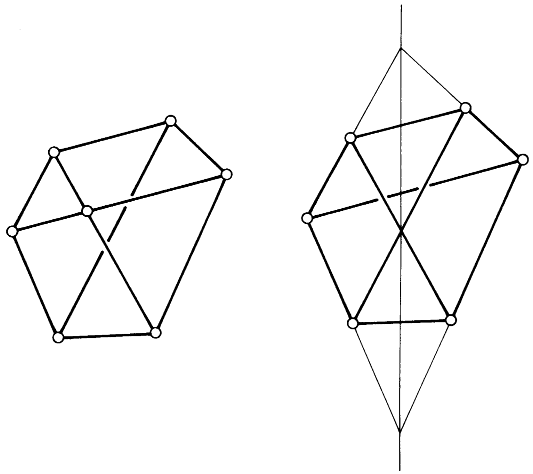


Figure 14. A dependent plane structure, and a statically equivalent structure with a planar graph, obtained by the use of Bow's notation. Note that the latter is a polyhedral projection.

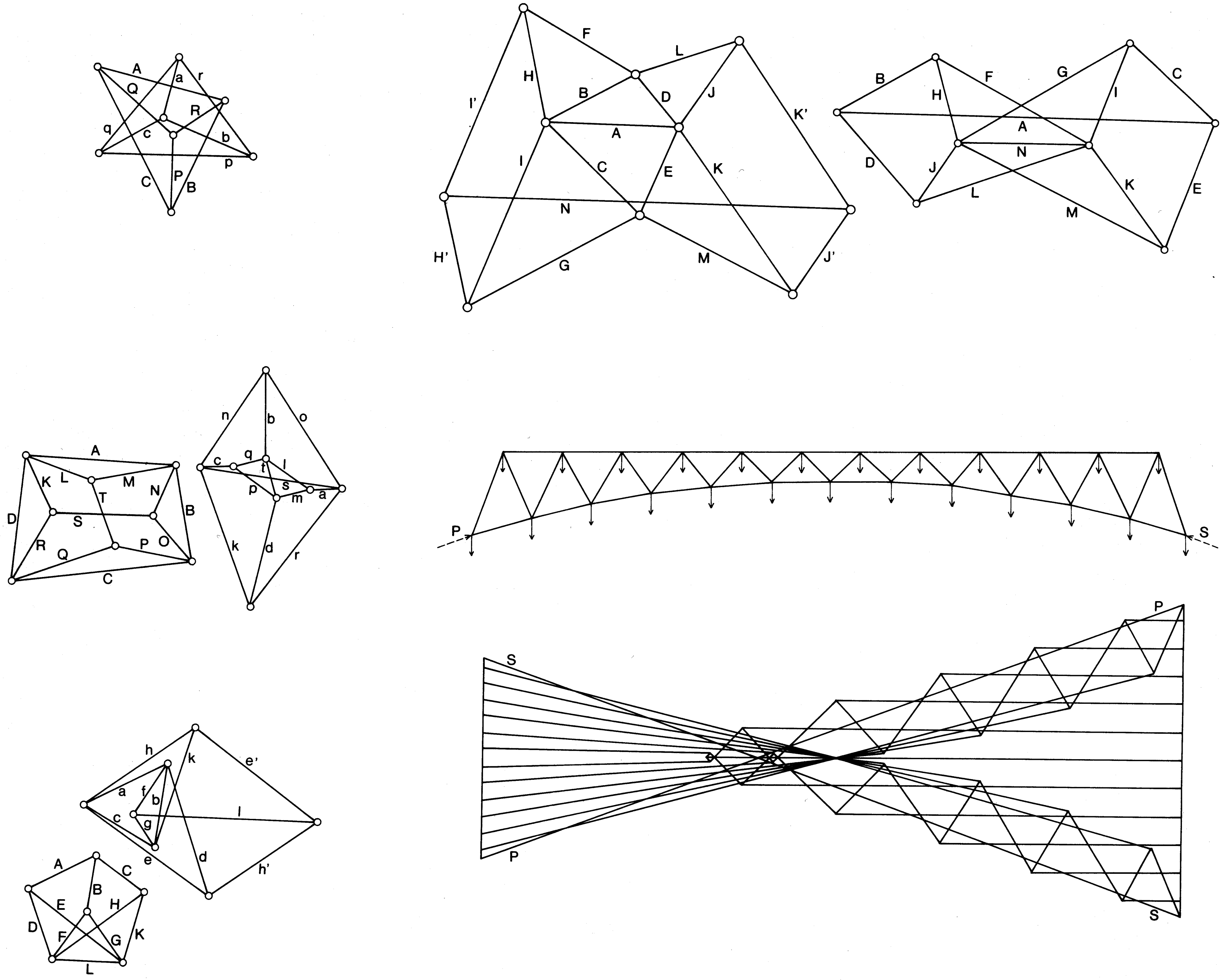


Figure 13. Pairs of reciprocal figures in the plane, from Maxwell's original papers.

