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An Orthogonality-based Classification of Conjectures in Ortholattices

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Abstract

A mathematical model for conjectures (including hypotheses, consequences and speculations), was recently introduced, in the context of ortholattices, by Trillas, Cubillo and Castiñeira (Artificial Intelligence 117, 2000, 255–257). The aim of the present paper is to further clarify the structure of this model by studying its relationships with one of the most important ortholattices' relation, the orthogonality relation. The particular case of orthomodular lattices -the framework for both boolean and quantum logics- is specifically taken into account.

1 Introduction

The importance of the classical categories of inference, namely deduction, induction and abduction, at least for common research and formal types of reasoning, does not need to be stressed. Although deduction has traditionally deserved a good deal of high quality research, mainly but not only in Mathematical Logic, induction and abduction have been comparatively less analyzed by means of clear-cut mathematical models. Ever since in the field of Artificial Intelligence computer systems either inducing or abducing appeared, clear mathematical models became even more necessary for a better understanding of the involved objects. From this point of view, the introduction of theoretical models of conjectures (general induction) and hypotheses (abduction), as well as the links between them and with consequences (deduction), is to be viewed as something valuable in itself.

A mathematical model for conjectures, hypotheses and consequences, in the framework of ortholattices, was recently introduced and discussed in the papers [5], [4] and [7], and its application to the case of fuzzy logic has also been studied (see [3] and [6]). The aim of this paper is to provide a deeper insight on the structure and properties of the afore-mentioned mathematical model, studying its relationships with one of the most important relations among those that may be defined within the elements of an ortholattice, namely the orthogonality relation. The importance of this relation relies on the fact that -as it will be recalled later- it allows to distinguish, among the whole set of ortholattices, the special class of orthomodular lattices -the algebraic counterpart of quantum logic-, and, among them, the case of boolean algebras.

The paper is organized as follows. In order to be self-contained, the next section presents a general account on both ortholattices and the mathematical model for conjectures presented in [5]. The following section (section 3) studies the relationships between orthogonality and conjectures, taking into account the different classes in which the later may be divided. Section 4 uses the results found in previous section in order to obtain a classification of conjectures based on orthogonality, both for general ortholattices as well as for the particular case of orthomodular lattices. Finally, the paper ends with some conclusions.

2 Preliminaries

This section is devoted to review the most important issues related to the two main concepts that this paper deals with. First of all, the basic results concerning *ortholattices* are briefly recalled, focussing on the *orthogonality* relations defined in such structures and their ability to characterize *orthomodular lattices*. Secondly, the mathematical model for *conjectures* in ortholattices that was introduced in [5] is summarized.

2.1 Ortholattices and Orthogonality

The well-known *orthocomplemented lattices*, or, for short, *ortholattices*, are quite general algebraic structures that encompass both classical and quantum logical calculi. For detailed discussions on this topic or proofs of the results that are presented below, see [1] and [2].

Definition 2.1 *A structure* $(L, \cdot, +, ', 0, 1)$ *is called an* ortholattice *whenever it satisfies the following properties:*

1. $(L, \cdot, +)$ *is a lattice where* $\cdot$ *and* $+$ *represent, respectively, the infimum and supremum operations,* 0 *is the least element of* L *and* 1 *is its greatest element. The partial order of the lattice is defined, for any* $a, b \in L$, *by*

$$a \leq b \quad \textit{if and only if} \quad a \cdot b = a \quad \textit{if and only if} \quad a + b = b$$

2. $' : L \to L$ *is a unary operation, called* orthocomplementation, *verifying, for any* $a, b \in L$:

   (a) $0' = 1$

   (b) $a \cdot a' = 0$

(c) $(a')' = a$

(d) If $a \leq b$*, then* $b' \leq a'$

Recall in addition that, from the above definition, it follows that ortholattices do also verify, for any $a, b \in L$, the law $a + a' = 1$ and the De Morgan laws $(a \cdot b)' = a' + b'$ and $(a + b)' = a' \cdot b'$.

On the other hand, the two following fundamental relations may be defined among the elements of any ortholattice:

Definition 2.2 *Let* $(L, \cdot, +, ', 0, 1)$ *be an ortholattice. For any* $a, b \in L$*, the following relations are defined:*

- *The* left-orthogonality *relation, denoted by* $\perp_l$*, and given by*

$$a \perp_l b \qquad \textit{if and only if} \qquad a = a \cdot b + a \cdot b'$$

- *The* right-orthogonality *relation, denoted by* $\perp_r$*, and given by*

$$a \perp_r b \qquad \textit{if and only if} \qquad b = b \cdot a + b \cdot a'$$

When it is $a \perp_l b$, it is said that the element a *commutes* with b. Clearly, the relation $\perp_r$ is nothing else than the inverse of the relation $\perp_l$ (it is $a \perp_r b$ if and only if $b \perp_l a$ for any $a, b \in L$), and, in general, these two relations are not coincidental, i.e., it is not always the case that $\perp_l = \perp_r$. If we define, for any $a \in L$, the sets

$$O_l(a) = \{b \in L; a \perp_l b\} \qquad \text{and} \qquad O_r(a) = \{b \in L; a \perp_r b\}$$

then it is $b \in O_l(a)$ if and only if $a \in O_r(b)$ for any $a, b \in L$, $O_l(a)$ and $O_r(a)$ are, in general, different sets, and the following properties may be easily proven:

Proposition 2.1 *Let* $(L, \cdot, +, ', 0, 1)$ *be an ortholattice. The following properties are verified for any* $a, b \in L$*:*

1. $\{0, 1, a, a'\} \subseteq O_l(a) \cap O_r(a)$
2. $b \in O_l(a) \Leftrightarrow b' \in O_l(a), \quad b \in O_r(a) \Leftrightarrow b \in O_r(a')$
3. *If* $a \leq b$*, then:*
   - *3.1.* $b \in O_l(a), \quad a \in O_r(b)$
   - *3.2.* $b \in O_r(a) \Leftrightarrow b = a + b \cdot a', \quad a \in O_l(b) \Leftrightarrow b = a + b \cdot a'$
4. *If* $a \leq b'$*, then* $b \in O_l(a) \cap O_r(a), \quad a \in O_l(b) \cap O_r(b)$

