Proceedings of the

3rd International Workshop on Optimal Networks Topologies IWONT 2010

Facultat de Matemàtiques i Estadística
Universitat Politècnica de Catalunya
Barcelona, 9-11 June 2010

This edition of IWONT is dedicated to
Professor Miquel Àngel Fiol
on the occasion of his 60th birthday

Editors
Josep M. Brunat, Josep Fàbrega, and Josef Širáň

Sponsored by

Ministerio de Ciencia e Innovación
Universitat Politècnica de Catalunya
Facultat de Matemàtiques i Estadística
Grup de recerca Combinatòria, Teoria de Grafs i Aplicacions

Primera edició: febrero de 2011

Disseny de la coberta: Iniciativa Digital Politècnica

Oficina de Publicacions Acadèmiques Digitals de la UPC
Jordi Girona Salgado 31,
Edifici Torre Girona, D-203, 08034 Barcelona
Tel.: 934 015 885 Fax: 934 054 101
www.upc.edu/idp
E-mail: info.idp@upc.edu

Producció: SERVICE POINT
Pau Casals, 161-163
08820 El Prat de Llobregat, (Barcelona)

Dipòsit legal:
ISBN: 978-84-7653-565-3

Chairmans

Josep Fàbrega
Universitat Politècnica de Catalunya, Barcelona

Josef Širáň
Open University, Milton Keynes
and Slovak University of Technology, Bratislava

Scientific Comitee

Josep Fàbrega
Universitat Politècnica de Catalunya, Barcelona

Joan Gimbert
Universitat de Lleida, Lleida

Anna Lladó
Universitat Politècnica de Catalunya, Barcelona

Zdeněk Ryáček
University of West Bohemia, Pilsen

Oriol Serra
Universitat Politècnica de Catalunya, Barcelona

Josef Širáň
Open University, Milton Keynes
and Slovak University of Technology, Bratislava

Organizing Comitee
UPC, Barcelona

Francesc Aguiló
Camino Balbuena
Josep M. Brunat
Francesc Comelles
Josep Fàbrega
Margarida Mitjana
Xavier Muñoz

Contents

Foreword

The International Workshop on Optimal Network Topologies (IWONT) provides researchers interested in theoretical problems arising from the design and analysis of interconnections networks with the opportunity to meet and discuss on different topics related to this general subject. The themes covered by the workshop came mainly from Graph Theory and include the degree-diameter problem and the associated design of large graphs and digraphs; spectral techniques; connectivity and vulnerability; symmetry and regularity in graphs; Cayley graphs; factors and graph decompositions; random graphs and probabilistic methods; permutation networks, and routing and protocols in communication networks.

The previous two IWONT meetings took place in 2005 at the University of Ballarat, Australia, organized by Prof. Mirka Miller, and in 2007 at the West-Bohemian University in Pilsen, Czech Republic, chaired by Prof. Zdeněk Ryjáček. This third edition has been held at the Universitat Politècnica de Catalunya, Barcelona, Spain, and we particularly wish to thank the previous organizers for this opportunity.

We are very proud to dedicate IWONT 2010 to Prof. Miquel Àngel Fiol on the occasion of his 60th birthday. Together with Prof. José Luís A. Yebra, he founded in the early 80's the research group CombGraph, which is hosting this IWONT edition. He is acknowledged as one of the founders of the Graph Theory community in Spain and his intense activity in building a strong research group in Barcelona is very much appreciated by his numerous students, colleagues and friends. Since then, Miquel Àngel Fiol has not only been a leading Spanish graph theorist, but has also received world-wide recognition. His most influential contributions include constructions of snarks; the use of the line digraph technique to deal with the degree-diameter problem in digraphs; the study of maximally connected and superconnected graphs and digraphs, and the Spectral Excess Theorem, as well as many other results concerning distance-regularity in graphs. He has published more than one hundred papers in international journals and several book chapters, many of which have been profusely cited. To find out how all this work began, we invite you to read the paper "The Beginnings" by Prof. José Luís A. Yebra, which due to its special nature we have placed at the beginning of these Proceedings.

A previous book containing the extended abstracts of all the contributions presented at IWONT 2010 was published prior to the conference. This volume of Proceedings now brings together the full-length papers of most of those contributions. We would like to thank the invited speakers and all the participants for attending the workshop and for the high quality of their talks, which made the IWONT 2010 conference such a success. Moreover, we are specially indebted to all authors contributing to this volume for their scientific work and for their efforts in preparing the articles.

On behalf of all members of the Programme and Organizing Committees we would like to express gratitude to the Spanish Ministerio de Ciencia e Innovación ("Acciones complementarias", grant MTM2009-08119-E/MTM); the Universitat Politècnica de Catalunya ("Ajuts per a l'organització de congressos 2009"), and the Facultat de Matemàtiques i Estadística for sponsoring IWONT 2010, and to the members of the CombGraph group for running the workshop.

Barcelona, December 2010

Josep M. Brunat, Josep Fàbrega and Jozef Širáň.
Editors, IWON 2010 Organization

CONTRIBUTIONS

The Beginnings

José Luis A. Yebra (invited speaker)
Universitat Politcnica de Catalunya
Castelldefels

Abstract

The simple title tells many things. Of course, it could tell almost nothing since a well-known motto says that 'everything must have a beginning'. However, another motto is more appropriate here: 'A journey of a thousand miles must begin with a single step'. Yes, we remember here the first steps of a journey of more than 60/2 years; steps that set the basis for the first group of people working on Graph Theory not only at Barcelona, but in the whole of Spain. And all this began when Miguel Angel Fiol ...

1 Once ... in 1975-1979

I was going to say "Once upon a time..." but I will be much more precise: I will speak of things that happened in the years 1975-79. This means more than 60/2 years ago. I am sure that some of you cannot remember this period and I see that some others were not yet among us. I think that to better understand the beginnings of the Graph Theory group at Barcelona we should travel back to those years. We need to become acquainted with the circumstances under which we worked; it is relevant to be conscious of the advantages but also the hardships of this period.

It was during the 1975-79 years that the Vietnam War ended and the Iran revolution took place, while in Spain Franco died and for the first time in the last forty years we enjoyed democratic elections under a new Constitution. In Mathematics, it was the age of public-key cryptography and the RSA encryption algorithm, as well as the appearance of Mandelbrot fractals. More generally the scientific world celebrated the appearance of the first personal computer, the first CD audio and the first human in vitro fertilization (which, incidentally, assures the older of us of our natural conception, perhaps one of the few advantages we have over the younger ones). Of course, there was no cellular phones, no DVD's and no Internet. These few sentences compose a very fast review of the 1975-79 period and are not enough to understand the limitations encountered. So, I will come back to this issue.

This flash-back is especially important for those under thirty years old, since they have not lived through this period. As the lyrics of La Bohème, the Charles Aznavour song goes:

La bohème	**Very free translation**
Je vous parle d'un temps	I tell you about a time
que les moins de vingt ans	That under-thirties
ne peuvent pas connaître	cannot know about
Montmartre, en ce temps-là ...	At that time, Miguel Angel ...

At that time Miguel Angel was a young student of Telecommunication Engineering At the UPC and a columnist of the satirical journal "La codorniz", surely unaware that new important activities were about to come. Its origin should be found in Martin Gardner's column in Scientific American "Snarks, Boojums and other conjectures related to the four-color-map theorem" ([2]), together with the providential launch late in 1976 of "Investigaciòn y Ciencia", the Spanish translation of the magazine, just in time to offer in the April 1977 issue the Snark's article to a broad Spanish speaking audience that included Miguel Angel.

But let us deviate a little from our theme to pay a small tribute to the immense figure of Martin Gardner, who has died just three weeks ago. He has been the most important writer in recreational mathematics. Surely all of us have enjoyed many of his writings. When we affirm that without him we would not be here, we can ask how many other mathematicians should also acknowledge an analogous origin. Thanks a million, Martin, for all the delightful moments you have given us and for the wonderful world you did open to us.

2 From the 4-colour problem to Snarks

In those years there was much research on the so-called 4-colour problem: Can every planar map be coloured using at most four colours in such a way that regions sharing a common boundary (other than a single point) get different colours? Fallacious proofs were given independently by Kempe (1879) and Tait (1880). Kempe's proof was accepted until 1890 when Heawood showed an error. It is easy to proof that five colours suffice, but reducing the number of colours to four proved to be very difficult. On April 1, 1975, Martin Gardner published a map with the claim that it required five colours if adjacent countries were to receive distinct colours. Of course, the map could be 4-coloured: It was just an April 1 Fool's joke.

To any planar map (where at most three countries meet at any boundary point) can be associated a 3-regular planar graph with boundaries between countries seen as edges and reciprocally. And it is very easy to proof that such planar map can be 4-coloured if and only if the edges of its associated 3-regular graph can be properly 3-coloured. Here properly means that the three edges meeting at every vertex get different colours. As a consequence, it became interesting to find 3-regular graphs that could not be properly 3-coloured. And if any such graph is planar the 4-colour problem is (negatively) solved. The positive result was finally obtained by Appel and Haken (1977), who constructed a computer-assisted proof that four colours sufficed. Some mathematicians did not accept it. However, the proof appeared valid and the result is accepted today as the 4-colour theorem. A shorter proof has since been constructed by Robertson et al. in [4].

Martin Gardner's proposed to use the term snarks for nontrivial 3-regular graphs that are not 3-coloreable. The name comes from Lewis Carroll's 'The hunting of the Snark' because of its elusive character. Just a few years before only a handful of snarks were known: The first one was the ubiquitous Petersen graph on 10 vertices, found in 1898; the next are two graphs on 18 vertices due to Blanusa (1946); the following one is a graph on 210 vertices found by B. Tutte, and the last one was a graph on 50 vertices found by G. Szekeres in 1973. Then Martin Gardner's article presented the work of Rufus Isaacs who discovered two infinite families of snarks in 1975, see [3].

As other Martin Gardner articles, it fostered much research on the subject. Miguel Angel generalized some of Isaacs's constructions by a quite new approach based on Boolean Algebra.

3 Communicating with the scientific community

Once the new constructions were ready, a new problem arose: What can man do with them? There was a simple answer: Communicate it to the scientific community. But remember that we were in 1977. There were no personal computers, nor any of those fancy things we can use with them. There was no such thing as e-mail. In those years communication meant to type a letter, to put it inside something called envelope, to stick in the front side a stamp, to put it in a mailbox and to wait several weeks for an answer. Perhaps I am exaggerating since every body knows what an envelope and a stamp are, but for how long? The most practical way of writing consisted in typing with a writing machine. We should remember that, since there were no personal computers, there was no such thing as word processors, style or Grammatik correctors, not even a delete key to correct the smallest errors, which were corrected by taking the sheet of paper out of the writing machine, creasing it (avoiding uttering four letter words) and beginning again with a new sheet.

Besides, at school people learnt some French, but communicating with the scientific community meant to write in English. Therefore, you can understand that the first two letters that Miguel Angel addressed to Martin Gardner were written in Spanish, beginning with

Barcelona, 6.XI.77

Sr. D. Martin Gardner
E.E.U.U.
Apreciado Sr.:
Soy estudiante ...

Barcelona, 22.XI.77

Sr. D. Martin Gardner
E.E.U.U.
Apreciado Sr.:
Como continuación ...

Well I do not know what most of us would do when receiving letters written in an alien language and related to some relatively aged work. But we know what Martin Gardner and Rufus Isaacs did through the answer of the latter, dated December 21, 1977:

> Dear Sr. Fiol
>
> Martin Gardner has forwarded me your two letters. Both look very interesting. I regret the delay in replying, but this was mainly due to my unsuccessful efforts to find a Spanish translator.

> By I think I understand you because of your beautiful drawings and because you are travelling a road I have been over myself. For example ...

Very nice from R. Isaacs but, since he has understood only the drawings, he had missed most of Miguel Angel work. It was imperative to communicate in English. How? Well, look at the way Miguel Angel did it: First type the letter in Spanish, double spaced:

> Agradezco su carta del 21.XII.77, así como también el folleto
>
> conteniendo su artículo sobre los Snarks. Ambos los he leído
>
> con mucho interés ...

Then look for someone that can translate it into English placing the handwritten translation between the Spanish lines, so as to obtain:

> Agradezco su carta del 21.XII.77, así como también el folleto
> *I want to thank you for your letter of December 21, 1977 as*
> conteniendo su artículo sobre los Snarks. Ambos los he leído
> *well as your paper about the Snarks. I've read both of them*
> con mucho interés ...
> *carefully ...*

Finally, typeset the English version to achieve some thing like

> I want to thank you for your letter of December 21, 1977 as
> well as your paper about the Snarks. I've read both of them
> carefully ...

The final result is a long typeset English letter ... without errors! And remember: without even a delete key to correct a single error.

But this effort pays: The answer from Rufus Isaacs says:

> Dear Senor Fiol,
>
> I owe you my deepest apologies on two grounds.
>
> First, ...
>
> Second, for under-estimating the depth of your work. I see now you have made contributions of insight and importance. Congratulations!
>
> I am enclosing a paper which I was arranging to publish in a graph theory journal. With your permission I should like to write a similar piece on your work.
>
> (...)
>
> I thing you should write a polished paper on your own about your fine Boolean logic method of generating snarks.
>
> (...)

And in a letter dated November 29, 1978 Rufus Isaacs offers:

> I recently spoke to Professor Gore, telling him my high opinion of you. He seems interested in your joining the faculty...

An offer to join the Johns Hopkins University that Miguel Angel could not accept because ... he was still a undergraduate student!

4 Conclusion

Under my light supervision Miguel Angel wrote his graduation thesis about the construction of snarks using Boolean operators. It is difficult to summarize here its more than two hundred pages. I will just restrict myself to showing below that the Petersen graph is a snark and refer to Miguel Angel's paper [1] for the way Boolean logic enters the snarks' world. The Boolean family contains the BDS family of Isaacs and some later additions by other authors as Loupekhine. But it also contains many more new snarks.

To see that the Petersen graph is a snark, notice that to properly colour the edges of a pentagon, any colour can be used at most twice. Thus, three colours are needed, using two of them twice and the other one just once. This is the only proper colouring of the pentagon except for permutation of the colours. As a consequence, the edges adjacent to those vertices (we will called them semi-edges) must be coloured as in figure . Notice that one colour appears three times in consecutive semi-edges while the other two colours appear just once in the remaining semi-edges. Again the colouring is unique except for permutations of the colours. When we draw the pentagon as the 'inside' pentagon in the usual representation of the Petersen graph this circular ordering is modified, so that the three semi-edges equally colored are not consecutive, see Figure 1. As a consequence, the Petersen graph cannot be properly coloured. It is possible to colour at most three of the five edges joining the 'inside' and the 'outside' pentagons, see Figure 1, where the remaining two edges sholud be coloured 1 at one end and 3 at the other one.

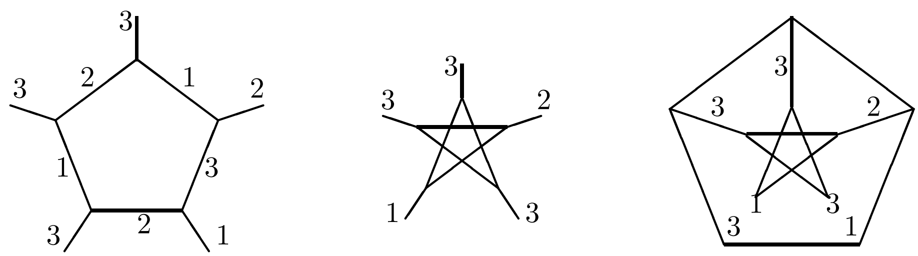


Figure 1: The Petersen graph

Today, thirty two years later, all small snarks are known. For instance, there are exactly 2 snarks on 18 vertices, 6 on 20 vertices, 20 on 22 vertices, 38 on 24 vertices, 280 on 26 vertices and 2900 on 28 vertices.

We could also speak of Miguel Angel's PhD Thesis (1981); we could speak of Minimal Connections, Double Loop Networks, Tessellations, Line Digraphs, the (Δ, D) problem, etc. But this will take us much longer.

Today, Miguel Angel is more than sixty year old. This is a consequence of the simple addition rule. But the most important thing is that instead of being alone there are more than 60/2 other people working in Combinatorics and Graph Theory at Barcelona, not counting those people that work at other places after been initiated here.

References

[1] M.A. Fiol. A Boolean algebra approach to the construction of snarks. *Graph Theory, Comb. and Appl.* Y. Alavi, G. Chartrand, O.R. Oellermann, and A.J. Schwenk., Eds. New York: John Wiley & Sons, Vol.1, 493–524, 1991.

[2] M. Gardner. Snarks, Boojums and other conjectures related to the four-colormap theorem. *Scientific American*, 234:126–130, 1976.

[3] R. Isaacs. Infinite families of nontrivial trivalent graphs which are not Tait colorable. *Amer. Math. Monthly*, 82:221-239, 1975.

[4] N. Robertson, D. Sanders, P. Seymour, and R. Thomas. The Four-Colour Theorem. *J. Comb. Theory B*, 70(1):2–44, 1997.

Infinite families of 3-numerical semigroups with arithmetic-like links

Francesc Aguiló-Gost
Universitat Politècnica de Catalunya
Barcelona

Abstract

Let $S = \langle a, b, N \rangle$ be a numerical semigroup generated by $a, b, N \in \mathbb{N}$ with $1 < a < b < N$ and $\gcd(a, b, N) = 1$. The *conductor* of S, denoted by $\mathrm{c}(S)$ or $\mathrm{c}(a, b, N)$, is the minimum element of S such that $\mathrm{c}(S) + m \in S$ for all $m \in \mathbb{N} \cup \{0\}$. Some arithmetic-like links between 3-numerical semigroups were remarked by V. Arnold. For instance he gave links of the form

$$\frac{\mathrm{c}(13,32,52)}{\mathrm{c}(13,33,51)} = \frac{\mathrm{c}(9,43,45)}{\mathrm{c}(9,42,46)} = \frac{\mathrm{c}(5,35,37)}{\mathrm{c}(5,34,38)} = 2 \quad \text{or} \quad \frac{\mathrm{c}(4,20,73)}{\mathrm{c}(4,19,74)} = 4.$$

In this work several infinite families of 3-numerical semigroups with similar properties are given. These families have been found using a plane geometrical approach, known as L-shaped tile, that can be related to a 3-numerical semigroup. This tile defines a plane tessellation that gives information on the related semigroup.

1 Introduction and known results

A 3-*semigroup* $S = \langle a, b, N \rangle$ with $a, b, N \in \mathbb{N}$ and $1 < a < b < N$, is defined as $\langle a, b, N \rangle = \{m \in \mathbb{N} \mid m = xa + yb + zN;\ x, y, z \in \mathbb{N}\}$. The values a, b and N are called the generators of S. The set $\overline{S} = \mathbb{N} \setminus S$ is called the set of *gaps* of S. If the cardinality of $\overline{S}$ is finite, then S is a 3-*numerical semigroup*. It is well known that S is a 3-numerical semigroup if and only if $\gcd(a, b, N) = 1$. The *Frobenius Number* of S is the value $\mathfrak{f}(S) = \max \overline{S}$. The *conductor* of S is the value $\mathrm{c}(S) = \mathfrak{f}(S) + 1$. Given $m \in S \setminus \{0\}$, the *Apéry set* of S with respect to m, $\mathrm{Ap}(S,m) = \{s \in S \mid s - m \notin S\}$, contains significant information of S. In particular, it is well known that $\mathfrak{f}(S) = \max \mathrm{Ap}(S,m) - m$. A 3-numerical semigroup $S = \langle a,b,N\rangle$ is *minimally generated* if the semigroups $\langle a,b\rangle$, $\langle a,N\rangle$ or $\langle b,N\rangle$ are proper subsets of S. You can find recent results on numerical semigroups in the book of Rosales and García-Sánchez [6]. Recent results mainly related on the Frobenius number can be found in the book of Ramírez Alfonsín [4].

The equivalence class of m modulo N will be denoted by $[m]_N$. A *weighted double-loop* digraph $G(N;a,b;\mathfrak{a},\mathfrak{b})$ is a directed graph with set of vertices $V(G) = \{[0]_N,...,[N-1]_N\}$ and set of weighted arcs $A(G) = \{[v]_N \xrightarrow{\mathfrak{a}} [v+a]_N, [v]_N \xrightarrow{\mathfrak{b}} [v+b]_N \mid [v]_N \in V(G)\}$. The idea of using weighted double-loop digraphs as a tool in the study of the Frobenius number of 3-numerical semigroups was already used by Selmer [8] in 1977 and Rødseth [5] in 1978.

Each weighted double-loop digraph G has related several *minimum distance diagrams* (MDD for short) that periodically tessellates the squared plane. Each vertex $[ia + jb]_N$ of G is associated with the unit square of the plane $(i,j) \in \mathbb{N}^2$, that is the interval $[i,i+1] \times [j,j+1] \in \mathbb{R}^2$. An MDD is composed by N unit squares and has a geometrical shape like the (capital) letter 'L' or it is a rectangle (that is considered a degenerated L-shape), see [5, 3] for more details. Sabariego and Santos [7] gave an algebraic characterization of these diagrams in any dimension. Here we include this characterization in two dimensions.

Definition 1 [Sabariego and Santos, [7]] A minimum distance diagram is any map $D : \mathbb{Z}_N \to \mathbb{N}^2$ with the following two properties:

(a) For every $[m]_N \in \mathbb{Z}_N$, $D([m]_N) = (i,j)$ satisfies $ia + jb \equiv m \pmod N$ and $\|D([m]_N)\|$ is minimum among all the vectors in $\mathbb{N}^2$ with that property ($\|(s,t)\| = s\mathfrak{a} + t\mathfrak{b}$).

(b) For every $[m]_N$ and for every $(s,t) \in \mathbb{N}^2$ that is coordinate-wise smaller than $D([m]_N)$, we have $(s,t) = D([n]_N)$ for some $[n]_N$ (with $n \equiv sa + tb \pmod N$).

An MDD $\mathcal{H}$ is denoted by the lengths of his sides, $\mathcal{H} = L(l,h,w,y)$, with $0 \le w < l$, $0 \le y < h$, $\gcd(l,h,w,y) = 1$ and $lh - wy = N$, as it is depicted in the Figure 1. The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ define the tessellation of the plane by the L-shaped tile $\mathcal{H}$. These lengths fulfill the *compatibility*

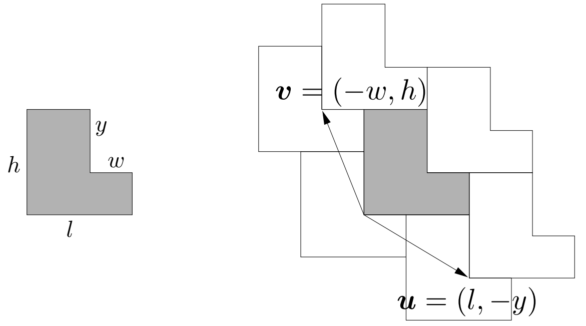


Figure 1: Generic MDD tessellating the plane

equations, stated by Fiol, Yebra, Alegre and Valero [3] in 1987, related to the tessellation

$$la - yb \equiv 0 \pmod N, \quad -wa + hb \equiv 0 \pmod N. \tag{1}$$

Definition 2 [Tessellation related to S] Let $S = \langle a, b, N\rangle$ be a 3-numerical semigroup. A tessellation related to S is a tessellation of the plane generated by an L-shaped MDD of the weighted double-loop digraph $G(N; a, b; a, b)$.

Let D be the map that appears in Definition 1 associated with $G = G(N; a, b; a, b)$, that is $\mathfrak{a} = a$ and $\mathfrak{b} = b$. Then

$$\mathrm{Ap}(S, N) = \{D([0]_N), ..., D([N-1]_N)\}$$

and $D([m]_N)$ can be though as the length of a minimum path from $[0]_N$ to $[m]_N$ in G. Definition 2 gives a metrical view of some properties of S. A geometrical characterization of MDD related to S is needed for practical reasons. This characterization is given in the following result.

Theorem 3 (A., Miralles and Zaragozá, [1]) *The L-shaped tile $\mathcal{H} = L(l, h, w, y)$ satisfying (1) with $lh - wy = N$ and $\gcd(l, h, w, y) = 1$ is related to $S = \langle a, b, N\rangle$ iff $la \geq yb$ and $hb \geq wa$ and both equalities are not satisfied.*

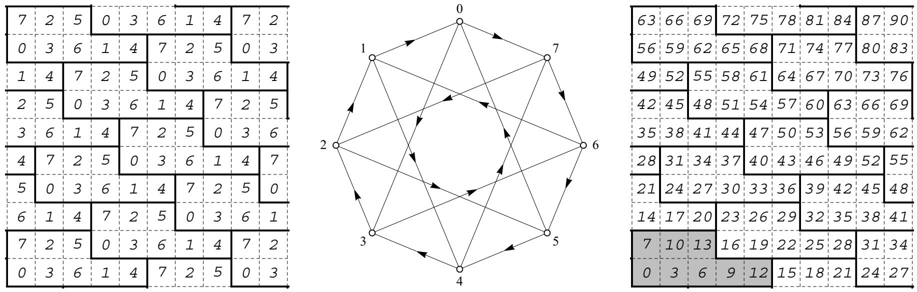


Figure 2: Minimum distance diagram related to $G(8;3,7;3,7)$

Example 4 Consider the weighted double-loop digraph $G = G(8;3,7;3,7)$ that is depicted in the Figure 2. An L-shaped MDD related to G is $\mathcal{H} = L(5,2,2,1)$. Note that the lengths of $\mathcal{H}$, $(l,h,w,y) = (5,2,2,1)$, fulfill the conditions $\gcd(l,h,w,y) = 1$ and $lh - wy = N$, the compatibility equations (1) and Theorem 3. The left-hand side of Figure 2 shows a piece of the first quadrant of the squared plane and how $\mathcal{H}$ tessellates the plane. It also shows the periodic distribution of the equivalence classes modulo 8, where each unit square (i,j) is labelled by the class $[3i+7j]_8$. The right-hand side of this figure shows the same piece of the first quadrant, however each unit square (i,j) is labelled now by $\|D([3i+7j]_8)\| = 3i+7j$ (D is the map of Definition 1). Note that the labels inside the grey L-shape (the one that contains the unit square $(0,0)$) form the set $\mathrm{Ap}(\langle 3,7,8\rangle, 8)$. In particular, we have $\mathfrak{f}(\langle 3,7,8\rangle) = 13-8 = 5$.

V. Arnold [2] in 2009 commented that his *1999 calculations of Frobenius numbers* provided hundreds of empirical properties. He remarked some *strange arithmetical facts* like

$$\frac{\mathrm{c}(13,32,52)}{\mathrm{c}(13,33,51)} = \frac{\mathrm{c}(9,43,45)}{\mathrm{c}(9,42,46)} = \frac{\mathrm{c}(5,35,37)}{\mathrm{c}(5,34,38)} = 2, \qquad \frac{\mathrm{c}(4,20,73)}{\mathrm{c}(4,19,74)} = 4. \tag{2}$$

It was shown in [1] that if $\mathcal{H} = L(l,h,w,y)$ is related to $S = \langle a,b,N\rangle$, then the Frobenius number is

$$\mathfrak{f}(\langle a,b,N\rangle) = \max\{(l-1)a + (h-y-1)b, (l-w-1)a + (h-1)b\} - N. \tag{3}$$

Therefore, from the identities $\mathrm{c}(S) = \mathfrak{f}(S)+1$ and (3), arithmetic-like links between conductors as those appearing in (2) can be though as geometrical-like relations between related L-shaped MDD tiles.

When the semigroup is 2-minimally generated, that is $S = \langle a, b\rangle$ with $\gcd(a, b) = 1$, it is well known that his Frobenius number is

$$\mathfrak{f}(\langle a, b\rangle) = ab - a - b. \tag{4}$$

Although this result was published by Sylvester [9] in 1884, it seems to be true that (4) was given first by Frobenius in his lectures. Therefore, the conductor is given by the expression $\mathrm{c}(a, b) = \mathfrak{f}(\langle a, b\rangle) + 1 = (a - 1)(b - 1)$.

In this work, several infinite families of pairs of 3-numerical semigroups are given such that each pair fulfills a (2)-like relation.

2 Computer assisted numerical remarks

Properties in (2) suggest looking for semigroups like

$$\frac{\mathrm{c}(\alpha, n, m)}{\mathrm{c}(\alpha, n - 1, m + 1)} = k, \tag{5}$$

where $\langle \alpha, n, m\rangle$ and $\langle \alpha, n - 1, m + 1\rangle$ are 2 and 3 minimally generated numerical semigroups respectively, for different natural numbers n and m and fixed values of α and k.

α \ k	1	2	3
4	11	0	0
5	0	109	0
6	4	0	1
7	0	55	6
8	4	0	1
9	5	13	3
10	2	0	1

Table 1: Cardinalities of some sets $P(\alpha, k, 100)$

Let us consider the set

$$P(\alpha, k, \ell) = \{\langle \alpha, n, m\rangle \mid \frac{\mathrm{c}(\alpha, n, m)}{\mathrm{c}(\alpha, n - 1, m + 1)} = k, \;\; m \leq \ell\}$$

where $\langle \alpha, n, m\rangle$ and $\langle \alpha, n-1, m+1\rangle$ are 2 and 3 minimally generated. A computer search reveals the cardinality of some sets $P(\alpha, k, 100)$. These cardinalities are included in Table 1.

Let us consider now the set $Q(\alpha, k, \ell)$, defined as $P(\alpha, k, \ell)$ but now both semigroups $\langle \alpha, n, m\rangle$ and $\langle \alpha, n-1, m+1\rangle$ are 3-minimally generated. The cardinalities of $Q(\alpha, 1, 100)$, with $\alpha = 4, ..., 10$, are $276, 5, 0, 15, 0, 218$ and 4, respectively. We have now $Q(\alpha, k, 100) = \emptyset$ for $(\alpha, k) \in \{4, ..., 10\} \times \{2, 3\}$. Let us denote the sets

$$P(\alpha, k) = \bigcup_{\ell \geq \alpha+2} P(\alpha, k, \ell) \text{ and } Q(\alpha, k) = \bigcup_{\ell \geq \alpha+2} Q(\alpha, k, \ell).$$

We use the numerical data of this section to search infinite families of pairs of semigroups belonging to $P(\alpha, k)$ or $Q(\alpha, k)$, for some values of α and k.

3 Infinite families

In this section we use the L-shaped tile technique included in Section 1 for finding infinite families of 3-numerical semigroups that belong to $P(4, 1)$, $P(7, 3)$ and $Q(9, 1)$.

Theorem 5 *Let us consider the 3-numerical semigroups* $S_t = \langle 4,\ 4t+3,\ 8t+6\rangle$ *for* $t \geq 1$. *Then* $\{S_t\}_{t\geq 1} \subset P(4, 1)$.

Proof: Let us consider S_t and $T_t = \langle 4, 4t+2, 8t+7\rangle$. First, we check that S_t and T_t are numerical semigroups for $t \geq 1$, that is $\gcd(4, 4t+3, 8t+6) = \gcd(4, 4t+2, 8t+7) = 1$,

$$\begin{aligned} \gcd(4, 4t+3, 8t+6) &= \gcd(4, 3, 6) = \gcd(3, 2) = 1, \\ \gcd(4, 4t+2, 8t+7) &= \gcd(4, 2, 7) = \gcd(2, 7) = 1. \end{aligned}$$

Second, we have to see that S_t and T_t are 2 and 3 minimally generated, respectively. To this end, note that $8t+6 = 2 \times (4t+3)$ and so $S_t = \langle 4, 4t+3, 8t+6\rangle = \langle 4, 4t+3\rangle$, that is a 2-minimally generated semigroup because $4t+3$ can not be a multiple of 4. Consider now $T_t = \langle 4, 4t+2, 8t+7\rangle$, we have that neither $4t+2$ nor $4t+7$ are multiples of 4; also $8t+7$ is not a multiple of $4t+2$. Let us see also that $8t+7 \notin \langle 4, 4t+2\rangle$, that is $8t+7 \neq c_t \times 4 + d_t \times (4t+2)$ with $c_t, d_t \in \mathbb{N}$, for $t \geq 1$; if so, the even number $c_t \times 4 + d_t \times (4t+2)$ would equalize the odd one $8t+7$, a contradiction.