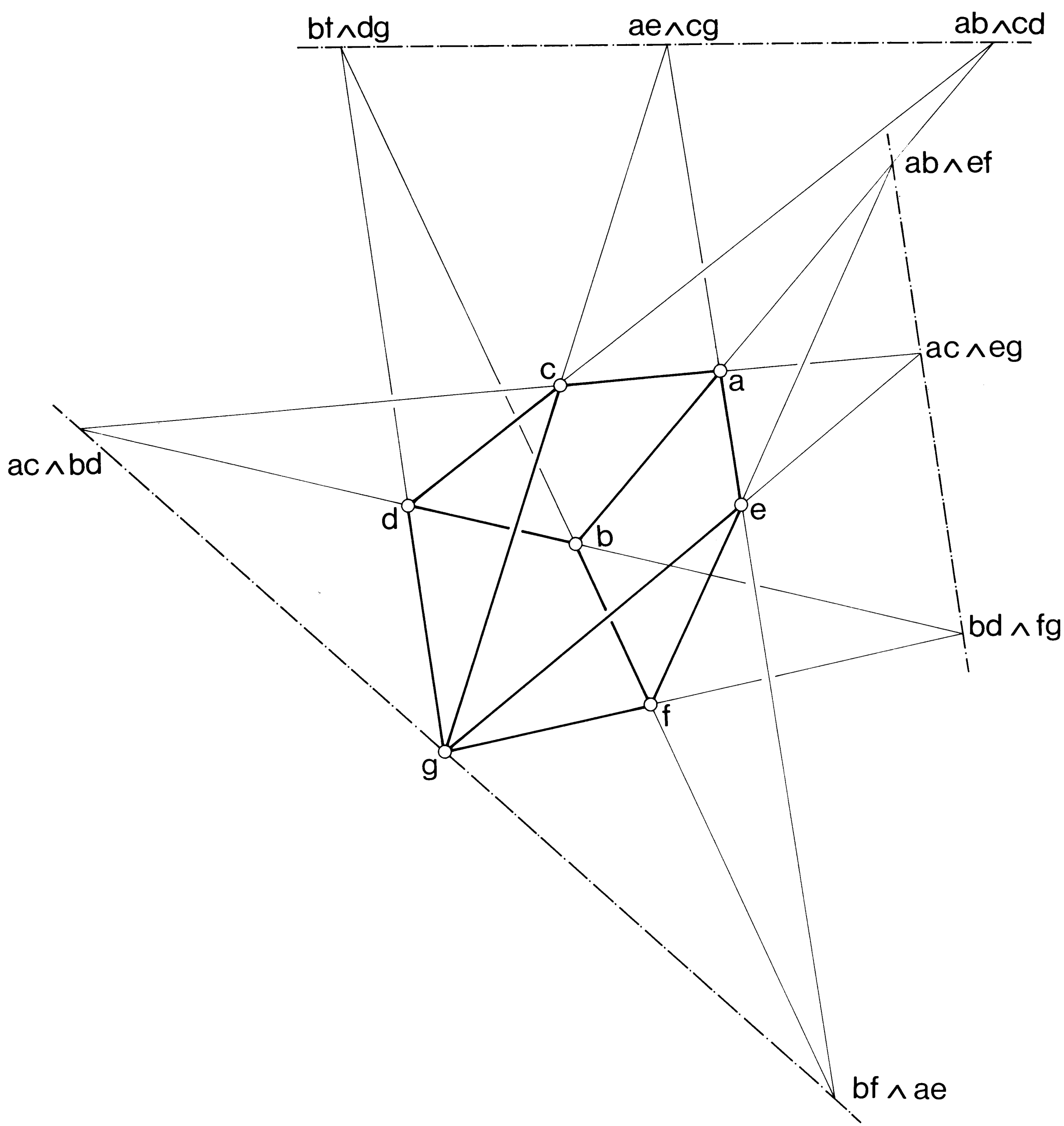


Figure 16. Three equivalent projective conditions guarantee dependence of this plane structure.

One such single projective condition is that which arises when a ring of plane faces surrounding a single face, all of whose nodes are trivalent, is projected into a plane. One statement of the condition is that a **roof** (or **calotte***) can be completed using the third bar at each node as the meeting of successive roof sections. **Figure 15** exhibits this condition for a ring of five planes. Equivalently, and also shown in **Figure 15,** a parallel redrawing of the polygon is possible with vertices displaced along the given bars. (Think of cutting a roof with a plane parallel to its base.) In treatises on statics this figure is called a **funicular polygon*** or **string polygon** , because a loop of string can be held taut in that position in the plane by forces in the directions of the third bar at each node. For plane structures with enough trivalent vertices, it suffices to verify dependence by checking this polygonal condition around a small number of faces. For instance, for the structure in **Figure 16,** it is sufficient to check the polygonal condition on any one quadrilateral face. There are four quadrilateral faces, but the conditions associated with them are equivalent. The plane structure as drawn has nullity 1, and since it has 2V-3 bars, it has one internal motion. The graph of the cube, in **Figure 17,** as a plane structure, must satisfy **two** independent projective conditions. These conditions may be stated in a variety of different but equivalent formulations, the most suggestive being that the corresponding bars in two opposite quadrilateral faces meet in four points on a line which may be viewed as the intersection of the planes of those two faces in space.