An important subclass of ortholattices are the so-called *orthomodular lattices*:

Definition 2.3 *An ortholattice* $(L, \cdot, +, ', 0, 1)$ *is called an* orthomodular lattice *if it verifies, for any* $a, b \in L$*, the law:*

$$a \leq b \qquad \textit{implies} \qquad b = a + b \cdot a'$$

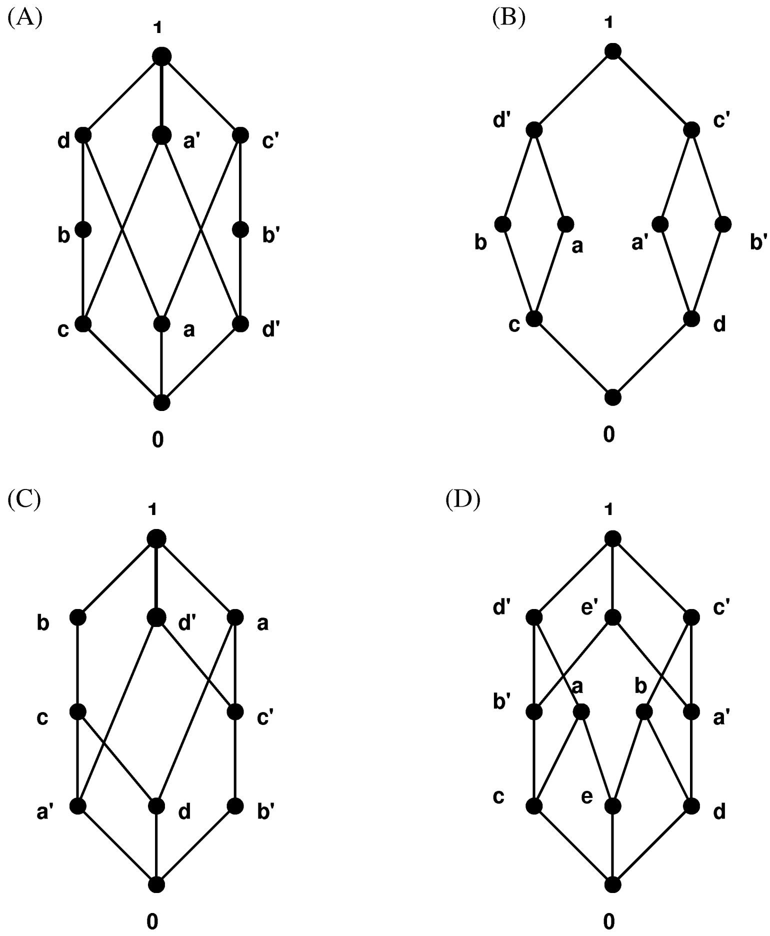

Figure 1: Some examples of non-orthomodular ortholattices

When $a \leq b$ and $b = a + b \cdot a'$ (i.e., $b \in O_r(a)$), the element $b \cdot a'$ is called the *relative complement* of a with respect to b, and it is usually written as $b - a$. Note that, since in any ortholattice $b = a + b \cdot a'$ implies $a \leq b$, orthomodular lattices are those ortholattices where the ordering $a \leq b$ is equivalent to $b = a + b \cdot a'$. In addition, boolean algebras, thanks to their distributivity property, are clearly particular cases of orthomodular lattices: indeed, it is $a + b \cdot a' = (a + b) \cdot (a + a') = a + b$ and, if $a \leq b$, since this is equivalent to $a + b = b$, the equality $a + b \cdot a' = b$ follows.

Figure 1 shows some examples of ortholattices which are not orthomodular (these will be called in the sequel *proper ortholattices*), and figure 2 includes two non-boolean orthomodular lattices.

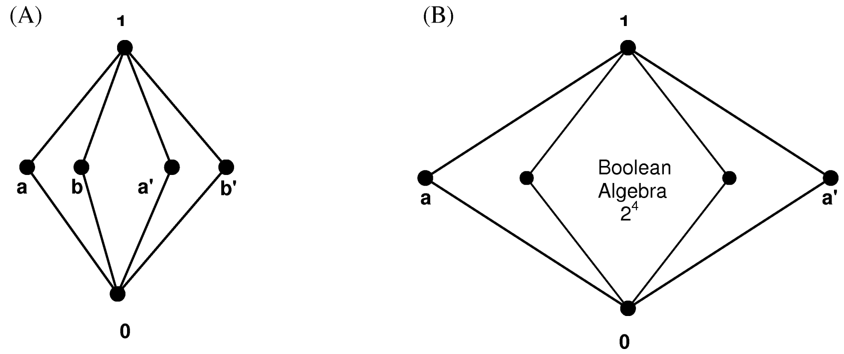

Figure 2: Some examples of non-boolean orthomodular lattices

An interesting feature of the orthogonality relations $\perp_l$ and $\perp_r$ is that they allow to distinguish, among the whole set of ortholattices, both orthomodular lattices and boolean algebras. Indeed, the following result is available:

Theorem 2.1 *Let* $(L, \cdot, +, ', 0, 1)$ *be an ortholattice and* $\perp_l, \perp_r$ *the relations given in definition 2.2. Then:*

- *L is an orthomodular lattice if and only if* $\perp_l = \perp_r$ *(in these cases the symbol* $\perp$ *will be used to denote the relation* $\perp_l = \perp_r$*).*
- *L is a boolean algebra if and only if* $\perp_l = \perp_r = L \times L$.

Note that, as a consequence of the above theorem, in the context of orthomodular lattices, the sets $O_l(a)$ and $O_r(a)$ coincide for any $a \in L$, and will be denoted by $O(a)$.

