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MINQUE OF VARIANCE COMPONENTS IN REPLICATED AND MULTIVARIATE LINEAR MODEL WITH LINEAR RESTRICTIONS

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The Minimum Norm Quadratic Unbiased Invariant Estimator of the estimable linear function of the unknown variance-covariance component parameter ϑ in the linear model with given linear restrictions of the type $R\vartheta = c$ is derived in two special structures: replicated and growth-curve model.

Key words: MINQUE, linear model with restrictions, replicated model, growth-curve model.

1. INTRODUCTION

A general linear model with variance-covariance components is often considered in the form (see e.g.Rao–Kleffe [2])

$$(1) \qquad y = X\beta + \varepsilon, \ E(\varepsilon) = 0, \quad E(\varepsilon\varepsilon') = V(\vartheta) = \sum_{i=1}^{p} \vartheta_i V_i$$

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where X, V_i are given matrices, V_i are symmetric for all $i = 1, \ldots, n$, and $\vartheta = (\vartheta_1, \ldots, \vartheta_p)' \in \Theta \subset \mathbb{R}^p$ is such that $\mathcal{V} = \{V(\vartheta) : \vartheta \in \Theta\}$ constitutes a closed convex cone. The vector parameter β is an unknown vector of fixed effects. The existence of the 3rd and the 4th moments of the vector ε is assumed.

The aim is to find an estimator of a given linear function $f'\vartheta$ as a function of the vector of observations y.

There are many authors investigating the problem (see e.g. [1]) considering a quadratic approach. The basic idea of this approach is to construct a quadratic form, say $y'Ay$, with a symmetric matrix A such that the statistic $y'Ay$ meets additional requirements as e. g. unbiasedness and invariance. If the distribution of the vector ε is given, it is known how to find a statistic which minimizes the variance in the class of quadratic unbiased estimators.

In the early seventies C. R. Rao introduced a MINQUE principle which is based on the idea to find a quadratic form $y'A_0y$ which is an unbiased and invariant estimator and the matrix A_0 minimizes the Euclidean norm $\operatorname{tr} AGAG$ for a suitable choice of the matrix G.

Let ϑ_0 be a preassigned vector from the parametric space such that $V(\vartheta_0)$ is a nonnegative definite matrix. In case that the vector y is normally distributed, to minimize the variance of the unbiased and invariant statistic $y'Ay$ at the point ϑ_0 means to minimize the norm $\operatorname{tr} AV(\vartheta_0)AV(\vartheta_0)$. One reasonable suggestion is to substitute the matrix $V(\vartheta_0)$ for the matrix G in the expression which should be minimized for getting the MINQUE.

To recall the known facts we give the following considerations which lead to the explicit form of the MINQUE under the model (1) for a nonnegative definite matrix $V(\vartheta_0)$. For simplicity we shall use the notation $V(\vartheta_0) = V_0$.

It is useful to analyze first the invariance principle. We can refer to e.g. Seely in [3] or, Rao and Kleffe in [2].

If the expectation of the vector y is an unknown vector $X\beta$ we can investigate a vector of observations in the form

$$y_* = X\beta_* + \varepsilon,$$

where $y_* = y - X\beta_0$ for a fixed vector $\beta_0 \in \mathbb{R}^k$ and $\beta_* = \beta - \beta_0$. The covariance matrix $V(\vartheta)$ of the vector y_* is the same as the one of the vector y. It is natural to require that the estimator of the function $f'\vartheta$ based on the vector y is the same as the estimator based on the vector y_*. That should be valid for all vectors $\beta \in \mathbb{R}^k$ what implies the next definition.

Definition 1

The statistic $T(y)$ is said to be invariant under the group of transformations $y \mapsto y - X\beta$ in model (1) if $T(y) = T(y - X\beta)$, for all $\beta \in \mathbb{R}^k$.

It is easy to see that the quadratic form $y'Ay$, $A = A'$ is an invariant statistic in the model (1) if and only if $AX = 0$. Referring to Rao and Kleffe in [2] p.78 we give the definition of a maximal invariant.

Definition 2

The statistic $T(y)$ is said to be maximal invariant with respect to the group of translations if $T(y_1) \neq T(y_2)$ whenever y_1 and y_2 are such that no translation maps y_1 into y_2.

The condition for the quadratic form to be an unbiased and invariant estimator for the function $f'\vartheta$ yields the necessary and sufficient condition for the matrix A of the form:

$$AX = 0, \quad \operatorname{tr} AV_i = f_i, \quad i = 1, \ldots, p. \tag{2}$$

Denote $T_0 = V_0 + XX'$. The matrix A satisfying the condition (2) meets the equality $\operatorname{tr} AV_0AV_0 = \operatorname{tr} AT_0AT_0$. The inclusion $R(X) \subseteq R(T_0)$ and the property

$$T_0^+X - T_0^+X(X'T_0^+X)^+X'T_0^+X = 0$$

imply the equalities

$$T_0^+ - T_0^+X(X'T_0^+X)^+X'T_0^+ = (MT_0M)^+ = (MV_0M)^+, \tag{3}$$

where "+" can be replaced by "−" in the first part of the equation. Here $M = I - XX^+$, and the superscript "−" denotes the g-inverse of a matrix and "+" denotes the Moore - Penrose inverse of a matrix.

The following result can be found in [2] page 94.

Proposition 1

(a) The *MINQUE* of the estimable function $f'\vartheta$ in the model (1) is given by

$$\widehat{f'\vartheta} = \sum_{i=1}^{p} \lambda_i q_i,$$

where

$$q_i = y'(MV_0M)^+V_i(MV_0M)^+y, i = 1,\dots,p. \tag{4}$$

The vector λ is any solution of the system $K\lambda = f$, where the matrix K is given by the entries

$$\{K\}_{i,j} = \operatorname{tr}(MV_0M)^+V_i(MV_0M)^+V_j$$

and represents the criterion matrix for the estimability of the function $f'\vartheta$.