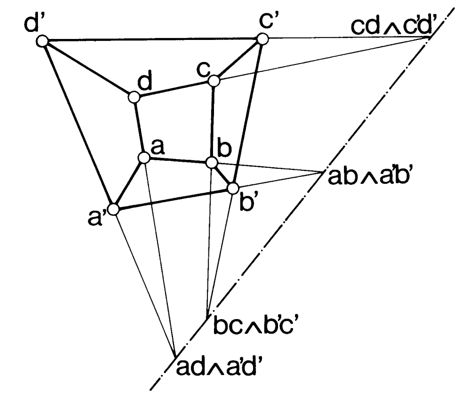


Figure 17. A projected cube, dependent in the plane, satisfying two independent projective conditions.

Generalizations to 3-Dimensional Space.

There is one well-known family of independent rigid spatial bar and joint structures: the triangulated convex spherical polyhedra. The Euler formula guarantees that all such structures have 3V-6 bars on V nodes, and because all their faces are plane-rigid, being triangles, Cauchy's theorem (Cauchy 1813) shows they are rigid structures.

Our study of dependent structures in three space began with the best-documented example of a deformable polyhedron' the Bricard octahedron (Bricard 1897), illustrated in **Figure 8.** Since this figure consists of two rigid bodies (two opposite triangular faces A, B, say) linked by 6 bars, we reasoned that any internal motion which fixed the face A would have to be, at least infinitesimally, a screw motion of the face B. The six bars would have to be perpendicular to the velocity vectors of that motion, at their points of contact with the face B. But available treatises on the geometry of lines in 3-space (Veblen 1900), (Semple 1949), (Klein 1939) are explicit on this point: a line perpendicular to a screw motion at any point is perpendicular to the motion at every point along the line, and the set of all lines perpendicular to a specified screw motion is precisely the set of lines satisfying one linear condition on their Plucker coordinates*. These families of lines are called (non-singular) **line complexes** . The geometry of non-singular line complexes is well-known, so it is not difficult to decide when the six bars linking opposite faces of an octahedron lie within a single complex: the four planes of alternate faces of the octahedron must meet in a point. (When this happens, the other set of four alternate faces will also meet in a point.) If aa', bb', cc' are pairs of opposite nodes of the octahedron, and d, d' are the two derived points of intersection of alternate faces, the eight points form eight planes

abcd', abc'd, ab'cd, a'bcd,
a'b'c'd, a'b'cd', a'bc'd', ab'c'd'.

This configuration of eight face planes of any dependent octahedral graph has an illustrious history: they are the face planes of **two mutually-inscribed tetrahedra** *(Mobius 1837). The vertices abcd of one tetrahedron lie on the face planes of the other, and its face planes pass through the vertices a'b'c'd' of the other.

The dependent octahedron is the first example of a spatial structure with 3V-6 bars on V nodes requiring one projective condition for dependence. There is a simpler dependent structure, the complete graph K_5, but it has 3V-5 bars, and is thus a generic dependent structure, dependent in any position. The complete graph K_5 is the 1-skeleton of the basic 4-polytope with 5 vertices, 10 edges and 5 tetrahedral 3-cells, so a K_5 in any position in 3-space is the projection of a 4-polytope. The dependent octahedron also seems to have some such connection with 4-space, but we have not succeeded in finding it.

We have made efforts in several directions to generalize the Maxwell-Cremona theorem to 3-dimensional space, but none as yet has produced a definitive theory. We shall now discuss three partial generalizations, mainly to cast some light on various facets of the remaining unsolved problem.

(a) Cones

If all nodes of a plane structure are joined to a single node q not lying in its plane P, the resulting spatial structure will be mechanically equivalent to the original plane structure in the following sense. The spatial structure is projectively equivalent to one in which the added node q is on the plane at infinity and all bars incident to that node are perpendicular to the plane P. These bars then serve only to hold the nodes of the plane structure within the plane. Any motion of the plane structure is a motion of the cone with the node q fixed, and conversely.