2.2 Conjectures in Ortholattices

As it has been recalled in the introduction, an algebraic model for conjectures, hypotheses and consequences, within the framework of ortholattices, was proposed in [5]. In the sequel the main ingredients of this model, that has been further studied in [4] and [7], are reviewed. Given a complete ortholattice $(L, \cdot, +, ', 0, 1)$ and a set of premises $P \subseteq L$ such that $P \neq \emptyset$ and $Inf P = p_\wedge \neq 0$, the following sets were defined in [5]:

- The set of *strict conjectures* of P: $Conj(P) = \{q \in L; \quad p_\wedge \nleq q'\}$.
- The set of *consequences* of P: $Cons(P) = \{q \in L; \quad p_\wedge \leq q\}$.
- The set of *hypotheses* of P: $Hyp(P) = \{q \in L - \{0\}; \quad q < p_\wedge\}$
- The set of *speculations* of P (or *speculative conjectures*, as they were called in [5]): $Spec(P) = \{q \in L; \quad qNCp_\wedge, p_\wedge \nleq q'\}$ (given $a, b \in L$, $aNCb$ will be

used to indicate that a and b are non-comparable with respect to the lattice order, i.e., it is neither $a \leq b$ nor $b < a$).

The above definitions allow for the following partition of the set $Conj(P)$ ([5]):

$$Conj(P) = Cons(P) \cup Hyp(P) \cup Spec(P)$$

Moreover, a further partitioning of the set $Spec(P)$, which will be of special interest for this paper, can be established. Indeed, since $p_\wedge \nleq q'$ means that it is either $q' < p_\wedge$ or, else, $q'NCp_\wedge$, we can define the disjoint sets

$$Spec1(P) = \{q \in L; \quad qNCp_\wedge, q' < p_\wedge\}$$
$$Spec2(P) = \{q \in L; \quad qNCp_\wedge, q'NCp_\wedge\}$$

in such a way that $Spec(P)$ may be written as $Spec1(P) \cup Spec2(P)$, and, therefore (see figure 3) it is

$$Conj(P) = Cons(P) \cup Hyp(P) \cup Spec1(P) \cup Spec2(P)$$

Finally, the following properties are easily proven:

Proposition 2.2 *Let $(L, \cdot, +, ', 0, 1)$ be a complete ortholattice, $P \subseteq L$ a set such that $P \neq \emptyset$, $Inf P = p_\wedge \neq 0$ and $Conj(P), Cons(P), Hyp(P), Spec1(P)$ and $Spec2(P)$ the sets defined above. Then, for any $q \in L$:*

1. $P \cup \{1, p_\wedge\} \subseteq Cons(P)$
2. $q \in Cons(P) \Leftrightarrow q' \in Conj(P)^c$
3. *If* $p_\wedge \neq 1$, *then* $q \in Hyp(P) \Leftrightarrow q' \in Spec1(P)$
4. $q \in Spec2(P) \Leftrightarrow q' \in Spec2(P)$
5. *If* $q \neq p_\wedge$, *then* $q \in Cons(P) \Leftrightarrow p_\wedge \in Hyp(\{q\})$
6. $q \in Spec1(P) \Leftrightarrow p_\wedge \in Spec1(\{q\})$
7. $q \in Spec2(P) \Leftrightarrow p_\wedge \in Spec2(\{q\})$

Note that item *(1)* in the above proposition shows that the sets $Cons(P)$ and, therefore, $Conj(P)$, are never empty, and the combination of *(1)* and *(2)* entails that the same happens with $Conj(P)^c$ (the complement of $Conj(P)$), since $P' \cup \{0, p'_\wedge\} \subseteq Conj(P)^c$, where $P' = \{p' : p \in P\}$.

3 Conjectures and Orthogonality

As it was stated in the introduction, the main goal of this paper is to study, within the framework that has been described in section 2, the relationships between conjectures and orthogonality. Unless something else is explicitly indicated,

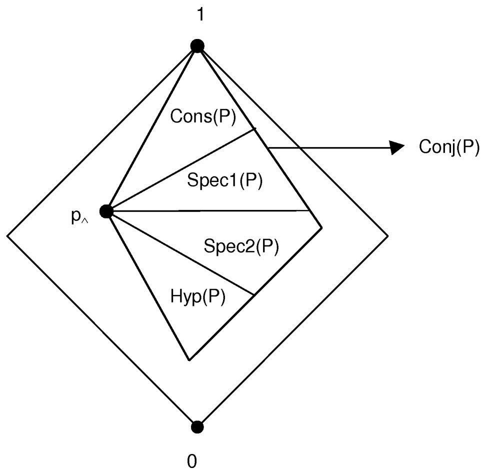

Figure 3: Partition of conjectures in ortholattices

in the remaining of this paper $P \subseteq L$ will be a subset of a complete ortholattice $(L, \cdot, +, ', 0, 1)$ such that $P \neq \emptyset$ and $Inf P = p_\wedge \neq 0$; $\perp_l, \perp_r, O_l(a)$ and $O_r(a)$ will be as defined in 2.1, and $Conj(P), Cons(P)$, $Hyp(P)$, $Spec1(P)$ and $Spec2(P)$ will represent the sets given in 2.2. Our goal is therefore to analyze which elements in $Conj(P)$ appear to have a left/right orthogonality relationship with the infimum $p_\wedge$, i.e., to find out which elements of $Conj(P)$ do belong to $O_l(p_\wedge)$ and/or to $O_r(p_\wedge)$. The centrality of the element $p_\wedge$ comes from the obvious equality $Conj(P) = Conj(\{p_\wedge\})$ (analogous equalities hold for the sets $Cons(P), Hyp(P), Spec1(P)$ and $Spec2(P)$).