Third, we have to see the identity $\mathrm{c}(S_t) = \mathrm{c}(T_t)$, for all $t \geq 1$. The conductor $\mathrm{c}(S_t)$ is easy to compute because S_t is 2-generated and we can apply (4), that is $\mathrm{c}(a,b) = \mathfrak{f}(a,b)+1 = (a-1)(b-1)$. So, $\mathrm{c}(S_t) = (4-1)(4t+3-1) = 12t+6$. To compute the conductor $\mathrm{c}(T_t)$, we use the expression (3). To this end, we have to find the related sequence of L-shaped minimum distance diagrams.

Let us see that T_t has related the L-shaped MDD $\mathcal{H}_t = L(5t+4, 2, 2t+1, 1)$, for all $t \geq 1$. Obviously $\gcd(5t+4, 2, 2t+1, 1) = 1$. Set $N_t = 8t+7$, $a_t = 4$, $b_t = 4t+2$, $l_t = 5t+4$, $h_t = 2$, $w_t = 2t+1$ and $y_t = 1$. It is easily checked that $l_t h_t - w_t y_t = (5t+4)\times 2 - (2t+1) = N_t$ and the compatibility equations (1)

$$\begin{aligned} l_t a_t - y_t b_t \equiv 0 \pmod{N_t} &\Leftrightarrow 20t+16-4t-2 = 16t+14 \equiv 0 \pmod{N_t}, \\ h_t b_t - w_t a_t \equiv 0 \pmod{N_t} &\Leftrightarrow 8t+4-8t-4 = 0 \equiv 0 \pmod{N_t}. \end{aligned}$$

$\mathcal{H}_t$ is also an MDD because Theorem 3 is fulfilled, that is $l_t a_t > y_t b_t$ and $h_t b_t = w_t a_t$, for all $t \geq 1$. Therefore $\mathcal{H}_t$ is related to T_t and we can use the expression (3) to compute the conductor $\mathrm{c}(T_t)$

$$\mathrm{c}(T_t) = \mathfrak{f}(T_t)+1 = \max\{(5t+3)\times 4+0, (3t+2)\times 4+4t+3\} - 8t-7+1 = 12t+6.$$

Hence, $\mathrm{c}(S_t) = \mathrm{c}(T_t)$ as it is stated. $\square$

Theorem 6 *Consider the 3-numerical semigroups $S_t = \langle 7, 7t+7, 14t+9\rangle$ for $t \geq 1$. Then $\{S_t\}_{t\geq 1} \subset P(7,3)$.*

Proof: Consider S_t and $T_t = \langle 7, 7t+6, 14t+10\rangle$. We have $\gcd(7, 7t+7, 14t+9) = \gcd(7, 7t+6, 14t+10) = 1$, so S_t and T_t are numerical semigroups. The semigroup S_t is minimally 2-generated and $S_t = \langle 7, 14t+9\rangle$, so his conductor is $\mathrm{c}(S_t) = (7-1)(14t+9-1) = 84t+48$.

Let us see that T_t is 3-minimally generated. We have $7 \nmid 7t+6$, $7 \nmid 14t+10$ and $7t+6 \nmid 14t+10$, for all $t \geq 1$. We have to see now $14t+10 \notin \langle 7, 7t+6\rangle$. If $7 \times m_t + (7t+6) \times n_t = 14t+10$ with $m_t, n_t \in \mathbb{N}$, then $0 \leq n_t \leq 1$ (if $n_t \geq 2$ then $n_t \times (7t+6) > 14t+10$). If $n_t = 0$, the identity can not be satisfied, hence $n_t = 1$. So the equality turns to be $7m_t = 7t+4$ that has no solution for $m_t \in \mathbb{N}$ because $7m_t \equiv 0 \pmod 7$ and $7t+4 \equiv 4 \pmod 7$. Therefore, the semigroup T_t is 3-minimally generated.

The semigroup T_t has related the L-shaped MDD $\mathcal{H}_t = L(5t+4, 4, 2t+2, 3)$, that is $\gcd(5t+4, 4, 2t+2, 3) = 1$, his area is $14t+10$ and $\mathcal{H}_t$ fulfills the compatibility equations (1) and Theorem 3. Therefore, by using (3), his conductor is

$$\mathrm{c}(T_t) = \max\{(5t+3)\times 7+0, (3t+1)\times 7+3\times(7t+6)\}-14t-10+1 = 28t+16.$$

So $\mathrm{c}(S_t) = 3\mathrm{c}(T_t)$ as it is stated. □

Theorem 7 *Consider the* 3*-numerical semigroups* $S_t = \langle 9, 9t+7, 9t+12\rangle$ *for* $t \geq 1$*. Then* $\{S_t\}_{t\geq 1} \subset Q(9, 1)$*.*

Proof: Consider S_t ad $T_t = \langle 9,\ 9t+6,\ 9t+13\rangle$. From the identities $\gcd(9,\ 9t+7,\ 9t+12) = \gcd(9, 9t+6, 9t+13) = 1$, the semigroups S_t and T_t are numerical semigroups. Let us see that both semigroups are 3-minimally generated.

From $9 \not| 9t+7, 9t+6, 9t+12, 9t+13$ and $9t+7 \not| 9t+12$ and $9t+6 \not| 9t+13$, we have to see $9t+12 \notin \langle 9, 9t+7\rangle$ and $9t+13 \notin \langle 9, 9t+6\rangle$. Let us assume that $9\times m_t + (9t+7)\times n_t = 9t+12$ with $m_t, n_t \in \mathbb{N}$ and $0 \leq n_t \leq 1$ (if $n_t \geq 2$ then $n_t \times (9t+7) > 9t+12$). Then $n_t = 1$ because $9 \not| 9t+12$ and so we have the identity $9m_t = 5$ for $m_t \in \mathbb{N}$, that is a contradiction. A similar argument proves that $9t+13 \notin \langle 9, 9t+6\rangle$.

It can be checked that S_t and T_t have related the L-shaped minimum distance diagrams $L(3t+4, 3, 2t+1, 0)$ and $L(4t+5, 3, 3t+2, 1)$, respectively. Therefore, from (3), we have $\mathrm{c}(S_t) = \mathrm{c}(T_t) = 36t+30$. □

Acknowledgement

Work supported by the Comisión Interministerial de Ciencia y Tecnología MCYT ref. MTM2008-06620-C03-01/MTM and the Catalan Research Council under the projects DURSI 2005SGR00256 and 2009SGR1387.

References

[1] F. Aguiló, A. Miralles and M. Zaragozá. Using Double-Loop digraphs for solving Frobenius' Problems. *ENDM*, 24:17–24, 2006.

[2] V.I. Arnold. Geometry of continued fractions associated with Frobenius numbers. *Funct. Anal. Other Math.*, 2:129–138, 2009.

[3] M.A. Fiol, J.L.A. Yebra, I. Alegre and M. Valero. A discrete optimization problem in local networks and data alignment. *IEEE Trans. Comput.*, C-36:702–713, 1987.

[4] J.L. Ramírez Alfonsín. *The Diophantine Frobenius Problem.* Oxford Univ. Press, Oxford, 2005, ISBN 978-0-19-856820-9.

[5] ***O***.J. Rødseth. On a linear diophantine problem of Frobenius. *J. Reine Angewandte Math.*, 301:171–178, 1978.

[6] Rosales, J. C. and García-Sánchez, P. A. *Numerical semigroups.* Developments in Mathematics, 20. Springer, New York, 2009, ISBN 978-1-4419-0159-0.

[7] Sabariego, P. and Santos, F. Triple-loop networks with arbitrarily many minimum distance diagrams. *Discrete Math.*, 309(6):1672–1684, 2009.

[8] E.S. Selmer. On the linear diophantine Problem of Frobenius. *J. Reine Angewandte Math.*, **293/294**(1):1–17, 1977.

[9] J. J. Sylvester. Mathematical questions and their solutions. *Educational Times*, 41:21, 1884.

On identifying codes in partial linear spaces

Gabriela Araujo-Pardo and Luis Montejano
Universidad Nacional Autonóma de México
México

Camino Balbuena
Universitat Politècnica de Catalunya
Barcelona

Juan Carlos Valenzuela
Universidad de Cádiz
Cádiz

Abstract

Let $(\mathcal{P}, \mathcal{L}, I)$ be a partial linear space and $X \subseteq \mathcal{P} \cup \mathcal{L}$. Let us denote by $(X)_I = \bigcup_{x \in X}\{y : yIx\}$ and by $[X] = (X)_I \cup X$. With this terminology a *partial linear space* $(\mathcal{P}, \mathcal{L}, I)$ *is said to admit a* $(1, \leq k)$*-identifying code* if the sets $[X]$ are mutually different for all $X \subseteq \mathcal{P} \cup \mathcal{L}$ with $|X| \leq k$. In this paper we give a characterization of k-regular partial linear spaces admitting a $(1, \leq k)$-identifying code. Equivalently, we give a characterization of k-regular bipartite graphs of girth at least six admitting a $(1, \leq k)$-identifying code. That is, k-regular bipartite graphs of girth at least six admitting a set C of vertices such that the sets $N[x] \cap C$ are nonempty and pairwise distinct for all vertex $x \in X$ where X is a subset of vertices of $|X| \leq k$. Moreover, we present a family of k-regular partial linear spaces on $2(k-1)^2+k$ points and $2(k-1)^2 + k$ lines whose incidence graphs do not admit a $(1, \leq k)$-identifying code. Finally, we show that the smallest $(k; 6)$-graphs known up to now for $k - 1$ not a prime power admit a $(1, \leq k)$-identifying code.

1 Introduction

We only consider undirected simple graphs without loops or multiple edges. Unless otherwise stated, we follow the book by Godsil and Royle [18] for terminology and definitions.

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *distance* between two vertices u, v in G, $d_G(u, v)$ or simply $d(u, v)$, is the length of a shortest path joining u and v. The *degree* of a vertex $v \in V$, denoted by $d_G(v)$ or $d(v)$, is the number of edges incident with v. The *minimum degree* of G is denoted by $\delta(G)$, and a graph is said to be *k-regular* if all its vertices have the same degree k. The *neighborhood* $N(v)$ of a vertex v is the set of all vertices that are adjacent to v. The closed neighborhood of v is defined by $N[v] = N(v) \cup \{v\}$. For a vertex subset $X \subseteq V$, the neighborhood of X is defined as $N(X) = \cup_{x \in X} N(x)$, and $N[X] = N(X) \cup X$. The *girth* of a graph G is the length of a shortest cycle and a $(k; g)$-graph is a k-regular graph with girth g. A $(k; g)$-*cage* is a smallest $(k; g)$-graph.

Let C be a nonempty subset of V. For $X \subseteq V$ the set of vertices $I(C) = I(C; X)$ is defined as follows

$$I(C) = \bigcup_{x \in X} N[x] \cap C.$$

If all the sets $I(C)$ are different for all subset $X \subseteq V$ where $|X| \leq k$, then C is said to be a $(1, \leq k)$-*identifying code* in G. In 1998, Karpovsky, Chakrabarty and Levitin [22] introduced $(1, \leq k)$-identifying codes in graphs. Identifying codes appear motivated by the problem of determining faulty processors in a multiprocessor system. We say that a graph G *admits a* $(1, \leq k)$-*identifying code* if there exists such a code $C \subseteq V$ in G. Not all graphs admit $(1, \leq k)$-identifying codes, for instance Laihonen [23] pointed out that a graph formed by a set of independent edges cannot admit a $(1, \leq 1)$-identifying code, because clearly for all $uv \in E$, $N[u] = \{u, v\} = N[v]$. It is not difficult to see that if G admits $(1, \leq k)$-identifying codes, then $C = V$ is also a $(1, \leq k)$-identifying code. Hence a graph admits $(1, \leq k)$-identifying codes if and only if the sets $N[X]$ are mutually different for all $X \subseteq V$ with $|X| \leq k$. Results on identifying codes in specific families on graphs as well as results on the smallest cardinality of an identifying code can be seen in [6, 9, 13, 14].

Laihonen and Ranto [24] proved that if G is a connected graph with at least three vertices admitting a $(1, \leq k)$-identifying code, then the minimum degree is $\delta(G) \geq k$. Gravier and Moncel [17] showed the existence of a graph with minimum degree exactly k admitting a $(1, \leq k)$-identifying code. Recently, Laihonen [23] proved the following result.

Theorem 1 [23] *Let $k \geq 2$ be an integer.*

(i) *If a k-regular graph has girth $g \geq 7$, then it admits a $(1, \leq k)$-identifying code.*

(ii) *If a k-regular graph has girth $g \geq 5$, then it admits a $(1, \leq k-1)$-identifying code.*

According to item (ii) of Theorem 1, all $(k; 6)$-graphs admit a $(1, \leq k-1)$-identifying code. The main aim of this paper is to approach the problem of characterizing bipartite $(k; g)$-graphs for $g \geq 6$ admitting $(1, \leq k)$-identifying codes. To do that we consider a bipartite graph as the incidence graph of a partial linear space $(\mathcal{P}, \mathcal{L}, I)$ [18]. A point $p \in \mathcal{P}$ and a line $L \in \mathcal{L}$ are said to be incident if $(p, L) \in I \subseteq \mathcal{P} \times \mathcal{L}$ and for short this is denoted by pIL or LIp. A *partial linear space* is an incidence structure in which any two points of $\mathcal{P}$ are incident with at most one line of $\mathcal{L}$. This implies that any two lines are incident with at most one point. The *incidence graph* $\mathcal{B}$ of a partial linear space $(\mathcal{P}, \mathcal{L}, I)$ is the graph with vertex set $V(\mathcal{B}) = \mathcal{P} \cup \mathcal{L}$ and edge set $E(\mathcal{B}) = I$, i.e., two vertices are adjacent if and only they are incident. It is easy to check that $\mathcal{B}$ is a bipartite graph of girth at least 6. A partial linear space $(\mathcal{P}, \mathcal{L}, I)$ is said to be *k-regular* if every line is incident with k points and every point is incident with k lines. Obviously the incidence graph of a k-regular partial linear space is a k-regular bipartite graph.

First, we define a partial linear space admitting a $(1, \leq k)$-identifying code. In our main theorem we give a characterization of k-regular partial linear spaces admitting a $(1, \leq k)$-identifying code. As a consequence of this result, we show that minimal $(k; 6)$-cages, which are the incidence graphs of projective planes of order $k-1$, do not admit a $(1, \leq k)$-identifying code. Moreover, we present a family of k-regular partial linear space on $2(k-1)^2 + k$ points and $2(k-1)^2 + k$ lines whose incidence graphs do not admit a $(1, \leq k)$-identifying code. Finally, we show that the smallest

$(k;6)$-graphs known up to now and constructed in [1, 2, 3, 4, 5, 7, 16] for $k-1$ not a prime power admit a $(1,\leq k)$-identifying code.

The paper is organized as follows. In the next section we present our main theorem and we give a construction of a family of k-regular partial linear spaces without $(1,\leq k)$-identifying codes. In the final section we apply the theorem to show certain families of small $(k;6)$-graphs that have $(1,\leq k)$-identifying codes.

2 Main theorem

Let $(\mathcal{P},\mathcal{L},I)$ be a partial linear space and $X \subseteq \mathcal{P}\cup\mathcal{L}$. Following Dembowski [10], let us denote by $(X)_I = \bigcup_{x\in X}\{y : yIx\}$ and by $[X] = (X)_I \cup X$. With this terminology we give the following definition.

Definition 2 A partial linear space $(\mathcal{P},\mathcal{L},I)$ is said to admit a $(1,\leq k)$-identifying code if and only if the sets $[X]$ are mutually different for all $X \subseteq \mathcal{P}\cup\mathcal{L}$ with $|X| \leq k$.

As an immediate consequence of Theorem 1 we can write the following corollary.

Corollary 3 *Let $k \geq 2$ be an integer. A k-regular partial linear space $(\mathcal{P},\mathcal{L},I)$ admits a $(1,\leq k-1)$-identifying code.*

Next, we present a characterization of k-regular partial linear spaces admitting a $(1,\leq k)$-identifying code as well as some consequences.

Theorem 4 *Let $k \geq 2$ be an integer. A k-regular partial linear space $(\mathcal{P},\mathcal{L},I)$ admits a $(1,\leq k)$-identifying code if and only if the following two conditions hold:*

(i) *For every two collinear points $u,p \in \mathcal{P}$ there exists a point $z \in \mathcal{P}$ which is collinear with just one of u,p. Equivalently, for every $u,p \in \mathcal{P}$ such that $|(u)_I \cap (p)_I| = 1$, there exists $z \in \mathcal{P}$ such that $|(u)_I \cap (z)_I| + |(p)_I \cap (z)_I| = 1$.*

(ii) *For every two concurrent lines $L,M \in \mathcal{L}$ there exists a line $\Lambda \in \mathcal{L}$ which is concurrent with just one of L,M. Equivalently, for every $L,M \in \mathcal{L}$ such that $|(L)_I \cap (M)_I| = 1$, there exists $\Lambda \in \mathcal{L}$ such that $|(L)_I \cap (\Lambda)_I| + |(M)_I \cap (\Lambda)_I| = 1$.*

Proof: Suppose that $(\mathcal{P},\mathcal{L},I)$ admits a $(1,\leq k)$-identifying code and that there exist two concurrent lines $M,L\in\mathcal{L}$ such that

$$\text{for every line } \Lambda\in\mathcal{L}, |(M)_I\cap(\Lambda)_I|=1 \text{ iff } |(L)_I\cap(\Lambda)_I|=1. \tag{1}$$

Let $(M)_I\cap(L)_I=\{p\}$ and consider the sets $X=\{M\}\cup((L)_I-p)\subset\mathcal{P}\cup\mathcal{L}$ and $Y=\{L\}\cup((M)_I-p)\subset\mathcal{P}\cup\mathcal{L}$. Observe that $X\neq Y$ and $|X|=|Y|=k$ because $(\mathcal{P},\mathcal{L},I)$ is k-regular. Then

$$\begin{aligned}
[X] &= [M]\cup((L)_I-p)\cup\textstyle\bigcup_{h\in(L)_I-p}\{\Lambda\in\mathcal{L}:\Lambda Ih\},\\
[Y] &= [L]\cup((M)_I-p)\cup\textstyle\bigcup_{h\in(M)_I-p}\{\Lambda\in\mathcal{L}:\Lambda Ih\}.
\end{aligned}$$

Clearly $[X]\cap\mathcal{P}=(M)_I\cup((L)_I-p)=[Y]\cap\mathcal{P}$; and $[X]\cap\mathcal{L}=\{M,L\}\cup\bigcup_{h\in(L)_I-p}\{\Lambda\in\mathcal{L}:\Lambda Ih\}$ and $[Y]\cap\mathcal{L}=\{M,L\}\cup\bigcup_{h\in(M)_I-p}\{\Lambda\in\mathcal{L}:\Lambda Ih\}$. Assumption (1) yields to $[X]\cap\mathcal{L}=[Y]\cap\mathcal{L}$ meaning that $[X]=[Y]$, which is a contradiction with the hypothesis that $(\mathcal{P},\mathcal{L},I)$ admits a $(1,\leq k)$-identifying code. We may reason analogously to prove that there are no two collinear points $p,q\in\mathcal{P}$ such that for every point $r\in\mathcal{P}$, $|(p)_I\cap(r)_I|=1$ iff $|(q)_I\cap(r)_I|=1$.

Conversely, suppose that $(\mathcal{P},\mathcal{L},I)$ does not admit a $(1,\leq k)$-identifying code and let us assume that for every two elements $u,v\in\mathcal{P}\cup\mathcal{L}$ such that $|(u)_I\cap(v)_I|=1$, there exists $z\in\mathcal{P}\cup\mathcal{L}$, for which

$$|(u)_I\cap(z)_I|+|(v)_I\cap(z)_I|=1.$$

By Corollary 5, $(\mathcal{P},\mathcal{L},I)$ admits $(1,\leq k-1)$-identifying codes and hence $[X]\neq[Y]$ holds for all $X,Y\subseteq\mathcal{P}\cup\mathcal{L}$ such that $|X|,|Y|\leq k-1$. According to our assumption, there must exist two different sets $X,Y\subseteq\mathcal{P}\cup\mathcal{L}$ such that $\max\{|X|,|Y|\}=k$ and $[X]=[Y]$. Without loss of generality, we may assume that $X,Y\subseteq\mathcal{P}\cup\mathcal{L}$, $X\neq Y$, $|X|=k$, $|Y|\leq k$ and $[X]=[Y]$.

First, let us see that $|Y|=k$. Let $x\in X\setminus Y$, then $(x)_I\subset[X]=[Y]$. Since $x\notin Y$ it follows that $([w]-x)\cap Y\neq\emptyset$ for all $w\in(x)_I$. Moreover as two points are incident with at most one line and two lines are incident with at most one point, we have $([w]-x)\cap([w']-x)=\emptyset$ for all $w,w'\in(x)_I$, $w\neq w'$. Therefore $|Y|\geq|(x)_I|=k$, giving $|Y|=k$.

Now let us see that each X and Y must contain both points and lines. Otherwise suppose that $X\subseteq\mathcal{P}$, then $[X]\cap\mathcal{P}=X$. In this case if $Y\subseteq\mathcal{P}$ then $[Y]\cap\mathcal{P}=Y$ yielding that $X=Y$ because $[X]=[Y]$, which is a contradiction. Therefore there exists $L \in Y \cap \mathcal{L}$, hence $(L)_I \subseteq [Y] \cap \mathcal{P} = [X] \cap \mathcal{P} = X$, which implies $(L)_I = X$ because $|(L)_I| = k$, as $(\mathcal{P}, \mathcal{L}, I)$ is k-regular, and $|X| = k$. As two lines have at most one common point and $k \geq 2$ we have $Y \cap \mathcal{L} = \{L\}$. Further, $Y \cap \mathcal{P} \subseteq [Y] \cap \mathcal{P} = [X] \cap \mathcal{P} = X$, hence we may assume that $Y = \{x_1, \ldots, x_{k-1}, L\}$ and $X = \{x_1, \ldots, x_k\} = (L)_I$. As $k \geq 2$ we can take $L' \neq L$ such that $(L')_I \cap (L)_I = \{x_k\}$, i.e., $L' \notin Y$ and $L' \notin (x_i)_I$ for $i = 1, \ldots, k-1$, yielding that $L' \in [X] \setminus [Y]$, a contradiction because $[X] = [Y]$. Thus $X \not\subseteq \mathcal{P}$. Analogously, $Y \not\subseteq \mathcal{P}$, and changing points for lines we may check that $X \not\subseteq \mathcal{L}$, and $Y \not\subseteq \mathcal{L}$.

Henceforth, let us assume that

$$X \cap \mathcal{P} = \{x_1, \ldots, x_s\}, X \cap \mathcal{L} = \{L_{s+1}, \ldots, L_k\},$$

$$Y \cap \mathcal{P} = \{y_1, \ldots, y_r\}, Y \cap \mathcal{L} = \{M_{r+1}, \ldots, M_k\}$$

and let us prove the following claim.

Claim 1 (i) $(x_i)_I \cap \{L_{s+1}, \ldots, L_k\} = \emptyset$ *for all* $i = 1, \ldots, s$.

(ii) $(y_i)_I \cap \{M_{r+1}, \ldots, M_k\} = \emptyset$ *for all* $i = 1, \ldots, r$.

Proof: First, suppose that $y_j \notin \{x_1, \ldots, x_s\}$ for some $j \in \{1, \ldots, r\}$. As $y_j \in Y$ we have

$$(y_j)_I \subseteq [Y] \cap \mathcal{L} = [X] \cap \mathcal{L} = \{L_{s+1}, \ldots, L_k\} \cup (x_1)_I \cup \cdots \cup (x_s)_I.$$

As $|(y_j)_I| = k$ and $|(y_j)_I \cap (x_i)_I| \leq 1$, then $\{L_{s+1}, \ldots, L_k\} \subset (y_j)_I$, $|(y_j)_I \cap (x_i)_I| = 1$ for all $i = 1, \ldots, s$, and $(y_j)_I \cap (x_i)_I \notin \{L_{s+1}, \ldots, L_k\}$. Hence $(x_i)_I \cap \{L_{s+1}, \ldots, L_k\} = \emptyset$, so item (i) of the claim is true in this case. Second, suppose $\{y_1, \ldots, y_r\} \subseteq \{x_1, \ldots, x_s\}$, then there exists a line $M_j \notin \{L_{s+1}, \ldots, L_k\}$ because $X \neq Y$. We have $(M_j)_I \subseteq [X] \cap \mathcal{P} = [Y] \cap \mathcal{P}$. Therefore changing points for lines and reasoning as before it follows that $\{x_1, \ldots, x_s\} \subset (M_j)_I$, $|(M_j)_I \cap (L_i)_I| = 1$ for all $i = s+1, \ldots, k$, and $(M_j)_I \cap (L_i)_I \notin \{x_1, \ldots, x_s\}$, hence $(x_i)_I \cap \{L_{s+1}, \ldots, L_k\} = \emptyset$, so item (i) of the claim holds. The proof of (ii) is analogous. ■

Now, suppose that $Y \cap \mathcal{L} = \{M_{r+1}, \ldots, M_k\} \subseteq \{L_{s+1}, \ldots, L_k\}$. Without loss of generality assume that $M_j = L_j$, $j = r+1, \ldots, k$. Hence $[X] \cap \mathcal{P} = \{x_1, \ldots, x_s\} \cup (L_{s+1})_I \cup \cdots \cup (L_k)_I = [Y] \cap \mathcal{P} = \{y_1, \ldots, y_r\} \cup (L_{r+1})_I \cup \cdots \cup (L_k)_I$. Claim 1, yields that $\{x_1, \ldots, x_s\} \subset \{y_1, \ldots, y_r\}$ and $\{L_{s+1}, \ldots, L_r\} \cap$

$Y = \emptyset$, otherwise $X = Y$ which is a contradiction. Therefore, $|(L_{s+1})_I \cap \{y_1, \ldots, y_r\}| \leq r-s$, and as $|(L_{s+1})_I \cap (L_j)_I| \leq 1$ for all $j = r+1, \ldots, k$, we have $|(L_{s+1})_I| \leq r-s+k-r = k-s < k$ which is a contradiction. Therefore $\{M_{r+1}, \ldots, M_k\} \not\subseteq \{L_{s+1}, \ldots, L_k\}$ and in analogous way it is proved that $\{L_{s+1}, \ldots, L_k\} \not\subseteq \{M_{r+1}, \ldots, M_k\}$.

Next, suppose that $s \geq 2$ and take $M \in \{M_{r+1}, \ldots, M_k\} \backslash \{L_{s+1}, \ldots, L_k\}$. We have $(M)_I \subset [Y] \cap \mathcal{P} = [X] \cap \mathcal{P} = \{x_1, \ldots, x_s\} \cup (L_{s+1})_I \cup \cdots \cup (L_k)_I$. As $|(M)_I| = k$, $\{x_1, \ldots, x_s\} \subset (M)_I$ and $|(M)_I \cap (L_i)_I| = 1$ for all $i = s+1, \ldots, k$; thus M must be unique because $s \geq 2$. Therefore $Y \cap \mathcal{L} = \{M_{r+1}, \ldots, M_k\} \subseteq \{L_{s+1}, \ldots, L_k\} \cup \{M\}$. Without loss of generality assume that $Y \cap \mathcal{L} = \{M, L_{r+2}, \ldots, L_k\}$. Again, $(y_j)_I \subseteq [X] \cap \mathcal{L} = \{L_{s+1}, \ldots, L_k\} \cup (x_1)_I \cup \cdots \cup (x_s)_I$. By Claim 1, $(y_j)_I \cap \{L_{r+2}, \ldots, L_k\} = \emptyset$ and as $|(y_j)_I \cap \bigcup_{i=1}^{s} (x_i)_I| \leq s$, then $k = |(y_j)_I| \leq (r+1-s)+s = r+1$, so $r \geq k-1$. Hence $Y = \{y_1, \ldots, y_{k-1}\} \cup \{M\}$. Now, take $L \in X \cap \mathcal{L}$, $L \neq M$. As $(L)_I \subseteq [Y] \cap \mathcal{P}$, reasoning as before we obtain that $(L)_I = \{y_1, \ldots, y_{k-1}\} \cup ((L)_I \cap (M)_I)$ yielding that L must be unique, so $X = \{x_1, \ldots, x_{k-1}\} \cup \{L\}$. As $[X] \cap \mathcal{P} = [Y] \cap \mathcal{P} = \{x_1, \ldots, x_{k-1}\} \cup (L)_I = \{y_1, \ldots, y_{k-1}\} \cup (M)_I$, it follows that $(M)_I = \{x_1, \ldots, x_{k-1}\} \cup ((L)_I \cap (M)_I)$. Hence L and M are two concurrent lines such that every line Λ is concurrent with L if and only if Λ is concurrent with M because $[X] \cap \mathcal{L} = [Y] \cap \mathcal{L}$. In other words, L and M satisfy (1), which is a contradiction with the hypothesis (ii).

It remains to study the case $s = 1$ so that $X = \{x_1, L_2, \ldots, L_k\}$. If $r \geq 2$ reasoning as for the case $s \geq 2$ we get that $s \geq k-1$ meaning that $k = 2$ which is a contradiction with the fact that $2 \leq r < k$. Thus we get that $r = 1$ and so $Y = \{y_1, M_2, \ldots, M_k\}$. By Claim 1, $(x_1)_I = \{M_2, \ldots, M_k\} \cup ((x_1)_I \cap (y_1)_I)$ and $(y_1)_I = \{L_2, \ldots, L_k\} \cup ((x_1)_I \cap (y_1)_I)$. Hence x_1 and y_1 are two collinear points such that every point z is collinear with x_1 if and only if z is collinear with y_1, contradicting the hypothesis (i). $\square$

As an immediate consequence of Theorem 4 we get the following theorem which is a characterization of k-regular bipartite graphs of girth at least 6 admitting a $(1, \leq k)$-identifying code.

Theorem 5 *A k-regular bipartite graph $\mathcal{B}$ of girth at least 6 admits a $(1, \leq k)$-identifying code if and only if for every two vertices $u, v \in V(\mathcal{B})$ such that $|N(u) \cap N(v)| = 1$, there exists $z \in V(\mathcal{B})$ in such a way that $|N(u) \cap N(z)| + |N(v) \cap N(z)| = 1$.*

3 Families of small $(k, 6)$-graphs without $(1, \leq k)$-identifying codes

A projective plane of order $k-1$ is a k-regular partial linear space such that any two distinct points are collinear and any two distinct lines are concurrent. A minimal $(k; 6)$-cage is a bipartite graph which can be obtained as the incidence graph of a projective plane of order $k-1$. Using the properties of projective planes it is not difficult to check that a projective plane of order $k-1$ does not admit a $(1, \leq k)$-identifying code as a consequence of Theorem 4. And in the same way it is shown that a minimal $(k; 6)$-cage has no $(1, \leq k)$-identifying code as a consequence of Theorem 5.

Corollary 6 (i) *A projective plane of order* $k-1$ *does not admit a* $(1, \leq k)$*-identifying code.*

(ii) *A minimal* $(k; 6)$*-cage does not admit a* $(1, \leq k)$*-identifying code.*

Projective planes are not the unique partial linear spaces which do not admit a $(1, \leq k)$-identifying code. For instance, Figure 1 depicts on the right side a partial linear space of 11 points and 11 lines which does not admit $(1, \leq 3)$-identifying codes. On the left side we can see the corresponding $(3; 6)$-bipartite graph on 22 vertices. It is easy to find two different lines L and M satisfying condition (1) of the proof of Theorem 4. So this graph does not admit $(1, \leq 3)$-identifying codes. In the next theorem we construct a family of k-regular partial linear spaces without $(1, \leq k)$-identifying codes. The partial plane of Figure 1 belongs to this family.