More generally, a spatial structure is a **cone*** if and only if it has one node, called its **apex,** which is joined to all other nodes. If we use the apex of a cone as centre of projection, and project the cone onto a plane, we obtain a plane figure with the same nullity as the cone itself. The conditions for dependence of a cone are quite revealing, and suggest principles on which a more thorough-going generalization may yet be based. A cone is dependent if and only if projection from the apex as centre results in a plane structure which is dependent. Note that as in **Figure 9,** above, the nodes other than the apex need not be coplanar. In **Figure 9,** we see the spatial structure is dependent if and only if the lines bc and de, ef and cg, which are normally two pairs of skew lines, **appear** in projection from centre a to meet in points which are collinear with the projection of the point p. That is, the condition illustrated in **Figure 16** holds in the projected figure, or holds from the point of view of an observer standing at the apex of the cone. Rephrasing this as a condition on the spatial structure, we find that through the point a we may draw unique lines L and M which meet the pairs of lines bc, de, and eg, ef, respectively. These lines L and M determine a plane (through a) which must contain the point p if the structure is to be dependent.

(b) Cremona Reciprocals

Maxwell's theorem on reciprocal figures and projected polyhedra was followed by an equivalent result due to Cremona (Cremona 1890). The latter observed what happens when 3-dimensional space is acted upon by a **skew polarity** * , which maps each point to an incident plane, each plane to an incident point, maps lines to lines, and reverses incidence. Under such a mapping, any polyhedron goes over into a polar polyhedron. For example, a pair of mutually-inscribed tetrahedra (Mobius 1837), discussed above, are polar to one another in this sense. If a polyhedron and its polar are simultaneously projected from a centre c onto a plane Q, their projected images will appear to be drawn with **corresponding lines parallel,** relative to an appropriate choice of line at infinity (namely Q intersected with the image of the centre c under the polarity). The Cremona theory gives no new information in the plane, since a Cremona reciprocal figure is easily obtained by rotating the Maxwell reciprocal by a right angle. But it has an interesting generalization to certain spatial structures. Say we have a spatial structure which is topologically a spherical polyhedron. If the structure is dependent, the stresses in the bars produce a vector sum zero at each vertex, which we may represent as a closed cycle of vectors, using the cyclic order around each node on the surface of the topological sphere. If we consistently choose, say, the vector at the "left" end of the edge as we travel on the "outside" of the sphere from a face A to an adjacent face B as the vector from the "reciprocal node A" to the "reciprocal node B", we will obtain a spatial structure (also dependent) with the dual of the original polyhedron as its topological structure, whose edges give the stresses in the original polyhedron. The relation between these two structures is symmetrical: the reciprocal structure has a stress given by the edges of the original polyhedron.

For example, consider the Cremona reciprocal of the dependent octahedron, namely the **cube graph***. in **Figure 11.** It is constructed in such a way that **the three bars incident at any node are coplanar,** yet the eight nodes do not lie in a single plane. Since three non-coplanar non-zero forces cannot sum to zero, the bars at every trivalent node of a minimal dependent structure (a circuit) must be coplanar. Thus the coplanarity of the bars incident at each node of the cube graph is necessary for the cube to be a circuit.

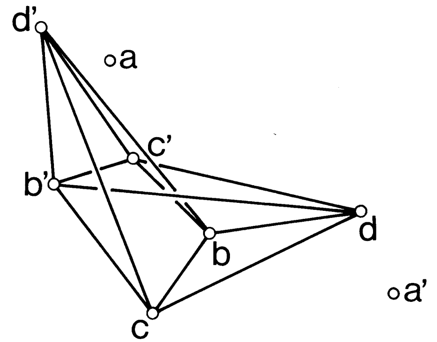


Fig. 18a.

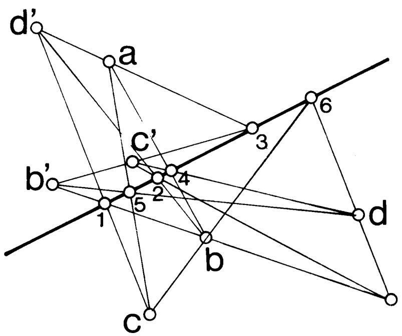


Fig. 18b.

Figure 18. A dependent octahedron, obtained from the dependent cube in Figure 11.

A further argument shows that by replacing two antipodal nodes of the cube by triangles on the nodes to which they were incident constructs a **dependent octahedron (Figure 18a)** . We see in this way that any such cube is dependent, and furthermore that **any such cube,** not lying entirely in one plane, **is a circuit.**

This cube graph is labelled systematically as a bipartite graph in which primed nodes are joined only to unprimed nodes, and nodes with the same letter are antipodal. The eight planes determined by the bars incident at the eight nodes thus each contain four nodes with distinct letters, either one or three of which are primed. Fixing our attention on two of these planes ab'c'd', a'bcd and on their line q of intersection, as in **Figure 11(a),** we find the remaining six planes must meet the line q in 6 collinear points. That is, the lines ab' and cd, ac' and bd, ad' and bc, b'c' and a'd, b'd' and a'c, c'd' and a'b are actually coplanar in pairs, and meet at the six collinear points. The plane projection of this **Figure 11(a)** commonly occurs in treatises on projective geometry (Veblen 1910, Figure 21, page 50). The six points on the line q form a **quadrilateral set*** determined by a complete quadrilateral in one plane, and an **oppositely placed** or polar quadrilateral in another plane. In fact, taking any pair (say d, d') of opposite nodes of this dependent cube graph, the planes of the bars at those nodes each contain three other nodes of the cube. The line of intersection of these two planes is shown in **Figure 18(b).** It contains the quadrilateral set of six points of intersection with the planes of the bars at the other six nodes. There are four figures like this, one for each pair of opposite nodes, one of which of course is **Figure 11(a).**