(b) The variance of $\widehat{f'\vartheta}$ at ϑ_0 under the normality assumption is given by

$$\operatorname{var}_{\vartheta_0}\widehat{f'\vartheta} = 2\lambda'K\lambda = 2f'K^-f.$$

Remark 1

Alternatively, the entries q_i defined by (4) are given as:

$$q_i = y'R_{T_0}V_iR'_{T_0}y, \quad R_{T_0} = T_0^-M_{T_0}, \quad M_{T_0} = I - X(X'T_0^-X)^-X'T_0^-,$$

The choice T_0^+ for T_0^- leads to (4). (See also [2].)

The estimation of $f'\vartheta$ can be investigated also as a special linear problem, as considered by e.g. Verdooren in [4]. The estimator $l'(y \otimes y)$ is a linear function of the vector $y \otimes y$, where l is an n^2 dimensional vector and '$\otimes$' stands for the Kronecker product. This approach enables to apply linear methods to the problem of quadratic estimation.

Now let us introduce a class of estimators which is wider than the class of quadratic forms in y. The class of quadratic functions including the linear term and a scalar term

$$\mathcal{E} = \{y'Ay + b'y + d : \ A = A', \ b \in \mathbb{R}^n, \ d \in \mathbb{R}^1\}$$

is investigated. A characterization of unbiasedness and invariance is given in the next theorem.

Theorem 1

Let $R(V(\vartheta)) = R(V_1 \vdots \dots \vdots V_p)$, for all $\vartheta \in \Theta$. The class of unbiased estimators for the function $f'\vartheta$ and invariant with respect to the group of translations $y \mapsto y - X\beta$ in the model (1) is given by

$$\mathcal{E}_{U,I} = \{y'Ay + b'y : \ A = A', \ b \in \ker X', \ AX = 0, \ \operatorname{tr} AV_i = f_i, \ i = 1,\dots,p\} \tag{5}$$

Proof

A condition for the unbiasedness of the estimator $y'Ay + b'y + d \in \mathcal{E}$ is expressed as

$$E(y'Ay + b'y + d) = f'\vartheta \text{ for all } \beta \in \mathbb{R}^k \text{ and } \vartheta \in \Theta. \tag{6}$$

The left hand side of the relation (6) is

$$\beta'X'AX\beta + \sum_{i=1}^{p} \vartheta_i \mathrm{tr}\, AV_i + b'X\beta + d = f'\vartheta$$

for all $\beta \in \mathbb{R}^k$ and $\vartheta \in \Theta$, which implies

$$\sum_{i=1}^{p} \vartheta_i \mathrm{tr}\, AV_i + d = f'\vartheta, \text{ and } \beta'X'AX\beta + b'X\beta = 0, \tag{7}$$

for all $\beta \in \mathbb{R}^k$ and $\vartheta \in \Theta$.

A necessary and sufficient condition for invariance is

$$(y + X\beta)'A(y + X\beta) + b'(y + X\beta) + d = y'Ay + b'y + d \tag{8}$$

for all $\beta \in \mathbb{R}^k$, which implies

$$\beta'X'AX\beta + b'X\beta + 2\beta'X'Ay = 0 \text{ for all } \beta \in \mathbb{R}^k. \tag{9}$$

The relations (7) and (9) imply the necessary and sufficient condition of the form

$$\begin{aligned} \sum_{i=1}^{p} \vartheta_i \mathrm{tr}\, AV_i + d &= f'\vartheta \\ \beta'X'AX\beta + b'X\beta &= 0 \\ \beta'X'Ay &= 0 \end{aligned} \tag{10}$$

for all $\beta \in \mathbb{R}^k$ and $\vartheta \in \Theta$.

Let T be an arbitrary matrix which fulfils the condition $R(T) = R(X \vdots V_1 \vdots \ldots \vdots V_p)$. Then (10) is equivalent with

$$\begin{aligned} \sum_{i=1}^{p} \vartheta_i \mathrm{tr}\, AV_i + d &= f'\vartheta \\ \beta'X'AX\beta + b'X\beta &= 0 \\ \beta'X'ATz &= 0 \end{aligned} \tag{11}$$

for all $\vartheta \in \Theta$, $\beta \in \mathbb{R}^k$, and $z \in \mathbb{R}^n$. The system of equations (11) leads to

$$\begin{aligned} d = 0, \quad & \operatorname{tr} AV_i = f_i \quad i = 1, \dots, p \\ X'AT = 0, \quad & b \in \ker X' \end{aligned} \tag{12}$$

It is clear that each estimator from the class $\mathcal{E}$ meets the condition (12).

Let $y'Ay+b'y$ be such that the matrix A and the vector b satisfy (12). Denote $A_* = T^{-\prime}T'ATT^-$. We have to prove that $A_*X = 0$. Since $y \in R(T)$ almost everywhere, the equality

$$y'A_*y + b'y = y'Ay + b'y$$

holds.

Further, $\operatorname{tr} A_*V_i = \operatorname{tr} T^{-\prime}T'ATT^-V_i = \operatorname{tr} AV_i = f_i$ for all $i = 1, \dots, p$, and $A_*X = T^{-\prime}T'ATT^-X = T^{-\prime}T'AX = 0$, hence the estimator $y'A_*y+b'y$ belongs to the class $\mathcal{E}$.

■

Remark 2

The following definition will prove to be very useful in more complicated situations when the parameters of the model do not belong to an open set, e. g. if linear restrictions on the parameters are present.

Definition 3

The linear-quadratic statistic $T(y)$ of the form $T(y) = y'Ay+b'y+d = l'(y\otimes y) + b'y + d$ which is unbiased for $f'\vartheta$, invariant under the group of translations $y \mapsto y+X\beta$ and minimizes the variance $\operatorname{var}_{\vartheta_0} T(y)$ under normality of the vector y at a given point ϑ_0 will be called the *MINQUE* of the estimable function $f'\vartheta$.