Before entering into details concerning conjectures, let us note that the elements which, according to the definition of the set $Conj(P)$, may not be considered as conjectures of P, are always both left and right orthogonal to $p_\wedge$. Indeed, the following result can be established:

Proposition 3.1 $Conj(P)^c \subseteq O_l(p_\wedge) \cap O_r(p_\wedge)$

Proof. If $q \in Conj(P)^c$, it is $p_\wedge \leq q'$, and then, by proposition 2.1, property number *(4)*, it is $q \in O_l(p_\wedge) \cap O_r(p_\wedge)$. ∎

Note that the above result is obviously equivalent to $O_l(p_\wedge)^c \cup O_r(p_\wedge)^c \subseteq Conj(P)$, that is, any element of the lattice that is not orthogonal to $p_\wedge$, either left or right, is necessarily a conjecture. In addition, if L is an orthomodular lattice, according to theorem 2.1, it is simply $Conj(P)^c \subseteq O(p_\wedge)$ or, equivalently, $O(p_\wedge)^c \subseteq Conj(P)$.

Regarding conjectures, since, as it was recalled before, these can be classified into several disjoint subsets representing the so-called consequences, hypotheses and speculations, next sub-sections deal with the orthogonal behavior of each of these conjectures' subclasses.

3.1 Consequences and Orthogonality

The relationships between the set of consequences and the sets of orthogonal elements may be summarized as follows:

Theorem 3.1

1. $Cons(P) \subseteq O_l(p_\wedge)$
2. $\forall q \in Cons(P): \quad q \in O_r(p_\wedge) \Leftrightarrow q = p_\wedge + q \cdot p'_\wedge$

Proof. If $q \in Cons(P)$, it is, by definition, $p_\wedge \leq q$, and then property number *(3)* of proposition 2.1 entails both *(1)* and *(2)*. ∎

In addition, the following results are obtained as a corollary:

Corollary 3.1

1. $\forall q \in Cons(P): \quad q \in O_l(p_\wedge) \cap O_r(p_\wedge) \Leftrightarrow q = p_\wedge + q \cdot p'_\wedge$
2. *If L is an orthomodular lattice, then* $Cons(P) \subseteq O(p_\wedge)$

It therefore appears that, when dealing with orthomodular lattices, consequences are always orthogonal to $p_\wedge$, whereas in the case of proper ortholattices, consequences are left-orthogonal but not necessarily right-orthogonal to $p_\wedge$. This last result naturally entails the following question regarding proper ortholattices: does any set of premises have consequences which are right-orthogonal to $p_\wedge$, and, hence, both left and right-orthogonal? The next proposition shows that this question has a positive answer:

Proposition 3.2 *For any complete ortholattice* $(L, \cdot, +, ', 0, 1)$ *and for any* $P \subseteq L$ *such that* $P \neq \emptyset$ *and* $Inf P = p_\wedge \neq 0$, *it is* $Cons(P) \cap O_r(p_\wedge) \neq \emptyset$.

Proof. The elements 1 and $p_\wedge$ belong both to $Cons(P)$ (property number *(1)* of proposition 2.1) and to $O_r(p_\wedge)$ (proposition 2.2.*(1)*). ∎

Therefore, for any $p_\wedge$, there are at least two consequences of $p_\wedge$, namely the elements 1 and $p_\wedge$, which are right-orthogonal to $p_\wedge$, and then it is $Cons(P) \cap O_r(p_\wedge) \neq \emptyset$. But 1 and $p_\wedge$ are very special elements, so a new question arises: is it possible to affirm that there are always right-orthogonal consequences, different from 1 and $p_\wedge$? The answer to this last question is, in general, negative: indeed, it suffices to think of a set with a unique premise, $P = \{p\}$, where p is an immediate predecessor of 1 in the lattice partial order. In this case it is $p_\wedge = p$ and $Cons(P) = \{1, p_\wedge\}$, i.e., there are no right-orthogonal consequences other than 1 and $p_\wedge$.

This last example also shows that it is possible to have sets of premises such that *all* their consequences are right-orthogonal to $p_\wedge$, and this means that, in general, sets P do not always verify $Cons(P) \cap O_r(p_\wedge) \neq Cons(P)$. Nevertheless, for any proper ortholattice, it is always possible to find sets P having at least one non-right-orthogonal consequence. Indeed, by definition, any non-orthomodular lattice contains elements a, b such that $a \leq b$ and $b \neq a + b \cdot a'$, and then, taking $P = \{a\}$, it is $b \in Cons(P)$ and $b \notin O_r(p_\wedge)$.

In general, proper ortholattices do exist where it is possible to find sets of premises some of whose consequences are right-orthogonal and some are not:

Example 3.1 *Let us consider the non-orthomodular lattice (A) given in Figure 1 and the set of premises* $P = \{b, a'\}$. *It is* $p_\wedge = c$, $Cons(P) = \{q \in L; c \leq q\} = \{c, b, d, a', 1\}$ *and:*

- $d \in O_r(c)$, *since* $c + d \cdot c' = c + a = d$.
- $b \notin O_r(c)$, *since* $c + b \cdot c' = c + 0 = c \neq b$

3.2 Hypotheses and Orthogonality

The orthogonality properties of hypotheses are symmetric to those of consequences, in the sense that hypotheses are always right-orthogonal to $p_\wedge$, but not necessarily left-orthogonal:

Theorem 3.2

1. $Hyp(P) \subseteq O_r(p_\wedge)$
2. $\forall q \in Hyp(P): \quad q \in O_l(p_\wedge) \Leftrightarrow p_\wedge = q + p_\wedge \cdot q'$

Proof. If $q \in Hyp(P)$, it is, by definition, $q < p_\wedge$, and then proposition 2.1.*(3)* provides both *(1)* and *(2)*. ∎

Again, the following results can be obtained as a corollary:

Corollary 3.2

1. $\forall q \in Hyp(P): \quad q \in O_l(p_\wedge) \cap O_r(p_\wedge) \Leftrightarrow p_\wedge = q + p_\wedge \cdot q'$
2. *If* L *is an orthomodular lattice, then* $Hyp(P) \subseteq O(p_\wedge)$