(c) Projected 4-polytopes

These first two efforts to generalize the Maxwell-Cremona theory to spatial structures are fairly closely related and are applicable to only a small family of spatial structures. Our third effort at generalization we have tended to regard as more central to the subject and more likely to lead to results of broader application. We know that if a spatial structure is the projection into 3-space of an oriented 4-polytope, it is dependent. For instance, if four quadrilateral faces of the structure in **Figure 9** are **flat** (that is their nodes are coplanar), the structure is dependent. In that event, the required collinearity of de meet bc, cg meet ef, and p occurs already in space, before projection. **Figure 10** provides another example. The structure on nine nodes a ... c" will be dependent if all nine quadrilateral faces are flat, because then the given structure will be the projection of a polytope with 6 cells, each of which is a triangular prism. But in neither case are these conditions necessary for dependence. We have seen this already for **Figure 9.** An analysis similar to that we have used for the cube (Tay 1976) reveals that the necessary and sufficient conditions for dependence in **Figure 10** are that the planes abc, a'b'c', a"b"c" meet in a line, as do the planes aa'a", bb'b", cc'c".

It is the main unsolved problem in this area of our work to find out how consistently to modify the conditions for 4-polytopial realizability to obtain the weaker conditicn for dependence. Presently we are investigating nonsingular mappings which preserve certain features of a structure, including dependence, yet are capable of destroying coplanarity of points, coincidence of lines, etc. For example, consider a mapping which "unwinds" a cone to produce a hyperboloid of one sheet, or "unwinds" the family of all lines through a point into the rulings of a one-parameter family of concentric hyperboloids. If such a mapping were to act on the bars of a dependent structure, the image would again be a dependent structure, but a projected polytope could be changed into something more general. (Such mappings are provided by non-singular linear transformations of the 6-dimensional vector space spanned by line segments in 3-space, namely of the space of wrenches or screws, described above.) This line of investigation is highly conjectural: the problem is still wide open.

There is also a wide range of structures about which nothing is known concerning necessary and sufficient conditions for dependence. Locally-flat polyhedral graphs, like the dependent cube graph, have relatively few bars; projected 4-polytopes have many more. The whole middle ground between these two types of structures remains to be explored.

Before leaving the subject of spatial bar and joint structures, we should observe that typically it is easier to state necessary and sufficient conditions for dependence than it is to state the corresponding conditions that a structure be a circuit. A structure in a certain special position may be a circuit, but in even more special position it may still have nullity 1, but certain bars may carry no stress, and are no longer in the circuit. The Bricard octahedron, for example, in its most symmetric position has two bars which carry no stress. The circuit is simply two plane-rigid structures hinged at two nodes, and set at an angle to one another in space. Furthermore, in even more special position, a structure may have nullity higher than one. For instance, a complete graph K_5 has nullity 1 in general position in 3-space, but has nullity 6 when all the nodes lie on a line.

Strut and Cable Structures

We close this article by discussing briefly two different types of spatial structures, tensegrity structures and hinged panel structures. In each case are a few facts which carry over from bar and joint structures, and a few new things to learn.

The bars of a bar and joint structure resist both tension and compression. In a tensegrity structure, bars are replaced by **cables***, which carry only tension, and **struts***, which are called upon to carry only compression. More generally, any dependent structure of rank r having a stress which is non-zero on every edge (a circuit, for example) can be rebuilt as a strut and cable structure which will carry an r-dimensional space of external loads. Since the stress is non-zero in every bar, the set of bars divides neatly into two classes according to the sign of the stress. One of these classes becomes the set of cables, the other the set of struts. Any finite load resolved by the bar and joint structure also has a resolution in a sufficiently tightened model of the associated strut and cable structure. To see this, observe that if a sufficiently large positive scalar multiple of the stress is added to the response of the bar and joint structure, all the scalars can be made to agree in sign with of the stress. That is, every cable will be in tension, every strut will be in compression' and the structure will be in equilibrium with the applied load. Thus any external load supportable by such a bar and joint structure can be carried by a sufficiently prestressed strut and cable structure of the same conformation. A strut and cable structure is rigid if and only its associated bar and joint structure is rigid and has a stress which is non-zero on every bar.