Theorem 7 *Let* $(\mathcal{P}, \mathcal{L}, I)$ *be a projective plane of order* $k-1 \geq 2$ *and consider a point* $p_0 \in \mathcal{P}$ *and a line* $L_0 \in (p_0)_I \cap \mathcal{L}$*. Let* $\mathcal{L}_0 = \mathcal{L} \setminus (p_0)_I$ *and* $\mathcal{P}_0 = \mathcal{P} \setminus (L_0)_I$ *and take* $\mathcal{L}_0'$*,* $\mathcal{P}_0'$ *disjoint copies of* $\mathcal{L}_0$ *and* $\mathcal{P}_0$*, respectively. Observe that* $|\mathcal{L}_0| = |\mathcal{P}_0| = (k-1)^2$*, thus we can consider a bijection* $f : \mathcal{P}_0' \to \mathcal{L}_0'$*. Let us define a new incidence structure* $\left(\mathcal{P} \cup \mathcal{P}_0', \mathcal{L} \cup \mathcal{L}_0', I_f'\right)$ *as follows.*

1. *For all* $(z', M) \in (\mathcal{P} \cup \mathcal{P}_0') \times (\mathcal{L} \setminus \mathcal{L}_0)$*,* $z' I_f' M$ *iff* $z' \in \mathcal{P}$ *and* $z' I M$*.*
2. *For all* $(z', M) \in (\mathcal{P} \cup \mathcal{P}_0') \times \mathcal{L}_0$*,* $z' I_f' M$ *iff*

$$\begin{cases} z' \in \mathcal{P} \setminus \mathcal{P}_0 & \text{and } z' I M; \\ z' \in \mathcal{P}_0' & \text{and } z I M, \text{ where } z \in \mathcal{P}_0 \text{ is the copy of } z'. \end{cases}$$

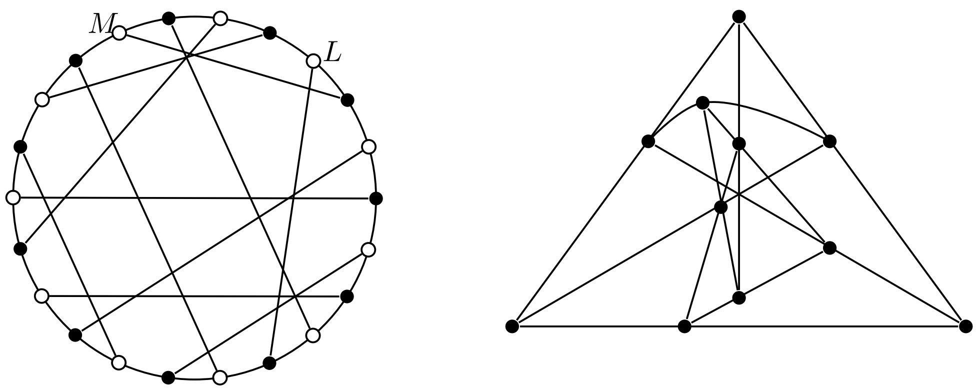


Figure 1: A (3,6)-bipartite graph on 22 vertices without $(1, \leq 3)$ codes and its corresponding partial linear space.

3 *For all* $(z', M') \in (\mathcal{P} \cup \mathcal{P}_0') \times \mathcal{L}_0'$, $z' I_f' M'$ *iff*

$$\begin{cases} z' \in \mathcal{P}_0 & \textit{and } z' I M \textit{ where } M \in \mathcal{L}_0 \textit{ is the copy of } M'; \\ z' \in \mathcal{P}_0' & \textit{and } f(z') = M'. \end{cases}$$

Then $\left(\mathcal{P} \cup \mathcal{P}_0', \mathcal{L} \cup \mathcal{L}_0', I_f'\right)$ *is a* k*-regular partial linear space on* $2(k-1)^2+k$ *points and* $2(k-1)^2+k$ *lines without* $(1, \leq k)$*-identifying codes.*

Proof: First let us see that $\left(\mathcal{P} \cup \mathcal{P}_0', \mathcal{L} \cup \mathcal{L}_0', I_f'\right)$ is a partial linear space. To do that let us show that two distinct lines $A', B' \in \mathcal{L} \cup \mathcal{L}_0'$ have at most one point in common. Let z' be a point such that $z' I_f' A'$ and $z' I_f' B'$. Due to the rules given in 1 and 2 and from the fact that $(\mathcal{P}, \mathcal{L}, I)$ is a projective plane it follows that z' is unique if both A' and B' are in $\mathcal{L}$. If both lines A' and B' are in $\mathcal{L}_0'$, then $z' \in \mathcal{P}_0$ because the rule 3, so z' is unique. And finally if $A' \in \mathcal{L}_0$ and $B' \in \mathcal{L}_0'$ the unique possible point in common is $z' \in \mathcal{P}_0'$ such that $f(z') = B'$ and $A' I z$ (in the projective plane) where z is the copy of z'. By duality it can be shown that there exists at most one line through two distinct points. (In Figure 2 it is depicted the incidence graph corresponding to $\left(\mathcal{P} \cup \mathcal{P}_0', \mathcal{L} \cup \mathcal{L}_0', I_f'\right)$, where $(\mathcal{P}, \mathcal{L}, I)$ is the projective plane of order 2. This graph is also depicted in Figure 1.)

Next let us see that $\left(\mathcal{P} \cup \mathcal{P}'_0, \mathcal{L} \cup \mathcal{L}'_0, I'_f\right)$ is k-regular. It is clear that $(p_0)_{I'_f} = (p_0)_I$, i.e., every line in the set $\{M \in \mathcal{L} \cup \mathcal{L}'_0 : MI'_f p_0\}$ is incident with the same k points as in the projective plane $(\mathcal{P}, \mathcal{L}, I)$. Moreover, a line $M \in \mathcal{L}_0$ is incident with one point from $\mathcal{P} \setminus \mathcal{P}_0$ and $k-1$ points from $\mathcal{P}'_0$ because the rule 2. And a line $M \in \mathcal{L}'_0$ is incident with $k-1$ points from $\mathcal{P}_0$ and one point from $\mathcal{P}'_0$ due to the rule 3.

Finally observe that $\left(\mathcal{P} \cup \mathcal{P}'_0, \mathcal{L} \cup \mathcal{L}'_0, I'_f\right)$ has no $(1, \leq k)$-identifying codes because any two lines from the set $\{M \in \mathcal{L} \cup \mathcal{L}'_0 : MI'_f p_0,\ M \neq L_0\}$ satisfy the property (1) given in the proof of Theorem 4. □

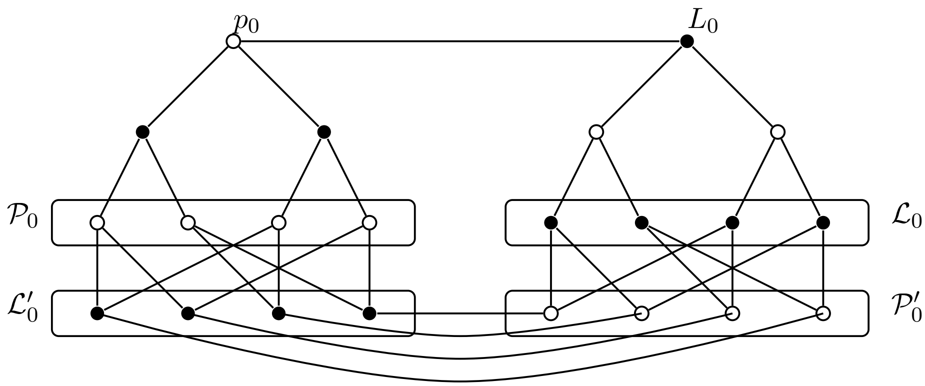


Figure 2: The incidence graph of $\left(\mathcal{P} \cup \mathcal{P}'_0, \mathcal{L} \cup \mathcal{L}'_0, I'_f\right)$, where $(\mathcal{P}, \mathcal{L}, I)$ is the projective plane of order 2.

4 Families of small $(k, 6)$-graphs with $(1, \leq k)$-identifying codes

Minimal $(k; 6)$-cages are known to exist when $k-1$ is a prime power. The order of any $(k; 6)$-cage is denoted by $n(k; 6)$. A new way for constructing projective planes via its incidence matrices is given in [5]. By removing some rows and columns from these matrices some new bipartite $(k; 6)$-graphs with $2(qk-1)$ vertices are obtained for all $k \leq q$ where q is a prime power [5]. The same result is also obtained in [3], but finding these graphs as subgraphs of the incidence graph of a known projective plane. For $k = q$ the same result is obtained in [1], also using incidence matrices. Moreover in [5] the incidence matrix of a $(q-1;6)$-regular balanced bipartite graph on $2(q(q-1)-2)$ vertices was obtained. When q is a square and is the smallest prime power greater than or equal to $k-1$, $(k;6)$-regular graphs with order $2(kq-(q-k)(\sqrt{q}+1)-\sqrt{q})$ have been constructed in [16]. Recently, these results have been improved finding new bipartite $(k;6)$-graphs with $2(qk-2)$ vertices for all $k \leq q$ where q is a prime power [2]. These graphs have the smallest number of vertices known so far among the regular graphs with girth 6 yielding that $n(k;6) \leq 2(qk-2)$ is the best upper bound known up to now. More details about constructions of cages can be found in the survey by Wong [25] or in the survey by Holton and Sheehan [21] or in the more recent dynamic cage survey by Exoo and Jajcay [12]. In this later survey some of the above mentioned constructions are described in a geometric way.

The main aim of this section is to prove that the mentioned new small bipartite $(k;6)$-graphs for all $k \leq q$ where q is a prime power constructed in [1, 2, 3, 4, 5, 7, 16] admit a $(1, \leq k)$-identifying code. With this aim we shall verify that the corresponding partial k-regular linear space admits $(1, \leq k)$-identifying code by means of Theorem 4. We recall some geometric notions introduced in [2, 16]. *A generalized d-gon of order $k-1$* is a partial linear space whose incidence graph is a k-regular bipartite graph with girth $2d$ and diameter d. Finite generalized d-gons exist only for $d \in \{3, 4, 6\}$ (see [8, 18]). When $d = 3$, a 3-gon of order $k-1$ is a projective plane of order $k-1$ (see [8, 18]). A *t-good* structure in a generalized d-gon (see [16]) is a pair $(\mathcal{P}^*, \mathcal{L}^*)$ consisting of a set of points $\mathcal{P}^*$ and a set of lines $\mathcal{L}^*$ satisfying the following conditions:

1. Any point not belonging to $\mathcal{P}^*$ is incident with t lines contained in $\mathcal{L}^*$.

2. Any line not belonging to $\mathcal{L}^*$ is incident with t points contained in $\mathcal{P}^*$.

Clearly, by removing the points and lines of a t-good structure from a $(q+1)$-regular generalized d-gon, we obtain a $(q+1-t)$-regular partial linear space. Its incidence graph is a balanced bipartite $(q+1-t)$-regular graph of girth at least $2d$.

Let $(\mathcal{P}, \mathcal{L}, I)$ be a partial linear space, we say that an incidence pIL is deleted if the point p is not removed from $\mathcal{P}$, but the line L of $\mathcal{L}$ is replaced with the new line $L - p$. The point p is said to be *separated* from the line L. In [2], $(t+1)$-*good* structures were generalized by defining $(t+1)$-*coregular* structures using this removal incidence. An ordered triple

$(\mathcal{P}_0, \mathcal{L}_0, \mathcal{I}_0)$, whose elements are a set of points $\mathcal{P}_0$, a set of lines $\mathcal{L}_0$ and a set of incidences $\mathcal{I}_0$, is said to be a $(t+1)$-*coregular* structure in a generalized d-gon (see [2]) if the removal from a $(q+1)$-regular d-gon of the points in $\mathcal{P}_0$, the lines in $\mathcal{L}_0$ and the incidences in I_0 leads to a new $(q-t)$-regular partial linear space. Obviously, its incidence graph is a bipartite $(q-t)$-regular graph with girth at least $2d$. More precisely, in [2] the following $(t+1)$-coregular structures in projective planes of order q for $t \leq q-2$ were found.

Theorem 8 [2] *Let $(\mathcal{P}, \mathcal{L}, I)$ be a projective plane of order q and $L^* \in \mathcal{L}$ such that $(L^*)_I = \{p, x_1, \ldots, x_q\}$. Let $(p)_I = \{L^*, L_p^1, \ldots, L_p^q\}$ be the set of lines passing through p. The following structures $(\mathcal{P}_0, \mathcal{L}_0, \mathcal{I}_0)$ are $(t+1)$-coregular for $0 \leq t \leq q-2$:*

$$t = 0 \ : \mathcal{P}_0 = \{x_1\} \cup (L_p^1)_I; \quad \mathcal{L}_0 = \{L_p^1\} \cup (x_1)_I; \quad \mathcal{I}_0 = \emptyset.$$

$$t \geq 1 \ : \mathcal{P}_0 = \{x_1, x_2, \ldots, x_{t+1}\} \cup (L_p^1)_I \cup (L_p^2)_I \cup \cdots \cup (L_p^t)_I \cup (M)_I \quad \textit{where } M \in (x_{t+2})_I - L^*;$$

$$\begin{array}{rcl}
\mathcal{L}_0 & = & \{L_p^1, L_p^2, \ldots, L_p^t, M\} \cup (x_1)_I \cup \cdots \cup (x_t)_I \\
 & & \cup \begin{cases} (x_2)_I & \textit{if } t = 1 \\ (x_{t+1})_I - \{A_1, \ldots, A_{t-1}\} & \textit{if } t \geq 2, \textit{ where } A_i \in (x_{t+1})_I - L^* \\ & \textit{is the line connecting } x_{t+1} \textit{ and } \\ & M \cap L_p^i, i = 1, \ldots, t-1; \end{cases} \\
\mathcal{I}_0 & = & \{x_j I L : L \in (x_j)_I \textit{ such that } M \cap L_p^i \in (L)_I \textit{ for some } i \in \{1, \ldots, t\}, \\
 & & j = t+3, \ldots, q\} \\
 & & \cup \{a_{ij} I L_p^j : a_{ij} = A_i \cap L_p^j, j = t+1, \ldots, q, i = 1, \ldots, t-1, \ t \geq 2\}.
\end{array}$$

It is not difficult to check that the partial linear spaces whose incidence graphs are the bipartite graphs constructed in [1, 2, 3, 4, 5, 7, 16] are obtained by removing $(t+1)$-good or $(t+1)$-coregular structures from projective planes. For all the constructions contained in these papers it is not difficult to verify the following remark:

Remark 9 If Π' is a partial linear space obtained by removing a t-good or a t-coregular structure from a projective plane Π and p is a removed or separated point, then p is incident to either $q-t+1$ or to $q-t+2$ lines in Π'. Moreover, in a special construction using Baer Subplanes and t-good structures in projective planes of order square prime powers (see [16]), the removed points are incident with exactly $q - \sqrt{q} - t + 1$ lines in Π'.

It is worth noting that in all the constructions of k-regular partial linear spaces contained in [1, 2, 3, 4, 5, 7, 16], the smallest prime power q with $k \leq q$ and an integer $t \geq 1$ such that $k = q + 1 - t$ are considered. Then, using the following result concerning with the existence of prime numbers in short intervals, we prove Theorem 11.

Theorem 10 [11]

(i) *If $k \geq 3275$ then the interval $[k, k(1 + \frac{1}{2ln^2(k)})]$ contains a prime number.*

(ii) *If $6 \leq k \leq 3276$ then the interval $[k, \frac{7k}{6}]$ contains a prime power.*

The Bertran's postulate states (see [19]) that for every $k > 2$ there exists a prime q verifying the inequality $k < q < 2k$. In this work we will take advantage from Theorem 10, because we only need to check the less restrictive inequality $q < 2k - 2$.

Theorem 11 *Let $q > 2$ be a prime power and $t < q+1$ an integer. Suppose that $2t < q$ or if q is a square prime power that $t \in (q', q)$ where q' is also a prime power such that there is no prime power in the interval (q', q). If Π' is a $(q+1-t)$-regular partial plane constructed by removing a t-good or a t-coregular structure from a projective plane Π of order q, then Π' admits a $(1, \leq k)$-identifying code.*

Proof: Assume that Π' does not admit a $(1, \leq k)$-identifying code and let L and M be two concurrent lines in Π' that satisfy the condition (1) in the proof of Theorem 4 with $\{p\} = (L)_I \cap (M)_I$. Let p_1 be a removed or separated point from $L - p$. Suppose that there are exactly a lines incident to p_1 in Π' (without considering L). If some of these lines had a common point with M in Π', then Π' would admit a $(1, \leq k)$–identifying code by Theorem 4 which is a contradiction with our assumption. Then any of these lines have in common with M points that are not in Π' or that have been separated from M. As M is incident to exactly t points in the projective plane which are not incident to M in Π' (they are removed or separated points), then a must be equal to t.Therefore, by Remark 9, we have the following three cases:

- If p_1 is incident to $q-t+1$ lines in Π', then $a = q-t$ (the number of lines in Π' except L). Hence $q-t=t$, i.e. $q=2t$. This is a contradiction with the hypothesis $2t<q$.

- If p_1 is incident to $q-t+1$ lines in Π', then $a = q-t+1 = t$, which is again a contradiction .

- If q is a square prime power, then p_1 is incident to $q-\sqrt{q}-t+1$ lines in Π' and $2t = q-\sqrt{q}$. Then $q = 2^{2\alpha}$ and $t = 2^{2\alpha-1} - 2^{\alpha-1}$, which is a contradiction to the hypothesis $t \in (\sqrt{q}, q)$, because $\sqrt{q} = 2^\alpha$ is also a prime power.

Reasoning as above and taking into account the dual of Remark 9 it is straightforward to prove that there are not two concurrent points p and q in Π' such that for any point r in Π' we have $|(p)_I \cap (r)_I| = 1$ iff $|(q)_I \cap (r)_I| = 1$.

Then, we can conclude that Π' admits a $(1, \leq k)$-identifying code. □

As an immediate consequence of Theorem 11, we can write the following corollary.

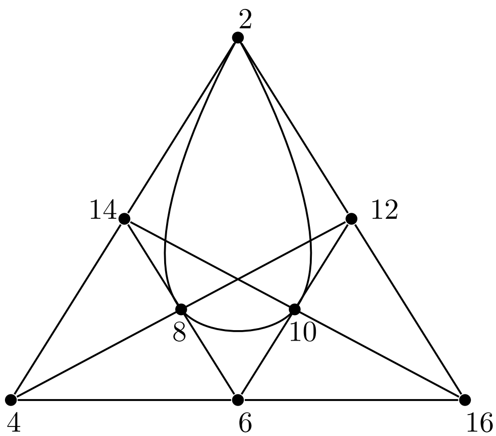
Corollary 12 (i) *The k-regular parcial linear spaces whose incidence graphs are the $(k;6)$-graphs constructed in [1, 2, 3, 4, 5, 7, 16] admit a $(1, \leq k)$-identifying code.*

(ii) *The $(k;6)$-graphs constructed in [1, 2, 3, 4, 5, 7, 16] admit a $(1, \leq k)$-identifying code.*

In Figure 3, a 3-regular linear space of 8 points and 8 lines is depicted. It is obtained by removing from a projective plane of order 3 a 1-coregular structure, see [2]. On the right side it is shown its corresponding bipartite graph on 16 vertices.

Acknowledgement

This research was supported by the Ministry of Education and Science, Spain, and the European Regional Development Fund (ERDF) under project MTM2008-06620-C03-02; by Catalonian Government 1298 SGR2009 and under the Andalusian Government project P06-FQM-01649. Also this research was supported by CONACyT-México under project 57371 and PAPIIT-México under project 104609-3.



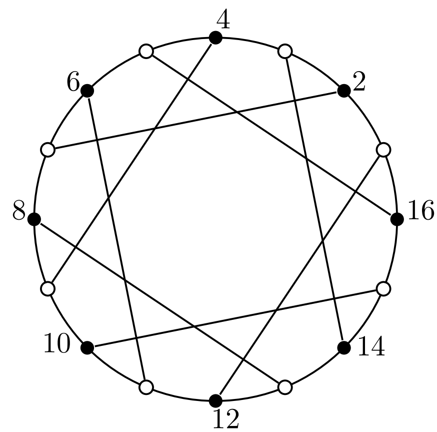


Figure 3: A 3-regular partial linear space of 8 points and 8 lines admitting $(1, \leq 3)$-identifying code and its corresponding (3,6)-bipartite graph on 16 vertices.

References

[1] M. Abreu, M. Funk, D. Labbate and V. Napolitano. On (minimal) regular graphs of girth 6. *Australas. J. Combin.*, 35:119–132, 2006.

[2] G. Araujo-Pardo and C. Balbuena. Constructions of small regular bipartite graphs of girth 6. *Networks*, (2010). Accepted.

[3] G. Araujo-Pardo, C. Balbuena and T. Héger. Finding small regular graphs of girth 6, 8 and 12 as subgraphs of cages. *Discrete Math.*, 310(8):1301–1306, 2010.

[4] G. Araujo, D. González Moreno, J.J. Montellano and O. Serra. On upper bounds and conectivity of cages. *Australas. J. Combin.*, 38:221–228, 2007.

[5] C. Balbuena. Incidence matrices of projective planes and other bipartite graphs of few vertices. *SIAM Journal on Discrete Math.*, 22(4):1351–1363, 2008.

[6] N. Bertrand, I. Charon, O. Hudry, A. Lobstein. Identifying and

locating-dominating codes on chains and cycles. *European J. Combin.*, 25:969–987, 2004.

[7] W. G. Brown. On hamiltoninan regular graphs of girth 6. *J. London Math. Soc.*, 42:644–648, 1967.

[8] N. Biggs. *Algebraic Graph Theory.* Cambridge University Press, New York, 1996.

[9] I. Charon, G. Cohen, O. Hudry and A. Lobstein. New identifying codes in the bynary Hamming space. *European J. Combin.*, 31:491–501, 2010.

[10] P. Dembowski. *Finite Geometries.* Springer-Verlag Berlin Heidelbert, 1997.

[11] P. Dusart. The k^{th} prime is greater than $k(lnk + lnlnk - 1)$ for $k \geq 2$. *Math. Comp.*, 68:411– 415, 1999.

[12] G. Exoo and R. Jajcay. Dynamic Cage Survey. *Electron. J. Combin.*, 15 (2008), #DS16.

[13] G. Exoo, V. Junnila, T. Laihonen and S. Ranto. Upper bounds for binary identifying codes. *Adv. in Appl. Math.*, 42:277–289, 2009.

[14] G. Exoo, V. Junnila, T. Laihonen and S. Ranto. Improved bounds on identifying codes in binary Hamming spaces. *European J. Conbin.*, 31:813–827, 2010.

[15] F. Füredi, À. Lazebnik, V. Seress, A. Ustimenko and J. Woldar. Graphs of prescribed girth and bi-degree. *J. Combin. Theory Ser B.*, 64(2):228–239, 1995.

[16] A. Gács and T. Héger. On geometric constructions of (k, g)-graphs. *Contrib. to Discrete Math.*, 3(1):63–80,2008.

[17] S. Gravier and J. Moncel. Constructions of codes identifying sets of vertices. *Electron. J. Combin.*, 12:R13, 2005.

[18] C. Godsil and G. Royle. *Algebraic Graph Theory.* Springer, New York, 2000.

[19] G.H. Hardy and E.M. Wright. *An introduction on the theory numbers, 5-th ed.* Clarendon, Oxford XVI, 1979.

[20] J.W.P Hirschfield. *Projective geometries over finite fields, 2nd Ed.* Clarendon, Oxford, 1995.

[21] D.A. Holton and J. Sheehan. *The Petersen graph.* Australian Mathematical Society Lecture Series 7. Cambridge University Press, Cambridge, 1993, Chapter 6: Cages, pp. 183–213.

[22] M. Karpovsky, K. Chakrabarty and L. Levitin. On a new class of codes for identifying vertices in graphs. *IEEE Trans. Inform. Theory*, 44:599–611, 1998.

[23] T. Laihonen. On cages admitting identifying codes. *European J. Combin.*, 29:737–741, 2008.

[24] T. Laihonen and S. Ranto. Codes identifying sets of vertices. *Lecture Notes in Comput. Sci.*, 2227:82–91, 2001.

[25] P. K. Wong. Cages-a survey. *J Graph Theory*, 6: 1–22, 1982.

Subdivisions in a bipartite graph

Camino Balbuena
Universitat Politècnica de Catalunya
Barcelona

Martín Cera and Pedro García-Vázquez
Universidad de Sevilla
Sevilla

Juan Carlos Valenzuela
Universidad de Cádiz
Cádiz

Abstract

Given a bipartite graph G with m and n vertices, respectively, in its vertices classes, and given two integers s, t such that $2 \leq s \leq t$, $0 \leq m-s \leq n-t$, and $m+n \leq 2s+t-1$, we prove that if G has at least $mn-(2(m-s)+n-t)$ edges then it contains a subdivision of the complete bipartite $K_{(s,t)}$ with s vertices in the m-class and t vertices in the n-class. Furthermore, we characterize the corresponding extremal bipartite graphs with $mn-(2(m-s)+n-t+1)$ edges for this topological Turan type problem.

1 Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow [5] for terminology and definitions.

Two well-known extensions of the Turán problem [19] are the Turán topological problem and the Zarankiewicz problem. The former one consists of estimating the extremal function $ex(n, TK_p)$ which denotes the maximum number of edges of a graph on n vertices free of a topological minor TK_p of a complete graph on p vertices (see Bollobás' excellent monograph [4] devoted to this subject and the contributions on this topic [1, 13, 11, 16, 15, 18]). The second was stated by Zarankiewicz [20] who studied the maximum size of a bipartite graph on (m, n) vertices, denoted by $z(m, n; s, t)$ that contains no bipartite complete $K_{(s,t)}$ subgraph with s vertices in the m-class and t vertices in the n-class. For a survey of this problem we also refer the reader to Section VI.2 of [4]. Most of the contributions are bounds for the function $z(m, n; s, t)$ when s, t are fixed and m, n are much larger than s, t (see, for example, [6, 7, 8]). Other contributions provide exact values of the extremal function [2, 9, 10].

Recent results on some problems involving the contention of a complete bipartite graph or a subdivision of a complete bipartite graph can be found in the literature [3, 12, 14, 17]. Böhme et al. [3] studied the size of a k-connected graph free of either an induced path of a given length or a subdivision of a complete bipartite graph. Kühn and Osthus [12] proved that for any graph H and for every integer s there exists a function $f = f(H, s)$ such that every graph of size at least f contains either a $K_{s,s}$ as a subgraph or an induced subdivision of H. Meyer [17] also relates the size of a graph with the property of containing a minor of $K_{s,t}$. Other problems involving the contention of maximum matching in graphs are considered in [14].

Combining the topological version of the Turán problem for complete graphs with the Zarankiewicz problem, we introduce the extremal function $tz(m, n; s, t)$ as a natural extension. The function $tz(m, n; s, t)$ is defined as *the maximum size of a (m, n)-bipartite graph free of a topological minor $TK_{(s,t)}$ of a complete bipartite $K_{(s,t)}$ with s vertices in the m-class and t vertices in the n-class.* The objective of this paper is to obtain exact values for this extremal function $tz(m, n; s, t)$ and to characterize the corresponding extremal bipartite graphs for infinitely many related values of m, n, s, t. Namely, we determine the exact value of $tz(m, n; s, t)$ and we characterize the family $TZ(m, n; s, t)$ of extremal graphs for any values of m, n, s, t satisfying $2 \le m - s \le n - t$ and $m + n \le 2s + t - 1$.

A subdivision of a graph H is a graph TH obtained from H by replacing the edges of H with internally disjoint paths. The *branch vertices* of TH are all those vertices that correspond to vertices of H. The complete bipartite graph $K_{(s,t)}$ is said to be *a topological minor* of a bipartite graph G if

$TK_{(s,t)} \subseteq G$.

Given two positive integers, m, n, a bipartite graph G with vertex classes X and Y of cardinalities $|X| = m$ and $|Y| = n$, is denoted by $G = (X,Y)$. The sets of vertices and edges of G are denoted by $V(G) = X \cup Y$ and $E(G)$, respectively, whereas $v(G)$ and $e(G)$ stand for the corresponding cardinalities.

For a bipartite graph $H = (X,Y)$, the degree of a vertex v in the graph H is denoted by $d_H(v)$ whereas $\Delta_X(H)$ (resp. $\Delta_Y(H)$) stand for the maximum degree among vertices in the first class (resp. second class). Thus, $\Delta(H) = \max\{\Delta_X(H), \Delta_Y(H)\}$ is the maximum degree of H. Let us consider two subsets of vertices $\{x_1, x_2, \ldots, x_p\} \subseteq X$ and $\{y_1, y_2, \ldots, y_p\} \subseteq Y$. Let us denote by $H_{0,0} = H$, $H_{1,0} = H - \{x_1\}$, $H_{1,1} = H_{1,0} - \{y_1\}$, and for all $i = 2, \ldots, p$, let us denoted by $H_{i,i-1} = H_{i-1,i-1} - \{x_i\}$ and $H_{i,i} = H_{i,i-1} - \{y_i\}$. Next we introduce the notion of *decreasing sequence of vertices* in a bipartite graph $H = (X,Y)$.

Definition 1 Given an integer $p \geq 1$ and a bipartite graph $H = (X,Y)$, a subset of vertices of H, $\{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$, with $\{x_1, \ldots, x_p\} \subseteq X$ and $\{y_1, \ldots, y_p\} \subseteq Y$, is called a decreasing sequence of H if the following assertions hold:

(i) $d_{H_{i-1,i-1}}(x_i) = \Delta_X(H_{i-1,i-1})$, for $i = 1, \ldots, p$.

(ii) $d_{H_{i,i-1}}(y_i) = \Delta_Y(H_{i,i-1})$, for $i = 1, \ldots, p$.

(iii) For each $i = 1, \ldots, p$, either $x_i y_i \notin E(H)$ or every vertex $y \in V(H_{i,i-1}) \cap Y$ with degree $d_{H_{i,i-1}}(y) = \Delta_Y(H_{i,i-1})$ is adjacent to vertex x_i in H.

Note that

$$d_{H_{0,0}}(x_1) \geq d_{H_{1,1}}(x_2) \geq \ldots \geq d_{H_{p-1,p-1}}(x_p) \geq \Delta_X(H_{p,p})$$

and

$$d_{H_{1,0}}(y_1) \geq d_{H_{2,1}}(y_2) \geq \ldots \geq d_{H_{p,p-1}}(y_p) \geq \Delta_Y(H_{p,p}),$$

and furthermore,

$$e(H) = \sum_{i=1}^{p} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) + e(H_{p,p}).$$

2 Exact values

Let G be a bipartite graph $G = (X, Y)$ on m and n vertices in X and Y respectively. We will henceforth use H to denote the bipartite complement of G, i.e., the bipartite graph $H = (X, Y) = K_{(m,n)} - E(G)$.

The problem of finding a $TK_{(s,t)}$ in a bipartite graph G can be formulated in terms of its bipartite complement H. Indeed, if $G = (X, Y)$ contains a $TK_{(s,t)}$ with set of branch vertices $S \cup T$, $S \subset X$, $T \subset Y$, then the edges of the graph $H[S \cup T]$ are missing in G and thus they must be replaced in G with internally disjoint paths passing through vertices of $X \setminus S$ and vertices of $Y \setminus T$. Since each of these paths must have odd length at least 3, it follows that $e\,(H[S \cup T]) \leq \min\{|X \setminus S|, |Y \setminus T|\}$. Hence, the following necessary but not sufficient condition on the induced subgraph $H[S \cup T]$ in order to determine whether $K_{(s,t)}$ is a topological minor of G is immediate.

Remark 2 Let $G = (X, Y)$ be with $|X| = m$ and $|Y| = n$ and let H be the bipartite complement of G. If G contains a $TK_{(s,t)}$, then there exist $S \subseteq X$ and $T \subseteq Y$ with $|S| = s$, $|T| = t$, such that the number of edges of the subgraph induced by $S \cup T$ in the bipartite complement of G satisfies

$$e\,(H[S \cup T]) \leq \min\{m - s, n - t\}.$$

By using Remark 2, the following proposition provides a lower bound on the maximum size of a (m, n)-bipartite graph free of a topological minor $TK_{(s,t)}$ of $K_{(s,t)}$.