Therefore, hypotheses in orthomodular lattices are always orthogonal to $p_\wedge$. When dealing with proper ortholattices, and contrary to the consequences' case, the existence of hypotheses which are left-orthogonal cannot be, in the general case, taken for granted, since, in fact, the set $Hyp(P)$ may even be, in some situations, empty (in particular, when $p_\wedge$ is an immediate successor, in the lattice partial order, of the least element 0). Notwithstanding, the existence of sets of premises having left-orthogonal hypotheses is closely related to the existence of right-orthogonal consequences. Indeed:

Proposition 3.3 *For any* $q \in L$, $q \neq p_\wedge$*:*

$$q \in Cons(P) \cap O_r(p_\wedge) \Leftrightarrow p_\wedge \in Hyp(\{q\}) \cap O_l(\{q\})$$

Proof. It suffices to remember that $\perp_l$ are $\perp_r$ are inverse relations and to apply proposition 2.2.*(5)*. ∎

This last result entails that, for any proper ortholattice, it is always possible to find a set P having at least one left-orthogonal hypothesis, since, as it was proven in the last section, any set P has always at least a right-orthogonal consequence. In addition, proposition 3.3 allows to easily find left and non-left-orthogonal hypotheses in proper ortholattices:

Example 3.2 *Applying the last proposition to the example 3.1: in the lattice given in figure 1 (A), c is an hypothesis of both b and d, c is left-orthogonal to d ($c \in Hyp(\{d\}) \cap O_l(\{d\})$) but it is not left-orthogonal to b ($c \notin Hyp(\{b\}) \cap O_l(\{b\})$). Note also that the set of premises $P = \{d\}$, in addition to the left-orthogonal hypothesis c, has also a non-left-orthogonal hypothesis, b (indeed, it is $b + d \cdot b' = b + 0 = b \neq d$).*

3.3 Speculations and Orthogonality

Section 2.2 has shown that the set of speculations, $Spec(P)$, may be further divided into two disjoint subsets, namely $Spec1(P) = \{q \in L; qNCp_\wedge, q' < p_\wedge\}$ and $Spec2(P) = \{q \in L; qNCp_\wedge, q'NCp_\wedge\}$. The two following subsections deal with the orthogonality properties of both classes of speculations, that we will call, respectively, *type1-speculations* and *type2-speculations*.

3.3.1 Type1-speculations

In general, speculations of this class are not necessarily orthogonal, neither left nor right, to the infimum of the premises. Indeed, the following result can be stated:

Theorem 3.3 *For any $q \in Spec1(P)$:*

1. $q \in O_l(p_\wedge) \Leftrightarrow p_\wedge = p_\wedge \cdot q + q'$
2. $q \in O_r(p_\wedge) \Leftrightarrow q = q \cdot p_\wedge + p'_\wedge$

Proof. It suffices to apply the definition of $\perp_l$ and $\perp_r$ (definition 2.2), taking into account that, due to the fact that $q' < p_\wedge$, it is $q' \cdot p_\wedge = q'$ and $p'_\wedge \cdot q = p'_\wedge$. ∎

As a corollary, we can characterize those type1-speculations which are both left and right-orthogonal, and establish their behavior in the particular case of orthomodular lattices:

Corollary 3.3

1. $\forall q \in Spec1(P)$:

$$q \in O_l(p_\wedge) \cap O_r(p_\wedge) \Leftrightarrow (p_\wedge = p_\wedge \cdot q + q' \text{ and } q = q \cdot p_\wedge + p'_\wedge)$$

2. *If L is an orthomodular lattice, then $Spec1(P) \subseteq O(p_\wedge)$*

Proof. *(1)* is obvious, and *(2)* follows from the definition of orthomodular lattice (definition 2.3), since, according to it, the ordering $q' < p_\wedge$ implies $p_\wedge = p_\wedge \cdot q + q'$, i.e., $q \in O_l(p_\wedge)$, and from theorem 2.1, which establishes that, in orthomodular lattices, $O_l(a) = O_r(a) = O(a)$. ∎

In summary, within the class of orthomodular lattices, type1-speculations are always orthogonal to the infimum of the premises, whereas in the case of proper ortholattices, this is not generally the case, neither for left nor for right-orthogonality. As a consequence, the question of the existence of type1-speculations being either left-orthogonal, right-orthogonal, both or none of them arises. In general, the existence of any of these type1-speculations cannot be ensured, since the set $Spec1(P)$ may be empty. Indeed, proposition 2.2 shows that this kind of conjectures are deeply related to hypotheses, since, if $p_\wedge \neq 1$, it is $q \in Hyp(P) \Leftrightarrow q' \in Spec1(P)$, and, therefore, the fact that $Hyp(P)$ may be empty -see previous section- implies that the set $Spec1(P)$ may also be empty. On the other hand, the next proposition shows that the existence of type1-speculations which are left or right-orthogonal relies on the existence of left-orthogonal hypotheses (or, equivalently -due to proposition 3.3-, on the existence of right-orthogonal consequences):

Proposition 3.4 *For any $q \in L$:*

1. $q \in Spec1(P) \cap O_l(p_\wedge) \Leftrightarrow q' \in Hyp(P) \cap O_l(p_\wedge)$
2. $q \in Spec1(P) \cap O_l(p_\wedge) \cap O_r(p_\wedge) \Leftrightarrow p_\wedge \in Spec1(\{q\}) \cap O_r(\{q\}) \cap O_l(\{q\})$

Proof. *(1)* is immediate taking into account proposition 2.1.*(2)* and proposition 2.2.*(2)*. The result given in *(2)* is proven by the the fact that $\perp_l^{-1} = \perp_r$ and by proposition 2.2.*(6)*. ∎

To end with this section, let us show examples with different classes of type1-speculations:

Example 3.3

- $q \in Spec1(P), \quad q \in O_l(p_\wedge), \quad q \in O_r(p_\wedge)$

  According to the last proposition, if an element is a left-orthogonal hypothesis, then its complement is a left-orthogonal type1-speculation. Therefore, following the example 3.2, the element c' in the lattice given in figure 1 (A) is such that $c' \in Spec1(\{d\}) \cap O_l(d)$. In addition, following theorem 3.3.(2), as it is $c' \cdot d + d' = a + d' = c'$, it is also $c' \in O_r(\{d\})$.