Lemma 1

The statistic $T(y) = y'Ay+b'y+d$ introduced in Definition 3 reduces in the model (1) to the simple form $y'Ay$.

Proof

We only need the decomposition

$$\operatorname{var}_{\vartheta_0} T(y) = \operatorname{var}_{\vartheta_0}(y'Ay + b'y) = \operatorname{var}_{\vartheta_0}(y'Ay) + \operatorname{var}_{\vartheta_0}(b'y),$$

as $\operatorname{cov}_{\vartheta_0}(y'Ay, b'y) = 0$ under normality. The minimization requirement yields $\operatorname{var}_{\vartheta_0}(b'y) = 0$, from which follows $b'y = 0$, as $E(b'y) = 0$.

■

Sometimes it is reasonable to consider the situation that linear restrictions on the parameter ϑ are given. They are often presented as $R\vartheta = c$, where the matrix R and the vector c are given. The model (1) together with $R\vartheta = c$ yields the model

$$(13) \qquad (y, X\beta, \sum_{i=1}^{p} \vartheta_i V_i | R\vartheta = c)$$

Utilizing the linear approach the MINQUE for the estimable function $f'\vartheta$ in model (13) has been derived by Volaufová and Witkovský, see [5]. In the next, certain special structures of the model (13) are considered.

2. REPLICATED MODEL

In practical work we come across situations where data from different sources contain information on the same set of parameters. In such cases we have the problem of pooling all the available information for an efficient estimation of parameters. In special cases one may have the replicated model

$$y_\alpha = X\beta + \varepsilon_\alpha \quad \alpha = 1, \ldots, m$$

$$E(\varepsilon_\alpha) = 0, \quad E(\varepsilon\varepsilon') = V(\vartheta) \quad \operatorname{cov}(\varepsilon_\alpha, \varepsilon_\beta) = 0 \quad \alpha \neq \beta$$

which can be written as a combined model

$$(14) \qquad \underline{y} = \underline{X}\beta + \underline{\varepsilon}$$

where $\underline{y} = (y_1', y_2', \ldots, y_m')'$, $\underline{X} = (\mathbf{1} \otimes X)$, $\mathbf{1} = (1, \ldots, 1)'$ and analogously $\underline{\varepsilon} = (\varepsilon_1', \ldots, \varepsilon_m')'$. The conditions $E(\underline{\varepsilon}) = 0$, $E(\underline{\varepsilon}\underline{\varepsilon}') = I \otimes V(\vartheta)$ hold.

Under the above given assumptions the estimators can be based on the sample mean vector and the sample variance matrix

$$(15) \qquad \bar{y} = \frac{1}{m} \sum_{\alpha=1}^{m} y_\alpha, \quad \hat{V} = \frac{1}{m-1} \sum_{\alpha=1}^{m} (y_\alpha - \bar{y})(y_\alpha - \bar{y})'.$$

Denote $W_i = I \otimes V_i$ and $W(\vartheta) = I \otimes V(\vartheta)$, respectively. In case we have linear restrictions on the parameter ϑ we get the model

$$(16) \qquad (\underline{y}, \underline{X}\beta, W(\vartheta) \mid R\vartheta = c)$$

Fix the value ϑ_0 and denote $T_0 = W(\vartheta_0) + \underline{X}\underline{X}'$. Denote by $T_0^{+\frac{1}{2}}$ a square root of the matrix T_0^+, for which the equalities $U_0'U_0 = T_0^+$, and $U_0 T_0 U_0' = I$ hold. The vector $\underline{y}$ transformed by the matrix U_0 imply the model

$$(17) \qquad U_0\underline{y} = U_0\underline{X}\beta + U_0\varepsilon.$$

The maximal invariant with respect to the translation $U_0\underline{y} \mapsto U_0\underline{y} + U_0\underline{X}\beta$ is then the vector $z = \underline{M}_0 U_0 \underline{y}$, where the matrix $\underline{M}_0$ of the form $\underline{M}_0 = I - U_0\underline{X}(\underline{X}'T_0^+\underline{X})^-\underline{X}'U_0'$ is the projection matrix onto the orthogonal complement of the column space of the matrix $U_0\underline{X}$.

Consider the vector $z \otimes z$. The expectation of this vector is

$$E_\vartheta(z \otimes z) = (\operatorname{vec} \underline{M}_0 U_0 W_1 U_0' \underline{M}_0, \ldots, \operatorname{vec} \underline{M}_0 U_0 W_p U_0' \underline{M}_0)\vartheta$$

The symbol "vec" of the matrix denotes the vector formed by the columns of the matrix one below the other. For the sake of simplicity we shall denote the matrix

$$(\operatorname{vec} \underline{M}_0 U_0 W_1 U_0' \underline{M}_0, \ldots, \operatorname{vec} \underline{M}_0 U_0 W_p U_0' \underline{M}_0)$$

by the symbol $\underline{Q}$.

In general the variance matrix $\Sigma(\vartheta)$ of the vector $z \otimes z$ depends on the 3rd and 4th moments of the vector y, i.e. not only on the vector parameter ϑ.

These simple considerations imply the model formed by the vector $z \otimes z$, its expectation $\underline{Q}\vartheta$, and the covariance matrix fixed at ϑ_0. Denote this model as

$$(18) \qquad (z \otimes z, \underline{Q}\vartheta, \Sigma(\vartheta_0))$$

A straightforward application of the linear theory offers the following lemmas.

Lemma 2

The linear function $f'\vartheta$ is unbiasedly invariantly estimable in model (18) iff $f \in R(\underline{Q}'\underline{Q})$ or equivalently iff $f \in R(H)$, where the matrix H has elements $H_{i,j} = \operatorname{tr} V_i V_j$.