This means that there exist rigid strut and cable structures with 3V-5 struts and cables on V nodes in space, for any V. There exist, however, strut and cable structures with many more than 3V-5 struts and cables, which are minimally rigid in the sense that removal of any strut or cable will introduce an internal motion. For instance, by a generalization of Cauchy's theorem (Whiteley 1977), a cube of struts in space, braced with cables along both diagonals in every face, is rigid. There are 24 = 3V struts and cables, but no member of either type can be removed without losing the rigidity.

Any pair of distinct points in the projective plane determine a line, and break the line into two line segments, one of which crosses the line at infinity in any affine drawing. Any strut in a strut and cable structure can be replaced by a cable in the other line segment connecting the same pair of nodes, and conversely. This observation permits us to generate from a given strut and cable structure a family of projectively equivalent but affinely distinct such structures **(Figure 19)**. Also, because the **negative** of a stress is also a stress, all struts can be replaced by cables, all cables by struts. This replacement does not affect rigidity, but may affect **stability** : the ability of the structure to restore itself following an infinitesimal motion. For instance, the structures in **Figure 19(a)** are plane rigid, spatially non-rigid, but the figure with four struts is spatially unstable, that with two struts is stable. This question of stability of strut and cable structures is a challenging area for future investigation, in which little is known.

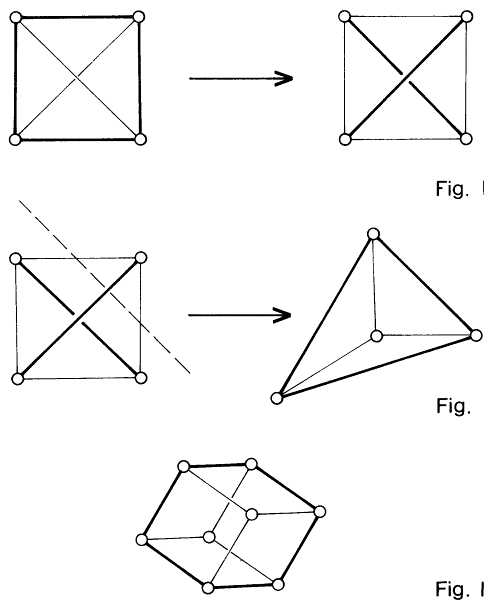


Fig. 19 a.
Fig. 19 b.
Fig. 19 c.

Figure 19. Strut and cable structures (cabled frameworks).

Hinged Panel Structures

A **spatial panel structure** consists of completely rigid polygonal pieces of plane, hinged to one another along edges (Baracs 1975). F separate panels floating in space have 6F degrees of freedom. A hinge between two panels reduces the six degrees of freedom of relative motion between the panels to one, namely rotation about the hinge. So each hinge can remove at most 5 degrees of freedom. A hinged panel structure will be rigid if in all 6F-6 = 6(F-I) degrees of freedom can be removed by the placing of hinges between panels, because the 6 isometries of space always remain as permitted motions. Thus the minimum number of hinges needed to rigidify a system of F panels is the least whole number greater than or equal to 6(F-I)/5. Thus F hinges are needed to rigidify F panels for F = 2, 3, 4, 5, 6, while F+1 hinges are needed for F panels for F = 7, 8, 9, 10, 11.

Cauchy's theorem guarantees that all convex sperical polyhedra are rigid, where all edges are hinged. Baracs and his students have done some investigation as to which hinges or even faces can be removed from such a structure, while retaining rigidity. For instance, there are two ways of choosing a cycle of six hinges on a cube, one of which produces a rigid cube, the other not (Baracs 1975).

Rigid spatial panel structures can be constructed step by step, using the principles that a ring of three panels is rigid unless they are coplanar with concurrent hinges, a ring of four or five panels is rigid unless the hinges go through a point (or are coplanar), and a ring of six panels is rigid unless they are part of a dependent octahedron, that is alternate panels meet at a point coplanar with the intersection points of the hinges on those panels. (If the ring is convex, as in **Figure 1(c)**, , it is surely rigid.)

Any motion of a hinged panel structure assigns to each hinge an extensor which is a multiple of the hinge, namely the relative angular velocity of the two panels at that hinge. These extensors sum to zero around every closed ring of panels, the motion of any panel with respect to itself being zero. In particular, these extensors sum to zero around every vertex, if the hinged panel structure is a polyhedron. In this way we see that a hinged panel structure with all edges hinged will be movable if and only if the 1-skeleton of the polyhedron is dependent as a bar and joint structure!

The subject of hinged panel structures forms a large new field of inquiry, with extensive practical application. The basic problem of rigidity of spatial panel structures in general remains unsolved.

Algebraic Proofs

Some proofs of rigidity theorems are best written out algebraically. For instance, a direct computation of the scalars of stress in the edges of a complete graph K_5 in 3-space reveals that each bar is stressed by an amount equal to the product of the volumes of the two tetrahedra formed by dropping the nodes at the ends of that bar one at a time.