- $q \in Spec1(P), \quad q \in O_l(p_\wedge), \quad q \notin O_r(p_\wedge)$

  An example of a type1-speculation which is left-orthogonal but not right-orthogonal may be found in the lattice (C) from figure 1. Indeed, taking $P = \{a\}$, since $bNCa$ and $b' < a$, it appears that $b \in Spec1(\{a\})$. In addition, it is $a \cdot b + b' = d + b' = a$ and $b \cdot a + a' = d + a' = c \neq b$, and this means, by theorem 3.3, that $b \in O_l(a)$ and $b \notin O_r(a)$.

- $q \in Spec1(P), \quad q \in O_r(p_\wedge), \quad q \notin O_l(p_\wedge)$

  An example of this kind can be immediately found from the last one and proposition 3.4.(2). Indeed, as we have just seen, in the lattice (C) of figure 1 it is $b \in Spec1(\{a\}), b \in O_l(a)$ *and* $b \notin O_r(a)$*, which is equivalent to* $a \in Spec1(\{b\}), a \in O_r(b)$ *and* $a \notin O_l(b)$*.*

- $q \in Spec1(P), \quad q \notin O_l(p_\wedge), \quad q \notin O_r(p_\wedge)$

  Finally, a type1-speculation which is neither left nor right-orthogonal may be found in lattice (B) from figure 1. Indeed, since $bNCc'$ *and* $b' < c'$*, it is* $b \in Spec1(\{c'\}$*. In addition, it is easy to check that* $c'\dot{b} + b' = 0 + b' = b' \neq c'$ *and* $b \cdot c' + c = 0 + c = c \neq b$*, that is,* $b \notin O_l(c')$ *and* $b \notin O_r(c')$*, respectively.*

3.3.2 Type2-speculations

Similarly to the case of type1-speculations, the elements belonging to this second class are not necessarily orthogonal to $p_\wedge$. Indeed:

Theorem 3.4 *For any* $q \in Spec2(P)$*:*

1. $q \in O_l(p_\wedge) \Leftrightarrow p_\wedge = p_\wedge \cdot q + p_\wedge \cdot q'$
2. $q \in O_r(p_\wedge) \Leftrightarrow q = q \cdot p_\wedge + q \cdot p'_\wedge$

Proof. Obvious from the definition of O_l and O_r. ∎

Again, next corollary is immediate:

Corollary 3.4 *For any* $q \in Spec2(P)$*:*

1. $q \in O_l(p_\wedge) \cap O_r(p_\wedge) \Leftrightarrow (p_\wedge = p_\wedge \cdot q + p_\wedge \cdot q'$ *and* $q = q \cdot p_\wedge + q \cdot p'_\wedge)$
2. *If* L *is an orthomodular lattice, then:*
   $q \in O(p_\wedge) \Leftrightarrow p_\wedge = p_\wedge \cdot q + p_\wedge \cdot q'$ $(\Leftrightarrow q = q \cdot p_\wedge + q \cdot p'_\wedge)$

Therefore, the main difference between type1 and type2-speculations appears in the case of orthomodular lattices, where the formers are always orthogonal whilst the laters are not.

Regarding the existence of orthogonal or non-orthogonal type2-speculations, now, contrary to the previous cases, this existence is not related to any of the other types of conjectures. Nevertheless, the following equivalences among type2-speculations can be established:

Proposition 3.5 *For any* $q \in L$*:*

1. $q \in Spec2(P) \cap O_l(p_\wedge) \Leftrightarrow q' \in Spec2(P) \cap O_l(p_\wedge)$
2. $q \in Spec2(P) \cap O_l(p_\wedge) \cap O_r(p_\wedge) \Leftrightarrow p_\wedge \in Spec2(\{q\}) \cap O_r(\{q\}) \cap O_l(\{q\})$

Proof. *(1)* is given by propositions 2.2.*(4)* and 2.1.*(2)*. Result *(2)* is easily proven by the the fact that $\perp_l^{-1} = \perp_r$ and by proposition 2.2.*(7)*. ∎

In the sequel it is shown, by means of some examples, that it is possible to find non-orthomodular lattices containing type2-speculations of different classes:

Example 3.4

- $q \in Spec2(P), \quad q \in O_l(p_\wedge), \quad q \in O_r(p_\wedge)$

  Lattice (D) of figure 1 provides a speculation of this type. Indeed, taking $P = \{b\}$, *it is clearly* $a \in Spec2(\{b\})$, *since it is* $aNCb$ *and* $a'NCb$. *In addition it is* $a \in O_l(b)$, *because* $b \cdot a + b \cdot a' = e + d = b$, *and* $a \in O_r(b)$ *because* $a \cdot b + a \cdot b' = e + c = a$. *Note that, thanks to proposition 3.5, it is easy to find another example of this kind:* b *is a type2-speculation of* a *which is both left and right-orthogonal.*

- $q \in Spec2(P), \quad q \in O_l(p_\wedge), \quad q \notin O_r(p_\wedge)$

  An example of this kind can be found in the lattice (A) from figure 1. Indeed, taking $P = \{a'\}$ *and* $q = b$, *it is* $b \in Spec2(\{a'\})$, *since neither* b *nor* b' *are comparable to* a'. *In addition, it is* $a' \cdot b + a' \cdot b' = c + d' = a'$ *and* $b \cdot a' + b \cdot a = c + 0 = c \neq b$, *and this, according to theorem 3.4, means that* $b \in O_l(a')$ *and* $b \notin O_r(a')$.

