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Two Approaches to Fuzzification of Payments in NTU Coalitional Game

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Abstract

There exist several possibilities of fuzzification of a coalitional game. It is quite usual to fuzzify, e. g., the concept of coalition, as it was done in [1]. Another possibility is to fuzzify the expected pay-offs, see [3, 4]. The latter possibility is dealt even here. We suppose that the coalitional and individual pay-offs are expected only vaguely and this uncertainty on the "input" of the game rules is reflected also by an uncertainty of the derived "output" concept like superadditivity, core, convexity, and others. This method of fuzzification is quite clear in the case of games with transferable utility, see [6, 3]. The not transferable utility (NTU) games are mathematically rather more complex structures. The pay-offs of coalitions are not isolated numbers but closed subsets of n-dimensional real space. Then there potentially exist two possible approaches to their fuzzification. Either, it is possible to substitute these sets by fuzzy sets (see, e. g., [3, 4]). This approach is, may be, more sophisticated but it leads to some serious difficulties regarding the domination of vectors from fuzzy sets, the concept of superoptimum, and others. Or, it is possible to fuzzify the whole class of (essentially deterministic) NTU games and to represent the vagueness of particular properties or components of NTU game by the vagueness of the choice of the realized game (see [5]). This approach is, perhaps, less sensitive regarding some subtile variations in the the fuzziness of some properties but it enables to transfer the study of fuzzy NTU coalitional games into the analysis of classes of deterministic games. These deterministic games are already well known, which fact significantly simplifies the demanded analytical procedures.

This brief contribution aims to introduce formal specifications of both approaches and to offer at least elementary comparison of their properties.

In all following sections we denote by I (non-empty and finite) set of players, and each its element $i \in I$ is a player. Any subset $K \subset I$ is called coalition. If $\boldsymbol{x}, \boldsymbol{y} \in R^I$ are real vectors, $\boldsymbol{x} = (x_i)_{i \in I}$, $\boldsymbol{y} = (y_i)_{i \in I}$, and K is a coalition then we say that $\boldsymbol{x}$ dominates $\boldsymbol{y}$ via K, and write $\boldsymbol{x} \operatorname{dom}_K \boldsymbol{y}$, iff

$$\begin{aligned} x_i &\geq y_i \quad \text{for all } i \in K, \\ x_j &> y_j \quad \text{for at least one } j \in K. \end{aligned} \tag{1}$$

1 Deterministic NTU Coalitional Game Model

Let us introduce, at least very briefly, the concept of the game which is to be fuzzified. The pair $(I, \mathcal{V})$, where I is a set of players, is called non-transferable utility coalitional game iff $\mathcal{V}$ is a mapping connecting any coalition $K \subset I$ with a set of real vectors $\mathcal{V}(K) \subset R^I$ such that

$$\mathcal{V}(K) \text{ is closed,} \tag{2}$$

$$\text{if } \boldsymbol{x} \in \mathcal{V}(K),\ \boldsymbol{y} \in R^I,\ \boldsymbol{x}\,\mathrm{dom}_K\,\boldsymbol{y},\ \text{then } \boldsymbol{y} \in \mathcal{V}(K), \tag{3}$$

$$\mathcal{V}(K) \text{ is non-empty,} \tag{4}$$

$$\mathcal{V}(K) = R^I \iff K = \emptyset. \tag{5}$$

It is very easy to verify the following statement (see, e. g., [2]).

Statement 1. If $K \subset I$, $\boldsymbol{x} \notin \mathcal{V}(K)$, $\boldsymbol{y} \in R^I$, $\boldsymbol{y}\,\mathrm{dom}_K\,\boldsymbol{x}$, then $\boldsymbol{y} \notin \mathcal{V}(K)$.

The following concept appears useful for the definition of some significant properties of the NTU games.