Since extensors at a point add just like vectors, the dependencies between the rows of the coordinatizing matrix M of a spatial bar and joint structure are unaffected if we use six-dimensional extensors ab instead of three-dimensional vectors a-b as its entries. Let M' be the corresponding matrix with extensor entries. Any dependence between the rows of M' holds also between the entries in any sum of columns of M'. In particular, if a dependent structure consists of two pieces linked by a set L of bars, we may add the columns belonging to nodes in one of the two pieces of the structure, and find that the same scalars are those of a dependence between the bars forming the **link** L between the two pieces. This is what is going on with the dependent octahedron, as discussed earlier: the six bars in any skew hexagon linking two opposite triangular faces are of rank 5 as extensors.

In general, the scalars of dependence in a circuit in 3-space can always be written as sums of products of volumes of tetrahedral cells. To find a manageable combinatorial algorithm for writing these scalars is one of the basic unsolved problems of structural topology.

Definitions

Structure. This term is used to indicate a very general class of geometric objects: any assemblage of rigid components (nodes, bars, panels, solids), in specified positions in projective space, joined to one another at articulated joints (articulated nodes, or hinges).

Rigid. Specifically: statically rigid, not permitting even any infinitesimal motion, save rigid motions of the entire space. A statically rigid structure will support any external equilibrium load.

Mechanism. A Structure having a finite (as opposed to a merely infinitesimal) motion.

Bar and joint structure. A structure composed of rigid bars, attached to one another at their ends, at nodes. The bars are articulated at the nodes, in the sense that two bars incident at a node are, in principal, free to change in angle relative to one another. External loading on such a structure will produce only axial reaction forces.

Cable. A member connecting two nodes, capable of resisting only tension.

Strut. A member connecting two nodes, capable of resisting only compression.

Strut and cable structure. A structure composed of cables and struts, joined to one another, and articulated, at nodes. Also called **cabled frameworks.**

Spatial panel structure. A structure composed of rigid and unbendable plane polygonal panels, hinged to one another (articulated) along edges. Such structures can be realized by using rigid bodies in space, as long as they join along edges forming plane polygons.

Graphical statics. The study of stress configurations in plane structures, using reciprocal figures and projected polyhedra. Graphical statics was the creation of Maxwell, Cremona, Culmann and others in the mid-19th century.

Linear dependence. (From linear algebra). A set of vectors is linearly dependent when there is some set of scalars, not all zero, which when multiplied by those vectors makes their sum equal to zero. For example, two collinear vectors are dependent, as are three coplanar or four cospatial vectors. Since forces in 3-space are representable as vectors only in 6-dimensional vector space, dependence of forces on a rigid body is not just a matter of coplanarity or collinearity of their lines of action. Bars in spatial structures are vectors in 3V-dimensional vector space, so their dependence is even less obvious.

Coordinatizing matrix. A matrix, thought of as having a vector entry corresponding to each incidence of a bar with a node. Linear dependence of rows in this matrix shows the dependence of the corresponding bars as structural constraints.

Stress. An assignment of tension and compression forces to the bars of a structure, in such a way that the forces in incident bars add to zero at every node. (We use unit cross-sections for the bars, to make this term agree with engineering usage.)

Motion. An assignment of velocity vectors to the nodes of a structure, such that the relative velocity of the two nodes on any bar is a vector perpendicular to that bar. That is: the velocities of the ends of the bar are such as to cause no infinitesimal change in the length of the bar.

Degree of freedom. The number of degrees of freedom of a structure is the dimension of the vector space of its motions.

Rigid motion. A motion of the entire space acting as one rigid body. An **isometry.** Such isometries of the plane are either **translations** or **rotations.** In space, translations and rotations may be combined to produce **screw motions.**

Screw motion. A general rigid motion of 3-space, representable as a rotation about an axis, combined with a translation along that axis.

Internal motion. An equivalence class of motions of a structure, modulo rigid motions. Loosely: any motion other than a rigid motion. Equivalently, for spatial structures, having fixed one bar and the plane of one adjacent bar, any remaining motion is internal.

Internal degree of freedom. The number of internal degrees of freedom of a structure is the dimension of its motion space, minus the dimension of the vector space of rigid motions.

Rank. The dimension of the vector space of equilibrium loads supportable by a given structure. For a spatial bar and joint structure, the rank is 3V-6 minus the dimension of the motion space of the structure. The rank is the number of degrees of freedom removed (from an unconnected set of nodes) by the bars.

Nullity. In a bar and joint structure, the number of bars minus the rank. It is the dimension of the vector space of dependencies between the bars, and equals the dimension of the vector space of spatial realizations, for planar polyhedral structure.

Dimension. For a vector space, the minimum number of vectors in terms of which all vectors can be expressed as linear combinations. Also the maximum size of any independent set of vectors. The number of components needed to express a vector quantity.

Couple. The sum of forces equal in magnitude and opposite in direction, acting on parallel but distinct lines.