(See also [2])

Lemma 3

Consider the model (18). Let the vector y be normally distributed. Then the ordinary least squares estimator of linear function $f'\vartheta$ with $f \in R(\underline{Q}'\underline{Q})$ is the BLUE in the sense that it is a linear function of $z \otimes z$.

Proof

The best linear (as a function of $z \otimes z$) unbiased estimator (BLUE) of the function $f'\vartheta$ in the model (18) is given in general as

$$f' \left(\underline{Q}'(\Sigma(\vartheta_0) + \underline{Q}\underline{Q}')^{-}\underline{Q}\right)^{-} \underline{Q}'(\Sigma(\vartheta_0) + \underline{Q}\underline{Q}')^{-}(z \otimes z).$$

The ordinary least squares estimator (OLS) of the estimable function $f'\vartheta$ is given as

$$f'(\underline{Q}'\underline{Q})^{-}\underline{Q}'(z \otimes z).$$

It is enough to show that the model (18) fulfils one of the necessary and sufficient conditions for the OLS to be the locally best linear (in $z \otimes z$) unbiased estimator of $f'\vartheta$, i. e. to show that the inclusion

$$R(\Sigma(\vartheta_0)\underline{Q}) \subseteq R(\underline{Q})$$

holds.

In the case that the vector y is normally distributed, the matrix $\Sigma(\vartheta_0)$ is of the form

$$\Sigma(\vartheta_0) = (\underline{M}_0 U_0 \otimes \underline{M}_0 U_0)(I + F)(W(\vartheta_0)U_0'\underline{M}_0 \otimes W(\vartheta_0)U_0'\underline{M}_0), \tag{19}$$

where the matrix F is uniquely determined by the relation $F\text{vec}\, A = \text{vec}\, A'$, for each matrix A of proper dimension. We show that

$$\Sigma(\vartheta_0)\text{vec}\, \underline{M}_0 U_0 W_i U_0'\underline{M}_0 \in R(\underline{Q}).$$

Substituting the expression from (19) for $\Sigma(\vartheta_0)$ we gradually get

$$\begin{aligned}
&\Sigma(\vartheta_0)\text{vec}\, \underline{M}_0 U_0 W_i U_0'\underline{M}_0 = \\
&\quad = (\underline{M}_0 U_0 \otimes \underline{M}_0 U_0)(I + F)\text{vec}\,(W(\vartheta_0)U_0'\underline{M}_0 U_0 W_i U_0'\underline{M}_0 U_0 W(\vartheta_0)) \\
&\quad = 2\text{vec}\,\left(\underline{M}_0 U_0 W(\vartheta_0)(\underline{M}T_0\underline{M})^{+} W_i (\underline{M}T_0\underline{M})^{+} W(\vartheta_0)U_0'\underline{M}_0\right) \\
&\quad = 2\text{vec}\,(\underline{M}_0 U_0 T_0 (\underline{M}T_0\underline{M})^{+} W_i (\underline{M}T_0\underline{M})^{+} T_0 U_0'\underline{M}_0) \\
&\quad = 2\text{vec}\,(\underline{M}_0 U_0 W_i U_0'\underline{M}_0).
\end{aligned}$$

Here we have used the first part of the equalities (3). The matrix $\underline{M}$ is defined as $\underline{M} = I - \underline{X}\underline{X}^+$.

The proof is complete.

■

Remark 3

The matrix $\underline{M}$ can be expressed as

$$\underline{M} = M_m \otimes I_n + P_m \otimes M, \tag{20}$$

where $P_m = \frac{1}{m}\mathbf{1}\mathbf{1}'$, $M_m = I - P_m$, I_n is the $n \times n$ identity matrix and the matrix M is given as $M = I - XX^+$.

Corollary 1

The ordinary least squares estimator in model (18) is the *MINQUE* of the estimable function $f'\vartheta$.

For the purpose of completeness we give the following lemma. (See also [2]).

Lemma 4

The *MINQUE* of the estimable function $f'\vartheta$ is given by

$$\widehat{f'\vartheta} = \sum_{i=1}^{p} \lambda_i \underline{q}_i = \operatorname{tr} G\hat{V} + \frac{m}{m-1}\bar{y}' A\bar{y}, \tag{21}$$

where the vector $\bar{y}$ and the matrix $\hat{V}$ are given by (15),

$$G = \sum_{i=1}^{p} \lambda_i^* V_0^+ V_i V_0^+,$$

$$A = \sum_{i=1}^{p} \lambda_i^* (MV_0M)^+ V_i (MV_0M)^+,$$

$$\underline{q}_i = (m-1)[\operatorname{tr} V_0^+ V_i V_0^+ \hat{V} + \frac{m}{m-1}\bar{y}'(MV_0M)^+ V_i (MV_0M)^+ \bar{y}] \tag{22}$$

$\lambda = (\lambda_1, \ldots, \lambda_p)'$ is any solution to the system $(\underline{Q}'\underline{Q})\lambda = f$, and $\lambda^* = (m-1)\lambda$.

Proof

Following the statement of Lemma 3 it is enough to express the estimator

$$f'(\underline{Q}'\underline{Q})^{-}\underline{Q}'z \otimes z$$

in the desired form. Let us denote by $\lambda = (\underline{Q}'\underline{Q})^{-}f$, where the matrix $(\underline{Q}'\underline{Q})^{-}$ is an arbitrary g-inverse of the matrix $\underline{Q}'\underline{Q}$.