Proposition 3 *Let m, n, s, t be integers such that $2 \leq s \leq t$, $0 \leq m - s \leq n - t$, and $m + n \leq 2s + t - 1$. Then the bipartite graph $G = K_{(m,n)} - M$, where M is any matching of cardinality $2(m - s) + n - t + 1$, does not contain $TK_{(s,t)}$ and therefore,*

$$tz(m, n; s, t) \geq mn - (2(m - s) + n - t + 1)\,.$$

Proof: First, let us see that $K_{(m,n)}$ has a matching of cardinality $2(m - s) + n - t + 1$. This is clear because from $2 \leq s \leq t$ and $0 \leq m - s \leq n - t$, it follows that $m \leq n$, and from the hypothesis $m + n \leq 2s + t - 1$ it follows that $2(m-s)+n-t+1 = (m+n)+m-2s-t+1 \leq m \leq n$. Therefore, we may consider the bipartite graph $G = (X, Y) = K_{(m,n)} - M$ where M is a matching of cardinality $2(m-s)+n-t+1$ in $K_{(m,n)}$. Next let us see that $K_{(s,t)}$ is not a topological minor of G. For that, from Remark 2 it is enough to prove that $e\left(H[S\cup T]\right) > m-s$ for any subsets $S\subseteq X$ and $T\subseteq Y$ of cardinalities s and t, respectively, with $s\le t$. Observe that the number of isolated vertices in the class Y of H is exactly $n-(2(m-s)+n-t+1)$. It follows that the number of edges of $H[X\cup T]$ is

$$e(H[X\cup T]) \ge t-(n-(2(m-s)+n-t+1)) = 2m-2s+1.$$

But since $e(H[(X\setminus S)\cup T]) \le m-s$, then we have

$$\begin{aligned} e(H[S\cup T]) &= e(H[X\cup T]) - e(H[(X\setminus S)\cup T]) \\ &\ge 2m-2s+1-(m-s) \\ &= m-s+1 > m-s. \end{aligned}$$

Thus the result holds. $\Box$

Lemma 4 *Let $p\ge 1$ be an integer and let $G=(X,Y)$ be a bipartite graph, with $|X|\ge p$ and $|Y|\ge p$, and denote by $H=(X,Y)$ the bipartite complement of G. Let $\{x_1,y_1,x_2,y_2,\dots,x_p,y_p\}$ be any decreasing sequence of H and denote by $r=e(H_{p,p})$. If $r\ge 1$ and $e(H)\le 3p$, then:*

(i) *$r\le p$.*

(ii) *$\Delta(H_{p,p})=1$.*

(iii) *$\{x_{p-(r-1)}y_{p-(r-1)},\dots,x_py_p\}\cap E(H)=\emptyset$.*

(iv) *$\{ay_{p-(r-1)},\dots,ay_p\}\cap E(H)=\emptyset$, for each $a\in X\setminus\{x_1,\dots,x_p\}$ of degree $d_{H_{p,p}}(a)=1$.*

(v) *If $r\ge 2$, then $\{x_{p-(r-2)}b,\dots,x_pb\}\cap E(H)=\emptyset$, for each $b\in Y\setminus\{y_1,\dots,y_p\}$ of degree $d_{H_{p,p}}(b)=1$.*

Proof: Since $e(H_{p,p})=r\ge 1$ we deduce $\Delta_X(H_{p,p})\ge 1$, $\Delta_Y(H_{p,p})\ge 1$, following that $d_{H_{i-1,i-1}}(x_i)\ge 1$ and $d_{H_{i,i-1}}(y_i)\ge 1$ for $i=1,\dots,p$, and therefore

$$e(H_{p,p}) = e(H) - \sum_{i=1}^{p}\left(d_{H_{i-1,i-1}}(x_i)+d_{H_{i,i-1}}(y_i)\right) \le 3p-2p = p,$$

thus item (i) is proved.

If $\Delta_X(H_{p,p}) \geq 2$, then $d_{H_{i-1,i-1}}(x_i) \geq 2$ for each $i = 1, \ldots, p$, hence,

$$e(H) = \sum_{i=1}^{p} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) + e(H_{p,p}) \geq 3p + r > 3p,$$

which is a contradiction. Analogously, we arrive at a contradiction if $\Delta_Y(H_{p,p}) \geq 2$. Thus, $\Delta_X(H_{p,p}) = \Delta_Y(H_{p,p}) = 1$, which implies $\Delta(H_{p,p}) = 1$, hence item (ii) is shown.

(iii) Let us denote the edges of $H_{p,p}$ by $e_1 = a_1b_1, \ldots, e_r = a_rb_r$, $a_i \in X \setminus \{x_1, \ldots, x_p\}$ and $b_i \in Y \setminus \{y_1, \ldots, y_p\}$, for $i = 1, \ldots, r$. By item (i) we know that $r \leq p$. We reason by way of contradiction supposing that there exists $j \in \{0, \ldots, r-1\}$ such that $x_{p-j}y_{p-j} \in E(H)$. First we claim that $d_{H_{p-j,p-j-1}}(y_{p-j}) = 1$. Otherwise, if $d_{H_{p-j,p-j-1}}(y_{p-j}) \geq 2$ then $d_{H_{i,i-1}}(y_i) \geq 2$, for $i = 1, \ldots, p-j$ and therefore, by (ii) we have

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) \\ &+ \sum_{i=p-j+1}^{p} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) + e(H_{p,p}) \\ &\geq 3(p-j) + 2j + r \\ &= 3p + (r-j) \\ &> 3p, \end{aligned}$$

the last inequality due to the fact that $j \leq r-1$. Since this is a contradiction with the hypothesis, then $\Delta_Y(H_{p-j,p-j-1}) = d_{H_{p-j,p-j-1}}(y_{p-j}) = 1$, yielding to $d_{H_{i,i-1}}(y_i) = 1$, for $i = p-j, \ldots, p$ and $d_{H_{p,p}}(b_i) = 1$, for $i = 1, \ldots, r$. As $\{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$ is a decreasing sequence of H, it follows that x_{p-j} is adjacent in H to each one of the vertices of the set $\{y_{p-j}, \ldots, y_p, b_1, \ldots, b_r\}$ because of point (iii) of Definition 1. That is, $d_{H_{p-j-1,p-j-1}}(x_{p-j}) \geq j+1+r$, which means that $d_{H_{i-1,i-1}}(x_i) \geq j+1+r \geq$

2, for $i = 1, \dots, p-j$ and therefore,

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &+ \sum_{i=p-j+1}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq 3(p-j) + 2j + r = 3p + (r-j) > 3p, \end{aligned}$$

again a contradiction. Thus $x_{p-j}y_{p-j} \notin E(H)$ for all $j \in \{0, \dots, r-1\}$, hence item (iii) is valid.
(iv) Note that $r \geq 1$ implies $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \dots, p$. We reason by way of contradiction supposing that there exists $j \in \{0, \dots, r-1\}$ such that $ay_{p-j} \in E(H)$ for a vertex $a \in X \setminus \{x_1, \dots, x_p\}$ of degree $d_{H_{p,p}}(a) = 1$. Then $d_{H_{p-j-1,p-j-1}}(a) \geq 2$ and hence, $d_{H_{i-1,i-1}}(x_i) \geq 2$, for $i = 1, \dots, p-j$. Thus,

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &+ \sum_{i=p-(j-1)}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq 3(p-j) + 2j + r = 3p + (r-j) > 3p, \end{aligned}$$

because $j \leq r-1$, against the hypothesis.
(v) Since $r \geq 2$ then $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \dots, p$. We reason by way of contradiction supposing that there exists $j \in \{0, \dots, r-2\}$ such that $x_{p-j}b \in E(H)$ for a vertex $b \in Y \setminus \{y_1, \dots, y_p\}$ of degree $d_{H_{p,p}}(b) = 1$. Then $d_{H_{p-j-1,p-j-2}}(b) \geq 2$ and hence, $d_{H_{i,i-1}}(y_i) \geq 2$, for $i = 1, \dots, p-j-1$. Thus,

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j-1} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &+ \sum_{i=p-j}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq 3(p-j-1) + 2(j+1) + r = 3p + (r-j-1) > 3p, \end{aligned}$$

because $j \leq r-2$, again a contradiction. $\square$

Lemma 5 *Let $p \geq 2$ be an integer. Let $G = (X, Y)$ be a bipartite graph with $|X| \geq p$ and $|Y| \geq p$, and denote by $H = (X, Y)$ the bipartite complement of G. Suppose that there exists a decreasing sequence of vertices $U = \{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$ of H such that $E(H_{p,p}) = \{ab\}$ with $a \in X$ and $b \in Y$. If $e(H) \leq 3p$ then there exists an (a, b)-path in G with its internal vertices belonging to U.*

Proof: Since $E(H_{p,p}) = \{ab\}$, then $\Delta_X(H_{p,p}) = \Delta_Y(H_{p,p}) = 1$, which implies that $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. If G contains the path a, y_p, x_p, b, then we are done. So assume that some of the edges ay_p, $x_p y_p$, $x_p b$ is an edge of H. We know by Lemma 4 (iii) that $x_p y_p \notin E(H)$. If $ay_p \in E(H)$, then $d_{H_{p-1,p-1}}(a) \geq 2$, because $\{ay_p, ab\} \subset E(H_{p-1,p-1})$. Then $d_{H_{i-1,i-1}}(x_i) \geq 2$ and we get

$$e(H) = \sum_{i=1}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \geq 3p + 1,$$

which is a contradiction. Therefore we can suppose that $x_p b \in E(H)$ and $ay_p \notin E(H)$. Then $\{x_p b, ab\} \subset E(H_{p-1,p-2})$, following that $d_{H_{p-1,p-2}}(b) \geq 2$, which implies that $d_{H_{i,i-1}}(y_i) \geq 2$, for $i = 1, \ldots, p-1$. Since $d_{H_{p,p-1}}(y_p) \geq 1$ and $d_{H_{i-1,i-1}}(x_i) \geq 1$ for $i = 1, \ldots, p$, it follows that

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-1} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &+ \big(d_{H_{p-1,p-1}}(x_p) + d_{H_{p,p-1}}(y_p)\big) + e(H_{p,p}) \\ &\geq 3(p-1) + 2 + 1 = 3p. \end{aligned}$$

This means that all the above inequalities become equalities, that is,

$$\begin{cases} d_{H_{i,i-1}}(y_i) = 2, \text{ for } i = 1, \ldots, p-1, \text{ and } d_{H_{p,p-1}}(y_p) = 1; \\ d_{H_{i-1,i-1}}(x_i) = 1, \text{ for } i = 1, \ldots, p. \end{cases} \tag{1}$$

Therefore we obtain that:

- $x_p y_{p-1} \notin E(H)$, because otherwise, $\{x_p b, x_p y_{p-1}\} \subset E(H_{p-2,p-2})$ and thus, $d_{H_{p-2,p-2}}(x_{p-1}) = \Delta_X(H_{p-1,p-1}) \geq 2$, contradicting (1).
- $x_{p-1} b \notin E(H)$, for if not, $\{x_{p-1} b, x_p b, ab\} \subset E(H_{p-2,p-3})$ and hence, $d_{H_{p-2,p-3}}(y_{p-2}) = \Delta_Y(H_{p-2,p-3}) \geq 3$, against (1).

- $x_{p-1}y_{p-1} \notin E(H)$, because otherwise, $d_{H_{p-2,p-3}}(y_{p-1}) \geq 3$ and therefore, $d_{H_{p-2,p-3}}(y_{p-2}) \geq 3$, contradicting (1).

Thus, it follows that $\{ay_p, x_py_p, x_py_{p-1}, x_{p-1}y_{p-1}, x_{p-1}b\} \cap E(H) = \emptyset$. Consequently, there exists in G the path $a, y_p, x_p, y_{p-1}, x_{p-1}, b$, hence the result holds. □

Lemma 6 *Let m, n, p be integers such that $p \geq 2$, $m > p$ and $n > p$. Let $G = (X, Y)$ be a bipartite graph with $|X| = m$ and $|Y| = n$, and denote by $H = (X, Y)$ the bipartite complement of G. If $e(H) \leq 3p$, then $K_{(m-p,n-p)}$ is a topological minor of G.*

Proof: Let $U = \{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$ be a decreasing sequence of H. The graph $H_{p,p}$ is a bipartite graph with vertex classes $X^* = X \setminus \{x_1, \ldots, x_p\}$ and $Y^* = Y \setminus \{y_1, \ldots, y_p\}$, so $|X^*| = m - p$ and $|Y^*| = n - p$. If $e(H_{p,p}) = 0$ then the bipartite complement of $H_{p,p}$ is $K_{(m-p,n-p)}$ and the result follows. We may henceforth assume that $e(H_{p,p}) > 0$, or in other words $\Delta_X(H_{p,p}) \geq 1$ and $\Delta_Y(H_{p,p}) \geq 1$, thus $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. Then by Lemma 4 we have $e(H_{p,p}) = r \leq p$ and $\Delta(H_{p,p}) = 1$. Let us denote the edges of $H_{p,p}$ by $e_1 = a_1b_1, \ldots, e_r = a_rb_r$, $a_i \in X^*$ and $b_i \in Y^*$, for $i = 1, \ldots, r$. In order to prove that G contains $TK_{(m-p,n-p)}$ with set of branch vertices $X^* \cup Y^*$, we will show the existence of vertex disjoint (a_i, b_i)-paths in G, $i = 1, \ldots, r$, with internal vertices from U. As $e(H) \leq 3p$, if $r = 1$ then the bipartite complement of $H_{p,p}$ is $K_{m-p,n-p} - e_1$. Thus, by Lemma 5, the bipartite graph G contains $TK_{m-p,n-p}$ and we are done. Hence assume that $2 \leq r \leq p$, then by Lemma 4 (iii), (iv), (v), for each $i = 1, \ldots, r$ and $j = 0, \ldots, r-2$, there exists in G the path $a_i, y_{p-j}, x_{p-j}, b_i$. Thus, we only must show that there exists $i \in \{1, \ldots, r\}$ such that the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ is contained in G. Otherwise, since $x_{p-(r-1)}y_{p-(r-1)} \in E(G)$ and $a_iy_{p-(r-1)} \in E(G)$ for all $i = 1, \ldots, r$, because of Lemma 4, we deduce that $x_{p-(r-1)}b_i \in E(H)$ for all $i = 1, \ldots, r$, that is, $d_{H_{p-r,p-r}}(x_{p-(r-1)}) \geq r$ and therefore, $d_{H_{i-1,i-1}}(x_i) \geq r$

for $i = 1, \ldots, p-(r-1)$. Then since $2 \le r \le p$ it follows that

$$\begin{aligned}
e(H) &= \sum_{i=1}^{p-(r-1)} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\
&+ \sum_{i=p-(r-2)}^{p} \big(d_{H_{i-1,i-1}}(i_k) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\
&\ge (r+1)(p-(r-1)) + 2(r-1) + r \\
&= 3p + 1 + (r-2)(p-r+1) \\
&> 3p
\end{aligned}$$

which is a contradiction. Hence there exists $i \in \{1, \ldots, r\}$ such that the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ is contained in G. Without loss of generality we may assume that $i = r$. Then there exist in G the vertex-disjoint paths $a_j, y_{p-(j-1)}, x_{p-(j-1)}, b_j$ for $j = 1, \ldots, r$. Thus, G contains $TK_{(m-p,n-p)}$ and this finishes the proof. $\square$

The following lemma gives a sufficient condition on the size of a bipartite graph in order to contain a complete bipartite graph as a topological minor.

Lemma 7 *Let m, n, s, t be integers such that $2 \le m - s \le n - t$. Let $G = (X, Y)$ be a bipartite graph with $|X| = m$, $|Y| = n$. If the bipartite complement H of G has size $e(H) \le 2(m-s) + n - t$, then $K_{(s,t)}$ is a topological minor of G.*

Proof: Set $p = m - s$ and $q = n - t$, then $2 \le p \le q$ and $e(H) \le 2p + q$. First, suppose that $p = q$. Thus the bipartite graph H has size at most $3p$, and by Lemma 6, we obtain that $K_{(m-p,n-p)} = K_{(s,t)}$ is a topological minor of G. Hence, assume that $p < q$. Without loss of generality, we may assume that the vertices of the partite set Y are ordered in such a way that $d_H(y_1) \ge d_H(y_2) \ge \cdots \ge d_H(y_n)$. Set $Y' = \{y_1, \ldots, y_{q-p}\} \subseteq Y$ and let us consider the bipartite graph $H' = (X, Y \setminus Y')$. Observe that $|X| = m$ and $|Y \setminus Y'| = n - (q-p) = t + p$. If $e(H') = 0$ then the bipartite complement G' of H' is the complete bipartite graph $K_{(m,t+p)}$. Since G' is a subgraph of G and $K_{(s,t)} \subseteq K_{(m,t+p)}$, then G contains a $K_{(s,t)}$ and we are done. So, we may assume that $e(H') > 0$, which implies that $d_H(y_i) \ge 1$ for $i = 1, \ldots, q-p$. Hence, $e(H') = e(H) - \sum_{i=1}^{q-p} d_H(y_i) \le 2p + q - (q-p) \le 3p$, and therefore, from Lemma 6, it follows that $K_{(m-p,t+p-p)} = K_{(s,t)}$ is a topological minor of G. $\square$

Combining Proposition 3 and Lemma 7 the following theorem is immediate.

Theorem 8 *Let m, n, s, t be integers such that $2 \le s \le t$, $2 \le m-s \le n-t$, and $m+n \le 2s+t-1$. Then*

$$tz(m, n; s, t) = mn - (2(m-s) + n - t + 1).$$

3 Family of extremal graphs

When an extremal problem is studied, it is not only important to know the exact value of the extremal function, but also characterize the family of extremal graphs. In this section we characterize the extremal family $TZ(m, n; s, t)$ for integers m, n, s, t such that $2 \le s \le t$, $2 \le m-s \le n-t$, and $m+n \le 2s+t-1$.

Lemma 9 *Let $p \ge 2$ be an integer and let $G = (X, Y)$ be a bipartite graph with $|X| \ge p$ and $|Y| \ge p$, and denote by $H = (X, Y)$ the bipartite complement of G. Let $\{x_1, y_1, x_2, y_2, \dots, x_p, y_p\}$ be any decreasing sequence of H and denote by $r = e(H_{p,p})$. If $e(H) \le 3p+1$ and $\Delta_X(H) \ge 2$ then*

(i) *$r \le p$.*

(ii) *$\Delta(H_{p,p}) \le 1$.*

(iii) *If $r = 1$ then $\{x_{p-(r-1)}y_{p-(r-1)}, \dots, x_p y_p\} \cap E(H) = \emptyset$.*

(iv) *If $r \ge 2$ then $\{ay_{p-(r-2)}, \dots, ay_p\} \cap E(H) = \emptyset$, for each $a \in X \setminus \{x_1, \dots, x_p\}$ of degree $d_{H_{p,p}}(a) = 1$, if any.*

(v) *If $r \ge 2$ then $\{x_{p-(r-2)}b, \dots, x_p b\} \cap E(H) = \emptyset$, for each $b \in Y \setminus \{y_1, \dots, y_p\}$ of degree $d_{H_{p,p}}(b) = 1$, if any.*

Proof: If $e(H_{p,p}) = r = 0$, then both items (i) and (ii) hold. Hence we may assume that $0 < r = e(H_{p,p}) \le 3p+1$, which implies $\Delta_X(H_{p,p}) \ge 1$, $\Delta_Y(H_{p,p}) \ge 1$, following that $d_{H_{i-1,i-1}}(x_i) \ge 1$ for $i = 2, \dots, p$ and

$d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. Moreover, $d_{H_{0,0}}(x_1) \geq 2$, because $\Delta_X(H) \geq 2$. Therefore

$$\begin{aligned} e(H_{p,p}) &= e(H) - \big(d_{H_{0,0}}(x_1) + d_{H_{1,0}}(y_1)\big) \\ &- \sum_{i=2}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &\leq 3p + 1 - 3 - 2(p-1) = p, \end{aligned}$$

thus item (i) is proved.

If $\Delta_X(H_{p,p}) \geq 2$, then $e(H_{p,p}) \geq 2$ and $d_{H_{i-1,i-1}}(x_i) \geq 2$ for each $i = 1, \ldots, p$, hence,

$$\begin{aligned} e(H) &= \sum_{i=1}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq 3p + e(H_{p,p}) \geq 3p + 2 > 3p + 1, \end{aligned}$$

which is a contradiction. Analogously, we arrive at a contradiction if $\Delta_Y(H_{p,p}) \geq 2$. Thus, $\Delta_X(H_{p,p}) = \Delta_Y(H_{p,p}) = 1$, which implies $\Delta(H_{p,p}) = 1$, hence item (ii) is shown.

(iii) From item (i) it follows that $r \leq p$. Let us denote the edges of $H_{p,p}$ by $e_1 = a_1b_1, \ldots, e_r = a_rb_r$, $a_i \in X \setminus \{x_1, \ldots, x_p\}$ and $b_i \in Y \setminus \{y_1, \ldots, y_p\}$, for $i = 1, \ldots, r$. Since $e(H_{p,p}) = r \geq 1$, then $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. We reason by way of contradiction supposing that there exists $j \in \{0, \ldots, r-1\}$ such that $x_{p-j}y_{p-j} \in E(H)$. Then $d_{H_{i,i-1}}(y_i) \geq 2$ for $i = 1, \ldots, p-j-1$, because $d_{H_{p-j,p-j-1}}(y_{p-j}) \geq 1$ and $x_{p-j}y_{p-j} \in E(H)$. We have two cases:

Case 1. Assume that $d_{H_{p-j,p-j-1}}(y_{p-j}) \geq 2$, then $d_{H_{i,i-1}}(y_i) \geq 2$ for $i = 1, \ldots, p-j$. Since $d_{H_{0,0}}(x_1) = \Delta_X(H) \geq 2$ and $j \leq r-1$ it follows that

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &+ \sum_{i=p-(j-1)}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq 4 + 3(p-j-1) + 2j + r \\ &= 3p + 1 + (r-j) \\ &> 3p + 1, \end{aligned}$$

which is a contradiction.

Case 2. Assume that $d_{H_{p-j,p-j-1}}(y_{p-j}) = 1$, then $d_{H_{i,i-1}}(y_i) = 1$, for $i = p-j, \dots, p$. Moreover, $d_{H_{p,p}}(b_i) = 1$, for $i = 1, \dots, r$, because $\Delta(H_{p,p}) = 1$. As $\{x_1, y_1, x_2, y_2, \dots, x_p, y_p\}$ is a decreasing sequence of H and $x_{p-j}y_{p-j} \in E(H)$, it follows that x_{p-j} is adjacent in H to each one of the vertices of the set $\{y_{p-j}, \dots, y_p, b_1, \dots, b_r\}$ because of point (iii) of Definition 1. That is, $d_{H_{p-j-1,p-j-1}}(x_{p-j}) \geq j+1+r$, which means that $d_{H_{i-1,i-1}}(x_i) \geq j+1+r$ for $i = 1, \dots, p-j$. If $j = 0$ then $d_{H_{i-1,i-1}}(x_i) \geq 1+r$ for $i = 1, \dots, p$, and therefore

$$
\begin{aligned}
e(H) &= \sum_{i=1}^{p-1} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i) \right) \\
&+ \left(d_{H_{p-1,p-1}}(x_p) + d_{H_{p,p-1}}(y_p) \right) + e(H_{p,p}) \\
&\geq (3+r)(p-1) + (r+2) + r \\
&= 3p + 1 + r(p+1) - 2 > 3p + 1,
\end{aligned}
$$

because $r \geq 1$ and $p \geq 2$, which is a contradiction. If $j = r-1$ then $d_{H_{i-1,i-1}}(x_i) \geq j+1+r = 2r$ for $i = 1, \dots, p-(r-1)$, and therefore

$$
\begin{aligned}
e(H) &= \sum_{i=1}^{p-r} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i) \right) \\
&+ \left(d_{H_{p-r,p-r}}(x_{p-(r-1)}) + d_{H_{p-(r-1),p-r}}(y_{p-(r-1)}) \right) \\
&+ \sum_{i=p-(r-2)}^{p} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i) \right) + e(H_{p,p}) \\
&\geq (2r+2)(p-r) + (2r+1) + 2(r-1) + r \\
&= 3p + 1 + (2rp - 2r^2 - p + 3r - 2) \\
&\geq 3p + 1 + (p-1) \\
&> 3p + 1,
\end{aligned}
$$

because $p \geq 2$, which also contradicts the hypothesis. Finally, if $1 \leq j \leq r-2$ then $d_{H_{i-1,i-1}}(x_i) \geq j+1+r \geq 3$ for $i = 1, \ldots, p-j$, and therefore

$$\begin{aligned}
e(H) &= \sum_{i=1}^{p-j-1} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) \\
&+ \left(d_{H_{p-j-1,p-j-1}}(x_{p-j}) + d_{H_{p-j,p-j-1}}(y_{p-j})\right) \\
&+ \sum_{i=p-(j-1)}^{p} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) + e(H_{p,p}) \\
&\geq 5(p-j-1) + 4 + 2j + r \\
&= 3p + 1 + (2p - 3j - 2 + r) \\
&\geq 3p + 1 + (3r - 3j - 2) > 3p + 1,
\end{aligned}$$

because $p \geq r$ and $j \leq r-2$, again a contradiction.

Thus $x_{p-j}y_{p-j} \notin E(H)$ for all $j \in \{0, \ldots, r-1\}$, hence item (iii) is valid.

(iv) Assume $e(H_{p,p}) = r \geq 2$. Then $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. We reason by way of contradiction supposing that there exists $j \in \{0, \ldots, r-2\}$ such that $ay_{p-j} \in E(H)$ for a vertex $a \in X \setminus \{x_1, \ldots, x_p\}$ of degree $d_{H_{p,p}}(a) = 1$. Then $d_{H_{p-j-1,p-j-1}}(a) \geq 2$ and hence, $d_{H_{i-1,i-1}}(x_i) \geq 2$, for $i = 1, \ldots, p-j$. Thus,

$$\begin{aligned}
e(H) &= \sum_{i=1}^{p-j} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) \\
&+ \sum_{i=p-(j-1)}^{p} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) + e(H_{p,p}) \\
&\geq 3(p-j) + 2j + r = 3p + (r-j) > 3p + 1,
\end{aligned}$$

because $j \leq r-2$, against the hypothesis.

(v) Assume $e(H_{p,p}) = r \geq 2$. Then $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. Moreover, $d_{H_{0,0}}(x_1) \geq 2$, due to the fact that $\Delta_X(H) \geq 2$. We reason by way of contradiction supposing that there exists $j \in \{0, \ldots, r-2\}$ such that $x_{p-j}b \in E(H)$ for a vertex $b \in Y \setminus \{y_1, \ldots, y_p\}$ of degree $d_{H_{p,p}}(b) = 1$. Then $d_{H_{p-j-1,p-j-2}}(b) \geq 2$ and hence, $d_{H_{i,i-1}}(y_i) \geq 2$, for

$i = 1, \dots, p - j - 1$. Thus,

$$\begin{aligned} e(H) &= (d_{H_{0,0}}(x_1) + d_{H_{1,0}}(y_1)) + \sum_{i=2}^{p-j-1} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &+ \sum_{i=p-j}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq 4 + 3(p-j-2) + 2(j+1) + r = 3p + (r-j) > 3p+1, \end{aligned}$$

because $j \leq r-2$, again a contradiction. □

Lemma 10 *Let $p \geq 4$ be an integer. Let $G = (X, Y)$ be a bipartite graph with $|X| \geq p$ and $|Y| \geq p$, and denote by $H = (X, Y)$ the bipartite complement of G. Suppose that $\Delta_X(H) \geq 2$ and there exists a decreasing sequence of vertices $U = \{x_1, y_1, x_2, y_2, \dots, x_p, y_p\}$ of H such that $E(H_{p,p}) = \{ab\}$ with $a \in X$ and $b \in Y$. If $e(H) \leq 3p + 1$ then there exists an (a, b)-path in G with its internal vertices belonging to U.*

Proof: Assume that $e(H) \leq 3p + 1$. Note that $d_{H_{0,0}}(x_1) = \Delta_X(H) \geq 2$. Since $E(H_{p,p}) = \{ab\}$, then $\Delta_X(H_{p,p}) = \Delta_Y(H_{p,p}) = 1$, which implies that $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \dots, p$. If G contains the path a, y_p, x_p, b, then we are done. So assume that some of the edges ay_p, x_py_p, x_pb is an edge of H. We know by Lemma 9 that $x_py_p \notin E(H)$. So, let us distinguish two cases.

Case 1. Suppose that $ay_p \in E(H)$. Then $d_{H_{p-1,p-1}}(a) \geq 2$, because $ab \in E(H)$. Then $d_{H_{i-1,i-1}}(x_i) \geq 2$ and we get

$$e(H) = \sum_{i=1}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \geq 3p + 1 \geq e(H).$$

Thus, all the inequalities become equalities, that is,

$$d_{H_{i-1,i-1}}(x_i) = 2 \text{ and } d_{H_{i,i-1}}(y_i) = 1, \text{ for } i = 1, \dots, p. \tag{2}$$

Hence, we obtain that:

- $x_{p-1}y_{p-1} \notin E(H)$. Otherwise, since

  $$\Delta_Y(H_{p-1,p-2}) = d_{H_{p-1,p-2}}(y_{p-1}) = 1$$

  and both y_p and b have also degree 1 in $H_{p-1,p-2}$, applying point (iii) of Definition 1, it follows that $\{x_{p-1}y_{p-1}, x_{p-1}y_p, x_{p-1}b\} \subset E(H)$ and therefore, $d_{H_{p-2,p-2}}(x_{p-1}) \geq 3$, which contradicts (2).

- $ay_{p-1} \notin E(H)$, because otherwise,

$$d_{H_{p-2,p-2}}(x_{p-1}) = \Delta_X(H_{p-2,p-2}) \geq d_{H_{p-2,p-2}}(a) \geq 3,$$

  contradicting (2).

- $x_{p-1}b \notin E(H)$, for if not,

$$d_{H_{p-2,p-3}}(y_{p-2}) = \Delta_Y(H_{p-2,p-3}) \geq d_{H_{p-2,p-3}}(b) \geq 2,$$

  against (2).

As a consequence, we get that the path a, y_{p-1}, x_{p-1}, b of G connects the vertices a and b.

Case 2. Suppose that $x_pb \in E(H)$ and $ay_p \notin E(H)$. Thus, $d_{H_{p-1,p-2}}(y_p) \geq 2$, which implies that $d_{H_{i,i-1}}(y_i) \geq 2$, for $i = 1, \ldots, p-1$. Since $d_{H_{p,p-1}}(y_p) \geq 1$, $d_{H_{0,0}}(x_1) \geq 2$ and $d_{H_{i-1,i-1}}(x_i) \geq 1$ for $i = 2, \ldots, p$, it follows that

$$\begin{aligned} e(H) &= (d_{H_{0,0}}(x_1) + d_{H_{1,0}}(y_1)) + \sum_{i=2}^{p-1} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) \\ &+ (d_{H_{p-1,p-1}}(x_p) + d_{H_{p,p-1}}(y_p)) + e(H_{p,p}) \\ &\geq 4 + 3(p-2) + 2 + 1 = 3p + 1 = e(H), \end{aligned}$$

which means that all the above inequalities become equalities, that is,

$$\begin{cases} d_{H_{0,0}}(x_1) = 2 \text{ and } d_{H_{i-1,i-1}}(x_i) = 1 \text{ for } i = 2, \ldots, p; \\ d_{H_{i,i-1}}(y_i) = 2 \text{ for } i = 1, \ldots, p-1, \text{ and } d_{H_{p,p-1}}(y_p) = 1. \end{cases} \tag{3}$$

Therefore, we have:

- $x_{p-1}b \notin E(H)$, because on the contrary,

$$d_{H_{p-2,p-3}}(y_{p-2}) = \Delta_Y(H_{p-2,p-3}) \geq d_{H_{p-2,p-3}}(b) \geq 3$$

  against (3).

