General Theory of the Fuzzy Integral

Pietro Benvenuti & Doretta Vivona*
Dip. Metodi e Mod. Matematici per Sci. Applicate
Università degli Studi "La Sapienza"
Via A. Scarpa, 16, 00161 Roma. Italy.

Abstract

By means of two general operations $\oplus$ and $\otimes$, called "pan-operations", we build a new kind of integral. This formulation contains, as particular cases, both Choquet's and Sugeno's integrals.
Keywords: Fuzzy functional analysis, fuzzy integral.
A.M.S.: 28E10.

1 Introduction

In the literature two kinds of fuzzy integral have been studied: Choquet's and Sugeno's integrals. They are defined, for measurable functions, with respect to any fuzzy measure, *i.e.* a monotone increasing set function μ, with $\mu(\emptyset) = 0$.

Both integrals have many applications in the generation of fuzziness, belief, plausibility, probability, possibility and necessity measures ([9], [2], [14]). Sugeno's integral, in particular, yields important applications in the theory of fuzzy control.

Choquet's and Sugeno's integrals enjoy characteristic properties which are similar; they are really different only for the use of the classical operations $+$ and $\cdot$ for Choquet's integral, $\vee$ and $\wedge$ for Sugeno's one.

An attempt to define a general kind of integral through two general operations $\oplus$ and $\otimes$, called *pan-operations*, is found in Wang-Klir's book [15], starting from Yang Qingji (1983) and Yang and Song (1985).

This attempt doesn't achieve the complete result, because it doesn't obtain Choquet's integral when specifically $\oplus = +$ and $\otimes = \cdot$, but only Lebesgue's integral if in particular the fuzzy measure is additive.

In this paper we build a new kind of integral by means of general operations and we generalize both Choquet's and Sugeno's integrals, which, in our construction, result two specific cases.

*Research partially supported by GNFM, CNR.

2 Preliminaries

Let Ω be an abstract space, $\mathcal{A}$ a σ-algebra of subsets of Ω and $\mu : \mathcal{A} \to [0,1]$ a fuzzy measure, *i.e.* a non-decreasing set function, continuous from below, with $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$. The triple $(\Omega, \mathcal{A}, \mu)$ will be called a *fuzzy measure space* [15].

We recall that a fuzzy measure μ is *additive* if

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B) \quad \forall\, A, B \in \mathcal{A}\,;$$

or in equivalent way if, for $A \cap B = \emptyset$, we have

$$\mu(A \cup B) = \mu(A) + \mu(B)\,.$$

Moreover, we say that μ is $\vee$-*additive* [15] if

$$\mu(A \cup B) \vee \mu(A \cap B) = \mu(A) \vee \mu(B) \quad \forall\, A, B \in \mathcal{A}\,,$$

which is equivalent to

$$\mu(A \cup B) = \mu(A) \vee \mu(B)\,.$$

Fixed a function $f : \Omega \to R^+$, set $\mathcal{M}_f = \{\{\omega \in \Omega / f(\omega) > x\}/x \in R\}$; we call this family the *chain associated* to f, because it is completely ordered with respect to set inclusion. We shall denote with $\mathcal{F}(\Omega)$ the set of all $\mathcal{A}$-measurable functions $f : \Omega \to R^+$: that means that the chain associated to f is a subset of $\mathcal{A}$. This definition of measurability coincides with the classic one, because $\mathcal{A}$ is a σ-algebra.

In order to obtain the additivity of the integral it is useful the concept of *comonotonic* (common monotonic) *functions*. This definition was introduced by Hardy-Littlewood-Pòlya in 1934 and then it was used by many authors: Dellacherie [5], Schmidler [12]. We refer to Dennenberg [6] for the definition and the properties.

Definition 1.1. *A class $\mathcal{F}' \subset \mathcal{F}(\Omega)$ is called comonotonic if*

$$\bigcup_{f \in \mathcal{F}'} \mathcal{M}_f \qquad \textit{is a chain.}$$

The most important properties are contained in the following proposition.