- $q \in Spec2(P), \quad q \in O_r(p_\wedge), \quad q \notin O_l(p_\wedge)$

  The last example, in combination with proposition 3.5, allows to find out that, in the lattice (A) from figure 1, it is $a' \in Spec2(\{b\}), a' \in O_r(b)$ *and* $a' \notin O_l(b)$.

- $q \in Spec2(P), \quad q \notin O_l(p_\wedge), \quad q \notin O_r(p_\wedge)$

  Finally, a type2-speculation that is neither left nor right-orthogonal may be found in lattice (B) from figure 1. Indeed, since $bNCa$ *and* $b'NCa$, *it is* $b \in Spec2(\{a\})$. *In addition, it is easy to check that* $a\dot{b} + a \cdot b' = c + 0 = c \neq a$ *and* $b \cdot a + b \cdot a' = c + 0 = c \neq b$, *that is,* $b \notin O_l(a)$ *and* $b \notin O_r(a)$, *respectively.*

In the particular case of orthomodular lattices, let us recall that it is $O_l(a) = O_r(a) = O(a)$ for any $a \in L$ (theorem 2.1), and, contrary to type1-speculations, type2-speculations do not appear to be necessarily orthogonal (corollary 3.4). Of course, if L is a boolean algebra, any type2-speculation will be orthogonal to $p_\wedge$ (theorem 2.1), but, what happens if L is a non-boolean orthomodular lattice? Next example shows that it is possible to find lattices of this kind holding both orthogonal and non-orthogonal type2-speculations.

Example 3.5

- *Lattice (A) from figure 2, which is a well-known example of non-boolean orthomodular lattice, holds type2-speculations which are not orthogonal to the premises' infimum. For example, it is easy to check that* $b \in Spec2(\{a\})$ *whereas* $b \notin O(a)$, *since* $a \cdot b + a \cdot b' = 0 + 0 = 0 \neq a$.

- *On the other hand, lattice (B) in figure 2 provides an example where both orthogonal and non-orthogonal type2-speculations may be found:*
  - *First, a (the same happens with a') is clearly a type2-speculation of any element b in the 2^4 boolean algebra, since it is $aNCb$ and $a'NCb$. In addition, $a \notin O(b)$, since $b \cdot a + b \cdot a' = 0 + 0 = 0 \neq b$.*
  - *Secondly, in order to find an orthogonal type2-speculation, let us suppose, without loss of generality, that the 2^4 boolean algebra belonging to L corresponds to $\mathcal{P}(E)$, where $E = \{e_1, e_2, e_3, e_4\}$. Now, let us choose $q = \{e_1, e_2\}$ and $P \subset L$ such that $p_\wedge = \{e_2, e_3\}$. Clearly, $q \in Spec2(P)$, since neither q nor $q' = \{e_3, e_4\}$ are comparable to $p_\wedge$. In addition, q and $p_\wedge$ are orthogonal, since they both belong to the same boolean algebra, where, according to theorem 2.1, any pair of elements is orthogonal.*

4 Classifying conjectures with respect to orthogonality

This section uses all the results obtained in previous section in order to classify conjectures with respect to orthogonality, focussing first of all on the case of proper ortholattices and secondly on orthomodular lattices.

4.1 Conjectures in proper ortholattices

When dealing with proper ortholattices, the set $O_l(p_\wedge)$, taking into account all the results given in previous section, may be written, in the general case, as follows:

$$\begin{aligned} O_l(p_\wedge) = \quad & Cons(P) \quad \cup \quad [Hyp(P) \cap \{q \in L; \quad p_\wedge = q + p_\wedge \cdot q'\}] \\ \cup \quad & [Spec1(P) \cap \{q \in L; \quad p_\wedge = p_\wedge \cdot q + q'\}] \\ \cup \quad & [Spec2(P) \cap \{q \in L; \quad p_\wedge = p_\wedge \cdot q + p_\wedge \cdot q'\}] \\ \cup \quad & Conj(P)^c \end{aligned}$$

This provides the following classification of the set of conjectures, $Conj(P)$, with respect to $O_l(p_\wedge)$:

$$Conj(P) = Cons(P) \cup [Hyp(P) \cap O_l(p_\wedge)] \cup [Spec(P) \cap O_l(p_\wedge)] \cup O_l(p_\wedge)^c$$

Similarly, the set $O_r(p_\wedge)$ is given by:

$$\begin{aligned} O_r(p_\wedge) = \quad & [Cons(P) \cap \{q \in L; \quad q = p_\wedge + q \cdot p'_\wedge\}] \quad \cup \quad Hyp(P) \\ \cup \quad & [Spec1(P) \cap \{q \in L; \quad q = q \cdot p_\wedge + p'_\wedge\}] \\ \cup \quad & [Spec2(P) \cap \{q \in L; \quad q = q \cdot p_\wedge + q \cdot p'_\wedge\}] \\ \cup \quad & Conj(P)^c \end{aligned}$$

and the corresponding classification of $Conj(P)$ with respect to $O_r(p_\wedge)$ is:

$$Conj(P) = [Cons(P) \cap O_r(p_\wedge)] \cup Hyp(P) \cup [Spec(P) \cap O_r(p_\wedge)] \cup O_r(p_\wedge)^c$$

The structure of $O_l(p_\wedge)$, $O_r(p_\wedge)$ and $Conj(P)$ in proper ortholattices is depicted in Figure 4. Their intersection (i.e., the elements of the lattice which are both left and right-orthogonal to $p_\wedge$) and their union (i.e., the elements of the lattice that are left or right-orthogonal to $p_\wedge$) are shown in Figure 5.