Let us concentrate now on the vector $\underline{Q}'(z \otimes z)$, which we denote by the symbol $\underline{q}$. The i-th entry of the vector $\underline{q}$ is then

$$\begin{aligned} \underline{q}_i &= (\operatorname{vec} \underline{M}_0 U_0 W_i U_0' \underline{M}_0)'(\underline{M}_0 U_0 \underline{y} \otimes \underline{M}_0 U_0 \underline{y}) \\ &= (\operatorname{vec} U_0' \underline{M}_0 U_0 W_i U_0' \underline{M}_0 U_0)' \operatorname{vec} \underline{y}\underline{y}' \\ &= \underline{y}'(\underline{M} W_0 \underline{M})^{+} W_i (\underline{M} W_0 \underline{M})^{+} \underline{y}, \end{aligned} \tag{23}$$

due to the fact that

$$U_0' \underline{M}_0 U_0 = (\underline{M} T_0 \underline{M})^{+} = (\underline{M} W_0 \underline{M})^{+}. \tag{24}$$

Substituting (20) for $\underline{M}$ in (24) we get

$$(\underline{M} W_0 \underline{M}) = M_m \otimes V_0 + P_m \otimes (M V_0 M)$$

and consequently

$$(\underline{M} W_0 \underline{M})^{+} = M_m \otimes V_0^{+} + P_m \otimes (M V_0 M)^{+}.$$

From that we get directly

$$\begin{aligned} \underline{q}_i &= \underline{y}'(M_m \otimes V_0^{+} V_i V_0^{+} + P_m \otimes (M V_0 M)^{+} V_i (M V_0 M)^{+})\underline{y} \\ &= (m-1) \operatorname{tr} V_0^{+} V_i V_0^{+} \hat{V} + m \bar{y}'(M V_0 M)^{+} V_i (M V_0 M)^{+} \bar{y} \\ &= (m-1) \left(\operatorname{tr} V_0^{+} V_i V_0^{+} \hat{V} + \frac{m}{m-1} \bar{y}'(M V_0 M)^{+} V_i (M V_0 M)^{+} \bar{y} \right). \end{aligned}$$

The i,j-th element of the matrix $\underline{Q}'\underline{Q}$ can be expressed as

$$\begin{aligned} \{\underline{Q}'\underline{Q}\}_{i,j} &= (\operatorname{vec} \underline{M}_0 U_0 W_i U_0' \underline{M}_0)'(\operatorname{vec} \underline{M}_0 U_0 W_j U_0' \underline{M}_0) \\ &= \operatorname{tr} (\underline{M} W_0 \underline{M})^{+} W_i (\underline{M} W_0 \underline{M})^{+} W_j \\ &= (m-1) \operatorname{tr} V_0^{+} V_i V_0^{+} V_j + \operatorname{tr} (M V_0 M)^{+} V_i (M V_0 M)^{+} V_j \\ &= (m-1) \left[\operatorname{tr} V_0^{+} V_i V_0^{+} V_j + (m-1)^{-1} \operatorname{tr} (M V_0 M)^{+} V_i (M V_0 M)^{+} V_j \right]. \end{aligned}$$

It means that

$$\underline{Q}'\underline{Q} = (m-1)[G_0 + (m-1)^{-1} Q'Q], \tag{25}$$

where

(26) $$(G_0)_{i,j} = \mathrm{tr}\, V_0^+ V_i V_0^+ V_j$$

(27) $$(Q'Q)_{i,j} = \mathrm{tr}\,(MV_0M)^+ V_i (MV_0M)^+ V_j .$$

Then the solutions of the systems $\underline{Q}'\underline{Q}\lambda = f$ and $(G_0 + (m-1)^{-1}Q'Q)\lambda^* = f$ are connected by the relation $\lambda = (m-1)^{-1}\lambda^*$. The statement of the Lemma is straightforward.

■

In case that there are linear restrictions on the vector ϑ we shall refer to the model

(28) $$(z \otimes z,\ \underline{Q}\vartheta | R\vartheta = c,\ \Sigma(\vartheta_0))$$

which can be treated as a linear model (in ϑ) with restrictions. The natural reparametrization of the model (28) is as follows: let $\vartheta = R^- c + B\eta$ be the general solution to the equation $R\vartheta = c$, where the matrix B fulfils $RB = 0$. Hence we get

(29) $$(z \otimes z - \underline{Q}R^- c,\ \underline{Q}B\eta,\ \Sigma(\vartheta_0))$$

It is clear that according to the relation

$$f'\vartheta = f'R^- c + f'B\eta$$

the estimability of the function $f'\vartheta$ is equivalent to the estimability of the function $f'B\eta$ in the model (29).

The MINQUE of the function $f'\vartheta$ would be then the estimator $\widehat{f'\vartheta} = f'R^- c + \widehat{f'B\eta}$, where $\widehat{f'B\eta}$ is the MINQUE derived in the reparametrized model (29).

The procedure avoiding the reparametrization is presented below.

The model (28) can be interpreted in the form

(30) $$\left[\begin{pmatrix} z \otimes z \\ c \end{pmatrix}, \begin{pmatrix} \underline{Q} \\ \underline{R} \end{pmatrix}, \begin{pmatrix} \Sigma(\vartheta_0) & 0 \\ 0 & 0 \end{pmatrix} \right].$$

We shall take into account Definition 3 and Lemma 1 . The following two theorems will conclude our considerations.

Theorem 2

The linear function $f'\vartheta$ is unbiasedly and invariantly estimable in model (16) iff $f \in R(U)$, where the matrix U is given as $U = \big(G_0 + (m-1)^{-1}(Q'Q+$

$R'R)$). The *MINQUE* of an estimable function $f'\vartheta$ is then $\widehat{f'\vartheta} = f'\hat{\vartheta}$, where $\hat{\vartheta}$ is the solution to the system

$$
\begin{aligned}
((m-1)G_0 + Q'Q)\vartheta + R'\nu &= q \\
R\vartheta &= c
\end{aligned}
\tag{31}
$$

The vector ν is the vector of the Lagrangian multipliers and the vector q is given by (4).