Wrench. The sum of forces acting along lines in space. It may be expressed as a couple in some plane, plus a force along a line perpendicular to the plane of the couple. The product of a wrench acting on a screw motion is a scalar quantity called **work**.

Plucker coordinates. Six-dimensional coordinates for line segments (for example, forces) in 3-space, with respect to which resolution of forces becomes simply vector addition. For a line segment from a point a to a point b, the Plucker coordinates can be arranged so that the first three give the free vector b-a, the last three the moment vector bxa relative to the origin of the given coordinate system for 3-space.

Independent. Having no redundancy. The rank of an independent structure is equal to its number of bars. Removal of any bar from an independent structure will introduce a new degree of freedom. Independent structures are statically determinate, in the sense that any supportable load has a unique response of tensions and compressions in the bars, but they are not necessarily rigid.

Dependent. Having redundancy. Dependent structures are statically indeterminate, but may be either rigid or non-rigid. A dependent structure has positive nullity, and has a number of bars in excess of its rank. Dependent structures have bars which can be removed without introducing new degrees of freedom.

Circuit. Any minimal dependent structure. For example, a collinear polygon, or any minimal projected spherical polyhedron. This terminology derives form combinatorial geometry, and thus has a meaning more general than it does in graph theory.

Structure geometry. That combinatorial geometry which describes the dependence of bars in a structure as linear constraints. By using the structure geometry, we may substitute counting arguments for algebraic computations. In this way, we may often find out not just that a structure is dependent, but also why it is dependent.

Statically indeterminate structure. A structure in which some external load can be resolved in more than one way by tensions and compressions in the bars. Dependent.

Hyperstatic structure. A structure which is rigid and dependent. Any external equilibrium load can be resolved in more than one way in tensions and compressions in the bars.

Isostatic structure. A structure which is rigid and independent. Every external equilibrium load has a unique resolution in the structure. For any rigid structure, its isostatic substructures are the bases (plural of basis) of its structure geometry.

Hinge. An articulation between rigid panels or other rigid bodies, permitting one degree of freedom of rotation about an axis (the hinge) in the other.

Generic structure. A structure in general position. A structure in such geometric position that it has maximum possible rank, given its topological makeup.

Generically — . To say a topological structure has a projective property generically is to say it has that property when realized in general position in the projective space.

Polarity. A type of duality of a projective space. A mapping of a projective space into its dual space. A polarity of the projective plane maps points to lines, lines to points, and reverses all incidences. Thus the line determined by two points is mapped to the intersection of the polars of those points. A polarity of space maps points to planes, lines to lines, and planes to points.

Skew polarity. A polarity of projective 3-space in which each point is mapped to some plane through that point. Under a skew polarity the image of a line is equal to or skew to that line. A line equal to its image is said to be **fixed** by the polarity. The set of lines fixed by a skew polarity forms a **line complex,** representable as all lines perpendicular to a certain screw motion (rigid motion) of space. Whenever a set of bars are contained in a line complex, they are incapable of bracing one rigid body relative to another.

Projective conditions. Statements concerning incidence of various points and lines derivable from a projective drawing jn the plane, or from a projective construction in space. Specifically, those conditions required for a given plane figure to be a projected polyhedron. These conditions arise from required incidences in the corresponding spatial structure.

Projective choices. In a projective construction, arbitrary one-parameter assignments made for the position of points, lines or planes, where the position of those points, lines or planes are not completely determined by the given data.

Calotte. A system of plane faces, edges and vertices in space which divides the space in two: the structure produced by omitting one face, with its incident edges and vertices, from a spherical polyhedron. Topologically: a simply-connected unbounded 2-manifold.

Funicular polygon. The position of a loop of string in equilibrium under the influence of a given set of equilibrium forces. The construction of a funicular polygon is a graphical method commonly used to compute the resultant of a given set of forces.

Bow's notation. The insertion of new nodes at the crossing points of bars, for the purpose of reducing structures with non-planar graphs to statically-equivalent planar structures.

Mobius pair. A pair of tetrahedra, arranged in space so that the nodes of each tetrahedron are on the face planes of the other. They are "mutually-inscribed". Every Mobius pair defines a skew polarity of the entire space.

Cone. A spatial structure formed at least topologically by joining a new node in space to every node of a given plane structure. For example, a pyramid.

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The code in the first block of each bibliographic item consists of three parts, separated by dashes. The first letter indicates whether the item is a

B ook
A rticle
P reprint, or
C ourse notes.

The middle letter(s) indicates whether the piece was intended primarily for an audience of

M athematicians,
A rchitects, or
E ngineers.

The final letter(s) indicates if the piece touches on one or more of the principal themes of our work:

P olyhedra,
J uxtaposition' or
R igidity.

The key words or other annotations in the third column are intended to show the relevance of the work to research in structural topology, and do not necessarily reflect its overall contents, or the intent of the author.

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