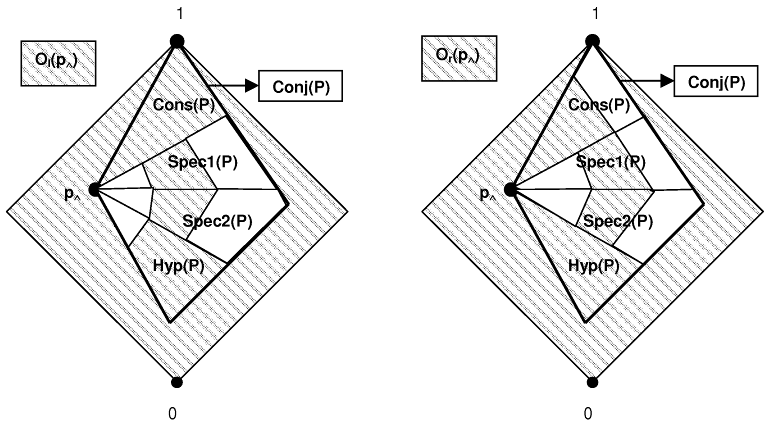

Figure 4: The sets $O_l(p_\wedge)$, $O_r(p_\wedge)$ and $Conj(P)$ in proper ortholattices

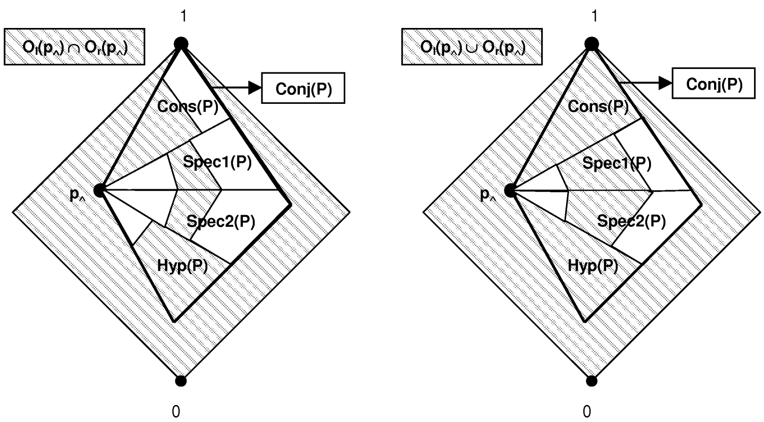

Figure 5: The sets $O_l(p_\wedge) \cap O_r(p_\wedge)$, $O_l(p_\wedge) \cup O_r(p_\wedge)$ and $Conj(P)$ in proper ortholattices

The main results regarding the relationships between orthogonality and conjectures in proper ortholattices may then be summarized as follows:

1. The only conjectures which are always left-orthogonal to $p_\wedge$ are consequences, whereas hypotheses are the only ones which are always right-orthogonal.

2. In general, it is possible to find conjectures of any kind (consequences, hy-

potheses or speculations of the two types) that are both left and right-orthogonal to $p_\wedge$.

3. Conjectures being neither left nor right-orthogonal to $p_\wedge$ have to be looked for among the set of speculations, i.e., if a conjecture is not orthogonal at all to $p_\wedge$, then it is necessarily a speculation.

4.2 Conjectures in orthomodular lattices

In the case of orthomodular lattices, the set $O(p_\wedge) = O_l(p_\wedge) = O_r(p_\wedge)$ is given by:

$$\begin{aligned} O(p_\wedge) = \; & Cons(P) \;\cup\; Hyp(P) \;\cup\; Spec1(P) \\ \cup \; & [Spec2(P) \cap \{q \in L; \quad p_\wedge = p_\wedge \cdot q + p_\wedge \cdot q'\}] \\ \cup \; & Conj(P)^c \end{aligned}$$

and this allows for the following classification of conjectures:

$$Conj(P) = Cons(P) \cup Hyp(P) \cup Spec1(P) \cup [Spec2(P) \cap O(p_\wedge)] \cup O(p_\wedge)^c$$

The later results are illustrated in figure 6, which shows that if a conjecture is not orthogonal to $p_\wedge$, then it is necessarily a type2-speculation. Of course, when dealing with boolean algebras, the above classification of $Conj(P)$ reduces to the already known equality (see section 2.2):

$$Conj(P) = Cons(P) \cup Hyp(P) \cup Spec1(P) \cup Spec2(P)$$

since it is $O(p_\wedge) = L$ and $O(p_\wedge)^c = \emptyset$.

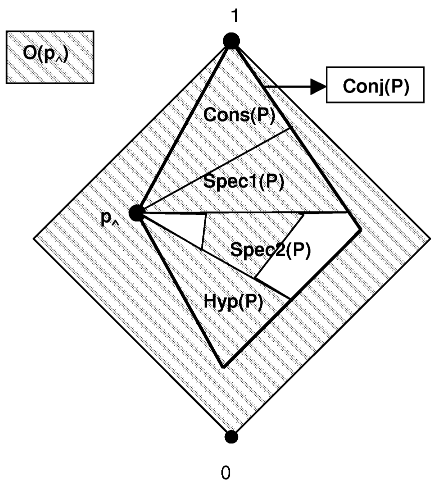

Figure 6: The sets $O(p_\wedge)$ and $Conj(P)$ in non-boolean orthomodular lattices

5 Conclusions

This paper has shown that the mathematical model for conjectures in ortholattices that was presented in [5] can be significantly clarified by studying its relationships with the ortholattices' orthogonality relations. Indeed, a finer classification of conjectures has been obtained, showing that the basic concepts for understanding conjectures, both in proper ortholattices and in orthomodular ones, are those of consequence, hypothesis, speculation (which includes non-comparability in the lattice's order) and orthogonality. In the particular case of proper orthomodular lattices, this new classification reveals that the so-called type2-speculations constitute a distinguished class of speculations, namely the only ones which may not be orthogonal to the infimum of the premises. Finally, it appears that in the limit case of boolean algebras, the classification of conjectures is not affected by orthogonality.

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