Proof

The linear function $f'\vartheta$ is linearly (in $(z'\otimes z', c')'$) unbiasedly estimable in model (30) iff $f \in R(\underline{Q}', R')\begin{pmatrix} \underline{Q} \\ R \end{pmatrix} = R(\underline{Q}'\underline{Q} + R'R)$. The i,j-th entry of the matrix $\underline{Q}'\underline{Q} + R'R$ is given from (25), (26), and (27) as

(32) $\{\underline{Q}'\underline{Q} + R'R\}_{i,j} = (m-1)\left[G_{0i,j} + (m-1)^{-1}(\{Q'Q\}_{i,j} + \{R'R\}_{i,j})\right].$

It is enough to denote $U = \underline{Q}'\underline{Q} + R'R$ and the first part of the theorem is proved.

Consider now two models:

$$(z\otimes z, \underline{Q}\vartheta | R\vartheta = c, I) \text{ and } (z\otimes z, \underline{Q}\vartheta | R\vartheta = c, \Sigma(\vartheta_0)).$$

According to Definition 3 it is enough to find an estimator which is unbiased and minimizes the variance at ϑ_0 under the normality assumption of the vector $\underline{y}$. The invariance is obvious since each estimator based on the vector $z\otimes z$ is a statistic which is a function of the maximal invariant.

As it was shown in the proof of Lemma 3, under the normality of the vector $\underline{y}$ the equality $\Sigma(\vartheta_0)\underline{Q} = 2\underline{Q}$ holds. Hence the minimization of the form $(z\otimes z - \underline{Q}\vartheta)'(z\otimes z - \underline{Q}\vartheta)$ under the restriction $R\vartheta = c$ is equivalent to the minimmization of $(z\otimes z - \underline{Q}\vartheta)'\Sigma(\vartheta_0)^-(z\otimes z - \underline{Q}\vartheta)'$ under $R\vartheta = c$. The statement of the theorem is then straightforward.

■

Theorem 3

One special choice of the *MINQUE* in model (16) of the *MINQUE*-estimable function $f'\vartheta$ is

$$\widehat{f'\vartheta} = \operatorname{tr} G^{(R)}\hat{V} + \frac{m}{m-1}\bar{y}'A^{(R)}\bar{y} + \sum_{i=1}^{p}\gamma_i c_i \tag{33}$$

where the vector $\bar{y}$ and the matrix $\hat{V}$ are given by (15), and the matrices $G^{(R)}$ and $A^{(R)}$ by the relations

$$G^{(R)} = \sum_{i=1}^{p} \kappa_i V_0^+ V_i V_0^+,$$

$$A^{(R)} = \sum_{i=1}^{p} \kappa_i (MV_0M)^+ V_i (MV_0M)^+,$$

respectively, where $\kappa = (\kappa_1, \ldots, \kappa_p)'$ is any solution to the system $(M_{R'}UM_{R'})^+\kappa = f$, and the vector γ is given by $\gamma = (RU^-R')^-RU^-f$. The matrix U is given in Theorem 2, and the matrix $M_{R'}$ is defined by $M_{R'} = I - R'(RR')^-R$.

Proof

The statement is a direct consequence of Lemma 3 taking into acount the special structure of the matrix $\underline{Q}'\underline{Q}$ and the criterion matrix U as well.

■

3. GROWTH CURVE

In the following we shall concentrate our attention on the multivariate model often referred to as the growth curve model, in the special form:

$$\boldsymbol{Y} = XBZ + \boldsymbol{e}, \tag{34}$$

where $\boldsymbol{Y}$ is the $n \times m$-matrix with expectation XBZ. The random matrix $\boldsymbol{e}$ satisfies the assumptions

$$E(\operatorname{vec} \boldsymbol{e}) = 0, \quad E\left((\operatorname{vec} \boldsymbol{e})(\operatorname{vec} \boldsymbol{e})'\right) = W(\vartheta) = \sum_{i=1}^{p} \vartheta_i (V_i \otimes \Sigma).$$

Both the matrices X, Z of the type $n \times r$ and $q \times m$, respectively are known, and the matrix B is an $r \times q$-matrix of unknown parameters of the expectation. The matrices V_i, $i = 1, \ldots, p$ and Σ are known and symmetric, and the parameter space $\Theta \subset \mathbb{R}^p$ for $\vartheta \in \Theta$ is such that the matrix $W(\vartheta)$ is p.s.d. for all $\vartheta \in \Theta$. At first we shall present the MINQUE of an estimable function $f'\vartheta$. We shall proceed analogously as in the replicated model.

Denote by W_i the matrices $V_i \otimes \Sigma$

Using the operation 'vec' we create the model

$$\text{vec}\, \boldsymbol{Y} = (Z' \otimes X)\text{vec}\, B + \text{vec}\, \boldsymbol{e} \tag{35}$$

Let us use the notation $y = \text{vec}\, \boldsymbol{Y}$, and $\varepsilon = \text{vec}\, \boldsymbol{e}$. Then the model (35) is in the vector form given as

$$y = (Z' \otimes X)\text{vec}\, B + \varepsilon \tag{36}$$

and together with the properties

$$E(\varepsilon) = 0, \quad E(\varepsilon\varepsilon') = \sum_{i=1}^{p} \vartheta_i W_i$$

it forms a special form of a linear model with variance-covariance components as given in (1).