Proposition 1.2. *Given two functions f_1 and f_2 in $\mathcal{F}$ the following conditions are equivalent:*

(i) *f_1 and f_2 are comonotonic;*

(ii) *there is no pair $\omega_1, \omega_2 \in \Omega$ such that $f_1(\omega_1) < f_1(\omega_2)$ and $f_2(\omega_1) > f_2(\omega_2)$;*

(iii) *there exist a function $h : \Omega \to R^+$ and two monotone non-decreasing functions φ_1 and φ_2 such that*

$$f_1 = \varphi_1 \circ h \quad \textit{and} \quad f_2 = \varphi_2 \circ h\,;$$

(iv) *there exist continuous and monotone non-decreasing functions* φ_1 *and* φ_2 *on* R *such that* $\varphi_1(z) + \varphi_2(z) = z$, $z \in R$ *and*

$$f_1 = \varphi_1(f_1 + f_2), \qquad f_2 = \varphi_2(f_1 + f_2).$$

3 Choquet's integral

Many authors have studied Choquet's integral: Choquet [3], De Giorgi-Letta [4], Greco [7], Dennenberg [6], Murofushi-Sugeno [10]. Now we recall the definition.

Given a fuzzy measure space $(\Omega, \mathcal{A}, \mu)$ *Choquet's integral* is the functional

$$L_c : \mathcal{F}(\Omega) \to \overline{R}^+$$

defined in this way [3]:

$$L_c(f) = \oint f d\mu =: \int_0^{+\infty} \mu(\{\omega \in \Omega / f(\omega) > x\}) dx. \tag{1}$$

The most significant properties of (1) are the following:

(i) $L_c(1_A) = \mu(A)$
where 1_A is the indicator function of the set $A \in \mathcal{A}$;

(ii) $L_c(a \cdot f) = a \cdot L_c(f)$ *homogeneity*;

(iii) $f_1 \leq f_2 \Longrightarrow L_c(f_1) \leq L_c(f_2)$ *monotonicity*;

(iv) $f_n \nearrow f \Longrightarrow L_c(f_n) \nearrow L_c(f)$ *continuity from below*;

(v) L_c is *additive* on any class $\mathcal{F}'$ of comonotonic functions:
given f_1 and f_2 in $\mathcal{F}'$, it holds
$L_c(f_1 + f_2) = L_c(f_1) + L_c(f_2)$ *comonotonic additivity*.

As, for every f in $\mathcal{F}(\Omega)$ and a in R^+, $f \wedge a$ and $f' = f - f \wedge a$ are comonotonic functions, a consequence of (v) is

(v') $L_c(f) = L_c(f \wedge a) + L_c(f')$ *horizontal additivity.*

Remark. The conditions (i), (ii) and (v′) imply the following property [4]:

(vi) $L_c\Big(\sum_{i=1}^{n} c_i \cdot 1_{C_i}\Big) = \sum_{i=1}^{n} c_i \cdot \mu(C_i)$,

where $(C_i)_{1 \leq i \leq n}$ is any finite decreasing sequence of sets in $\mathcal{A}$ and (c_i) are n real positive numbers.

This remark allows us to recognize that the properties above (i)-(v) are characteristic of Choquet's integral among all functionals defined in $\mathcal{F}(\Omega)$ [8].

4 Sugeno's integral

Let a fuzzy measure space $(\Omega, \mathcal{A}, \mu)$ be given. Let $\mathcal{F}_1(\Omega)$ be the set of all $\mathcal{A}$-measurable functions from Ω to $[0,1]$.

