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A Short Note on Lattices Allowing Disjunctive Reasoning*

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Abstract

This short note shows that the scheme of disjunctive reasoning, a *or* b, not $b : a$, does not hold neither in proper ortholattices nor in proper de Morgan algebras. In both cases the scheme, once translated into the inequality $b' \cdot (a + b) \leq a$, forces the structure to be a boolean algebra.

Keywords. Disjunctive reasoning, ortholattices, de Morgan algebras, boolean algebras.

1 Introduction

The classical scheme of reasoning by Modus Ponens, $a, a \to b : b$, is usually represented by means of the inequality $a \cdot (a \to b) \leq b$, with $a \to b = a' + b$, and all a, b in the domain of statements. Once such domain is taken to be either an ortholattice or a De Morgan algebra, the symbols $\cdot, +, '$ are those of the corresponding lattice $\mathfrak{L} = (L, \cdot, +; '; 0, 1)$.

In [2] it was proven that if $\mathfrak{L}$ is an ortholattice, the validity of the inequality $a \cdot (a \to b) \leq b$ forces the ortholattice to be a boolean algebra. It was also proven that if $\mathfrak{L}$ is a De Morgan algebra, the validity of that inequality forces the algebra to be a boolean algebra. It is shown in what follows that $a \cdot (a \to b) \leq b$ is equivalent to $b' \cdot (a + b) \leq a$ and, hence, that in no proper ortholattices and in no proper De Morgan algebras, the scheme: $a + b, b' : a$, can be used to perform disjunctive reasonings.

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2 Examples

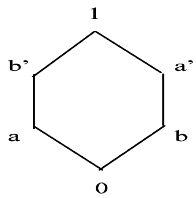
Example 2.1. If $\mathfrak{L}$ is a boolean algebra, it is $b' \cdot (a+b) = b' \cdot a + b' \cdot b = b' \cdot a + 0 = b' \cdot a \leq a$. Hence, disjunctive reasoning can be performed if the structure of statements is a boolean algebra.

Example 2.2. If $\mathfrak{L}$ is the De Morgan algebra $([0,1], \min, \max, 1-\mathrm{id})$, it is

$$\min(\max(0.5, 0.3), 1-0.5) = \min(0.5, 0.5) = 0.5 > 0.3.$$

Hence, disjunctive reasoning can't be generally made when the structure of statements is a De Morgan algebra.

Example 2.3. In the hexagonal ortholattice



is: $b' \cdot (a+b) = b' \cdot 1 = b' > a$. Hence, disjunctive reasoning can't be generally made when the structure of statements is an ortholattice.

3 The results

Proposition 3.1. *The law $b' \cdot (a+b) \leq a$ implies, either in ortholattices and in De Morgan algebras, the law $a \cdot (a \to b) \leq b$ with $a \to b = a' + b$.*

Proof. It is $b' \cdot (a+b) = b' \cdot (b'' + a) = b' \cdot (b' \to a) \leq a$, for all a, b in L. Hence, it holds $a \cdot (a' + b) = a \cdot (a \to b) \leq b$. □

Remark 3.2. Notice that in the case of DMA , a shorter proof is obtained with $a = 0$ in $b' \cdot (a+b) \leq a$ that gives $b \cdot b' = 0$ for all b in L

Proposition 3.3. *The law $a \cdot (a \to b) \leq b$ with $a \to b = a' + b$ implies, either in ortholattices and in De Morgan algebras, the law $b' \cdot (a+b) \leq a$.*

Proof. It is $a \cdot (a \to b) = a \cdot (a' + b) = a'' \cdot (b + a') \leq b$, for all a, b in L. Hence, $b' \cdot (a' + b) \leq a'$, or $b' \cdot (a+b) \leq a$. □

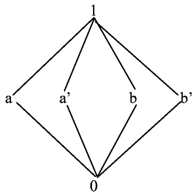
Theorem 3.4.

1) $\mathfrak{L}$ is a boolean algebra if and only if $\mathfrak{L}$ is an ortholattice where it holds the law $b' \cdot (a+b) \leq a$.

2) $\mathfrak{L}$ is a boolean algebra if and only if $\mathfrak{L}$ is an De Morgan algebra where it holds the law $b' \cdot (a+b) \leq a$.

Proof. Follows from propositions 3.1 and 3.3, and from [2] □

Hence, since disjunctive reasoning can be only made properly in boolean algebras, it seems to be an absolutely classical scheme of reasoning. Notice that it also does not hold in orthomodular lattices, for example, in the orthomodular *Chineese Lantern*



it is $b' \cdot (a + b) = b' \cdot 1 = b'$, an element that is not comparable with a. Then, if the statements in the field of Quantum Mechanics are taken to be, as it is usual, in an orthomodular lattice, disjunctive reasoning can't be generally made with them.

Remark 3.5. If instead of taking the inclusive or $+$, it is taken the exclusive or $\Delta(a, b) = (a + b) \cdot (a \cdot b)'$, since

$$b' \cdot \Delta(a, b) = b' \cdot (a + b) \cdot (a \cdot b)' = b' \cdot (a + b) \cdot (a' + b') = b' \cdot (a + b),$$

the same conclusions below follow.

Remark 3.6. Notice that the propositions 3.1 and 3.3 strongly depend on the model $a \to b = a' + b$ for the implication arrow.

4 Conclusion

This note aims to study the scheme of disjunctive reasoning in the domain of lattices with negation.

It is proven that, in the subdomain of lattices with negation obtained by the union of ortholattices and De Morgan algebras, the validity of the inequality $b' \cdot (a + b) \leq a$, that translates that of disjunctive reasoning

$$a \textit{ or } b, \text{ not } b : a,$$

forces the structure of boolean algebras. That is, in that subdomain, disjunctive reasoning only can be performed in boolean algebras and, in this sense, it appears as a very classical mode of reasoning.

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