- $x_py_{p-1} \notin E(H)$, for if not,

$$d_{H_{p-3,p-3}}(x_{p-2}) = \Delta_X(H_{p-3,p-3}) \geq d_{p-3,p-3}(x_p) \geq 2$$

  and this contradicts (3), since $p \geq 4$.

- $x_{p-1}y_{p-1} \notin E(H)$, because otherwise, taking into account that $d_{H_{p-1,p-2}}(y_{p-1}) = 2$, we have

$$d_{H_{p-2,p-3}}(y_{p-2}) = \Delta_Y(H_{p-2,p-3}) \geq d_{H_{p-2,p-3}}(y_{p-1}) \geq 3,$$

contradicting (3).

Thus, in this case, it follows that $\{ay_p, x_py_p, x_py_{p-1}, x_{p-1}y_{p-1}, x_{p-1}b\} \cap E(H) = \emptyset$. Consequently, there exists in G the path $a, y_p, x_p, y_{p-1}, x_{p-1}, b$, and the result also holds in this case. $\square$

Lemma 11 *Let m, n, p be integers such that $p \geq 4$, $m > p$ and $n > p$. Let $G = (X, Y)$ be a bipartite graph with $|X| = m$ and $|Y| = n$, and denote by $H = (X, Y)$ the bipartite complement of G. If $\Delta(H) \geq 2$ and $e(H) \leq 3p+1$, then $K_{(m-p,n-p)}$ is a topological minor of G.*

Proof: Without loss of generality we may assume that $\Delta(H) = \Delta_X(H)$ (otherwise it is enough to interchange the classes X with Y). Let $U = \{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$ be a decreasing sequence of H. The graph $H_{p,p}$ is a bipartite graph with vertex classes $X^* = X \setminus \{x_1, \ldots, x_p\}$ and $Y^* = Y \setminus \{y_1, \ldots, y_p\}$, so $|X^*| = m - p$ and $|Y^*| = n - p$. If $e(H_{p,p}) = 0$ then the bipartite complement of $H_{p,p}$ is $K_{(m-p,n-p)}$ and the result follows. So, we may henceforth assume that $e(H_{p,p}) \geq 1$ or in other words, $\Delta_X(H_{p,p}) \geq 1$ and $\Delta_Y(H_{p,p}) \geq 1$, thus $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. Then by Lemma 9 we have $e(H_{p,p}) = r \leq p$ and $\Delta(H_{p,p}) = 1$. Let us denote the edges of $H_{p,p}$ by $e_1 = a_1b_1, \ldots, e_r = a_rb_r$, $a_i \in X^*$ and $b_i \in Y^*$, for $i = 1, \ldots, r$. In order to prove that G contains a $TK_{(m-p,n-p)}$ with set of branch vertices $X^* \cup Y^*$, we will show the existence of vertex disjoint (a_i, b_i)-paths in G, $i = 1, \ldots, r$, with internal vertices in U. As $e(H) \leq 3p + 1$, we are done if $r = 1$ by applying Lemma 10, hence assume that $2 \leq r \leq p$.

First, suppose that $2 \leq r \leq p - 1$. Then, by Lemma 9 (iii), (iv), (v), for each $i = 1, \ldots, r$ and $j = 0, \ldots, r - 2$, there exists in G the path $a_i, y_{p-j}, x_{p-j}, b_i$. Thus, we only must show that there exists $i \in \{1, \ldots, r\}$ such that the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ is contained in G. We reason by way of contradiction supposing that for all $i = 1, \ldots, r$ the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ does not exist in G. From Lemma 9 it follows that $x_{p-(r-1)}, y_{p-(r-1)} \in E(G)$, thus $a_iy_{p-(r-1)} \in E(H)$ or $x_{p-(r-1)}b_i \in E(H)$ for each $i = 1, \ldots, r$. We will distinguish three possible cases:

Case 1. Assume that $x_{p-(r-1)}b_i \in E(H)$ for all $i = 1, \ldots, r$, then $d_{H_{p-r,p-r}}(x_{p-(r-1)}) \geq r$ and thus, $d_{H_{j-1,j-1}}(x_j) \geq r$ for $j = 1, \ldots, p-(r-1)$. Moreover, $d_{H_{p-r,p-(r+1)}}(y_{p-r}) = \Delta_Y(H_{p-r,p-(r+1)}) \geq d_{H_{p-r,p-(r+1)}}(b_i) \geq 2$, which means that $d_{H_{j,j-1}}(y_j) \geq 2$ for $j = 1, \ldots, p-r$. Thus,

$$\begin{aligned}
e(H) &= \sum_{j=1}^{p-r} \big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big) \\
&+ \Big(d_{H_{p-r,p-r}}(x_{p-(r-1)}) + d_{H_{p-(r-1),p-r}}(y_{p-(r-1)})\Big) \\
&+ \sum_{j=p-(r-2)}^{p} \big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big) + e(H_{p,p}) \\
&\geq (r+2)(p-r) + (r+1) + 2(r-1) + r \\
&= 3p + 1 + (r-2)(p-r) + p - 2 \\
&> 3p + 1,
\end{aligned}$$

since $2 \leq r < p$ and $p > 2$, which is a contradiction.

Case 2. Assume that $a_i y_{p-(r-1)} \in E(H)$ for all $i = 1, \ldots, r$, then, reasoning as in Case 1, we have $d_{H_{j,j-1}}(y_j) \geq r$ for $j = 1, \ldots, p-(r-1)$, and $d_{H_{j-1,j-1}}(x_j) \geq 2$ for $j = 1, \ldots, p-(r-1)$. Thus,

$$\begin{aligned}
e(H) &= \sum_{j=1}^{p-(r-1)} \big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big) \\
&+ \sum_{j=p-(r-2)}^{p} \big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big) + e(H_{p,p}) \\
&\geq (r+2)(p-(r-1)) + 2(r-1) + r \\
&= 3p + 1 + (r-2)(p-r) + p - 1 \\
&> 3p + 1,
\end{aligned}$$

since $2 \leq r < p$ and $p > 1$, which is a contradiction.

Case 3. Assume that there exist $i_0, j_0 \in \{1, \ldots, r\}$ such that $x_{p-(r-1)}b_{i_0} \notin E(H)$ and $a_{j_0}y_{p-(r-1)} \notin E(H)$. Clearly $i_0 \neq j_0$, because $x_{p-(r-1)}y_{p-(r-1)} \notin E(H)$ (by Lemma 9) and by hypothesis, the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ does not exist in G for all $i = 1, \ldots, r$. Since

$x_{p-(r-1)}y_{p-(r-1)} \notin E(H)$, it follows that $x_{p-(r-1)}b_{j_0} \in E(H)$, for if not, we find in G the path $a_{j_0}, y_{p-(r-1)}, x_{p-(r-1)}, b_{j_0}$ against our assumption. Analogously, $a_{i_0}x_{p-(r-1)} \in E(H)$. Observe that $\{a_{i_0}x_{p-(r-1)}, a_{i_0}b_{i_0}\} \subset E(H_{p-r,p-r})$ and therefore,

$$d_{H_{p-r,p-r}}(x_{p-(r-1)}) = \Delta_X(H_{p-r,p-r}) \geq d_{H_{p-r,p-r}}(a_{i_0}) \geq 2$$

, which implies that $d_{H_{i-1,i-1}}(x_i) \geq 2$ for $i = 1, \ldots, p-(r-1)$. Moreover, observe also that $\{y_{p-(r-1)}b_{j_0}, a_{j_0}b_{j_0}\} \subset E(H_{p-r,p-(r+1)})$ and therefore, $d_{H_{p-r,p-(r+1)}}(y_{p-r}) = \Delta_X(H_{p-r,p-(r+1)}) \geq d_{H_{p-r,p-(r+1)}}(b_{j_0}) \geq 2$, which means that $d_{H_{i,i-1}}(y_i) \geq 2$ for $i = 1, \ldots, p-r$. Hence,

$$\begin{aligned}
e(H) &= \sum_{j=1}^{p-r} \left(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\right) \\
&+ \Big(d_{H_{p-r,p-r}}(x_{p-(r-1)}) + d_{H_{p-(r-1),p-r}}(y_{p-(r-1)})\Big) \\
&+ \sum_{j=p-(r-2)}^{p} \left(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\right) + e(H_{p,p}) \\
&\geq 4(p-r) + 3 + 2(r-1) + r \\
&= 4p + 1 - r \\
&= 3p + 1 + (p-r) \\
&> 3p + 1,
\end{aligned}$$

since $r \leq p-1$. Then, if $2 \leq r \leq p-1$, in all the possible cases, we arrive at a contradiction with the assumption that the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ does not exist in G for all $i = 1, \ldots, r$. Thus, if $2 \leq r \leq p-1$ there exists $i \in \{1, \ldots, r\}$ such that the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ is contained in G. Without loss of generality we may assume that $i = r$. Then there exist in G the vertex-disjoint paths $a_j, y_{p-(j-1)}, x_{p-(j-1)}, b_j$ for $j = 1, \ldots, r$.

Second, assume that $r = p$. Then, from Lemma 9 it follows that

$$\left\{\begin{aligned}
\{x_1y_1, \ldots, x_py_p\} \cap E(H) &= \emptyset; \\
\{a_iy_2, \ldots, a_iy_p\} \cap E(H) &= \emptyset \text{ for } i = 1, \ldots, p; \\
\{x_2b_i, \ldots, x_pb_i\} \cap E(H) &= \emptyset \text{ for } i = 1, \ldots, p.
\end{aligned}\right. \tag{4}$$

This means that for each $i = 1, \ldots, p$ and $j = 0, \ldots, p-2$, there exists in G the path $a_i, y_{p-j}, x_{p-j}, b_i$. Thus, we only must show that there exists $i \in$

$\{1,\dots,p\}$ such that the path a_i, y_1, x_1, b_i is contained in G. We reason by way of contradiction supposing that for all $i = 1,\dots,p$ the path a_i, y_1, x_1, b_i does not exist in G. Since $x_1y_1 \in E(G)$ we deduce that for each $i = 1,\dots,p$, $a_iy_1 \in E(H)$ or $x_1b_i \in E(H)$. If $\{a_iy_1, a_{i^*}y_1\} \subset E(H)$ for two indices $i, i^* \in \{1,\dots,p\}$, with $i \neq i^*$, then $d_{H_{1,0}}(y_1) \geq 2$. Since $d_{H_{0,0}}(x_1) = \Delta_X(H) \geq 2$ we have

$$\begin{aligned} e(H) &= \big(d_{H_{0,0}}(x_1) + d_{H_{1,0}}(y_1)\big) \\ &+ \sum_{j=2}^{p} \big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big) + e(H_{p,p}) \\ &\geq 4 + 2(p-1) + p \\ &= 3p + 2 \\ &> 3p + 1, \end{aligned}$$

a contradiction. Thus, in the set $\{a_1,\dots,a_p\}$ there is at most one vertex adjacent to y_1 in H, which means that x_1 must be adjacent in H to at least $p-1$ vertices of the set $\{b_1,\dots,b_p\}$, due to the fact that for each $i = 1,\dots,p$, $a_iy_1 \in E(H)$ or $x_1b_i \in E(H)$. Then $d_{H_{0,0}}(x_1) \geq p-1$ and therefore,

$$\begin{aligned} e(H) &= \big(d_{H_{0,0}}(x_1) + d_{H_{1,0}}(y_1)\big) \\ &+ \sum_{j=2}^{p} \big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big) + e(H_{p,p}) \\ &\geq p + 2(p-1) + p \\ &= 3p + 1 + (p-3) \\ &> 3p + 1, \end{aligned}$$

since $p \geq 4$, again a contradiction with the hypothesis. Hence, there exists $i \in \{1,\dots,r\}$ such that the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ is contained in G. Without loss of generality we may assume that $i = r$. Then there exist in G the vertex-disjoint paths $a_j, y_{p-(j-1)}, x_{p-(j-1)}, b_j$ for $j = 1,\dots,r$, and the result holds. $\square$

Theorem 12 *Let m, n, s, t be integers such that $2 \le s \le t$, $4 \le m - s \le n - t$, and $m + n \le 2s + t - 1$. Then $G = (X, Y) \in TZ(m, n; s, t)$ iff $G = K_{(m,n)} - M$ where M is any matching of cardinality $2(m-s)+n-t+1$.*

Proof: By Proposition 3 and Theorem 8, if $G = K_{(m,n)} - M$ where M is any matching of cardinality $2(m - s) + n - t + 1$, then $G \in TZ(m, n; s, t)$. Thus, we only must show that there are no more extremal bipartite graphs. For that, it is enough to prove that the bipartite complement $H = (X, Y)$ of every extremal bipartite graph $G = (X, Y) \in TZ(m, n; s, t)$ has maximum degree $\Delta(H) = 1$.

Let $G = (X, Y) \in TZ(m, n; s, t)$ satisfy the hypothesis of the theorem and let us denote by $H = (X, Y)$ the bipartite complement of G. Set $p = m - s$ and $q = n - t$, then $4 \le p \le q$ and $e(H) = 2p + q + 1$. If $p = q$ then $\Delta(H) = 1$, follows from Lemma 11. Thus, assume that $p < q$. Without loss of generality, we may assume that the vertices of the partite set Y are ordered in such a way that $d_H(y_1) \ge d_H(y_2) \ge \cdots \ge d_H(y_n)$. Set $Y' = \{y_1, \ldots, y_{q-p}\} \subseteq Y$ and let us consider the bipartite graph $H' = (X, Y \setminus Y')$. Observe that $|X| = m$ and $|Y \setminus Y'| = n - (q - p) = t + p$. If $e(H') = 0$ then the bipartite complement G' of H' is the complete bipartite graph $K_{(m,t+p)}$. Since G' is a subgraph of G and $K_{(s,t)} \subseteq K_{(m,t+p)}$, then G contains a $K_{(s,t)}$, against the assumption. So, we may assume that $e(H') > 0$, which means that $d_H(y_i) \ge 1$ for $i = 1, \ldots, q - p$. Hence,

$$e(H') = e(H) - \sum_{i=1}^{q-p} d_H(y_i) \le 2p + q + 1 - (q - p) \le 3p + 1. \qquad (5)$$

Then the following facts can be concluded:

- $E(H') = 3p + 1$. Otherwise if $E(H') < 3p + 1$ then, from Lemma 6, it follows that G' contains $TK_{(m-p,n-(q-p)+p)} = TK_{(m-p,n-q)} = TK_{(s,t)}$, but this contradicts the fact that $G \in TZ(m, n; s, t)$.
- $d_H(y_i) = 1$, for $i = 1, \ldots, q - p$, thus $\Delta_Y(H) = 1$, because $\Delta_Y(H) = d_H(y_1)$. This is directly derived because all the inequalities (5) become equalities since $E(H') = 3p + 1$.

Next let us see that $\Delta_X(H) = 1$. Otherwise, there is a vertex $x \in X$ having two distinct neighbors $y, y^* \in N_H(x)$. Since $\Delta_Y(H) = 1$, then $N_H(y) = N_H(y^*) = \{x\}$, and besides, there are exactly $e(H) = 2p+q+1 > q-p+2$

vertices of degree 1 in the class Y. Let us consider the bipartite graph $G^* = (X^*, Y^*)$ whose bipartite complement $H^* = (X^*, Y^*)$ is obtained from H by removing any $q-p$ vertices of $Y \setminus \{y, y^*\}$ of degree 1. The graph H^* satisfies that $|X^*| = |X| = m > p$, $|Y^*| = |Y| - (q-p) = t + p > p$, $e(H^*) = e(H) - (q-p) = 3p+1$. Further, observe that $d_{H^*}(x) \geq 2$, because $\{y, y^*\} \subset Y^*$ and $\{xy, xy^*\} \subset E(H^*)$, which means that $\Delta(H^*) \geq 2$. Then, by applying Lemma 11, the bipartite complement G^* of H^* contains a $TK_{(m-p,t+p-p)} = TK_{(s,t)}$. Since G^* is a subgraph of G, we deduce that G contains $TK_{(s,t)}$, and this contradicts the fact that $G \in TZ(m, n; s, t)$. Hence, $\Delta(H) = \min\{\Delta_X(H), \Delta_Y(H)\} = 1$ and this proves the result. $\square$

Acknowledgement

Research supported by the Ministry of Education and Science, Spain, and the European Regional Development Fund (ERDF) under project MTM2008-06620-C03-02; and under the Andalusian Government project P06-FQM-01649.

References

[1] C. Balbuena, M. Cera, A. Dinez and P. García-Vázquez. New exact values of the maximum size of graphs free of topological complete subgraphs. *Discrete Math.*, 307(9-10):1038–1046, 2007.

[2] C. Balbuena, P. García-Vázquez, X. Marcote and J.C. Valenzuela. New results on the Zarankiewicz problem. *Discrete Math.*, 307(17-18):2322–2327, 2007.

[3] T. Böhme, B. Mohar, R. Škrekovski and M. Stiebitz. Subdivisions of large complete bipartite graphs and long induced paths in k-connected graphs. *J. Graph Theory*, 45:270–274, 2004.

[4] B. Bollobás. *Extremal Graph Theory.* Academic Press, New York, 1978.

[5] R. Diestel. *Graph Theory.* Springer-Verlag Berlin Heildeberg, 2005.

[6] Z. Furëdi. New asymptotics for Bipartite Turán Numbers. *J. Combin. Theory Ser. A*, 75:141–144, 1996.

[7] A.P. Godbole and H.C. Graziano. Contributions to the problem of Zarankiewicz. *J. Statist. Plann. Inference*, 95:197–208, 2001.

[8] A.P. Godbole, B. Lamorte and E.J. Sandquist. Threshold functions for the bipartite Turán property. *Electron. J. Combin.*, 4:R18, 1997.

[9] W. Goddard, M.A. Henning and O.R. Oellermann. Bipartite Ramsey numbers and Zarankiewicz numbers. *Discrete Math.*, 219:85–95, 2000.

[10] J. Griggs and H. Chih-Chang. On the half-half case of the Zarankiewicz problem. *Discrete Math.*, 249:95–104, 2002.

[11] D. Kühn and D. Osthus. Improved bounds for topological cliques in graphs of large girth. *SIAM J. Discrete Math.*, 20(1):62–78, 2006.

[12] D. Kühn and D. Osthus. Induced subdivisions in $K_{s,s}$-free graphs of large average degree. *Combinatorica*, 24(2):287–304, 2004.

[13] D. Kühn and D. Osthus. Topological minors in graphs of large girth. *J. Combin. Theory Ser. B*, 86(2):364–380, 2002.

[14] Y. Liu and G.Y. Yan. Graphs isomorphic to their maximum matching graphs. *Acta Mathematica Sinica, English Series,* 29(9):15071516, 2009.

[15] W. Mader. Graphs with $3n-6$ edges not containing a subdivision of K_5. *Combinatorica*, 25(4):425–438, 2005.

[16] W. Mader. $3n-5$ edges do force a subdivision of K_5. *Combinatorica*, 18(4):569–595, 1998.

[17] J.S. Myers. The extremal function for unbalanced bipartite minors. *Discrete Math.*, 271:209–222, 2003.

[18] M. Simonovits. *Extremal graph theory, Selected topics in graph theory, 2.* Academic Press, London, 1983.

[19] P. Turán. On an extremal problem in graph theory. *Mat. Fiz. Lapok*, 48:436–452, 1941.

[20] K. Zarankiewicz. Problem P 101. *Colloq. Math.*, 2:301, 1951.

On the λ'-optimality of s-geodetic digraphs

Camino Balbuena
Universitat Politècnica de Catalunya
Barcelona

Pedro García-Vázquez
Universidad de Sevilla
Sevilla

Abstract

For a strongly connected digraph D the restricted arc-connectivity $\lambda'(D)$ is defined as the minimum cardinality of an arc-cut over all arc-cuts S satisfying that $D - S$ has a non trivial strong component D_1 such that $D - V(D_1)$ contains an arc. Let S be a subset of vertices of D. We denote by $\omega^+(S)$ the set of arcs uv with $u \in S$ and $v \notin S$, and by $\omega^-(S)$ the set of arcs uv with $u \notin S$ and $v \in S$. A digraph $D = (V, A)$ is said to be λ'-optimal if $\lambda'(D) = \xi'(D)$, where $\xi'(D)$ is the minimum arc-degree of D defined as $\xi(D) = \min\{\xi'(xy) : xy \in A\}$, and $\xi'(xy) = \min\{|\omega^+(\{x,y\})|, |\omega^-(\{x,y\})|, |\omega^+(x) \cup \omega^-(y)|, |\omega^-(x) \cup \omega^+(y)|\}$. In this paper a sufficient condition for a s-geodetic strongly connected digraph D to be λ'-optimal is given in terms of its diameter. Further we see that the h-iterated line digraph $L^h(D)$ of a s-geodetic digraph is λ'-optimal for certain iteration h.

1 Introduction

We consider finite digraphs without loops and multiple edges. Let $D = (V, A)$ be a strongly connected digraph, with vertex set $V = V(D)$ and arc set $A = A(D)$. For any pair F, F' of proper vertex subsets of a digraph D, we define $[F, F'] = \{xy \in A : x \in F, y \in F'\}$. If $F' = \overline{F} = V \setminus F$, we write

$\omega^+(F)$ or $\omega^-(\overline{F})$ instead of $[F, \overline{F}]$. When $F = \{x\}$ we abbreviate $\omega^+(\{x\})$ and $\omega^-(\{x\})$ to $\omega^+(x)$ and $\omega^-(x)$, respectively. Clearly, $d^+(x) = |\omega^+(x)|$ and $d^-(x) = |\omega^-(x)|$.

A subset $S \subseteq A$ of arcs is an arc-cut if $D - S$ is not strongly connected. Each minimum arc-cut has the form $\omega^+(F)$, where F is a proper subset of V. Thus, the *arc-connectivity* of a digraph D can be defined as

$$\lambda(D) = \min\{|\omega^+(F)| : F \subset V,\ F \neq \emptyset,\ F \neq V\}.$$

It is well-known that for any digraph D, $\lambda(D) \le \delta(D)$ [10]. Hence, D is said to be *maximally arc-connected* if $\lambda(D) = \delta(D)$. Following Hamidoune [12, 13], a subset F of vertices of a strongly connected digraph D with arc-connectivity λ is a *positive α-fragment* if $|\omega^+(F)| = \lambda$ and, similarly, F is a *negative α-fragment* if $|\omega^-(F)| = \lambda$. Note that F is a positive α-fragment if an only if $\overline{F} = V(D) \setminus F$ is a negative α-fragment.

When the underlying topology of an interconnection network is modeled by a connected graph or a strongly connected digraph D, where $V(D)$ is the set of processors and $A(D)$ is the set of communication links, the edge-connectivity or arc-connectivity of D are important measurements for fault tolerance of the network. However, one might be interested in more refined indices of reliability. Even two graphs or digraphs with the same edge/arc-connectivity λ may be considered to have different reliabilities, since the number or type of minimum arc-cuts is different.

The study of fault-tolerance of networks modeled by an undirected graph has been intense in recent years. By restricting the forbidden fault set to be the sets of neighboring edges of any spanning subgraph with no more than k-vertices in the faulty networks, Fàbrega and Fiol [9, 8] introduced the k-extra-edge-connectivity of interconnection networks (where k is a positive integer) as follows. Given a graph G and a non-negative integer k, the k-extra-edge-connectivity $\lambda_k(G)$ of G is the minimum cardinality of a set of edges of G, if any, whose deletion disconnects G and every remaining component contains at least k vertices. More information and results on the k-extra-edge-connectivity can be found [3, 6]. The *restricted edge-connectivity* $\lambda'(G)$, introduced by Esfahanian and Hakimi [7] for a graph G, corresponds to the 2-extra-edge-connectivity and it is the minimum cardinality over all *restricted edge-cuts* S, i.e., those such that there are no isolated vertices in $G - S$. A restricted edge-cut S is called a *λ'-cut* if $|S| = \lambda'(G)$. Obviously for any λ'-cut S, the graph $G - S$ consists of exactly two components. A connected graph G is called *λ'-connected* if $\lambda'(G)$ exists. Esfahanian and Hakimi [7] showed that each connected graph G of order $n(G) \geq 4$ except a star, is λ'-connected and satisfies $\lambda'(G) \leq \xi(G)$, where $\xi(G)$ denotes the minimum *edge-degree* of G defined as $\xi(G) = \min\{d(u) + d(v) - 2 : uv \in E(G)\}$. More information and recent results on restricted edge-connectivity of graphs can be found in the survey by Hellwig and Volkmann, [15]. All these concepts of the extraconnectivity and restricted connectivity were inspired by the definition of conditional connectivity introduced by Harary [14] who asked for the minimum cardinality of a set of edges of G, if any, whose deletion disconnects G such that every remaining component satisfies some prescribed property.

Volkmann [17] extended the notion of *restricted edge-connectivity* to digraphs. Given a strongly connected digraph D, an arc set S of D is a *restricted arc-cut* of D if $D-S$ has a non-trivial strong component D_1 such that $D-V(D_1)$ contains an arc. The restricted arc-connectivity $\lambda'(D)$ is defined as the minimum cardinality over all restricted arc-cuts S. A strongly connected digraph D is called λ'-connected if $\lambda'(D)$ exists. A restricted arc-cut S is called a *λ'-cut* if $|S| = \lambda'(D)$. In the same paper, Volkmann proved that each strong digraph D of order $n \geq 4$ and girth $g = 2$ or $g = 3$ except some families of digraphs is λ'-connected and satisfies $\lambda(D) \leq \lambda'(D) \leq \xi(D)$, where $\xi(D)$ is defined as follows. If $C_g = u_1 u_2 \ldots u_g u_1$ is a shortest cycle of D, then $\xi(C_g) = \min\{|\omega^+(C_g)|, |\omega^-(C_g)|\}$, and $\xi(D) = \min\{\xi(C_g) : C_g \text{ is a shortest cycle of } D\}$.

More recently, Wang and Lin [18] have focused in studying the λ'-optimal digraphs by considering the notion of *arc-degree*. For any arc $xy \in A(D)$, the arc-degree of xy is defined as

$$\xi'(xy) = \min\{|\omega^+(\{x,y\})|, |\omega^-(\{x,y\})|, |\omega^+(x) \cup \omega^-(y)|, |\omega^-(x) \cup \omega^+(y)|\}.$$

The minimum arc-degree of D is $\xi'(D) = \min\{\xi'(xy) : xy \in A(D)\}$. Similar to the definition of λ'-optimal graphs, in [18] a λ'-connected digraph D is called λ'-optimal if $\lambda'(D) = \xi'(D)$. In the aforementioned paper [18], Wang and Lin proved the following useful theorem.

Theorem A [18] *Let D be a strongly connected digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. Then D is λ'-connected and $\lambda'(D) \leq \xi'(D)$.*

Starting from this result, Wang and Lin introduced the notion of λ'-*optimality* to denote the digraphs D for which $\lambda'(D) = \xi'(D)$. Then, they provided an example of a digraph having a λ'-cut which can not be written as $\omega^+(F)$ for any proper subset $F \subset V(D)$. And further in the same paper they proved that if D has no minimum restricted arc-cut of the form $\omega^+(F)$ where F is a proper subset of $V(D)$, then D is λ'-optimal.

In this paper we prove that every λ'-cut S of a λ'-connected digraph D with cardinality $|S| < \xi'(D)$ is necessarily of the form $S = \omega^+(F)$. Furthermore, both induced subdigraphs $D[F]$ and $D[\overline{F}]$ of D are shown to have an arc. These structural results allows us to give a sufficient condition for a s-geodetic digraph to have $\lambda'(D) = \xi'(D)$, i.e. to be λ'-optimal. A digraph D with diameter $diam(D)$ is said to be s-geodetic if for any two (not necessarily different) vertices $x, y \in V$, there exists at most one $x \to y$ path of length at most s. Obviously, if $d(x, y) \leq s$ then there exists exactly one such path. Note that $1 \leq s \leq g - 1 \leq diam(D)$, where $g \geq 2$ is the girth of D. Our interest is in the maximum integer s for which D is s-geodetic. If $s = diam(D)$, the digraph D is called strongly geodetic [16]. In this reference it was proved that all strongly geodetic digraphs are either complete digraphs or cycles.

Sufficient conditions for a s-geodetic digraph with minimum degree δ to be maximally arc connected have been given in terms of its diameter $diam(D)$ and the parameter s. In this regard, the following result is contained in [4]:

$$\lambda = \delta \text{ if } diam(D) \leq 2s.$$

The k-extra-connectivity was studied for s-geodetic digraphs in [2]. In this work we prove that $\lambda'(D) = \xi'(D)$ if $diam(D) \leq 2s - 1$, and we also show that D is λ'-optimal if $diam(D) = 2s$ when D satisfies an additional hypothesis. Furthermore, we see that the h-iterated line digraph $L^h(D)$ of a s-geodetic digraph is λ'-optimal for certain iteration h.

2 Results

Following the book by Harary (see [11], pg. 199), each vertex of a digraph is in exactly one strong component and an arc lies in one strong component depending on whether or not it is in some cycle. It follows from the maximality of strong components that the strong components of a digraph D can be labeled $D_1, \ldots, D_k$ such that there is no arc from D_j to D_i unless $j < i$. Such an ordering is called an *acyclic ordering* of the strong components of D. In order to obtain our main result we require the following lemma.

Lemma 1 *Let $D=(V,A)$ be a λ'-connected digraph and S a λ'-cut of D such that $|S|<\xi'(D)$. Then the set V can be partitioned into two subsets, F, $\overline{F}$ such that $S=\omega^+(F)=\omega^-(\overline{F})$ and both induced subdigraphs $D[F]$ and $D[\overline{F}]$ of D contain an arc.*

Proof: Let S be a λ'-cut of D and let $D_1,\dots,D_k$, $k\geq 2$, be an acyclic ordering of the strong components of $D-S$. Since S is a restricted arc-cut some strong component D_j of $D-S$ must be non trivial, i.e. $|V(D_j)|\geq 2$ and $D-V(D_j)$ contains an arc. Suppose that D_j is the unique non-trivial strong component of $D-S$. As $D-V(D_j)$ contains an arc yz, then $k\geq 3$. If $j=1$ then by considering $F=\cup_{i=2}^{k}V(D_i)$ and $\overline{F}=V(D_1)$, it follows that $\omega^+(F)$ is a restricted arc-cut of D. Since $\omega^+(F)\subseteq S$ and S is a λ'-cut, then $\omega^+(F)=S$ and clearly both induced subdigraphs $D[F]$ and $D[\overline{F}]$ of D contain an arc. The prof is analogous if $j=k$, hence, assume that $2\leq j\leq k-1$. If $\{y,z\}\subseteq\cup_{i=1}^{j-1}V(D_i)$ then it is enough to consider $F=\cup_{i=j}^{k}V(D_i)$ and $\overline{F}=\cup_{i=1}^{j-1}V(D_i)$ and clearly $S=\omega^+(F)$ and both induced subdigraphs $D[F]$ and $D[\overline{F}]$ of D contain an arc. The prof is also analogous if $\{y,z\}\subseteq\cup_{i=j+1}^{k}V(D_i)$. Thus, we may assume that $y\in\cup_{i=1}^{j-1}V(D_i)$ and $z\in\cup_{i=j+1}^{k}V(D_i)$, yielding that $\omega^+(z)\cup\omega^-(y)\subseteq S$ or is there the previous situation for another arc. Clearly $\omega^+(z)\cup\omega^-(y)$ is a restricted arc-cut of D because D_j is a strong component of $D-(\omega^+(z)\cup\omega^-(y))$ and the arc yz belongs to $D-D_j$. Then $\omega^+(z)\cup\omega^-(y)=S$ and hence $\xi'(D)\leq|\omega^+(z)\cup\omega^-(y)|=|S|$ which is a contradiction with the hypothesis. Therefore $D-S$ has at least two distinct non-trivial strong components D_t and D_j, meaning that $D[\cup_{i=1}^{j-1}V(D_i)]$ contains an arc or $D[\cup_{i=j+1}^{k}V(D_i)]$ contains an arc. In the former case let $F=\cup_{i=j}^{k}V(D_i)$ and $\overline{F}=\cup_{i=1}^{j-1}V(D_i)$. Since there is no arc from F to $\overline{F}$ in $D-S$, then $\omega^+(F)=S$ and we are done because clearly both $D[F]$ and $D[\overline{F}]$ contain an arc. Similarly if $D[\cup_{i=j+1}^{k}V(D_i)]$ contains an arc, then $F=\cup_{i=j+1}^{k}V(D_i)$ and $\overline{F}=\cup_{i=1}^{j}V(D_i)$ satisfy the lemma. □

We will henceforth denote the set of arcs $\omega^+(F)$ by $[X,\overline{X}]$, where $X\subseteq F$ and $\overline{X}\subseteq\overline{F}$ are, respectively, the sets of out and in vertices of the arcs of $\omega^+(F)$.