According with Sugeno [13], we consider the so-called *Sugeno's integral*, as the functional

$$L_s : \mathcal{F}_1(\Omega) \to [0,1],$$

defined in the following way

$$L_s(f) = \oint f \, d\mu =: \bigvee_{x\in[0,1]} [x \wedge \mu(\{\omega \in \Omega / f(\omega) > x\})] \,. \tag{2}$$

Sugeno's integral enjoys similar properties as Choquet's integral:

(i) $L_s(1_A) = \mu(A) \quad \forall\, A \in \mathcal{A}\,;$

(ii) $L_s(a \wedge f) = a \wedge L_s(f)$ *homogeneity*;

(iii) $f_1 \leq f_2 \Longrightarrow L_s(f_1) \leq L_s(f_2)$ *monotonicity*;

(iv) $f_n \nearrow f \Longrightarrow L_s(f_n) \nearrow L_s(f)$ *continuity from below*;

(v) L_s is $\vee$-additive on any class $\mathcal{F}'$ of comonotonic functions, given f_1 and f_2 in $\mathcal{F}'$, it holds $L_s(f_1 \vee f_2) = L_s(f_1) \vee L_s(f_2)$ *comonotonic $\vee$-additivity*.

Let f' be a function defined by: $f'(\omega) = 0$ if $f(\omega) \leq a$ and $f'(\omega) = f(\omega)$ if $f(\omega) > a$. It is $f = (f \wedge a) \vee f'$ and the two functions $f \wedge a$ and f' are comonotonic. Then, a consequence of (v) is:

(v') $L_s(f) = L_s(f \wedge a) \vee L_s(f')$ *horizontal additivity*.

Remark. The conditions (i), (ii) and (v$'$) imply:

(vi) $$L_s\left[\bigvee_{i=1}^{\infty} (c_i \wedge 1C_i)\right] = \bigvee_{i=1}^{\infty} [c_i \wedge \mu(C_i)]\,,$$

where $(C_i)_i$ is any finite decreasing sequence of elements in $\mathcal{A}$ and (c_i) are n real numbers in $[0,1]$.

From equation (vi) we recognize that properties (i)-(v) characterize Sugeno's integral among all functionals defined in $\mathcal{F}_1(\Omega)$ [1].

All properties, except (v), are known [13]; now we prove (v).

Proof. As f_1 and f_2 are comonotonic, for every $x \in [0,1]$ the sets $F_{1x} = \{f_1(\omega) > x\}$ and $F_{2x} = \{f_2(\omega) > x\}$ belong to the same chain and then it is $F_{1x} \supset F_{2x}$ or $F_{2x} \supset F_{1x}$. In every case

$$\mu\big(\{f_1(\omega) \vee f_2(\omega) > x\}\big) = \mu(F_{1x} \cup F_{2x}) = \mu(F_{1x}) \vee \mu(F_{2x})\,.$$

So we have the result:

$$\begin{aligned}
\oint (f_1 \vee f_2) d\mu &= \bigvee_x \left[x \wedge \big(\mu(F_{1x}) \cup \mu(F_{2x})\big)\right] = \\
&= \bigvee_x \left[\big(x \wedge \mu(F_{1x})\big) \vee \big(x \wedge \mu(F_{2x})\big)\right] = \\
&= \Big[\bigvee_x \big(x \wedge \mu(F_{1x})\big)\Big] \vee \Big[\bigvee_x \big(x \wedge \mu(F_{2x})\big)\Big] = \\
&= \oint f_1 \, d\mu \vee \oint f_2 \, d\mu\,.
\end{aligned}$$

5 Pan-operations and pan-integral

Fixed a real number $M > 0$, let I be the interval $[0, M]$. Let two operations $\oplus$ and $\otimes$ be given in I, called *pan-addition* and *pan-multiplication*, respectively.

We assume that I is a *semigroup* with respect to $\oplus$, *i.e.* $\oplus$ is an operation $\oplus : I^2 \to I$, which enjoys the following properties:

1) $a \oplus b = b \oplus a \quad \forall\, a, b\,;$

2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c \quad \forall\, a, b, c\,;$

3) $a' < a'' \Longrightarrow a' \oplus c \leq a'' \oplus c \quad \forall\, c\,;$

4) $a \oplus 0 = a \quad \forall\, a\,;$

5) $\oplus$ is continuous.