If $K \subset I$ then the set $\mathcal{V}^*(K) \subset R^I$ such that

$$\begin{aligned}&\mathcal{V}^*(K) = \{y \in R^I : \text{for any } \boldsymbol{x} \in \mathcal{V}(K) \text{ the domination } \boldsymbol{x}\,\mathrm{dom}_K\,\boldsymbol{y}\\ &\text{is not fulfilled}\} = \{\boldsymbol{y} \in R^I : \forall\,\boldsymbol{x} \in \mathcal{V}(K) \text{ either } \exists\, i \in K,\ y_i > x_i, \text{ or}\\ &\forall\, j \in K,\ y_k = x_j\},\end{aligned} \tag{6}$$

is called superoptimum of the coalition K (see [2]). The following statements are immediate consequences of (6).

Statement 2. For any $K \subset I$, $\mathcal{V}(K) \cup \mathcal{V}^*(K) = R^I$.

Statement 3. For empty coalition, $\mathcal{V}^*(\emptyset) = R^I$.

Statement 4. If for some $K \subset I$, $\boldsymbol{x} \in \mathcal{V}^*(K)$ and $\boldsymbol{y} \in R^I$, $\boldsymbol{y}\,\mathrm{dom}_K\,\boldsymbol{x}$, then $\boldsymbol{y} \in \mathcal{V}^*(K)$.

In the next sections we briefly describe some interesting fuzzifications of the expected payments, and discuss the validity of some analogies of the previous statements for them.

2 Fuzzification of Sets $\mathcal{V}(K)$

If we aim to fuzzify the concept of NTU game then it is most natural to consider some fuzzy subsets of R^I instead of the crisp sets of admissible pay-offs $\mathcal{V}(K)$. Let us consider for every coalition $K \subset I$ a fuzzy subset $\mathcal{W}(K)$ of R^I with membership function $\mu_K : R^I \to [0,1]$ fulfilling the following conditions. For every coalition $K \subset I$:

$$\text{the set } \{\boldsymbol{x} \in R^I : \mu_K(\boldsymbol{x}) = 1\} \text{ is closed,} \tag{7}$$

$$\text{if } \boldsymbol{x} \,\mathrm{dom}_K\, \boldsymbol{y} \text{ then } \mu_K(\boldsymbol{x}) \le \mu_K(\boldsymbol{y}), \tag{8}$$

$$\text{if } K \neq \emptyset \text{ then } \exists\, \boldsymbol{x} \in R^I : \mu_K(\boldsymbol{x}) = 0, \text{ and } \exists\, \boldsymbol{y} \in R^I : \mu_K(\boldsymbol{y}) = 1, \tag{9}$$

$$\text{for empty coalition } \mu_\emptyset(\boldsymbol{x}) = 1 \text{ for all } \boldsymbol{x} \in R^I. \tag{10}$$

Remark 1. If the membership function μ_K, $K \subset R$, achieves only two values 0, 1, then the sets $\mathcal{W}(K)$ fulfil the properties (2), (3), (4), (5) and the pair $(I, \mathcal{W})$ forms a classical NTU coalitional game.

It can be easily seen that if we for any $K \subset I$ denote

$$\mathcal{V}(K) = \{\boldsymbol{x} \in R^I : \mu_K(\boldsymbol{x}) = 1\}$$

then the pair $(I, \mathcal{V})$ forms a deterministic NTU game in the sense of (2) – (5). The fuzzy NTU game $(I, \mathcal{W})$ can be considered for fuzzy extension of $(I, \mathcal{V})$ or, vice versa, $(I, \mathcal{V})$ can be considered for deterministic reduction of $(I, \mathcal{W})$. If we use in the following sections the symbol $\mathcal{V}(K)$ in connection with $\mathcal{W}(K)$, it always means the deterministic reduction in the above sense.