The following remark is immediate from the definition of s-geodetic digraphs.

If uv is an arc of a s-geodetic digraph D with $s \geq 2$, then $N_i^+(u) \cap N_i^+(v) = \emptyset$ and $N_{i+1}^+(u) \cap N_i^+(v) = \emptyset$ for all $i = 1, \ldots, s-1$.

Some properties on the s-geodetic digraphs not being λ'-optimal are provided in the following results.

Lemma 2 *Let D be a λ'-connected s-geodetic digraph and $\omega^+(F) = [X, \overline{X}]$ a λ'-cut. If $\lambda'(D) < \xi'(D)$ then there exists some vertex $u \in F$ such that $d(u, X) \geq s-1$ and there exists some vertex $\overline{u} \in \overline{F}$ such that $d(\overline{X}, \overline{u}) \geq s-1$.*

Proof: When $s = 1$ the assertion is obvious, hence assume $s \geq 2$. Let us denote by $\mu = \max\{d(u, X) : u \in F\}$. We reason by way of contradiction by supposing $\mu \leq s-2$. First assume that $\mu = 0$. This implies that every vertex of F is an initial of some arc of $[X, \overline{X}]$, that is $F = X$. By Lemma 1, we can consider an arc uv in $D[F]$ and since $N^+(u) \cap N^+(v) = \emptyset$ because $s \geq 2$, then

$$\begin{aligned}
\lambda'(D) = \left|[X, \overline{X}]\right| &\geq |[\{u, v\}, \overline{X}]| + \left|[(N^+(u) - v) \cap X, \overline{X}]\right| \\
&\quad + \left|[(N^+(v) - u) \cap X, \overline{X}]\right| \\
&\geq |N^+(u) - v| + |N^+(v) - u| \\
&= |\omega^+(\{u, v\})| \geq \xi'(uv) \geq \xi'(D),
\end{aligned}$$

which is a contradiction. Hence, assume that $1 \leq \mu \leq s-2$, which means that $s \geq 3$.

Case 1: There exists an arc uv in $D[F]$ such that $d(u, X) = d(v, X) = \mu$.

Let us denote by $A_u = (N^+(u) - v) \cap N_\mu^-(X)$, $A_v = (N^+(v) - u) \cap N_\mu^-(X)$, $B_u = N^+(u) \cap N_{\mu-1}^-(X)$ and $B_v = N^+(v) \cap N_{\mu-1}^-(X)$ and observe that $N^+(u) - v = A_u \cup B_u$ and $N^+(v) - u = A_v \cup B_v$. It is clear by Remark 1 that $N^+(u) \cap N^+(v) = \emptyset$ because $s \geq 3$, and therefore, the sets A_u, A_v, B_u, B_v are pairwise disjoint. Let us see that the sets $N_\mu^+(A_u) \cap X$, $N_\mu^+(A_v) \cap X$, $N_\mu^+(u) \cap X$ and $N_\mu^+(v) \cap X$ are pairwise disjoint. Note that every vertex x belonging to any of the previous sets is at distance at most $\mu + 2 \leq s$ from u. Hence, the existence of some vertex x belonging to two of these sets implies the existence of two paths $u \to x$ of length at most s, which contradicts the hypothesis that D is s-geodetic. The same argument justifies that $|N_\mu^+(A_u) \cap X| \geq |A_u|$, $|N_\mu^+(A_v) \cap X| \geq |A_v|$, $|N_\mu^+(u) \cap X| \geq |N^+(u) \cap N_{\mu-1}^-(X)| = |B_u|$ and $|N_\mu^+(v) \cap X| \geq |N^+(v) \cap N_{\mu-1}^-(X)| = |B_v|$, since D is s-geodetic. Hence,

$$\begin{aligned}
\lambda'(D) = \left|[X,\overline{X}]\right| \geq |X| &\geq |N^+_\mu(A_u)\cap X| + |N^+_\mu(A_v)\cap X| \\
&\quad +|N^+_\mu(u)\cap X| + |N^+_\mu(v)\cap X| \\
&\geq |A_u| + |A_v| + |B_u| + |B_v| \\
&= |N^+(u)-v| + |N^+(v)-u| \\
&= |\omega^+(\{u,v\})| \geq \xi'(uv) \geq \xi'(D),
\end{aligned}$$

against the fact that $\lambda'(D) < \xi'(D)$.

Case 2: There is no arc uv in $D[F]$ such that $d(u,X) = d(v,X) = \mu$.

Let $u \in N^-_\mu(X)$ and take any $v \in N^+(u)\cap N^-_{\mu-1}(X)$. Let us denote by $A = (N^+(v)-u)\cap N^-_\mu(X)$, $B = N^+(v)\cap N^-_{\mu-1}(X)$ and $C = (N^+(A)-v)\cap N^-_{\mu-1}(X)$. As $s \geq 3$ the girth of D is at least 4, and thus it is clear by Remark 1 that the sets $N^+(u)$, B, C are pairwise disjoint. Since $s \geq 3$ and the induced subdigraph $D[N^-_\mu(X)\cap F]$ contains no arc, then $|C| \geq |A|$. Let us see that the sets $N^+_{\mu-1}(N^+(u))\cap X$, $N^+_{\mu-1}(B)\cap X$ and $N^+_{\mu-1}(C)\cap X$ are pairwise disjoint. Note that every vertex x belonging to any of these sets is at distance at most $\mu+2 \leq s$ from u. Hence, the existence of some vertex x belonging to two of these sets implies the existence of two paths of length at most s from u to x, which contradicts the hypothesis that D is s-geodetic. The same argument justifies that $|[N^+_{\mu-1}(N^+(u))\cap X,\overline{X}]| \geq |N^+(u)-v| + (|N^+(v)-u| - |B| - |A|)$, $|[N^+_{\mu-1}(B)\cap X,\overline{X}]| \geq |N^+_{\mu-1}(B)\cap X| \geq |B|$ and $|[N^+_{\mu-1}(C)\cap X,\overline{X}]| \geq |N^+_{\mu-1}(C)\cap X| \geq |C| \geq |A|$, since D is s-geodetic. Hence,

$$\begin{aligned}
\lambda'(D) = \left|[X,\overline{X}]\right| &\geq |[N^+_{\mu-1}(N^+(u))\cap X,\overline{X}]| + |[N^+_{\mu-1}(B)\cap X,\overline{X}]| \\
&\quad +|[N^+_{\mu-1}(C)\cap X,\overline{X}]| \\
&\geq |N^+(u)-v| + (|N^+(v)-u| - |B| - |A|) + |B| + |A| \\
&= |N^+(u)-v| + |N^+(v)-u| \\
&= |\omega^+(\{u,v\})| \geq \xi'(uv) \geq \xi'(D),
\end{aligned}$$

against the fact that $\lambda'(D) < \xi'(D)$.

Thus, $\mu \geq s-1$. The prof of $\overline{\mu} \geq s-1$, being $\overline{\mu} = \max\{d(\overline{X}, \overline{u}) : \overline{u} \in \overline{F}\}$ is analogous. So the result holds. $\square$

Lemma 3 *Let $D = (V, A)$ be a λ'-connected s-geodetic digraph and $\omega^+(F) = [X, \overline{X}]$ a λ'-cut such that $\max\{d(u, X) : u \in F\} = s-1$ and $\max\{d(\overline{X}, \overline{u}) : \overline{u} \in F\} = s-1$. If $\lambda'(D) < \xi'(D)$ then the following assertions hold:*

(i) *The induced subdigraphs $D[N^-_{s-1}(X) \cap F]$ and $D[N^+_{s-1}(\overline{X}) \cap \overline{F}]$ contain some arc.*

(ii) *There exist $u_0 \in N^-_{s-1}(X) \cap F$, $\overline{u}_0 \in N^+_{s-1}(\overline{X}) \cap \overline{F}$ such that $|[N^+_{s-1}(u_0) \cap X, \overline{X}]| = 1$ and $|[X, N^-_{s-1}(\overline{u}_0) \cap \overline{X}]| = 1$.*

Proof: (i) Clearly, the result holds if $s = 1$, because of Lemma 1. Therefore, assume that $s \geq 2$ and reason by way of contradiction supposing that $D[N^-_{s-1}(X) \cap F]$ has no arc. Then every vertex $u \in N^-_{s-1}(X) \cap F$ satisfies that $N^+(u) \cap N^-_{s-1}(X) = \emptyset$ which means that $d(u, X) = s-1$, and $|N^+_{s-1}(u) \cap X| \geq d^+(u)$ because D is s-geodetic. Let us consider a vertex $u \in N^-_{s-1}(X) \cap F$ such that $d^+(u) \leq d^+(u')$ for all $u' \in N^-_{s-1}(X) \cap F$. Take any vertex $v \in N^+(u)$ and let us consider the subsets of F, $A = (N^+(v) - u) \cap N^-_{s-1}(X)$, $B = N^+(v) \cap N^-_{s-2}(X)$. It is clear by Remark 1 and due to $s \geq 2$ that the sets $N^+(u) - v$, A, B are pairwise disjoint. Moreover, since $2 \leq s \leq g-1$ where g is the girth of D, then there is no symmetric arc in D, yielding that $v \notin N^+(a)$ for all $a \in A$. As $d^+(a) \geq d^+(u)$ and $D[N^-_{s-1}(X) \cap F]$ has no arc, then $|(N^+(a) \setminus N^+(u)) \cap N^-_{s-2}(X)| \geq 1$ for all $a \in A$, hence the set $C = (N^+(A) \setminus N^+(u)) \cap N^-_{s-2}(X)$ satisfies that $|C| \geq |A|$. Let us see that the sets $N^+_{s-2}(N^+(u) - v) \cap X$, $N^+_{s-2}(B) \cap X$, $N^+_{s-2}(C) \cap X$ and $N^+_{s-2}(v) \cap X$ are pairwise disjoint. Note that every vertex x belonging to any of these sets is at distance at most $s-2+2 = s$ from v. Hence, if some vertex x belongs to two of these sets, then two distinct directed paths of length at most s from v to x exist, which contradicts the hypothesis that D is s-geodetic. The same argument justifies that $|[N^+_{s-2}(N^+(u) - v) \cap X, \overline{X}]| \geq |N^+(u) - v|$, $|[N^+_{s-2}(B) \cap X, \overline{X}]| \geq |N^+_{s-2}(B) \cap X| \geq |B|$, $|[N^+_{s-2}(C) \cap X, \overline{X}]| \geq |N^+_{s-2}(C) \cap X| \geq |C| \geq |A|$ and $|[N^+_{s-2}(v) \cap X, \overline{X}]| \geq |N^+_{s-2}(v) \cap X| \geq |N^+(v) - u| - |A| - |B|$, since D is

s-geodetic. Hence,

$$\begin{aligned}
\lambda'(D) = \big|[X,\overline{X}]\big| &\geq |[N^+_{s-2}(N^+(u)-v)\cap X,\overline{X}]| + |[N^+_{s-2}(B)\cap X,\overline{X}]| \\
&\quad +|[N^+_{s-2}(C)\cap X,\overline{X}]| + |[N^+_{s-2}(v)\cap X,\overline{X}]| \\
&\geq |N^+(u)-v| + |B| + |A| + |N^+(v)-u| - |A| - |B| \\
&= |N^+(u)-v| + |N^+(v)-u| \\
&= |\omega^+(\{u,v\})| \geq \xi'(uv) \geq \xi'(D),
\end{aligned}$$

against the fact that $\lambda'(D) < \xi'(D)$. Thus, $D[N^-_{s-1}(X)\cap F]$ must contain some arc. Analogously it is proved that $D[N^+_{s-1}(\overline{X})\cap\overline{F}]$ contains some arc.

(ii) First assume that $s=1$, which means that $F=X$. Let uv be an arc of $D[F]$ and suppose that $|[\{z\},\overline{X}]| \geq 2$ for all $z\in X$. Then

$$\begin{aligned}
\lambda'(D) &= |[X,\overline{X}]| \\
&\geq |[\{u\},\overline{X}]| + |[\{v\},\overline{X}]| + |[((N^+(u)\cup N^+(v))\setminus\{u,v\})\cap X,\overline{X}]| \\
&\geq |[\{u\},\overline{X}]| + |[\{v\},\overline{X}]| + 2|\,((N^+(u)\cup N^+(v))\setminus\{u,v\})\cap X| \\
&\geq |[\{u\},\overline{X}]| + |[\{v\},\overline{X}]| + |(N^+(u)-v)\cap X| \\
&\qquad\qquad\qquad\qquad\qquad + |(N^+(v)-u)\cap X| \\
&\geq \xi'(uv) \geq \xi'(D),
\end{aligned}$$

which is a contradiction. Hence assume that $s\geq 2$. Let $u\in F\cap(N^-_{s-1}(X)\cup N^-_{s-2}(X))$ be such that $|N^+(u)\cap N^-_{s-1}(X)|\geq 1$ is maximum in $F\cap(N^-_{s-1}(X)\cup N^-_{s-2}(X))$. Two cases need to be distinguished:

Case 1. Assume that $u\in N^-_{s-1}(X)\cap F$. Take any $v\in N^+(u)\cap N^-_{s-2}(X)$ and denote by $U=N^+(u)\cap N^-_{s-1}(X)$ and $W=(N^+(v)-u)\cap N^-_{s-1}(X)$. Since $2\leq s\leq g-1$ then D has no symmetric arc, hence $W=N^+(v)\cap N^-_{s-1}(X)$. Observe that $|U|\geq|W|$ because the way that vertex u has been selected. Further notice that $|U|+|W|\geq 1$ for if not, $\lambda'(D)=|[X,\overline{X}]|\geq |X|\geq |N^+_{s-1}(u)\cap X|+|N^+_{s-1}(v)\cap X|\geq |N^+(u)-v|+|N^+(v)-u|\geq\xi'(uv)\geq \xi'(D)$ and this is a contradiction.

Suppose that $|[N^+_{s-1}(z)\cap X,\overline{X}]|\geq 2$ for all $z\in U\cup W$. Since D is s-geodetic, then the sets $N^+_{s-1}(u)\cap X$, $N^+_{s-2}\left(N^+(v)\cap N^-_{s-2}(X)\right)\right)\cap X$ and $N^+_{s-1}(U)\cap X$ are pairwise disjoint. Furthermore, the inequalities

$$\begin{aligned}
|[N^+_{s-1}(u)\cap X,\overline{X}]| &\geq |N^+_{s-1}(u)\cap X|\\
&= |N^+_{s-2}(N^+(u)-v)\cap X| + |N^+_{s-2}(v)\cap X|\\
&\geq |N^+(u)-v| - |U| + |N^+(v)-u|\\
&\quad -|N^+(v)\cap N^-_{s-2}(X)| - |W|,
\end{aligned}$$

$$\begin{aligned}
|[N^+_{s-2}\left(N^+(v)\cap N^-_{s-2}(X)\right)\cap X,\overline{X}]| &\geq |N^+_{s-2}\left(N^+(v)\cap N^-_{s-2}(X)\right)\cap X|\\
&\geq |N^+(v)\cap N^-_{s-2}(X)|
\end{aligned}$$

and

$$|[N^+_{s-1}(U)\cap X,\overline{X}]| \geq 2|U| \geq |U|+|W|$$

hold. Hence,

$$\begin{aligned}
\lambda'(D) &\geq |X|\\
&\geq |[N^+_{s-1}(u)\cap X,\overline{X}]| + |[N^+_{s-2}\left(N^+(v)\cap N^-_{s-2}(X)\right)\cap X,\overline{X}]|\\
&\quad +|[N^+_{s-1}(U)\cap X,\overline{X}]|\\
&\geq |N^+(u)-v| - |U| + |N^+(v)-u| - |N^+(v)\cap N^-_{s-2}(X)| - |W|\\
&\quad +|N^+(v)\cap N^-_{s-2}(X)| + |U| + |W|\\
&= |N^+(u)-v| + |N^+(v)-u|\\
&= |\omega^+(\{u,v\})| \geq \xi'(uv) \geq \xi'(D),
\end{aligned}$$

a contradiction. Then there must exists a vertex $u_0 \in U\cup W$ such that $|[N^+_{s-1}(u_0)\cap X,\overline{X}]| = 1$.

Case 2. Assume that $u\in N^-_{s-2}(X)\cap F$. Note that $|N^+(u)\cap N^-_{s-1}(X)| \geq |N^+(v)\cap N^-_{s-1}(X)|+1$ for all $v\in N^-_{s-1}(X)\cap F$ may be assumed because if for some $v\in N^-_{s-1}(X)\cap F$, $|N^+(u)\cap N^-_{s-1}(X)| = |N^+(v)\cap N^-_{s-1}(X)|$ the result follows from Case 1. Take any $v\in N^+(u)\cap N^-_{s-1}(X)$ and denote by $U=(N^+(u)-v)\cap N^-_{s-1}(X)$ and $W=N^+(v)\cap N^-_{s-1}(X)$ and observe that $u\notin W$ because $u\in N^-_{s-2}(X)\cap F$. Observe that $|U| = |(N^+(u)-v)\cap N^-_{s-1}(X)| = |N^+(u)\cap N^-_{s-1}(X)|-1 \geq |W|$ because the way that vertex u has been selected. Further notice that $|U|+|W|\geq 1$ because otherwise

$|N^+(u)\cap N^-_{s-1}(X)|=1$ and $|N^+(v)\cap N^-_{s-1}(X)|=0$ for all $v\in N^-_{s-1}(X)\cap F$ yielding that the subdigraph $D[N^-_{s-1}(X)\cap F]$ has no arc, which contradicts item (i).

As in the above case suppose that $|[N^+_{s-1}(z)\cap X,\overline{X}]|\geq 2$ for all $z\in U\cup W$. The sets $N^+_{s-2}(u)\cap X$, $N^+_{s-2}\left(N^+(u)\cap N^-_{s-2}(X)\right)\cap X$, $N^+_{s-1}(U)\cap X$ and $N^+_{s-1}(v)\cap X$ are pairwise disjoint, since D is s-geodetic. Furthermore, the inequalities

$$|[N^+_{s-2}(u)\cap X,\overline{X}]|\geq|N^+_{s-2}(u)\cap X|\geq|N^+(u)-v|-|U|-|N^+(u)\cap N^-_{s-2}(X)|,$$

$$\begin{aligned}|[N^+_{s-2}\left(N^+(u)\cap N^-_{s-2}(X)\right)\cap X,\overline{X}]| &\geq |N^+_{s-2}\left(N^+(u)\cap N^-_{s-2}(X)\right)\cap X|\\ &\geq |N^+(u)\cap N^-_{s-2}(X)|,\end{aligned}$$

$$|[N^+_{s-1}(U)\cap X,\overline{X}]|\geq 2|U|\geq|U|+|W|$$

and

$$|[N^+_{s-1}(v)\cap X,\overline{X}]|\geq|N^+_{s-1}(v)\cap X|\geq|N^+(v)-u|-|W|$$

hold. Hence,

$$\begin{aligned}\lambda'(D)\geq|X| &\geq |[N^+_{s-2}(u)\cap X,\overline{X}]|\\ &\quad+|[N^+_{s-2}\left(N^+(u)\cap N^-_{s-2}(X)\right)\cap X,\overline{X}]|\\ &\quad+|[N^+_{s-1}(U)\cap X,\overline{X}]|+|[N^+_{s-1}(v)\cap X,\overline{X}]|\\ &\geq |N^+(u)-v|-|U|-|N^+(u)\cap N^-_{s-2}(X)|\\ &\quad+|N^+(u)\cap N^-_{s-2}(X)|+|U|+|W|+|N^+(v)-u|-|W|\\ &= |N^+(u)-v|+|N^+(v)-u|\\ &= |\omega^+(\{u,v\})|\geq\xi'(uv)\geq\xi'(D),\end{aligned}$$

again a contradiction. Then there must exists a vertex $u_0\in U\cup W$ such that $|[N^+_{s-1}(u_0)\cap X,\overline{X}]|=1$.

The prof of the existence of a vertex $\overline{u}_0\in\overline{F}$ such that $|[X,N^-_{s-1}(\overline{u})\cap\overline{X}]|=1$ is analogous. $\Box$

As a consequence of all the above previous result, a sufficient condition for a s-geodetic digraph to be λ'-optimal is given in the following theorem.

Theorem 4 *Let D be a strongly connected s-geodetic digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. Then D is λ'-optimal if the diameter is $diam(D) \leq 2s-1$.*

Proof: From Theorem A it follows that D is λ'-connected and $\lambda'(D) \leq \xi'(D)$. Suppose that D is non λ'-optimal and let S be a λ'-cut. Then from Lemma 1 it follows that $S = \omega^+(F) = [X, \overline{X}]$. Moreover, from Lemma 2 there is a vertex $u_0 \in F$ such that $d(u_0, X) \geq s-1$ and there is a vertex $\overline{u}_0 \in \overline{F}$ such that $d(\overline{X}, \overline{u}_0) \geq s-1$. Hence $diam(D) \geq d(u_0, X)+1+d(\overline{X}, \overline{u}_0) \geq 2s-1$, which is a contradiction unless $diam(D) = 2s-1$. In this case, all the former inequalities become equalities, that is, $\max\{d(u,X) : u \in F\} = \max\{d(\overline{X}, \overline{u}) : \overline{u} \in F\} = s-1$. Then by Lemma 3 we may assume that $|[N^+_{s-1}(u_0) \cap X, \overline{X}]| = 1$ and $|[X, N^-_{s-1}(\overline{u}_0) \cap \overline{X}]| = 1$. Let us denote by $[N^+_{s-1}(u_0) \cap X, \overline{X}] = [x_0, \overline{x}_0]$, for some $x_0 \in X$, $\overline{x}_0 \in \overline{X}$; and let us denote by $[X, N^-_{s-1}(\overline{u}_0) \cap \overline{X}] = [y_0, \overline{y}_0]$, for some $y_0 \in X$, $\overline{y}_0 \in \overline{X}$.

From $d(u_0, \overline{u}_0) = 2s-1$, it follows that $x_0 = y_0$ and $\overline{x}_0 = \overline{y}_0$. Notice also that $|N^+(u_0) \cap N^-_{s-1}(X)| \geq d^+(u_0) - 1$ because $|N^+(u_0) \cap N^-_{s-2}(X)| \leq |N^+_{s-1}(u_0) \cap X| = 1$; analogously $|N^-(\overline{u}_0) \cap N^+_{s-1}(X)| \geq d^-(\overline{u}_0) - 1$. First, suppose $\delta^+(D) \geq 3$, then there exists a vertex $v \in N^+(u_0) \cap N^-_{s-1}(X)$. Observe that $d(v, \overline{u}_0) = 2s-1$, yielding that $|[N^+_{s-1}(v) \cap X, \overline{x}_0]| \geq 1$, that is, $x_0 \in N^+_{s-1}(v) \cap X$. Therefore the shortest $u_0 \to x_0$ path together with the arc u_0v and the shortest $v \to x_0$ path are two distinct $u_0 \to x_0$ directed paths of length at most s, which is a contradiction. A similar contradiction is obtained supposing $\delta^-(D) \geq 3$. Hence, D is λ'-optimal. $\square$

Our next goal is to study sufficient conditions for λ'-optimality in s-geodetic digraphs of diameter $diam(D) = 2s$.

Theorem 5 *Let D be a strongly connected s-geodetic digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$ and diameter $diam(D) = 2s$. Then D is λ'-optimal if $|N^+_s(u) \cap N^-_s(v)| \geq 3$ for all pair u, v of vertices at distance $d(u,v) = 2s$.*

Proof: From Theorem A it follows that D is λ'-connected and $\lambda'(D) \leq \xi'(D)$. Let S be a λ'-cut of D and suppose that D is non λ'-optimal, that is, $|S| < \xi'(D)$. A contradiction will be obtained by proving the existence of two vertices $u, v \in V(D)$ such that $d(u,v) = 2s$ and $|N^+_s(u) \cap N^-_s(v)| < 3$. From Lemma 1 it follows that $S = \omega^+(F) = [X, \overline{X}]$. Let us denote by $\mu = \max\{d(u,X) : u \in F\}$ and $\overline{\mu} = \max\{d(\overline{X}, \overline{u}) : \overline{u} \in \overline{F}\}$. From Lemma 2 it follows that $\mu \geq s-1$ and $\overline{\mu} \geq s-1$. If $\mu + \overline{\mu} \geq 2s$ then it is enough to take two vertices u (at distance μ to X) and $\overline{u}$ (at distance $\overline{\mu}$ from $\overline{X}$), yielding that $2s = diam(D) \geq d(u,\overline{u}) \geq d(u,X)+1+d(\overline{X},\overline{u}) = \mu+\overline{\mu}+1 \geq 2s+1$, which is a contradiction, hence, $\mu = s-1$ or $\overline{\mu} = s-1$.

First assume that $\mu = s-1$ and $\overline{\mu} = s$. By Lemma 3, there exists a vertex $u_0 \in N^-_{s-1}(X) \cap F$ such that $[N^+_{s-1}(u_0) \cap X, \overline{X}] = \{x\overline{x}\}$. Given any vertex $\overline{u} \in \overline{F}$ at distance $\overline{\mu} = s$ from $\overline{X}$, we have $2s = diam(D) \geq d(u_0,\overline{u}) \geq d(u_0,X)+1+d(\overline{X},\overline{u}) = s-1+1+s = 2s$, following that $d(u_0,\overline{u}) = 2s$. Notice that $N^+_s(u_0) \cap \overline{F} = \{\overline{x}\}$ whereas $F \cap N^-_s(\overline{u}) = \emptyset$, since $d(\overline{X},\overline{u}) = s$. Hence $N^+_s(u) \cap N^-_s(\overline{u}) = \{\overline{x}\}$ which contradicts the hypothesis that $|N^+_s(u) \cap N^-_s(v)| \geq 3$ for all pair u, v of vertices at distance $d(u,v) = 2s$.

Second assume that $\mu = s-1$ and $\overline{\mu} = s-1$. By Lemma 3, there exists a vertex $u_0 \in N^-_{s-1}(X) \cap F$ such that $[N^+_{s-1}(u_0) \cap X, \overline{X}] = \{x\overline{x}\}$, and there is a vertex $\overline{u}_0 \in N^+_{s-1}(\overline{X}) \cap \overline{F}$ such that $[X, N^-_{s-1}(\overline{u}_0) \cap \overline{X}] = \{y\overline{y}\}$. Notice that $N^+_s(u_0) \cap \overline{F} = \{\overline{x}\}$ and $N^-_s(\overline{u}_0) \cap F = \{y\}$. Then

$$N^+_s(u_0) \cap N^-_s(\overline{u}_0) \subseteq \left(N^+_s(u_0) \cap \overline{F}\right) \cup \left(N^-_s(\overline{u}_0) \cap F\right) = \{\overline{x}, y\}$$

and therefore, $|N^+_s(u_0) \cap N^-_s(\overline{u}_0)| < 3$, again a contradiction.

Hence, D is λ'-optimal and the result holds. $\square$

We recall here that in the line digraph $L(D)$ of a digraph D, each vertex represents an arc of D. Thus, $V(L(D)) = \{uv : (u,v) \in A(D)\}$; and a vertex uv is adjacent to a vertex xz if and only if $v = x$, that is, when the arc (u,v) is adjacent to the arc (x,z) in D. For any $h \geq 1$ the *h-iterated line digraph*, $L^h(D)$, is defined recursively by $L^h(D) = L(L^{h-1}(D))$. From the definition it follows that the minimum degrees $\delta(L(D)) = \delta(D) = \delta$. Moreover, the diameter of any strongly connected digraph other than a directed cycle [1] satisfies

$$diam(L^h(D)) = diam(D) + h. \tag{1}$$

Moreover, if D is s-geodetic then $L^h(D)$ is s'-geodetic with $s' = \min\{s+h, g-1\}$, where g denotes the girth of D [5].

Theorem 6 *Let D be a strongly connected s-geodetic digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$ and girth $g \geq s+1$. Then $L^{g-1-s}(D)$ is λ'-optimal if $diam(D) \leq g+s-2$.*

Proof: Observe that the iterated line digraph $L^{g-1-s}(D)$ is s'-geodetic with $s' = s+g-1-s = g-1$. From (1) and the hypothesis $diam(D) \leq$

$g + s - 2$ it follows that

$$diam(L^{g-1-s}(D)) = diam(D) + g - 1 - s \leq 2(g-1) - 1 = 2s' - 1.$$

Hence the result follows directly from Theorem 4. $\square$

Acknowledgement

Research supported by the Ministry of Education and Science, Spain, and the European Regional Development Fund (ERDF) under project MTM2008-06620-C03-02 and under the Andalusian Government project P06-FQM-01649.

References

[1] M. Aigner. On the linegraph of a directed graph. *Math. Z.*, 102:56–61, 1967.

[2] C. Balbuena. Extraconnectivity of s-geodetic digraphs and graphs. *Discrete Math.*, 195:39–52, 1999.

[3] C. Balbuena, A. Carmona, J. Fàbrega and M.A. Fiol. Extraconnectivity of graphs with large minimum degree and girth. *Discrete Math.*, 167/168:85–100, 1997.

[4] M.C. Balbuena, A. Carmona, J. Fàbrega and M.A. Fiol. On the order and size of s-geodetic digraphs with given connectivity. *Discrete Math.*, 174:19–27, 1997.

[5] M.C. Balbuena, A. Carmona and M.A. Fiol. Distance connectivity in graphs and digraphs. *J. Graph Theory*, 22(4):281–292, 1996.

[6] C. Balbuena, M. Cera, A. Diánez, P. García-Vázquez and X. Marcote. Diameter-girth sufficient conditions for optimal extraconnectivity in graphs. *Discrete Math.*, 308:3526–3536, 2008.

[7] A.H. Esfahanian and S.L. Hakimi. On computing a conditional edge-connectivity of a graph. *Inform. Process. Lett.*, 27:195–199, 1988.

[8] J. Fàbrega and M.A. Fiol. On the extraconnectivity of graphs. *Discrete Math.*, 155:49–57, 1996.

[9] J. Fàbrega and M.A. Fiol. Extraconnectivity of graphs with large girth. *Discrete Math.*, 127:163–170, 1994.

[10] D. Geller and F. Harary. Connectivity in digraphs. *Lecture Notes in Math.*, 186:105–114, 1970.

[11] F. Harary. *Graph Theory,* 3rd edition. Addison-Wesley Series in Mathematics, 1972.

[12] Y.O. Hamidoune. Sur les atomes d'un graphe orienté, abéliens. *Discrete Math.*, 55:323–326, 1985.

[13] Y.O. Hamidoune. A property of α-fragments of a digraph. *Discrete Math.*, 31:105–106, 1980.

[14] F. Harary. Conditional connectivity. *Networks*, 13:347–357, 1983.

[15] A. Hellwig and L. Volkmann. Maximally edge-connected and vertex-connected graphs and digraphs: a survey. *Discrete Math.*, 308(15):3265–3296, 2008.

[16] J. Plesník and S. Znám. Strongly geodetic directed graphs. *Acta Fac. Rerum Natur. Univ. Comenian., Math. Publ.*, 29:29-34, 1974.

[17] L. Volkmann. Restricted arc-connectivity of digraphs. *Inform. Process. Lett.*, 103:234–239, 2007.