Moreover, we suppose that in I there is another operation $\otimes : I^2 \to I$, which satisfies the following properties:

6) $a' \leq a'', b' \leq b'' \Longrightarrow a' \otimes b' \leq a'' \otimes b''\,;$

7) $a \otimes (b \otimes c) = (a \otimes b) \otimes c\,;$

8) $a \otimes 0 = 0 \otimes a = 0 \quad \forall\, a\,;$

9) $\exists u \in I : a \otimes u = a \quad \forall a\,;$

10) $\otimes$ is continuous;

11) $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$;

12) $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$.

So, the triple $(I, \oplus, \otimes)$ is a *distributive semiring* with respect to $\oplus$ and $\otimes$.

Through these operations, we would like to define an integral, called *pan-integral*, in such a way that if $\oplus$ and $\otimes$ are the common addition and the common multiplication the pan-integral coincides with Choquet's one and if $\oplus = \vee$ and $\otimes = \wedge$ the pan-integral equals Sugeno's one.

Let us assume, furthermore, that

13) For every a and b in I, the equation

$$a \oplus x = b\,,$$

which admits solution for $b \geq a$, has a *unique* solution for $b > a$.

Now, let $(\Omega, \mathcal{A}, \mu)$ be a fuzzy measure space with $\mu : \mathcal{A} \to I$ and $\mathcal{F}_I(\Omega)$ be the family of all $\mathcal{A}$-measurable fuctions $f : \Omega \to I$. We are defining the pan-integral, which is characterized by properties that are similar to the ones characterizing Choquet's and Sugeno's integrals. Pan-integral is a functional

$$L_p : \mathcal{F}_I(\Omega) \to R^+\,,$$

such that:

(i) $L_p(U_A) = \mu(A) \quad \forall\, A \in \mathcal{A}$,
where $U_A(\omega) = u$ if $\omega \in A$ and $U_A(\omega) = 0$ if $\omega \notin A$;

(ii) $L_p(a \otimes f) = a \otimes L_p(f)$ *homogeneity*;

(iii) $f_1 \leq f_2 \Longrightarrow L_p(f_1) \leq L_p(f_2)$ *monotonicity*;

(iv) $f_n \nearrow f \Longrightarrow L_p(f_n) \nearrow L_p(f)$ *continuity from below*;

(v) L_p is $\oplus$ *−additive* on any class $\mathcal{F}'$ of comonotonic functions:
given f_1 and f_2 in $\mathcal{F}'$, it holds:
$L_p(f_1 \oplus f_2) = L_p(f_1) \oplus L_p(f_2)$ *comonotonic $\oplus$-additivity.*

Given a function $f \in \mathcal{F}_I(\Omega)$, setting $f = (f \wedge a) \oplus f'$, with $f'(\omega) = 0$ for ω such that $f(\omega) \leq a$, we have that f' is uniquely determined, $f \wedge a$ and f' are comonotonic, so a consequence of (v) is:

(v') $L_p(f) = L_p(f \wedge a) \oplus L_p(f')$ *horizontal additivity*.

From (i) and (ii), setting $f(\omega) = a \otimes U_A(\omega) \;\; \forall \;\; \omega \in \Omega$, we obtain $L_p(a \otimes U_A) = a \otimes \mu(A)$.

Fixed c_1, c_2 in I and C_1, C_2 in $\mathcal{A}$ with $C_1 \supset C_2$, we consider

$$f = c_1 \otimes U_{C_1} \oplus c_2 \otimes U_{C_2} \quad ^{(*)}. \tag{3}$$

Applying the (v′) with $a = c_1$, we have $f \wedge a = c_1 \otimes U_{C_1}$, $f' = c_2 \otimes U_{C_2}$ and then it holds:

$$L_p(f) = c_1 \otimes \mu(C_1) \oplus c_2 \otimes \mu(C_2). \tag{4}$$

We can note that, if $c_1 \oplus c_2 = c_1$, we get also $f = c_1 \otimes U_{C_1}$ and

$$\begin{aligned} c_1 \otimes \mu(C_1) &\leq c_1 \otimes \mu(C_1) \oplus c_2 \otimes \mu(C_2) \leq c_1 \otimes \mu(C_1) \oplus c_2 \otimes \mu(C_1) = \\ &= (c_1 \oplus c_2) \otimes \mu(C_1) = c_1 \otimes \mu(C_1). \end{aligned}$$

We thus recognize that, with the two possible expressions of the function f, we obtain the same value for $L_p(f)$. Trivially, if in (3) $C_2 = \emptyset$, again we find two possible expressions for f, to which the same value of the functional is assigned.