The previous definition of fuzzy NTU game $(I, \mathcal{W})$ is natural. Nevertheless, it is necessary to work with this model and to derive further concepts characterizing the game and its solutions. As it can be easily seen, e. g., in [3], the fuzzy counterpart of the superoptimum set $\mathcal{V}^*$ (we will denote it by $\mathcal{W}^*$) is one of the most important notions used for the development of those advanced concepts of fuzzy NTU. The properties of fuzzy sets $\mathcal{W}^*$ are determined by the domination relation generating it. There are at least two approaches to its construction. In the following subsection we recall both of them and briefly discuss their advantages and disadvantages.

Both approaches follow from the intuitively natural expectation that fuzzy set of possible pay-offs $\mathcal{W}(K)$ is to be connected with also vague (i. e. fuzzy) set of indominable vectors $\mathcal{W}^*(K)$. The method of the construction of the deterministic set $\mathcal{V}^*(K)$, formalized in (6) is based on the relation of domination via a coalition. Even in the fuzzy case it would be so. It means that the two approaches to the definition of fuzzy $\mathcal{W}^*(K)$ will be fully determined by corresponding approaches to the fuzzification of the originally deterministic relation $\cdot\mathrm{dom}_K\cdot$.

2.1 Natural Fuzzy Domination

The probably most natural way of transmission of the deterministic relation dom_K into the environment of fuzzy NTU game $(I, \mathcal{W})$ is based on the assumption that the domination is as strong as much the dominating vector belongs to the set $\mathcal{W}(K)$ (note, please, that the terminal purpose of the domination via K is to define the fuzzy superoptimum set $\mathcal{W}^*(K)$).

Hence, for every coalition $K \subset I$ we define fuzzy ordering relation dom_K^F on the set R^I with membership function $\nu_K(\cdot,\cdot)$ where the value $\nu_K(\boldsymbol{x}, \boldsymbol{y})$ defines the possibility with which $\boldsymbol{x}$ fuzzy dominates $\boldsymbol{y}$ via coalition K. The values $\nu_K(\boldsymbol{x}, \boldsymbol{y})$ are defined by

$$\begin{aligned} \nu_K(\boldsymbol{x}, \boldsymbol{y}) &= \mu_K(\boldsymbol{x}) && \text{if } \boldsymbol{x} \,\mathrm{dom}_K\, \boldsymbol{y}, \\ &= 0 && \text{in the opposite case.} \end{aligned} \tag{11}$$

Remark 2. Condition (8) immediately implies that for $K \subset I$, $\boldsymbol{z} \,\mathrm{dom}_K\, \boldsymbol{y}$, always $\nu_K(\boldsymbol{z}, \boldsymbol{x}) \leq \nu_K(\boldsymbol{y}, \boldsymbol{x})$.

It is worth mentioning that for $\mathcal{W}(K)$ such that $\mu_K(\boldsymbol{x}) \in \{0, 1\}$ the relation dom_K^F does not turn into dom_K, as dom_K is much more universal, related to the whole R^I, meanwhile dom_K^F is specialized for the construction of $\mathcal{W}^*(K)$ as shown below.

In the fuzzy NTU game the set $\mathcal{W}^*(K)$, $K \subset I$, is to represent the set of all vectors in R^I which are not (fuzzy) dominated by any vector from the fuzzy set $\mathcal{W}(K)$. It means that $\mathcal{W}^*(K)$, for any $K \subset I$, is a fuzzy subset of R^I. We denote its membership function μ_K^* and define by

$$\mu_K^*(\boldsymbol{x}) = 1 - \sup\left(\nu_K(\boldsymbol{y}, \boldsymbol{x}) : \boldsymbol{y} \in R^I\right). \tag{12}$$

Remark 3. If for $K \subset I$ μ_K achieves only values 0 and 1 then also $\mu_K^*(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \notin \mathcal{W}^*(K)$ and $\mu_K^*(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in \mathcal{V}^*(K)$.