[18] S. Wang and S. Lin. λ'-optimal digraphs. *Inform. Process. Lett.*, 108:386–389, 2008.

On the connectivity and restricted edge-connectivity of 3-arc graphs

Camino Balbuena and Luis Pedro Montejano
Universitat Politècnica de Catalunya
Barcelona

Pedro García-Vázquez
Universidad de Sevilla
Sevilla

Abstract

A $3-arc$ of a graph G is a 4-tuple (y,a,b,x) of vertices such that both (y,a,b) and (a,b,x) are paths of length two in G. Let $\overleftrightarrow{G}$ denote the symmetric digraph of a graph G. The *3-arc graph* $X(G)$ of a given graph G is defined to have vertices the arcs of $\overleftrightarrow{G}$. Two vertices (ay), (bx) are adjacent in $X(G)$ if and only if (y,a,b,x) is a 3-arc of G. The purpose of this work is to study the edge-connectivity and restricted edge-connectivity of 3-arc graphs. We prove that the 3-arc graph $X(G)$ of every connected graph G of minimum degree $\delta(G) \geq 3$ has edge-connectivity $\lambda(X(G)) \geq (\delta(G)-1)^2$; and restricted edge- connectivity $\lambda_{(2)}(X(G)) \geq 2(\delta(G)-1)^2 - 2$ if $\kappa(G) \geq 2$. We also provide examples showing that all these bounds are sharp.

1 Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow [10] for terminology and definitions.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For every $v \in V(G)$, $N_G(v)$ denotes the neighborhood of v, that is, the set of all vertices adjacent to v. The *degree* of a vertex v is $d(v) = |N_G(v)|$ and the minimum degree $\delta = \delta(G)$ of the graph G is the minimum degree over all vertices of G.

A graph G is called *connected* if every pair of vertices is joined by a path. If $S \subset V(G)$ and $G - S$ is not connected, then S is said to be a *cutset*. A *component* of a graph G is a maximal connected subgraph of G. A (noncomplete) connected graph is called *k-connected* if every cutset has cardinality at least k. The *connectivity* $\kappa(G)$ of a (noncomplete) connected graph G is defined as the maximum integer k such that G is k-connected. The *minimum* cutsets are those having cardinality $\kappa(G)$. The *connectivity* of a complete graph $K_{\delta+1}$ on $\delta + 1$ vertices is defined as $\kappa(K_{\delta+1}) = \delta$. Analogously, for edge connectivity an *edge-cut* in a graph G is a set W of edges of G such that $G - W$ is nonconnected. If W is a minimum edge-cut of a connected graph G, then $G - W$ contains exactly two components. Every connected graph on at least two vertices has an edge-cut. The *edge-connectivity* $\lambda(G)$ of a graph G is the minimum cardinality of an edge-cut of G. A classic result due to Whitney is that for every graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$. A graph is *maximally connected* if $\kappa(G) = \delta(G)$, and *maximally edge-connected* if $\lambda(G) = \delta(G)$.

Though the parameters κ, λ of connectivities give the minimum cost to disrupt the network, they do not take into account what remains after deletion. Even two graphs with the same connectivity κ, λ may be considered to have different reliabilities, since the number of minimum cutsets or edge-cuts is different. Superconnectivity is a stronger measure of connectivity, introduced by Boesch and Tindell in [8], whose study has deserved some attention in the last years, see for instance, [1, 6, 7, 19, 20]. A maximally connected [edge-connected] graph is called *super-κ* [*super-λ*] if for every cutset [edge-cut] W of cardinality $\delta(G)$ there exists a component C of $G - W$ of cardinality $|V(C)| = 1$. The study of super-κ [super-λ] graphs has a particular significance in the design of reliable networks, mainly due to the fact that attaining superconnectivity implies minimizing the number of minimum cutsets [edge-cuts] (see [7, 20]).

In order to measure the super edge-connectivity we use the following parameter introduced by Esfahanian and Hakimi [11]. The *restricted edge-connectivity* $\lambda_{(2)} = \lambda_{(2)}(G)$ is the minimum cardinality over all *restricted edge-cuts* W, i.e., those such that there are no isolated vertices in $G - W$. A restricted edge-cut W is called a *$\lambda_{(2)}$-cut* if $|W| = \lambda_{(2)}$. Obviously for any $\lambda_{(2)}$-cut W, the graph $G - W$ consists of exactly two components

$C, \overline{C}$ and clearly $|V(C)| \geq 2, |V(\overline{C})| \geq 2$. A connected graph G is called $\lambda_{(2)}$-*connected* if $\lambda_{(2)}$ exists. Esfahanian and Hakimi [11] showed that each connected graph G of order $n(G) \geq 4$ except a star, is $\lambda_{(2)}$-connected and satisfies $\lambda_{(2)} \leq \xi$, where $\xi = \xi(G)$ denotes the *minimum edge-degree* of G defined as $\xi(G) = \min\{d(u) + d(v) - 2 : uv \in E(G)\}$. Furthermore, a $\lambda_{(2)}$-connected graph is said to be $\lambda_{(2)}$-*optimal* if $\lambda_{(2)} = \xi$. Recent results on this property are obtained in [2, 5, 12, 13, 18, 21, 23]. Notice that if $\lambda_{(2)} \leq \delta$, then $\lambda_{(2)} = \lambda$. When $\lambda_{(2)} > \delta$ (that is to say, when every edge cut of order δ isolates a vertex) the graph must be super-λ. Therefore, by means of this parameter we can say that a graph G is super-λ if and only if $\lambda_{(2)} > \delta$. Thus, we can measure the super edge-connectivity of the graph as the value of the restricted edge-connectivity $\lambda_{(2)}$.

Let $\overleftrightarrow{G}$ denote the symmetric digraph of a graph G. For adjacent vertices u, v of $V(G)$ we use (u, v) to denote the arc from u to v, and $(v, u)(\neq (u, v))$ to denote the arc from v to u. A *3-arc* is a 4-tuple (y, a, b, x) of vertices such that both (y, a, b) and (a, b, x) are paths of length two in G. The *3-arc graph* $X(G)$ of a given graph G is defined to have vertices the arcs of $\overleftrightarrow{G}$ and they are denoted as (uv). Two vertices $(ay), (bx)$ are adjacent in $X(G)$ if and only if (y, a, b, x) is a 3-arc of G, see [17, 22]. Equivalently, two vertices $(ax), (by)$ are adjacent in $X(G)$ if and only if $d_G(a, b) = 1$; that is, the tails a, b of the arcs $(a, x), (b, y) \in A(\overleftrightarrow{G})$ are at distance one in G. Thus the number of edges of $X(G)$ is $\sum_{uv \in E(G)}(d(u) - 1)(d(v) - 1)$ so that the minimum degree of $X(G)$ is $(\delta(G) - 1)^2$. There is a bijection between the edges of $X(G)$ and those of the 2-path graph $P_2(G)$, which is defined to have vertices the paths of length two in G such that two vertices are adjacent if and only if the union of the corresponding paths is a path or a cycle of length three, see [9]. Since $P_2(G)$ is a spanning subgraph of the second iterated line graph $L_2(G) = L(L(G))$ (see e.g. [14]), we have a relation between 3-arc graphs and line graphs. Some results on the connectivity of P_2-path graphs are studied e.g. in [3, 4, 15].

The purpose of this paper is to study the edge-connectivity, the restricted edge-connectivity and vertex-connectivity of the 3-arc graph $X(G)$ of a given graph G. The following theorem gather together the results on connectivity of 3-arc-graph $X(G)$ obtained by Knor and Zhou [16].

Theorem 1 [16] *Let G be a graph with minimum degree $\delta(G)$.*

(i) *$X(G)$ is connected if G is connected and $\delta(G) \geq 3$.*

(ii) $\kappa(X(G)) \geq (\kappa(G)-1)^2$ *if* $\kappa(G) \geq 3$.

The main results contained in this paper are the following:
Let G be a connected graph with minimum degree $\delta(G) \geq 3$.

(i) $\lambda(X(G)) \geq (\delta(G)-1)^2$.

(ii) $\lambda_{(2)}(X(G)) \geq 2(\delta(G)-1)^2 - 2$ *if* $\kappa(G) \geq 2$.

(iii) $\kappa(X(G)) \geq \min\{\kappa(G)(\delta(G)-1), (\delta(G)-1)^2\}$.

(iv) $X(G)$ *is super-κ if* $\kappa(G) = \delta(G)$ *and* $\delta(X(G)) = (\delta(G)-1)^2$.

2 Results on the edge-connectivity and restricted edge-connectivity of 3-arc graphs

Let $X(G)$ be the 3-arc graph of a graph G. If (ay) and (bx) are adjacent in $X(G)$ then the edge $(ay)(bx)$ will be called an *ab-edge* (or *ba*-edge). Observe that $(ay)(bx) = (bx)(ay)$ but $(ay) \neq (ya)$ and $(bx) \neq (xb)$. For any edge $ab \in E(G)$ let $\mathcal{V}^a_{ab} = \{(ay) \in V(X(G)) : y \in N_G(a) - b\}$. Observe that the induced subgraph of $X(G)$ by the set $\mathcal{V}^a_{ab} \cup \mathcal{V}^b_{ba}$ is the complete bipartite graph $K_{|\mathcal{V}^a_{ab}|,|\mathcal{V}^b_{ba}|} = K_{d(a)-1,d(b)-1}$.

If W is a minimal edge cut of a connected graph G, then, $G - W$ necessarily contains exactly two components C and $\overline{C}$, so it is usual to denote an edge cut W as $[C,\overline{C}]$ where $[C,\overline{C}]$ denotes the set of edges between C and its complement $\overline{C}$.

Lemma 2 *Let G be a graph and $[C,\overline{C}]$ an edge-cut of $X(G)$. Let $ab \in E(G)$, if $[C,\overline{C}]$ contains ab-edges, then it contains at least* $\min\{d(a)-1, d(b)-1\}$ *ab-edges.*

Proof: Suppose that $(ay)(bx)$ is an edge of $[C,\overline{C}]$ such that $(ay) \in V(C)$ and $(bx) \in V(\overline{C})$. Then $\mathcal{V}^a_{ab} \cap V(C) \neq \emptyset$ and $\mathcal{V}^b_{ba} \cap V(\overline{C}) \neq \emptyset$. Let denote by $|\mathcal{V}^a_{ab} \cap V(C)| = r_a \geq 1$, $|\mathcal{V}^b_{ba} \cap V(C)| = r_b \geq 0$, $|\mathcal{V}^a_{ab} \cap V(\overline{C})| = \overline{r}_a \geq 0$ and $|\mathcal{V}^b_{ba} \cap V(\overline{C})| = \overline{r}_b \geq 1$. Moreover, these numbers must satisfy $r_a + \overline{r}_a = d(a) - 1$ and $r_b + \overline{r}_b = d(b) - 1$. Furthermore, the number of *ab*-edges contained in $[C,\overline{C}]$ is $r_a\overline{r}_b + r_b\overline{r}_a$, that is,

$$|[C,\overline{C}]| \geq r_a\overline{r}_b + r_b\overline{r}_a. \tag{1}$$

If $r_b = 0$, then $\overline{r}_b = d(b) - 1$. As $r_a \geq 1$, (1) implies $|[C, \overline{C}]| \geq d(b) - 1$ and the lemma follows. Similarly, if $\overline{r}_a = 0$, the result is also true. Therefore, we can assume that $r_a, r_b, \overline{r}_a, \overline{r}_b \geq 1$. In this case $r_a\overline{r}_b + r_b\overline{r}_a \geq r_a + \overline{r}_a = d(a) - 1$, and $r_a\overline{r}_b + r_b\overline{r}_a \geq r_b + \overline{r}_b = d(b) - 1$, and the result holds. □

Suppose that $[C, \overline{C}]$ is an edge-cut of $X(G)$. Let denote by $\omega(\alpha) = \{e \in E(G) : e = \alpha\beta\}$ and define $\mathcal{A} = \{\alpha\beta \in E(G) : (\alpha y)(\beta x) \in [C, \overline{C}]\}$. Then, as a consequence of the above lemma, we have $|[C, \overline{C}]| \geq |\mathcal{A}|(\delta(G) - 1)$. Next we prove that $|[C, \overline{C}]| \geq (\delta(G) - 1)^2$.

Lemma 3 *Let G be a graph and $[C, \overline{C}]$ an edge-cut of $X(G)$. Let $ab \in E(G)$ and suppose that $ab \in \mathcal{A}$. Then $|(\omega(a) \cup \omega(b)) \cap \mathcal{A}| \geq (\delta - 1)^2$.*

Proof: Suppose that for all $y \in N(a) - b$, $ay \in \mathcal{A}$. Then there are at least δ different ay-edges in $[C, \overline{C}]$, and by Lemma 2 the number of ay-edges in $[C, \overline{C}]$ is at least $\delta(\delta - 1) > (\delta - 1)^2$. The same occurs if for every $x \in N(b) - a$, $bx \in \mathcal{A}$. Therefore we may assume that there exists $y_0 \in N_G(a) - b$ such that $ay_0 \notin \mathcal{A}$ and there exists $x_0 \in N_G(b) - a$ such that $bx_0 \notin \mathcal{A}$.

As $ab \in \mathcal{A}$, $(ay')(bx') \in [C, \overline{C}]$ for some $y' \in N(a) - b$ and $x' \in N(b) - a$, and without loss of generality we may suppose that $(ay') \in V(C)$, $(bx') \in V(\overline{C})$. Suppose that $(ay_0)(bx_0) \notin [C, \overline{C}]$. Without loss of generality we may assume that $(ay_0), (bx_0) \in V(\overline{C})$ in which case $(ay')(bx_0) \in [C, \overline{C}]$ because $(ay') \in V(C)$. Then we can continue the proof assuming that there is an edge $(ay)(bx) \in [C, \overline{C}]$ such that $bx \notin \mathcal{A}$, i.e., there are no bx-edges in $[C, \overline{C}]$.

First suppose that $\mathcal{V}^x_{xb} \cap V(C) \neq \emptyset$. Let $B = \{x' \in N_G(b) \setminus \{x, a\} : (x'z) \in V(C)\}$ and $\overline{B} = \{x' \in N_G(b) \setminus \{x, a\} : (x'z) \in V(\overline{C})\}$. Observe that for all $x' \in B \cup \overline{B}$, $(x'z)$ is adjacent to $(bx) \in V(\overline{C})$, and $(x'z)$ is adjacent to (ba). Hence the edge-cut $[C, \overline{C}]$ must contain $|B|$ different bx'-edges. Moreover, since (ba) is adjacent to every $(xb') \in \mathcal{V}^x_{xb}$ and $bx \notin \mathcal{A}$, then $(ba) \in V(C)$ because our assumption $\mathcal{V}^x_{xb} \cap V(C) \neq \emptyset$. Hence $[C, \overline{C}]$ also contains $|\overline{B}|$ different bx'-edges yielding that $[C, \overline{C}]$ contains at least $|B| + |\overline{B}| + |\{ab\}| = d(b) - 1 \geq \delta - 1$ different bv-edges with $v \in N(b)$ and by Lemma 2, the result holds.

Second suppose that $\mathcal{V}^x_{xb} \subset V(\overline{C})$. Hence $\mathcal{V}^b_{ba} \subset V(\overline{C})$ because every $(bx') \in \mathcal{V}^b_{ba}$ is adjacent to every $(xb') \in \mathcal{V}^x_{xb}$ and $[C, \overline{C}]$ does not contain bx-edges. If $ay \notin \mathcal{A}$, reasoning for ay in the same way as for bx we get that $\mathcal{V}^a_{ab} \subset V(C)$. Thus as $\mathcal{V}^b_{ba} \subset V(\overline{C})$ it follows that $[C, \overline{C}]$ contains at least

$(d(a)-1)(d(b)-1) \geq (\delta-1)^2$ ab-edges and the lemma holds. Therefore, suppose that $ay \in \mathcal{A}$.

We know that there exists $v \in N_G(a) - y$ such that $av \notin \mathcal{A}$. As (va') is adjacent to (ay) for all $(va') \in \mathcal{V}_{va}^v$ it follows that $\mathcal{V}_{va}^v \subset V(C)$ (because $(ay) \in V(C)$ and $av \notin \mathcal{A}$). Hence $\mathcal{V}_{av}^a \subset V(C)$ because every $(ay') \in \mathcal{V}_{av}^a$ is adjacent to $(va') \in \mathcal{V}_{va}^v$. As $\mathcal{V}_{ba}^b \subset V(\overline{C})$ it follows that $[C, \overline{C}]$ contains at least $(d(a)-2)(d(b)-1)$ ab-edges. Further, as $ay \in \mathcal{A}$, by Lemma 2, $[C, \overline{C}]$ also contains at least $\delta - 1$ ay-edges, yielding that the number of au-edges contained $|[C, \overline{C}]|$ is at least $(\delta-2)(\delta-1) + (\delta-1) = (\delta-1)^2$, and the lemma holds. $\square$

Theorem 4 *Let G be a connected graph with minimum degree $\delta \geq 3$. Then*

$$\lambda(X(G)) \geq (\delta-1)^2.$$

Proof: Let $[C, \overline{C}]$ be a minimum edge-cut of $X(G)$ and $\mathcal{A} = \{ab \in E(G) : (ay)(bx) \in [C, \overline{C}]\}$. As G is connected and $\delta \geq 3$, then $X(G)$ is connected yielding that $|\mathcal{A}| \geq 1$. So considering $ab \in \mathcal{A}$, and using Lemma 3 we get $|[C, \overline{C}]| \geq (\delta-1)^2$, following the theorem. $\square$

The following corollary is an immediate consequence from Theorem 4, and from the fact that if G is a graph of minimum degree δ having an edge xy such that $d(x) = \delta$ and $d(y') = \delta$ for all $y' \in N_G(x) - y$, then the minimum degree of $X(G)$ is $\delta(X(G)) = (\delta-1)^2$.

Corollary 5 *Let G be a connected graph of minimum degree $\delta \geq 3$ having an edge xy such that $d(x) = \delta$ and $d(y') = \delta$ for all $y' \in N_G(x) - y$. Then the 3-arc graph $X(G)$ of G is maximally edge-connected.*

Figure 1 shows a 3-regular graph G with $\lambda(G) = 1$ and its 3-arc graph $X(G)$ which has $\lambda(X(G)) = 4 = \delta(X(G))$. However $X(G)$ is not super-λ and hence is not $\lambda_{(2)}$-optimal. And Figure 2 shows a 3-regular graph G with $\lambda(G) = \kappa(G) = 2$, and its 3-arc graph $X(G)$ which has $\lambda(X(G)) = 4$ and $\lambda_{(2)}(X(G)) = 6 = \xi(X(G))$, i.e., this graph is $\lambda_{(2)}$-optimal. In what follows we give a lower bound on the restricted edge-connectivity $\lambda_{(2)}(X(G))$ where G is a graph having connectivity $\kappa(G) \geq 2$.

Two edges which are incident with a common vertex are *adjacent*.

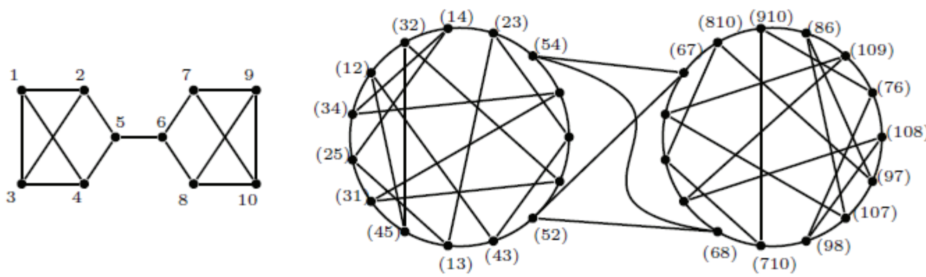


Figure 1: A 3-regular graph with $\lambda = 1$ and its 3-arc graph.

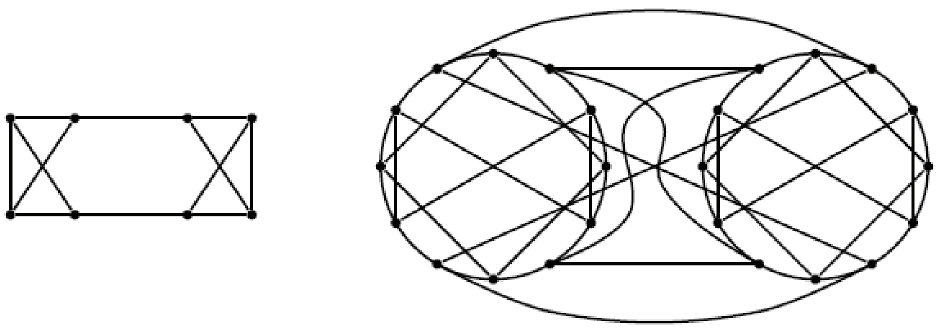


Figure 2: A 3-regular graph with $\lambda = 2$ ($\kappa = 2$) and its 3-arc graph.

Lemma 6 *Let G be a graph with minimum degree $\delta \geq 3$ and vertex connectivity $\kappa \geq 2$. Let $[C, \overline{C}]$ be a restricted edge-cut of $X(G)$ and consider the set $\mathcal{A} = \{ab \in E(G) : (ay)(bx) \in [C, \overline{C}]\}$. Then there are at least two nonadjacent edges in $\mathcal{A}$.*

Proof: Clearly $\mathcal{A} \neq \emptyset$, because $X(G)$ is connected. Thus let $(ay) \in V(C)$ and $(bx) \in V(\overline{C})$ be two adjacent vertices in $X(G)$, which implies that $ab \in \mathcal{A}$. Since $[C, \overline{C}]$ is a restricted edge-cut, then there exist $(uy') \in V(C)$ and $(wx') \in V(\overline{C})$ adjacent to (ay) and (bx) in $X(G)$, respectively. Observe that we may assume that $u \neq w$ because $\delta \geq 3$. Since G is 2-connected we can find a path $R : u = r_0, r_1, \ldots, r_k = w$ from u to w in $G - a$. As $\delta \geq 3$, there exists $v_i \in N(r_i) \setminus \{r_{i-1}, r_{i+1}\}$ for each $i = 1, \ldots, k-1$. Moreover we may choose $v_0 = y'$ and $v_k = x'$. Then the path R induces in $X(G)$ the path $R^* : (uy'), (r_1v_1), \dots, (r_{k-1}v_{k-1}), (wx')$ (observe that if $k = 1$ then $R^* : (uy'), (wx')$). Since $(uy') \in V(C)$ and $(wx') \in V(\overline{C})$, it follows that $[C, \overline{C}] \cap E(R^*) \neq \emptyset$, hence $r_ir_{i+1} \in \mathcal{A}$ for some $i \in \{0, \dots, k\}$. Since $a \notin V(R)$ then $a \notin \{r_i, r_{i+1}\}$.

Now reasoning analogously, we can find a path $S : u = s_0, s_1, \dots, s_\ell = w$ from u to w in $G - b$ that induces a path S^* from $(uy') \in V(C)$ to $(wx') \in V(\overline{C})$. This implies that $[C, \overline{C}] \cap E(S^*) \neq \emptyset$, hence $s_js_{j+1} \in \mathcal{A}$ for some $j \in \{0, \dots, \ell\}$. Since $b \notin V(S)$ then $b \notin \{s_j, s_{j+1}\}$.

As $ab, r_ir_{i+1}, s_js_{j+1} \in \mathcal{A}$, $a \notin \{r_i, r_{i+1}\}$ and $b \notin \{s_j, s_{j+1}\}$, it follow that al least two of the edges of $\{ab, r_ir_{i+1}, s_js_{j+1}\}$ are nonadjacent. $\square$

Theorem 7 *Let G be a graph with minimum degree $\delta \geq 3$ and vertex connectivity $\kappa \geq 2$. Then $X(G)$ has restricted edge-connectivity $\lambda_{(2)}(X(G)) \geq 2(\delta - 1)^2 - 2$.*

Proof: Let $[C, \overline{C}]$ be a restricted edge-cut of $X(G)$ and consider the set $\mathcal{A} = \{ab \in E(G) : (ay)(bx) \in [C, \overline{C}]\}$. From Lemma 6, $\mathcal{A}$ contains two nonadjacent edges ab and cd. By Lemma 3, the number of au-edges and bv-edges, $u, v \in N(a) \cup N(b)$ contained in $[C, \overline{C}]$ is at least $(\delta - 1)^2$, and the number of cu-edges and dv-edges, $u, v \in N(c) \cup N(d)$ contained in $[C, \overline{C}]$ is at least $(\delta - 1)^2$. If $|[\{a, b\}, \{c, d\}] \cap \mathcal{A}| \leq 2$ then $|[C, \overline{C}]| \geq 2(\delta - 1)^2 - |[\{a, b\}, \{c, d\}] \cap \mathcal{A}| \geq 2(\delta - 1)^2 - 2$. If $3 \leq |[\{a, b\}, \{c, d\}] \cap \mathcal{A}| \leq 4$ then we may assume without loss of generality that $ac, bd \in \mathcal{A}$, hence, by applying Lemma 3, the number of au-edges and cv-edges, $u, v \in N(a) \cup N(c)$ contained in $[C, \overline{C}]$ is at least $(\delta - 1)^2$, and the number of bu-edges and dv-edges, $u, v \in N(b) \cup N(d)$ contained in $[C, \overline{C}]$ is at least $(\delta - 1)^2$. Thus,

$$\begin{aligned} |[C, \overline{C}]| &\geq 2(\delta - 1)^2 - |[\{a, b\}, \{c, d\}] \cap \mathcal{A}| + 2(\delta - 1)^2 - |[\{a, c\}, \{b, d\}] \cap \mathcal{A}| \\ &\geq 4(\delta - 1)^2 - 8 \\ &\geq 2(\delta - 1)^2 - 2, \end{aligned}$$

since $\delta \geq 3$. Hence the theorem is valid. $\square$

Figure 3 shows that $\lambda(G) \geq 2$ is not enough to guarantee that $\lambda_{(2)}(X(G)) \geq 2(\delta - 1)^2 - 2$. In this example G is a 4-regular graph with $\lambda = 2$ and $\kappa = 1$, but $\lambda_{(2)}(X(G)) = 12 < 16$.

The following corollary is an immediate consequence from Theorem 7, and from the fact that if G is graph of minimum degree δ having an edge

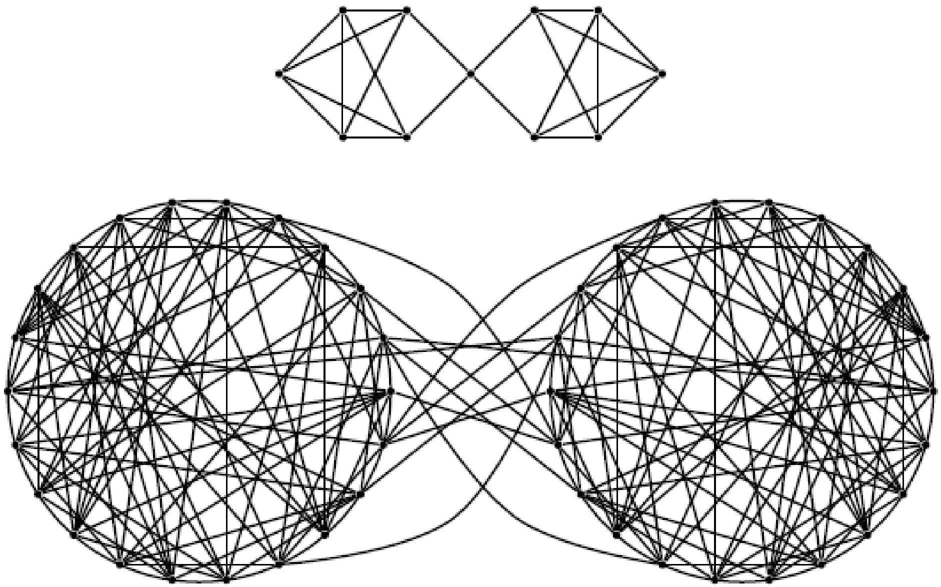


Figure 3: The 3-arc graph of a 4-regular graph with $\kappa = 1$ and $\lambda = 2$ with $\lambda_{(2)}(X(G)) = 12$.

xy such that $d(x) = \delta$, $d(y) = \delta$ and such that every $w \in (N_G(x) - y) \cup (N_G(y) - x)$ also has degree δ, then the minimum edge degree of $X(G)$ is $\xi(X(G)) = 2(\delta - 1)^2 - 2$.

Corollary 8 *Let G be a graph of minimum degree $\delta \geq 3$ and vertex connectivity $\kappa \geq 2$ having an edge xy such that $d(x) = \delta$, $d(y) = \delta$ and such that every $w \in (N_G(x) - y) \cup (N_G(y) - x)$ also has degree δ. Then the 3-arc graph $X(G)$ has restricted edge connectivity $\lambda_{(2)}(X(G)) = \xi(X(G)) = 2(\delta - 1)^2 - 2$.*

Acknowledgement

Research supported by the Ministry of Science and Innovation, Spain, and the European Regional Development Fund (ERDF) under project MTM2008-06620-C03-02/MTM; also by Catalonian goverment under proyect 2009 SGR 1298.

References

[1] C. Balbuena, M. Cera, A. Diánez, P. García-Vázquez and X. Marcote. On the restricted connectivity and superconnectivity in graphs with given girth. *Discrete Math.*, 307(6):659–667, 2007.

[2] C. Balbuena, M. Cera, A. Diánez, P. García-Vázquez and X. Marcote. Sufficient conditions for λ'-optimality of graphs with smaill conditional diameter. *Inf. Process. Lett.*, 95:429–434, 2005.

[3] C. Balbuena and D. Ferrero. Edge-connectivity and super edge-connectivity of P_2-path graphs. *Discrete Math.*, 269(1-3):13–20, 2003.

[4] C. Balbuena and P. García-Vázquez. A sufficient condition for P_k-path graphs being r-connected. *Discrete Appl. Math.*, 155:1745–1751, 2007.

[5] C. Balbuena, P. García-Vázquez and X. Marcote. Sufficient conditions for λ'-optimality in graphs with girth g. *J. Graph Theory*, 52(1):73–86, 2006.

[6] C. Balbuena, K. Marshall and L.P. Montejano. On the connectivity and superconnected graphs with small diameter. *Discrete Appl. Math.*, 158(5):397–403, 2010.

[7] F.T Boesch. Synthesis of reliable networks–A survey. *IEEE Trans. Reliability*, 35:240–246, 1986.

[8] F. Boesch and R. Tindell. Circulants and their connectivities. *J. Graph Theory*, 8:487–499, 1984.

[9] H.J. Broersma and C. Hoede. Path graphs. *J. Graph Theory*, 13:427–444, 1989.

[10] G. Chartrand and L. Lesniak. Graphs and Digraphs. Third edition. Chapman and Hall, London, UK, 1996.

[11] A.H. Esfahanian and S.L. Hakimi. On computing a conditional edge-connectivity of a graph. *Inf. Process. Lett.*, 27:195–199, 1988.

[12] A. Hellwig and L. Volkmann. Sufficient conditions for graphs to be λ'-optimal, super-edge- connected, and maximally edge-connected. *J. Graph Theory*, 48:228–246, 2005.

[13] A. Hellwig and L. Volkmann. Sufficient conditions for λ'-optimality in graphs of diameter 2. *Discrete Math.*, 283:113–120, 2004.

[14] M. Knor and L. Niepel. Connectivity of iterated line graphs. *Discrete Appl. Math.*, 125:255–266, 2003.

[15] M. Knor, L. Niepel and M. Malah. Connectivity of path graphs. *Australas. J. Combin.*, 25:175–184, 2002.

[16] M. Knor and S. Zhou. Diameter and connectivity of 3-arc graphs. *Discrete Math.*, 310:37–42, 2010.

[17] C.H. Li, C.E. Praeger and S. Zhou. A class of finite symmetric graphs with 2-arc transitive quotients. *Math. Proc. Cambridge Phil. Soc.*, 129:19–34, 2000.

[18] S. Lin and S. Wang. Super p-restricted edge connectivity of line graphs. *Information Sciences*, 179:3122–3126, 2009.