Given a simple function s, *i.e.* a function with a finite numbers of values, it admits a *standard representation* through a finite partition $A_1, A_2, \ldots, A_n$ of Ω and a corresponding finite number of values, $a_1, a_2, \ldots, a_n$ in the following way:

$$s(\omega) = a_i \quad \text{if} \quad \omega \in A_i.$$

Without loss of generality, let us assume that $a_1 < a_2 < \ldots < a_n$. Then the function s can be also expressed, and uniquely, as

$$s(\omega) = \bigoplus_{i=1}^{n} c_i \otimes U_{C_i}(\omega) \qquad (\textit{step representation}),$$

where $C_1 = \Omega$, $C_i = \bigcup_{j=i}^{n} A_j$, $C_n = A_n$, and $c_1 = a_1$, while the next c_i are given by the equations $a_i = a_{i-1} \oplus c_i$.

Chosen in (v') $a = a_{n-1}$, it holds $s \wedge a = \bigoplus_{i=1}^{n-1} c_i \otimes U_{C_i}$ and $s' = c_n \otimes U_{C_n}$; then, as a consequence of the horizontal additivity we obtain

$$L_p(s) = L_p\left[\bigoplus_{i=1}^{n-1} c_i \otimes U_{C_i}\right] \oplus c_n \otimes \mu(c_n).$$

By induction, we get the following expression:

0(*) In the expression of f and later on, one agrees to do first the multiplications and then the additions, in accordance with usual conventions and, then, it is possible to omit the brackets.

(vi) $L_p(s) = \bigoplus_{i=1}^{n} [c_i \otimes \mu(C_i)]\,.$

Proposition 1.5. *The functional defined by* (vi) *is monotone on the family of the simple functions.*

Proof. First, we prove that we can also apply the definition (vi) when it is not necessarily $a_i > a_{i+1}$ for every i, but in general, $a_i \geq a_{i+1}$. In fact, setting $a_i = a_{i+1}$, let c_{i+1} be such that $a_{i+1} = a_i \oplus c_{i+1}$. It holds, also, $a_{i-1} \oplus c_i \oplus c_{i+1} = a_{i+1}$ and moreover $c_i \oplus c_{i+1} = c_i$.

Then it follows:

$$c_i \otimes \mu(C_i) \leq \big[c_i \otimes \mu(C_i)\big] \oplus \big[c_{i+1} \otimes \mu(C_{i+1})\big] \leq (c_i \oplus c_{i+1}) \otimes \mu(C_i) = c_i \otimes \mu(C_i)\,.$$

In the expression of $L_p(s)$ the addend $[c_{i+1} \otimes \mu(C_{i+1})]$ can be omitted and the functional has the same form of that obtained with $A_i \cup A_{i+1}$.

When $s' \geq s$, the comparison between $L_p(s')$ and $L_p(s)$ can be obtained, by supposing that the sets A_i are the same and $a'_i \geq a_i$ for every i. Therefore let $a'_i = a_i$ if $i \neq h$ and $a'_h > a_h$. The difference between $L_p(s')$ and $L_p(s)$ is in the addends $[c'_h \otimes \mu(C_h)]$ and $[c'_{h+1} \otimes \mu(C_{h+1})]$; they are different from the corresponding terms $[c_h \otimes \mu(C_h)]$ and $[c_{h+1} \otimes \mu(C_{h+1})]$ because $c'_h > c_h$ and $c'_{h+1} < c_{h+1}$. Setting $c'_h = c_h \oplus d$, it holds $c_{h+1} = c'_{h+1} \oplus d$. Finally we get:

$$\begin{aligned}&[c'_h \otimes \mu(C_h)] \oplus [c'_{h+1} \otimes \mu(C_{h+1})] = \\ &\quad = [c_h \otimes \mu(C_h)] \oplus [d \otimes \mu(C_h)] \oplus [c'_{h+1} \otimes \mu(C_{h+1})] \geq \\ &\quad \geq [c_h \otimes \mu(C_h)] \oplus [c_{h+1} \otimes \mu(C_{h+1})]\,.\end{aligned}$$

Therefore, we recognize that it is, really,

$$L_p(s') \geq L_p(s)\,.$$

So, it is possible to extend the functional L_p in the family $\mathcal{F}_I(\Omega)$ setting, in according with the continuity property (iv),

$$L_p(f) = \oint f \, d\mu =: \sup\{L_p(s)/\, s \quad \text{simple and} \quad s \leq f\}\,.$$

The functional defined above satisfies properties (i)-(v)$^{(**)}$, and therefore it is characterized by them.

This functional coincides with Choquet's and Sugeno's one, when the structure of semiring $(I, \oplus, \otimes)$ equals to that seen in the paragraphs 3 and 4, respectively.

Choquet's integral satisfies the comonotonic additivity, but in general it is not additive. It is additive only if the fuzzy measure is additive. Sugeno's integral is $\vee$-additive only if the fuzzy measure is $\vee$-additive.

$^{0(**)}$ The proof is standard and so it is omitted.

In the same way, the pan-integral enjoys the property of comonotonic $\oplus$-additivity. It is also $\oplus$-additive if the fuzzy measure μ is $\oplus$-additive, that means that $\forall\ A, B \in \mathcal{A}$ with $A \cap B = \emptyset$, it is $\mu(A \cup B) = \mu(A) \oplus \mu(B)$, or equivalently $\forall\ A, B \in \mathcal{A}$ it is $\mu(A \cup B) \oplus \mu(A \cap B) = \mu(A) \oplus \mu(B)$.

6 Examples

In this section we present some examples of pan-integral.

Let $(\Omega, \mathcal{A}, \mu)$ be a fuzzy measure space.

1) Let $I = [0, +\infty]$ and $\oplus = \vee, \otimes = \wedge$; it is $u = +\infty$. The pan-integral is defined by

$$\oint f\, d\mu = \bigvee_{x \in R^+} [x \wedge \mu(\{\omega \in \Omega / f(\omega) > x\})] := \oint f\, d\mu\,.$$

This integral is also known as *Sugeno's integral* because it is a generalized kind of the original Sugeno's definition.

2) Let $I = [0, +\infty]$ and $\oplus = \vee$, $\otimes = \cdot$, with $u = 1$. Then we obtain

$$\oint f\, d\mu = \bigvee_{x \in R^+} [x \cdot \mu(\{\omega \in \Omega / f(\omega) > x\})]\,.$$

3) Fixed a real number $M \leq +\infty$, let $I = [0, M]$ with $\oplus = \vee$, $\otimes = \mathcal{T}$ a norm in the sense of Schweizer and Sklar [11]. We have then

$$\oint f\, d\mu = \bigvee_{x \in R^+} [x\, \mathcal{T}\, \mu(\{\omega \in \Omega / f(\omega) > x\})]\,.$$

4) Fixed a real number $M \leq +\infty$, let $I = [0, M]$ and $h : [0, M] \to [0, +\infty]$ any strictly increasing and continuous function. Setting $x \oplus y = h^{-1}[h(x) + h(y)]$ and $x \otimes y = h^{-1}[h(x) \cdot h(y)]$, u is determined by the condition $h(u) = 1$. In this case, we get

$$\oint f\, d\mu = h^{-1}\Big[\oint (h \circ f) d(h \circ \mu)\Big]\,.$$

If μ is $\oplus$-additive, the pan-integral coincides with Weber's one [16].