Remark 4. Definition (12) immediately implies that the membership function μ_K^* is monotonous in the sense that

$$\mu_K^*(\boldsymbol{x}) \geq \mu_K^*(\boldsymbol{y}) \quad \text{if} \quad \boldsymbol{x} \,\mathrm{dom}_K\, \boldsymbol{y}.$$

Lemma 1. If there exists $\boldsymbol{x} \in R^I$ such that $\mu_K(\boldsymbol{x}) \in (0, 1)$ then for all $\boldsymbol{y} \in R^I$,

$$\mu_K^*(\boldsymbol{y}) = 1 - \mu_K(\boldsymbol{y}).$$

Proof. The statement immediately follows from (8), (11) and (12).

Corollary. Under the assumption of Lemma 1

$$\min\left(\mu_K(\boldsymbol{x}), \mu_K^*(\boldsymbol{x})\right) \leq 1/2$$

for all $\boldsymbol{x} \in R^I$.

The interpretation of the previous result is rather unpleasant. It means that many important objects of the game (see, e. g., the core [3]) can contain some elements with a limited (and not very significant) possibility, only. This contradicts with our intuitive idea about their importance, and devaluates the concept of natural fuzzy domination dom_K^F.

2.2 Proper Fuzzy Domination

The definition of the fuzzy domination used in the previous subsection can be modified in the following sense.

We introduce fuzzy relation over R^I $\boldsymbol{x}\,\mathrm{Dom}_K^F\,\boldsymbol{y}$, and read $\boldsymbol{x}$ properly fuzzy dominates $\boldsymbol{y}$ via K, with membership function $\overline{\nu}_K(\boldsymbol{x},\boldsymbol{y}) : R^I \times R^I \to [0,1]$, where

$$\begin{aligned} \overline{\nu}(\boldsymbol{x},\boldsymbol{y}) &= \mu_K(\boldsymbol{x}) && \text{if } \boldsymbol{x}\,\mathrm{dom}_K\,\boldsymbol{y},\ \mu_K(\boldsymbol{x}) \geq \mu_K(\boldsymbol{y}), \\ &= 0 && \text{else.} \end{aligned} \tag{13}$$

Obviously, even the definition of proper fuzzy domination is, analogously to the case presented in Subsection 2.1, exclusively aimed to the construction of some kind of superoptimum set for $K \subset I$.

Remark 5. Due to (8) the previous definition is equivalent to

$$\begin{aligned} \overline{\nu}(\boldsymbol{x},\boldsymbol{y}) &= \mu_K(\boldsymbol{x}) && \text{if } \boldsymbol{x}\,\mathrm{dom}_K\,\boldsymbol{y},\ \mu_K(\boldsymbol{x}) = \mu_K(\boldsymbol{y}), \\ &= 0 && \text{else.} \end{aligned}$$

Then, evidently, fuzzy relation Dom_K^F can be used, analogously to (12), to the definition of a modified form of superoptimum for $K \subset I$. It is a fuzzy subset of R^I which we call proper fuzzy superoptimum (c. f. [3]), denote by $\overline{\mathcal{W}}^*(K)$, whose membership function is denoted by $\overline{\mu}_K^*$ and for any $\boldsymbol{x} \in R^I$

$$\overline{\mu}_K^*(\boldsymbol{x}) = 1 - \sup\left(\overline{\nu}_K(\boldsymbol{y},\boldsymbol{x}) : \boldsymbol{y} \in R^I\right). \tag{14}$$

Lemma 2. If $(I,\mathcal{V})$ is deterministic reduction of $(I,\mathcal{W})$ then for any $\boldsymbol{x} \in \mathcal{V}(K) \cap \mathcal{V}^*(K)$ is $\overline{\mu}_K^*(\boldsymbol{x}) = 1$.

Proof. The statement immediately follows from the concept of deterministic reduction, and from (13) and (14).

Corollary. The previous lemma means that the proper fuzzy superoptimum $\overline{\mathcal{W}}^*$ fulfils

$$\sup\left[\min\left(\mu_K(\boldsymbol{x}), \overline{\mu}_K^*(\boldsymbol{x})\right) : \boldsymbol{x} \in R^I\right] = 1$$

for the vectors $\boldsymbol{x} \in \mathcal{V}(K) \cap \mathcal{V}^*(K)$.