Fix the value ϑ_0. Let the matrix $W(\vartheta_0)$ be denoted by W_0. Let us use the notation $T_0 = W_0 + (Z'Z \otimes XX')$. Transform the model (36) by the matrix U_0, a square root of the matrix T_0, analogously as in the previous section. The resulting equality is

$$U_0 y = U_0(Z' \otimes X)\text{vec}\, B + U_0 \varepsilon.$$

If we are interested in invariant estimation with respect to the translations of the type $U_0 y \mapsto U_0 y + U_0(Z' \otimes X)\text{vec}\, B$ the maximal invariant is the statistic $\boldsymbol{M}_0 U_0 y$, with

$$M_0 = I - U_0(Z' \otimes X)\left((Z \otimes X')T_0^{+}(Z' \otimes X)\right)^{-}(Z \otimes X')U_0'$$

Lemma 5

The matrix $\boldsymbol{M}_0$ can be expressed as

$$\boldsymbol{M}_0 = M_{V_0^{+\frac{1}{2}} Z'} \otimes I + P_{V_0^{+\frac{1}{2}} Z'} \otimes M_{\Sigma^{+\frac{1}{2}} X}, \tag{37}$$

where the matrix

$$P_{V_0^{+\frac{1}{2}} Z'} = V_0^{+\frac{1}{2}} Z'(Z V_0^{+} Z')^{-} Z (V_0^{+\frac{1}{2}})',$$

$$M_{V_0^{+\frac{1}{2}} Z'} = I - P_{V_0^{+\frac{1}{2}} Z'},$$

and analogously the matrix

$$M_{\Sigma^{+\frac{1}{2}} X} = I - P_{\Sigma^{+\frac{1}{2}} X},$$

with

$$P_{\Sigma^{+\frac{1}{2}}X} = \Sigma^{+\frac{1}{2}}X(X'\Sigma^{+}X)^{-}X'(\Sigma^{+\frac{1}{2}})'.$$

Denote $S = M_0 U_0 y$. Then the corresponding linear model will be considered in the form

$$(38) \qquad (\operatorname{vec} SS', E(\operatorname{vec} SS'), \Sigma_0(\operatorname{vec} SS')),$$

where $\Sigma_0(\operatorname{vec} SS')$ is the covariance matrix of the vector $\operatorname{vec} SS'$ fixed at the point ϑ_0.

Lemma 6

The expectation of the vector $E(\operatorname{vec} SS')$ is

$$E(\operatorname{vec} SS') = \bar{Q}\vartheta,$$

where the columns of the matrix $\bar{Q}$ are given by the relation

$$\begin{aligned} \bar{Q}_{\cdot,j} &= \operatorname{vec}\Big[M_{V_0^{+\frac{1}{2}}Z'} V_0^{+\frac{1}{2}} V_i V_0^{+\frac{1}{2}'} M_{V_0^{+\frac{1}{2}}Z'} \otimes I \\ &+ P_{V_0^{+\frac{1}{2}}Z'} V_0^{+\frac{1}{2}} V_i V_0^{+\frac{1}{2}'} M_{V_0^{+\frac{1}{2}}Z'} \otimes M_{\Sigma^{+\frac{1}{2}}X} \\ &+ M_{V_0^{+\frac{1}{2}}Z'} V_0^{+\frac{1}{2}} V_i V_0^{+\frac{1}{2}'} P_{V_0^{+\frac{1}{2}}Z'} \otimes M_{\Sigma^{+\frac{1}{2}}X} \\ &+ P_{V_0^{+\frac{1}{2}}Z'} V_0^{+\frac{1}{2}} V_i V_0^{+\frac{1}{2}'} P_{V_0^{+\frac{1}{2}}Z'} \otimes M_{\Sigma^{+\frac{1}{2}}X}\Big]. \end{aligned}$$

As before we want to utilize the general results of the theory derived for model (13).

Lemma 7

The linear function of the parameters of the form $f'\vartheta$ is estimable in model (38) iff $f \in R(\bar{Q}'\bar{Q})$. The ordinary least squares estimator of $f'\vartheta$ is given as

$$\widehat{f'\vartheta} = f'(\bar{Q}'\bar{Q})^{-}\bar{Q}'\operatorname{vec} SS'.$$

Using the same argumentation as in the previous section it is easy but tedious to show that the OLSE of $f'\vartheta$ given in Lemma 7 is the MINQUE defined in Definition 3.

Consider the linear restrictions $R\vartheta = c$ on the parameter ϑ. Then we get the model

$$(39) \qquad (\operatorname{vec} SS', \bar{Q}\vartheta | R\vartheta = c, \Sigma_0(\operatorname{vec} SS')).$$

Denote $\bar{q} = \bar{Q}'\text{vec}\, SS'$. The next two lemmas give the expressions for the entries of the vector $\bar{q}$ and the matrix $\bar{Q}'\bar{Q}$.

Lemma 8

The i-th element of the vector $\bar{q}$ is given as

$$\bar{q}_i = \text{tr}\, \Sigma^+(\boldsymbol{Y} - \hat{\boldsymbol{Y}}_0)V_0^+ V_i V_0^+(\boldsymbol{Y}' - \hat{\boldsymbol{Y}}_0{}') . \tag{40}$$

Proof

The result is the consequence of the following calculations. Let us denote by U_i the matrix

$$M_{V_0^{+\frac{1}{2}} Z'} V_0^{+\frac{1}{2}} V_i V_0^{+\frac{1}{2}\prime} M_{V_0^{+\frac{1}{2}} Z'} \otimes I + P_{V_0^{+\frac{1}{2}} Z'} V_0^{+\frac{1}{2}} V_i V_0^{+\frac{1}{2}\prime} M_{V_0^{+\frac{1}{2}} Z'} \otimes M_{\Sigma^{+\frac{1}{2}} X}$$

$$+ M_{V_0^{+\frac{1}{2}} Z'} V_0^{+\frac{1}{2}} V_i V_0^{+\frac{1}{2}\prime} P_{V_0^{+\frac{1}{2}} Z'} \otimes M_{\Sigma^{+\frac{1}{2}} X} + P_{V_0^{+\frac{1}{2}} Z'} V_0^{+\frac{1}{2}} V_i V_0^{+\frac{1}{2}\prime} P_{V_0^{+\frac{1}{2}} Z'} \otimes M_{\Sigma^{+\frac{1}{2}} X}.$$

Then

$$\bar{q}_i = \bar{Q}'_{\cdot,i}\text{vec}\, SS' = (\text{vec}\, U_i)'\text{vec}\, SS'.$$