5) Fixed two numbers a and M with $0 < a < M$, let $I = [0, M]$ and $h : [a, M] \to [0, +\infty]$ be an increasing and continuous function. Setting

$x \oplus y = h^{-1}[h(x) + h(y)]$,	$x \otimes y = h^{-1}[h(x) \cdot h(y)]$,	if $(x, y) \in [a, M] \times [a, M]$,
$x \oplus y = x \vee y$,	$x \otimes y = x \wedge y$,	elsewhere, (see pict.).

The unit element u belongs to $[a, M]$ and it is determined by $h(u) = 1$.

The pan-integral has the following expression:

$$\int\!\!\!\!\!\!\not{P}\; f \, d\mu = \begin{cases} \int\!\!\!\!\!\!\!\;\mathcal{S}\; f \, d\mu & \text{if} \quad \mu(\{f > a\}) \leq a \\ h^{-1}\Big[\int\!\!\!\!\!\!\;\mathcal{C}\; f_h \, d\mu_h \Big] & \text{if} \quad \mu(\{f > a\}) > a \,, \end{cases}$$

where

$$f_h(\omega) = \begin{cases} 0 & \text{if} \quad f(\omega) \leq a \\ h(f(\omega)) & \text{if} \quad f(\omega) > a \end{cases} \,, \quad \mu_h(A) = \begin{cases} o & \text{if} \quad \mu(A) \leq a \\ h(\mu(A)) & \text{if} \quad \mu(A) \geq a \,. \end{cases}$$

References

[1] Bassanenzi R.C. and Greco G.H., On functionals representable by fuzzy measures, *J. Math. An. Appl.,* **133** (1988) 44-56.

[2] Benvenuti P., Vivona D. and Divari M., On fuzziness measure via Sugeno's integral, *Fuzzy Logic and Soft Computing.* Ed. B. Bouchon-Meunier, R.R. Yager and L.A. Zadeh, Advances in Fuzzy Systems, **4** (1995), 330-336.

[3] Choquet G., Theory of capacities, *Ann. Ist. Fourier,* **5** (1955) 131-295.

[4] De Giorgi E. and Letta G., Une notion générale de convergence faible pour des functions croissantes d'ensemble, *Ann. Sc. Norm. Pisa,* **4** (1977) 61-99.

[5] Dellacherie C., *Capacités: Propriété de Lusin et theoréme d'extension à la Caratheodory*, Preprint 1993.

[6] Dennenberg D., *Non-additive measure and integral*, Kluver Academic Publisher 1994.

[7] Greco G.H., L'integrale monotono, *Rend. Mat. Padova*, **57** (1977) 149-166.

[8] Greco G.H., Sulla rappresentazione di funzionali mediante integrali, *Rend. Sem. Mat. Univ. Padova,* **66** (1982) 21-42.

[9] Klir G.J. and Folger T.A., *Fuzzy sets, uncertainty and information*, Prentice Hall, New Jersey 1988.

[10] Murofushi T. and Sugeno M., A theory of fuzzy measures: representations, the Choquet's integral, and null sets, *J. Math. An. Appl.,* **159** (1991) 532-549.

[11] Schweizer B. and Sklar A., *Probabilistic Metric Space*, North-Holland, New York, Amsterdam, Oxford, 1983.

[12] Shmeidler D., Integral representation without additivity, *Proc. Am. Math. Soc.,* **97** *(1986) 255-261.*

[13] Sugeno M., Theory of fuzzy integrals and Applications, PhD. Doctoral dissertation, Tokyo Inst. of Tech. 1974.

[14] Vivona D., Mathematical aspects of the theory of Measures of Fuzziness, *in this issue*

[15] Wang Z. and Klir G., *Fuzzy measure theory*, Plenum Press 1992.

[16] Weber S., Decomposable Measures and Integrals for Archimedean t-Conorm, J. Math. An. Appl., **101** (1984) 114-138.