Lemma 3. The values of $\overline{\nu}_K(\boldsymbol{x},\boldsymbol{y})$ are different from 0 iff $\mu_K(\boldsymbol{x}) = \mu_K(\boldsymbol{y}) > 0$. Then also $\overline{\nu}_K(\boldsymbol{x},\boldsymbol{z}) = \overline{\nu}(\boldsymbol{x},\boldsymbol{y})$ for all $\boldsymbol{z} \in R^I$ such that

$$\boldsymbol{x}\,\mathrm{dom}_K\,\boldsymbol{z}, \quad \boldsymbol{z}\,\mathrm{dom}_K\,\boldsymbol{y}.$$

Proof. Also this statement immediately follows from (8), (13) and (14).

Lemma 4. The membership function $\overline{\mu}^*_K$ achieves the following values:

$$\begin{array}{rcll} \overline{\mu}^*_K(\boldsymbol{x}) & = & 1 & \text{for } \boldsymbol{x} \in \mathcal{V}(K) \cap \mathcal{V}^*(K), \\ & > & 0 & \text{if there exists } \boldsymbol{y} \in R \text{ with } 0 < \mu_K(\boldsymbol{y}) < 1 \text{ such that} \\ & & & \quad \mu_K(\boldsymbol{y}) = \mu_K(\boldsymbol{x}) \quad \text{and} \quad \boldsymbol{y} \notin \mathcal{V}(K), \\ & = & 0 & \text{for all } \boldsymbol{x} \in \mathcal{V}(K). \\ & = & 1 & \text{in other cases.} \end{array}$$

Proof. The statement summarizes the previous results with respect to (13) and (14).

The concept of proper fuzzy superoptimum $\overline{\mathcal{W}}^*(K)$ avoids the main discrepancy of the previous fuzzy superoptimum $\mathcal{W}^*(K)$ – its values are not limited by 1/2. On the other hand, its membership function $\overline{\mu}^*_K$ need not be monotonous as we can see in Lemma 4. This lack of monotonicity can be interpreted with serious problems and in some cases it can lead to rather strange results.

3 Fuzzy Classes of NTU Coalitional Games

The method of fuzzification of NTU games, used in the previous section can be characterized as fuzzification "from below" – the fuzzy sets $\mathcal{W}(K)$ compose the game. It is also possible to use another method – let us call it fuzzification "from above". This method (c. f. [5]) is based on the idea that there exists a fuzzy subclass of the class of all NTU games containing (with lower or higher degree of membership) the set of games whose realization is vaguely expectable. The fuzzy NTU game under the consideration can be constructed from this fuzzy class of games. This approach will be briefly described in the following paragraphs.

Let us denote by $\boldsymbol{\mathcal{V}}$ the class of all NTU games with the set of players I. By $\boldsymbol{\mathcal{W}}$ we denote a fuzzy subclass of $\boldsymbol{\mathcal{V}}$ with membership function $\pi : \boldsymbol{\mathcal{V}} \to [0, 1]$.

Then it is easy to define a fuzzy NTU game $(I, \mathcal{W})$ with fuzzy sets $\mathcal{W}(K)$, where the membership functions $\mu_K : R^I \to [0, 1]$ are given for every $\boldsymbol{x} \in R^I$

$$\mu_K(\boldsymbol{x}) = \sup(\pi(\mathcal{V}) : \boldsymbol{x} \in \mathcal{V}(K)), \quad K \subset I. \tag{15}$$

Remark 6. It can be easily verified that the game $(I, \mathcal{W})$ fulfills the properties of fuzzy NTU coalitional game (7) – (10).