For any two matrices A, B with appropriate dimensions the relation

$$(\text{vec}\, A)'\text{vec}\, B = \text{tr}\, A'B$$

holds. From that we get

$$\bar{q}_i = \text{tr}\, U_i' SS' = \text{tr}\, S' U_i S.$$

Substituting the vector $\boldsymbol{M}_0 U_0 y$ for S we get

$$\bar{q}_i = y' U_0 \boldsymbol{M}_0 U_i \boldsymbol{M}_0 U_0 y = (\text{vec}\, \boldsymbol{Y})' U_0 \boldsymbol{M}_0 U_i \boldsymbol{M}_0 U_0 \text{vec}\, \boldsymbol{Y},$$

what after substitution for $\boldsymbol{M}$, T_0 , and U_i leads to

$$\begin{aligned}\bar{q}_i \;=\;& \text{tr}\, \Sigma^+ \boldsymbol{Y} V_0^+ V_i V_0^+ \boldsymbol{Y}' \\ & -2\text{tr}\, \Sigma^+ X(X'\Sigma^+ X)^- X'\Sigma^+ \boldsymbol{Y} V_0^+ Z'(Z V_0^+ Z')^- Z V_0^+ V_i V_0^+ \boldsymbol{Y}' \\ & +\text{tr}\, \Sigma^+ X(X'\Sigma^+ X)^- X'\Sigma^+ \boldsymbol{Y} V_0^+ Z'(Z V_0^+ Z')^- Z V_0^+ \\ & \times V_i V_0^+ Z'(Z V_0^+ Z')^- Z V_0^+ \boldsymbol{Y}'\Sigma^+ X(X'\Sigma^+ X)^- X'.\end{aligned}$$

If we denote by $\hat{\boldsymbol{Y}}_0$ the estimator of $\widehat{XBZ}$ which is given by

$$\widehat{XBZ} = X(X'\Sigma^+ X)^- X'\Sigma^+ \boldsymbol{Y} V_0^+ Z'(Z V_0^+ Z')^- Z,$$

the resulting formula follows from

$$\bar{q}_i = \operatorname{tr}\Sigma^+ \boldsymbol{Y} V_0^+ V_i V_0^+ \boldsymbol{Y}' - 2\operatorname{tr}\Sigma^+ \hat{\boldsymbol{Y}}_0 V_0^+ V_i V_0^+ \boldsymbol{Y}' + \operatorname{tr}\Sigma^+ \hat{\boldsymbol{Y}}_0 V_0^+ V_i V_0^+ \hat{\boldsymbol{Y}}'_0.$$

■

By the analogous procedure we get the result of the following lemma

Lemma 9

The i,j-th entrie of the matrix $\bar{Q}'\bar{Q}$ is given by

$$\begin{aligned}\bar{Q}'\bar{Q}_{i,j} &= r(\Sigma)\operatorname{tr} M_{V_0^{+\frac{1}{2}}Z'} V_0^{+\frac{1}{2}} V_i V_0^{+\frac{1}{2}'} M_{V_0^{+\frac{1}{2}}Z'} V_0^{+\frac{1}{2}} V_j V_0^{+\frac{1}{2}'} \\ &+ \operatorname{tr} M_{\Sigma^{+\frac{1}{2}}X} \Big[\operatorname{tr} P_{V_0^{+\frac{1}{2}}Z'} V_0^{+\frac{1}{2}} V_i V_0^{+\frac{1}{2}'} M_{V_0^{+\frac{1}{2}}Z'} V_0^{+\frac{1}{2}} V_j V_0^{+\frac{1}{2}'} \\ &+ \operatorname{tr} M_{V_0^{+\frac{1}{2}}Z'} V_0^{+\frac{1}{2}} V_i V_0^{+\frac{1}{2}'} P_{V_0^{+\frac{1}{2}}Z'} V_0^{+\frac{1}{2}} V_j V_0^{+\frac{1}{2}'} \\ &+ \operatorname{tr} P_{V_0^{+\frac{1}{2}}Z'} V_0^{+\frac{1}{2}} V_i V_0^{+\frac{1}{2}'} P_{V_0^{+\frac{1}{2}}Z'} V_0^{+\frac{1}{2}} V_j V_0^{+\frac{1}{2}'}\Big]. \end{aligned} \tag{41}$$

The statement of the next theorem leads to the MINQUE of the estimable function $f'\vartheta$ in the growth curve model with linear restrictions on ϑ.

Theorem 4

The linear function $f'\vartheta$ is unbiasedly and invariantly estimable under the model (39) iff $f \in R(\bar{Q}'\bar{Q} + R'R)$. The MINQUE of an estimable function $f'\vartheta$ is then $f'\hat{\vartheta}$, where $\hat{\vartheta}$ is any solution to system of equations

$$\begin{aligned}\bar{Q}'\bar{Q}\vartheta + R'\nu &= \bar{q} \\ R\nu &= c.\end{aligned}$$

The matrix $\bar{Q}'\bar{Q}$ and the vector $\bar{q}$ are given by (41) and (40), respectively. The vector ν is the vector of Lagrangian multipliers.

The proof of the theorem goes on the same lines as the proof of Theorem 2. The last theorem gives the explicite form of the MINQUE in case that the matrix R is of full rank in rows and the relation $R(R') \subseteq R(\bar{Q}'\bar{Q})$ holds, what is equivalent to the existence of a matrix, say C, for which the equality $R = C\bar{Q}'\bar{Q}$

is valid. The statement of the theorem is then the direct consequence of the linear theory applied to model (39).

Theorem 5

The MINQUE of a MINQUE-estimable function $f'\vartheta$ in model (39) under the assumptions given above is given by

$$\widehat{\widehat{f'\vartheta}} = \widehat{f'\vartheta} + f'(\bar{Q}'\bar{Q})^{-}R'\left(R(\bar{Q}'\bar{Q})^{-}R'\right)^{-1}(c - C\bar{q}),$$

where $\widehat{f'\vartheta}$ is the MINQUE of $f'\vartheta$ in model (38).

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