The class $\boldsymbol{\mathcal{W}}$ is a fuzzy set whose elements are deterministic games for which the superoptima $\mathcal{V}^*(K)$ are defined in the usual way (6). It means that, analogously to the previous procedure, we may define fuzzy sets $\mathcal{W}^*(K)$ with membership functions μ^*_K by

$$\mu^*_K(\boldsymbol{x}) = \sup(\pi(\mathcal{V}) : \boldsymbol{x} \in \mathcal{V}^*(K)), \quad \boldsymbol{x} \in R^I, \ K \subset I. \tag{16}$$

In the previous paragraphs, we have constructed an individual fuzzy NTU game $(I, \mathcal{W})$ with fuzzy sets $\mathcal{W}(K)$ and $\mathcal{W}^*(K)$. As the method of construction of the sets $\mathcal{W}^*(K)$ significantly differs from the one used in Section 2, it can be obviously expected that also its properties, namely the relations between $\mathcal{W}(K)$ and $\mathcal{W}^*(K)$, are different from those presented in Section 2.

Lemma 5. Let $\mathcal{V}_1$, V_2 be two games from the fuzzy se $\boldsymbol{\mathcal{W}}$ with positive values of membership functions $\pi(\mathcal{V}_1)$, $\pi(\mathcal{V}_2)$. Let $K \subset I$ be a coalition such that $\mathcal{V}_1(K) \neq \mathcal{V}_2(K)$. Then there exists $\boldsymbol{x} \in R^I$ such that

$$\min\left(\mu_K(\boldsymbol{x}), \mu_K^*(\boldsymbol{x})\right) > 0.$$

Proof. Without lack of generality, we may suppose that there exists $\boldsymbol{x} \in R^I$ such that $\boldsymbol{x} \in \mathcal{V}_1(K)$ and $\boldsymbol{x} \notin \mathcal{V}_2(K)$, i. e., $\boldsymbol{x} \in \mathcal{V}_2^*(K)$. Hence,

$$\min\left(\mu_K^*(\boldsymbol{x}), \mu_K(\boldsymbol{x})\right) \geq \max\left(\pi(\mathcal{V}_1), \pi(\mathcal{V}_2)\right) > 0.$$

Remark 7. The membership function μ_K^*, $K \subset I$, is obviously monotonous in the sense that

$$\mu_K^*(\boldsymbol{x}) \geq \mu_K^*(\boldsymbol{y}) \quad \text{for } \boldsymbol{x}, \boldsymbol{y} \text{ such that } \boldsymbol{x} \operatorname{dom}_K \boldsymbol{y}.$$

The above Lemma 5 can be extended in the following way.

Lemma 6. Let $\mathcal{V}_1$, $\mathcal{V}_2$ be two games from the fuzzy set $\boldsymbol{\mathcal{W}}$ with positive values of membership $\pi(\mathcal{V}_1)$, $\pi(\mathcal{V}_2)$. Let $K \subset I$ and let $\mathcal{V}_1(K)$ differs from $\mathcal{V}_2(K)$. Then

$$\sup\left(\min(\mu_K^*(\boldsymbol{x}), \mu_K(\boldsymbol{x})) : \boldsymbol{x} \in R^I\right) \geq \max\left(\pi(\mathcal{V}_1), \pi(\mathcal{V}_2)\right).$$

Proof. The statement follows from Lemma 5.

Another lower limit for the membership function of the intersection $\mathcal{W}(K) \cap \mathcal{W}^*(K)$ is presented in the next result.

Lemma 7. Let $\mathcal{V}$ be a member of the fuzzy set $\boldsymbol{\mathcal{W}}$ with positive value of the membership function $\pi(\mathcal{V})$, let $K \subset I$ and let $\mathcal{V}(K) \cap \mathcal{V}^*(K) \neq \emptyset$. Then

$$\sup\left(\min(\mu_K^*(\boldsymbol{x}), \mu_K(\boldsymbol{x})) : \boldsymbol{x} \in R^I\right) \geq \pi(\mathcal{V}).$$

Proof. The inequality immediately follows from (15) and (16).

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