[19] J. Meng. Connectivity and super edge-connectivity of line graphs. *Graph Theory Notes of New York*, XL:12–14, 2001.

[20] T. Soneoka. Super edge-connectivity of dense digraphs and graphs. *Discrete Appl. Math.*, 37/38:511–523, 1992.

[21] Z. Zhang. Sufficient conditions for restricted-edge-connectivity to be optimal. *Discrete Math.*, 307(22):2891–2899, 2007.

[22] S. Zhou. Imprimitive symmetric graphs, 3-arc graphs and 1-designs. *Discrete Math.*, 244:521–537, 2002.

[23] Q. Zhu, J.M. Xu, X. Hou and M. Xu. On reliability of the folded hypercubes. *Information Sciences* 177:1782-1788, 2007.

Edge-superconnectivity of semiregular cages with odd girth

Camino Balbuena and Julián Salas
Universitat Politècnica de Catalunya
Barcelona

Diego González-Moreno
Universidad Nacional Autónoma de México
México

Abstract

A graph is said to be edge-superconnected if each minimum edge-cut consists of all the edges incident with some vertex of minimum degree. A graph G is said to be a $\{d, d+1\}$-semiregular graph if all its vertices have degree either d or $d+1$. A smallest $\{d, d+1\}$-semiregular graph G with girth g is said to be a $(\{d, d+1\}; g)$-cage. We show that every $(\{d, d+1\}; g)$-cage with odd girth g is edge-superconnected.

1 Introduction

We only consider undirected simple graphs without loops or multiple edges. Unless otherwise stated, we follow [9] for basic terminology and definitions. Let G stand for a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *distance* $d_G(u, v) = d(u, v)$ between two vertices of the graph G is the length of a shortest path between u and v, and the *diameter* of G denoted by $diam(G)$ is the maximum distance between any pair of vertices; when G is not connected, then $diam(G) = +\infty$. For $w \in V$ and $S \subset V$, $d(w, S) = d_G(w, S) = \min\{d(w, s) : \ s \in S\}$ denotes the *distance* between w and S. For every $S \subset V$ and every nonnegative integer $r \geq 0$, $N_r(S) = \{w \in V : d(w, S) = r\}$ denotes the *neighborhood of S at distance r*. Thus the set of vertices adjacent to a vertex v is $N(v) = N_1(\{v\})$, and the *degree* of a vertex v in G is $deg_G(v) = deg(v) = |N(v)|$, whereas the *minimum degree* $\delta = \delta(G)$ is the minimum degree over all vertices of G. A graph is called *r-regular* if every vertex of the graph has degree r.

A graph G is called *connected* if every pair of vertices is joined by a path. An *edge-cut* in a graph G is a set W of edges of G such that $G - W$ is disconnected. A graph is *k-edge-connected* if every edge-cut contains at least k edges. If W is a minimal edge-cut of a connected graph G, then necessarily, $G-W$ contains exactly two components. The *edge-connectivity* $\lambda = \lambda(G)$ of a graph G is the minimum cardinality of an edge-cut of G. A classic result is $\lambda \leq \delta$ for every graph G. A graph is *maximally edge-connected* if $\lambda = \delta$.

One might be interested in more refined indices of reliability. Even two graphs with the same edge-connectivity λ may be considered to have different reliabilities. As a more refined index than the edge-connectivity, edge-superconnectivity is proposed in [6, 7]. A subset of edges W is called *trivial* if it contains the set of edges incident with some vertex of the graph. Clearly, if $|W| \leq \delta - 1$, then W is nontrivial. A graph is said to be *edge-superconnected* if $\lambda = \delta$ and every minimum edge-cut is trivial.

The *degree set* D of a graph G is the set of distinct degrees of the vertices of G. The *girth* $g(G)$ is the length of a shortest cycle in G. A *$(D; g)$-graph* is a graph having degree set D and girth g. Let $n(D; g)$ denote the least order of a $(D; g)$-graph. Then a $(D; g)$-graph with order $n(D; g)$ is called a *$(D; g)$-cage*. If $D = \{r\}$ then a $(D; g)$-cage is a $(r; g)$-cage. When $D = \{r, r+1\}$, we refer to $(D; g)$-cages as *semiregular cages*.

The existence of $(r; g)$-cages was proved by Erdös and Sachs [10] in the decade of the 60's, and using this result Chartrand et al. [8] proved the existence of $(D; g)$-cages. Some of the structural properties of $(r; g)$-cages that have been studied are the vertex and the edge connectivity; concerning this problem Fu, Huang and Rodger [11] conjectured that every $(r; g)$-cage is r-connected, and they proved the statement for $r = 3$. Other contributions supporting this conjecture can be seen in [15, 16, 17, 20]. Moreover, some structural properties of $(r; g)$-cages have been extended for $(D; g)$-cages, for example the monotonicity of the order with respect to the girth (see Theorem 1) and the upper bound for the diameter (see Theorem 2). The edge-superconnectivity of cages was established in [18, 19]. For semiregular cages, it has been proved in [3] that they are maximally edge connected. The main objective of this work is to prove that every $(\{d, d+$

$1\}; g)$-cage with odd girth $g \geq 5$ is edge-superconnected. With this aim we need the following two results.

Theorem 1 [4] *Let g_1, g_2 be two integers such that $3 \leq g_1 < g_2$. Then $n(\{d, d+1\}; g_1) < n(\{d, d+1\}; g_2)$.*

Theorem 2 [5] *The diameter of a $(\{d, d+1\}; g)$-cage is at most g.*

2 Main theorem

In order to study the edge-superconnectivity of a graph in terms of its diameter and its girth, the following results were established [1, 2, 13].

Proposition 3 *Let $G = (V, E)$ be a connected graph with minimum degree $\delta \geq 2$ and girth g. Let $W \subset E$ be a minimum nontrivial edge-cut, let H_i be a component of $G - W$, and let $W_i \subset V(H_i)$ be the set of vertices of H_i which are incident with some edge in W, $i = 0, 1$. Then there exists some vertex $x_i \in V(H_i)$ such that*

(a) [1, 13] *$d(x_i, W_i) \geq \lfloor (g-1)/2 \rfloor$, if $|W_i| \leq \delta - 1$.*

(b) [2] *$d(x_i, W_i) \geq \lceil (g-3)/2 \rceil$, if $|W| \leq \xi - 1$, where $\xi = \min\{deg(u) + deg(v) - 2 : uv \in E\}$ is the minimum edge-degree of G.*

For every minimum edge-cut W of G such that H_0, H_1 are the two components of $G - W$, we will write henceforth $W = [W_0, W_1]$ with $W_0 \subset V(H_0)$ and $W_1 \subset V(H_1)$ containing all endvertices of the edges in W. Note that $|W_i| \leq |W|$, $i = 0, 1$. From now on, let

$$\mu_i = \max\{d(x, W_i) : x \in V(H_i)\}, \quad i = 0, 1.$$

When W is nontrivial and $|W| \leq \xi - 1$, it follows from Proposition 3 that $\mu_i \geq \lceil (g-3)/2 \rceil$. Likewise, μ_0 and μ_1 satisfy some other basic properties shown in next lemma.

Lemma 4 *Let $G = (V, E)$ be a connected graph with minimum degree $\delta \geq 3$ and odd girth $g \geq 5$. Let $W = [W_0, W_1] \subset E$ be a minimum nontrivial edge-cut with cardinality $|W| \leq \delta$. Let $G - W = H_0 \cup H_1$, where $W_i \subset V(H_i)$. If $\mu_i = (g-3)/2$ the following statements hold:*

(i) $|W_i| = |W| = \delta$, *and every* $a \in W_i$ *is incident to a unique edge of* W.

(ii) *Every vertex* $z \in V(H_i)$ *such that* $d(z, W_i) = \mu_i$ *has* $deg(z) = \delta$.

(iii) *For every* $a \in W_i$ *there exists a vertex* $x \in V(H_i)$ *such that* $d(x, W_i) = d(x, a) = \mu_i$ *and* $N_{(g-3)/2}(x) \cap W_i = \{a\}$. *Further,* $N(x)$ *can be labeled as* $\{u_1, u_2, \ldots, u_\delta\}$, *and* W_i *can be labeled as* $\{a_1, a_2, \ldots, a_\delta\}$, *where* $a_1 = a$, *so that* $N_{(g-5)/2}(u_1) \cap W_i = \{a_1\}$ *and* $N_{(g-3)/2}(u_k) \cap W_i = \{a_k\}$ *for every* $k > 1$. *Consequently* $|[N_{(g-3)/2}(x) \cap W_i, W_{i+1}]| = 1$ *and* $|[N_{(g-3)/2}(u_k) \cap W_i, W_{i+1}]| = 1$ *(with subscripts taken mod 2). See Figure 2.*

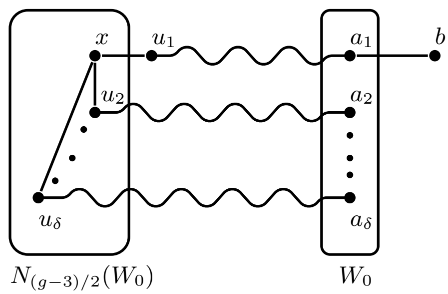


Figure 1: Lemma 4.

Proof: (i) Since $\mu_i = (g-3)/2$, $d(x, W_i) \le \mu_i = (g-3)/2 < (g-1)/2$ for all $x \in V(H_i)$. Hence from Proposition 3 (a), it follows that $|W_i| \ge \delta$, yielding $|W_i| = \delta$ because $|W_i| \le |W| \le \delta$. Observe that $\delta = |W_i| = |W|$ means that $|N(a) \cap W_{i+1}| = 1$ for each vertex $a \in W_i$ (taking the subscripts mod 2).

(ii) First observe that $\mu_i = (g-3)/2 \ge 1$ since $g \ge 5$. Let us define the following partition of $N(v)$ for all $v \in V(H_i)$

$$S^-(v) = \begin{cases} \{z \in N(v) : d(z, W_i) = d(v, W_i) - 1\} & \text{if } v \notin W_i; \\ W_{i+1} \cap N(v) & \text{if } v \in W_i. \end{cases}$$

$$S^+(v) = \{z \in N(v) : d(z, W_i) = d(v, W_i) + 1\}$$
$$S^=(v) = \{z \in N(v) : d(z, W_i) = d(v, W_i)\}.$$

Let z be a vertex of H_i such that $d(z, W_i) = \mu_i = (g-3)/2$. Then we have

$$\begin{array}{rcl} N(z) & = & S^{=}(z) \cup S^{-}(z); \\ |N_{(g-3)/2}(S^{=}(z)) \cap W_i| & \geq & |S^{=}(z)|; \\ |N_{(g-5)/2}(S^{-}(z)) \cap W_i| & \geq & |S^{-}(z)|; \\ N_{(g-3)/2}(S^{=}(z)) \cap N_{(g-5)/2}(S^{-}(z))) & = & \emptyset, \end{array} \tag{1}$$

because otherwise cycles of length less than the girth g appear. Since

$$\begin{array}{rcl} \delta \leq deg(z) & = & |S^{=}(z)| + |S^{-}(z)| \\ & \leq & |N_{(g-3)/2}(S^{=}(z)) \cap W_i| + |N_{(g-5)/2}(S^{-}(z)) \cap W_i| \\ & \leq & |W_i| = \delta \end{array}$$

it follows that $\delta = deg(z)$. Therefore item (ii) holds.

(iii) First let us prove that there exists an edge zz' such that $d(z, W_i) = d(z', W_i) = (g-3)/2$. Otherwise, $S^{=}(z) = \emptyset$ for all z with $d(z, W_i) = (g-3)/2$. This implies that for all $u \in N(z)$, $u \in S^{-}(z)$ and $S^{=}(S^{+}(u)) = \emptyset$. Further, $|N_{(g-5)/2}(u) \cap W_i| = 1$ for all $u \in N(z)$, because $\delta = |W_i| = \sum_{u \in N(z)} |N_{(g-5)/2}(u) \cap W_i| \geq \delta$. Hence $|S^{-}(u)| = 1$, and so $|S^{+}(u)| + |S^{=}(u)| = deg(u) - 1 \geq \delta - 1 \geq 2$. Suppose that $|S^{=}(u)| \geq 1$ for some $u \in N(z)$. Then as $N_{(g-3)/2}(z) \cap W_i$ and $N_{(g-5)/2}(S^{=}(u)) \cap W_i$ are two vertex disjoint sets we have $|W| \geq |N_{(g-3)/2}(z) \cap W_i| + |N_{(g-5)/2}(S^{=}(u)) \cap W_i| \geq \delta + 1$ which is a contradiction because $|W| = \delta$. Then we must assume that for all $u \in N(z)$, $|S^{+}(u)| = deg(u) - 1 \geq \delta - 1 \geq 2$. Let $t \in S^{+}(u) - z$, according to our first assumption $S^{=}(t) = \emptyset$ meaning that $N(t) = S^{-}(t)$. Since t has the same behavior as z we have $W_i = N_{(g-3)/2}(S^{-}(z)) = N_{(g-3)/2}(S^{-}(t))$, and as $2 < \delta \leq deg(z) = deg(t)$, there exist cycles through $\{z, u, t, w\}$ for some $w \in W_i$ of length less than g which is a contradiction.

Hence we may assume that there exists an edge zz' such that $d(z, W_i) = d(z', W_i) = (g-3)/2$. Since $N_{(g-5)/2}(S^{-}(z)) \cap W_i$, $N_{(g-5)/2}(S^{-}(z')) \cap W_i$ and $N_{(g-3)/2}(S^{=}(z') - z) \cap W_i$ are three pairwise disjoint sets because $g \geq 5$, and taking into account (1) we have

$$\begin{array}{rcl} \delta = |W| & \geq & |N_{(g-5)/2}(S^{-}(z)) \cap W_i| + |N_{(g-5)/2}(S^{-}(z')) \cap W_i| \\ & & + |N_{(g-3)/2}(S^{=}(z') - z) \cap W_i| \\ & \geq & |S^{-}(z)| + |S^{-}(z')| + |S^{=}(z') - z| \\ & = & deg(z) - 1 + |S^{-}(z)| \geq \delta. \end{array}$$

Therefore, all inequalities become equalities, i.e., $|S^-(z)| = 1 = |N_{(g-5)/2}(S^-(z)) \cap W_i|$. So $S^-(z) = \{z_1\}$ and $N(z) - z_1 = S^=(z)$ yielding a partition of W_i:

$$W_i = \left(N_{(g-5)/2}(z_1) \cap W_i\right) \cup \left(\cup_{z' \in N(z)-z_1} N_{(g-3)/2}(z') \cap W_i\right),$$

because for all $z' \in N(z) - z_1$ the sets $N_{(g-3)/2}(z') \cap W_i$ and the set $N_{(g-5)/2}(z_1) \cap W_i$ are mutually disjoint. Thus, $|N_{(g-3)/2}(z') \cap W_i| = 1$ for all $z' \in N(z) - z_1$. Therefore, for every vertex $a \in W_i$ there exists a vertex $x \in (N(z) - z_1) \cup \{z\}$ such that $d(x, W_i) = d(x, a) = (g-3)/2$ and $N_{(g-3)/2}(x) \cap W_i = \{a\}$. Furthermore, since every vertex $z' \in N(z) - z_1$ has the same behavior as z, $N(x)$ can be labeled as $\{u_1, u_2, \ldots, u_\delta\}$, and W_i can be labeled as $\{a_1, a_2, \ldots, a_\delta\}$, where $a_1 = a$, so that $N_{(g-5)/2}(u_1) \cap W_i = \{a_1\}$ and $N_{(g-3)/2}(u_k) \cap W_i = \{a_k\}$ for every $k > 1$. Finally, using (i) we obtain $|[N_{(g-3)/2}(x) \cap W_i, W_{i+1}]| = 1$ and $|[N_{(g-3)/2}(u_k) \cap W_i, W_{i+1}]| = 1$, which finishes the proof. □

A semiregular cage is known to be maximally edge-connected [3]. Now, we are ready to prove that semiregular cages with odd girth are edge-superconnected. As will be seen, *Hall's Theorem* is a key point of this study. Recall that if S is a set of vertices in a graph G, the set of all neighbors of the vertices in S is denoted by $N(S)$.

Theorem 5 ([12] **Hall's Theorem**) *A bipartite graph with bipartition (X_1, X_2) has a matching which covers every vertex in X_1 if and only if*

$$|N(S)| \geq |S| \text{ for all } S \subset X_1.$$

Using Hall's Theorem Jiang [14] proved the following result.

Lemma 6 [14] *Let G be a bipartite graph with bipartition (X_1, X_2) where $|X_1| = |X_2| = r$. If G contains at least $r^2 - r + 1$ edges, then G contains a perfect matching.*

The following lemma is an stronger version of Lemma 6, which is also proved using Hall's Theorem.

Lemma 7 *Let $\mathcal{B}$ be a bipartite graph with bipartition (X_1, X_2) where $|X_1| = |X_2| = r$. If $\delta(\mathcal{B}) \geq 1$ and $|E(\mathcal{B})| \geq r^2 - r$, then $\mathcal{B}$ contains a perfect matching.*

Proof: Let $\mathcal{B} = (X_1, X_2)$ be a bipartite graph with $|X_1| = |X_2| = r$, $\delta(\mathcal{B}) \geq 1$ and $|E(\mathcal{B})| \geq r^2 - r$. We shall apply Hall's Theorem to prove the lemma; we shall show that for a subset $S \subset X_1$, $|N(S)| \geq |S|$. Notice that if $|S| = 1$, then $|N(S)| \geq 1 = |S|$ because $\delta(\mathcal{B}) \geq 1$; and if $S = X_1$, $N(S) = X_2$ because $\delta(\mathcal{B}) \geq 1$ implies that each vertex $u \in X_2$ must have a neighbor in S, hence $|S| = |N(S)|$.

Therefore we continue the proof reasoning by contradiction and so assuming that $1 \leq |N(S)| < |S| = t \leq r - 1$. Then the number of edges in $\mathcal{B}$ is at most

$$|E(\mathcal{B})| = |[S, N(S)]| + |[X_1 \setminus S, X_2]| \leq t(t-1) + (r-t)r,$$

and by hypothesis $|E(\mathcal{B})| \geq r^2 - r$. Thus $r^2 - r \leq t(t-1) + (r-t)r$, yielding $0 \leq (t-r)(t-1)$, which is an absurdity because $1 < t < r$. Therefore $|N(S)| \geq |S|$ for all $S \subset X_1$, and by Hall's Theorem the lemma follows. $\square$

Theorem 8 *Let G be a $(\{d, d+1\}; g)$-cage with odd girth $g \geq 5$, and $d \geq 3$. Then G is edge-superconnected.*

Proof: Let us assume that G is a non edge-superconnected $(\{d, d+1\}; g)$-cage, and we will arrive at a contradiction. To this end, let us take a minimum nontrivial edge-cut $W = [W_0, W_1] \subset E(G)$ such that $|W| \leq \delta$. Let $G - W = H_0 \cup H_1$, and let $W_i \subset V(H_i)$ be the set of vertices of H_i which are incident with some edge in W, $i = 0, 1$. From Proposition 3 it follows that $\mu_i = \max\{d(x, W_i) : x \in V(H_i)\} \geq (g-3)/2$, $i = 0, 1$. Let $x_i \in V(H_i) \cap N_{\mu_i}(W_i)$. As G is a $(\{d, d+1\}; g)$-cage, the diameter is at most $diam(G) \leq g$ by Theorem 2, so we get the following chain of inequalities:

$$g \geq diam(G) \geq d(x_0, x_1) \geq d(x_0, W_0) + 1 + d(x_1, W_1) = \mu_0 + 1 + \mu_1 \geq g - 2.$$

If we assume henceforth $\mu_0 \leq \mu_1$ (without loss of generality), then either $(g-3)/2 = \mu_0 \leq \mu_1 \leq (g+1)/2$, or $\mu_0 = \mu_1 = (g-1)/2$. We proceed to study each one of these cases.

In what follows, let X_0, X_1 be two subsets of $V(G)$ such that $|X_0| = |X_1|$. Let $\mathcal{B}_\Gamma$ denote the bipartite graph with bipartition (X_0, X_1) and $E(\mathcal{B}_\Gamma) = \{u_i v_j : u_i \in X_0, v_j \in X_1, d_\Gamma(u_i, v_j) \geq g - 1\}$, where Γ is a certain subgraph of G.

Case (a): $\mu_0 = (g-3)/2$.

From Lemma 4 (i), $|W_0| = d = |W|$ so that each vertex of W_0 is incident to a unique edge of W, yielding that every vertex $a \in W_0$ has $deg_{H_0}(a) \in \{d-1, d\}$. Also by Lemma 4 (ii), every vertex $x \in N_{(g-3)/2} \cap V(H_0)$ has $deg(x) = d$. And by Lemma 4 (iii), for every $a \in W_0$ there exists a vertex $x_0 \in N_{(g-3)/2} \cap V(H_0)$ such that $N(x_0) = \{u_1, u_2, \ldots, u_d\}$ and $W_0 = \{a_1, a_2, \ldots, a_d\}$, where $a_1 = a$, in such a way that $d(u_1, a_1) = d(u_1, W_0) = (g-5)/2$, $d(u_j, W_0) = d(u_j, a_j) = (g-3)/2$, and by (ii), $deg(u_j) = d$ for every $j \geq 2$. This implies that $d_{G-x_0}(u_1, a_j) \geq (g-1)/2$ for all $j \geq 2$, because the shortest (u_1, a_j)-path in $G - x_0$, the shortest (u_j, a_j)-path in G, and the path $u_j x_0 u_1$ in G of length two, form a closed walk containing a cycle. Reasoning analogously, $d_{G-x_0}(u_j, a_1) \geq (g+1)/2$ for all $j \geq 2$ and $d_{G-x_0}(u_j, a_i) \geq (g-1)/2$ for $j \neq i$, $j, i \in \{2, \ldots, d\}$. Furthermore, $[N_{(g-3)/2}(x_0) \cap W_0, W_1] = \{a_1 b_1\}$ for some $b_1 \in W_1$.

Subcase (a.1): $\mu_1 = (g+1)/2$.

Let $x_1 \in V(H_1)$ be any vertex such that $d(W_1, x_1) = (g+1)/2$. Let $X_0 = \{u_2, \ldots, u_d\} \cup \{x_0\}$ and $X_1 = \{v_1, v_2, \ldots, v_d\} \subseteq N(x_1)$. As $d(u_i, W_0) = (g-3)/2$ for $i \geq 2$ and $d_{G-x_1}(W_1, N(x_1)) \geq (g-1)/2$, then $d_{G-x_1}(X_0, X_1) \geq g-1$, so $|E(\mathcal{B}_\Gamma)| = d^2$, where $\Gamma = G - x_1$. Clearly $\mathcal{B}_\Gamma$ is a complete bipartite graph, so there is a perfect matching M which covers every vertex in X_0 and if $deg(x_1) = d$, also covers $N(x_1)$. Hence, in this case the graph $G^* = (G - \{x_1\} - \{x_0 u_d\}) \cup M$ has girth at least g and the vertices $u_2, \ldots, u_{d-1}$ have degree $d+1$ in G^* as they had degree d in G; for the same reason x_0 and u_d have degree d in G^*. The remaining vertices have the same degree they had in G. As G^* is a $(\{d, d+1\}; g^*)$-graph with girth $g^* \geq g$ and $|V(G^*)| < |V(G)|$, we get a contradiction to the monotonocity Theorem 1. If $deg(x_1) = d+1$, since $d_{G^*}(u_d, v_{d+1}) \geq g-1$ where $v_{d+1} \in N(x_1) \setminus X_1$, we can add the new edge $u_d v_{d+1}$ to G^* without decreasing the girth. Then $G^* \cup \{u_d v_{d+1}\}$ gives us again a contradiction.

Subcase (a.2): $\mu_1 = (g-3)/2$.

By Lemma 4, given $b_1 \in W_1$ there exists $x_1 \in V(H_1) \cap N_{(g-3)/2}(W_1)$ of $deg(x_1) = d$ such that $N(x_1) = \{v_1, v_2, \ldots, v_d\}$, $W_1 = \{b_1, b_2, \ldots, b_d\}$ and each vertex of W_1 is incident to a unique edge of W, hence $W = \{a_1 b_1, a_2 b_2, \ldots, a_d b_d\}$. Also, $d(b_1, v_1) = d(W_1, v_1) = (g-5)/2$, and $d(W_1, v_j) = d(b_j, v_j) = (g-3)/2$ for every $j \geq 2$ and besides $deg(v_j) = d$. Then $d(x_0, x_1) = d(x_0, a_1) + 1 + d(b_1, x_1) = g-2$, and if $g = 5$ it is easy to see that the shortest (x_0, x_1)-path of length three is unique, clearly $x_0 a_1 b_1 x_1$.

Now let $\Gamma = G - \{x_0, x_1\}$. We have

$$\begin{aligned} d_\Gamma(u_1, N(x_1) - v_1) &= \min\{d_\Gamma(u_1, a_1) + 1 + d_\Gamma(b_1, N(x_1) - v_1); \\ &\qquad d_\Gamma(u_1, a_j) + 1 + d_\Gamma(b_j, N(x_1) - v_1), j \geq 2\} \\ &\geq \min\{\frac{g-5}{2} + 1 + \frac{g+1}{2}; \frac{g-1}{2} + 1 + \frac{g-3}{2}\} = g - 1, \end{aligned}$$

since $d_\Gamma(b_1, v_j) \geq (g+1)/2$ for all $j \geq 2$, because the shortest (b_1, v_j)-path in Γ, the shortest (b_1, v_1)-path in Γ, and the path $v_j x_1 v_1$ in G of length two, form a closed walk containing a cycle. Reasoning in the same way, it follows for all $j \geq 2$ that

$$\begin{aligned} & d_\Gamma(u_j, N(x_1) - v_j) = \\ &= \min\{d_\Gamma(u_j, a_j) + 1 + d_\Gamma(b_j, N(x_1) - v_j); d_\Gamma(u_j, a_h) + 1 \\ &\quad + d_\Gamma(b_h, N(x_1) - v_j), h \neq j\} \\ &\geq \min\left\{ \begin{array}{l} \left\{\frac{g-3}{2} + 1 + \frac{g-1}{2}; \frac{g-1}{2} + 1 + \frac{g-3}{2}\right\} \text{ if } h \geq 2, h \neq j \\ \left\{\frac{g-3}{2} + 1 + \frac{g-1}{2}; \frac{g+1}{2} + 1 + \frac{g-5}{2}\right\} \text{ if } h = 1 \end{array} \right\} \\ &= g - 1. \end{aligned}$$

Analogously, $d_\Gamma(N(x_0) - u_1, v_1) \geq g - 1$ and $d_\Gamma(N(x_0) - u_j, v_j) \geq g - 1$ for all $j \geq 2$. Let $X_0 = N(x_0)$ and $X_1 = N(x_1)$. The bipartite graph $\mathcal{B}_\Gamma = (X_0, X_1)$ has $|E(\mathcal{B}_\Gamma)| = d^2 - d$ and $deg_{\mathcal{B}_\Gamma}(w) \geq 1$ for all $w \in X_0 \cup X_1$. From Lemma 7, there is a perfect matching M between $X_0 = N(x_0)$ and $X_1 = N(x_1)$. Hence $G^* = (G - \{x_0, x_1\}) \cup M$ is a $(\{d, d+1\}; g^*)$-graph (because every vertex in G^* has the same degree it had in G and the removed vertices x_0, x_1 had degree d, as well as the vertices u_j, v_k for every $j, k \geq 2$) with $g^* \geq g$ and $|V(G^*)| \leq |V(G)|$, which contradicts the monotonocity Theorem 1, and we are done.

Subcase (a.3): $\mu_1 = (g-1)/2$. In this case we distinguish two other possible subcases.

Subcase (a.3.1): There exists $x_1 \in V(H_1) \cap N_{(g-1)/2}(W_1)$ such that $d(b, v) \leq (g-1)/2$ for all $b \in W_1$ and for all $v \in N(x_1)$.

Then, every $b \in W_1$ has $deg_{H_1}(b) = deg(x_1) \in \{d, d+1\}$ because $d(b, v) \leq (g-1)/2$ and $|N_{(g-3)/2}(v) \cap N(b)| \leq 1$ for all $v \in N(x_1)$ (otherwise cycles of length less than g appear). Hence $deg(x_1) = d$ and $deg(b) = d+1$ for all $b \in W_1$. Thus $N(x_1) = \{v_1, \ldots, v_d\}$ and $W = [W_0, W_1]$ is a matching, i.e., $W = \{a_1b_1, \ldots, a_db_d\}$. Therefore the subgraph H_1 gives a contradiction unless H_1 is d-regular. In this case let us consider the graph $\hat{G} = (G - x_1 - W) \cup \{a_1v_1, \ldots, a_dv_d\}$ which clearly has girth at least g. Moreover $deg_{\hat{G}}(b_i) = deg(b_i) - 1 = d$ and every vertex different from b_i has the same degree it had in G. Thus we may suppose that $\hat{G}$ is d-regular because otherwise $\hat{G}$ would be a $(\{d, d+1\}; g^*)$-graph with girth $g^* \geq g$ and smaller than G, a contradiction. Moreover, we may assume that $d_{H_1}(b_1, v_1) = (g-3)/2$ and $d_{H_1}(b_1, N(x_1) - v_1) = (g-1)/2$. Thus we have

$$\begin{aligned} d_{\hat{G}}(b_1, u_2) &\geq \min\{d_{H_1}(b_1, v_2) + |\{v_2a_2\}| \\ &\quad + d_{H_0}(a_2, u_2); d_{H_1}(b_1, v_1) + |\{v_1a_1\}| + d_{H_0}(a_1, u_2)\} \\ &\geq \min\{\frac{g-1}{2} + 1 + \frac{g-3}{2}; \frac{g-3}{2} + 1 + \frac{g+1}{2}\} \\ &= g - 1, \end{aligned}$$

which implies that we can add to $\hat{G}$ the edge u_2b_1 to obtain a graph without decreasing the girth g. As this new graph is smaller than G and has degrees $\{d, d+1\}$ we get a contradiction to the monotonicity Theorem 1, and we are done.

Subcase (a.3.2): For all $z \in V(H_1) \cap N_{(g-1)/2}(W_1)$ there exists $v \in N(x_1)$ and $b \in W_1$ such that $d(b, v) \geq (g+1)/2$.

Let $x_1 \in V(H_1) \cap N_{(g-1)/2}(W_1)$, $v_1 \in N(x_1)$ and $b^* \in W_1$ be such that $d(b^*, v_1) \geq (g+1)/2$. By Lemma 4, there exists a unique edge $a^*b^* \in W$ to which the vertex $a^* \in W_0$ is incident, and there exists a vertex $x^* \in V(H_0)$ of $deg(x^*) = d$ such that $d(x^*, W_0) = d(x^*, a^*) = (g-3)/2$ and $N_{(g-3)/2}(x^*) \cap W_0 = \{a^*\}$. Further, $N(x^*)$ can be labeled as $\{z_1, z_2, \ldots, z_d\}$, and W_0 can be labeled as $\{a_1, a_2, \ldots, a_d\}$, where $a_1 = a^*$, so that $N_{(g-5)/2}(z_1) \cap W_i = \{a_1\}$, $N_{(g-3)/2}(z_k) \cap W_i = \{a_k\}$ and $deg(z_k) = d$ for every $k > 1$. Furthermore, $[N_{(g-3)/2}(x^*) \cap W_0, W_1] = \{a_1b^*\}$

Let $\Gamma = G - \{x^*, x_1\}$. We obtain

$$\begin{aligned} &d_\Gamma(z_1, v_1) \\ &= \min\{d_\Gamma(z_1, a_1) + 1 + d_\Gamma(b^*, v_1);\ d_\Gamma(z_1, a_j) + 1 + d_\Gamma(b', v_1), j \geq 2, a_jb' \in W\} \\ &\geq \min\{\frac{g-5}{2} + 1 + \frac{g+1}{2};\ \frac{g-1}{2} + 1 + \frac{g-3}{2}\} = g - 1. \end{aligned}$$

Moreover, $d_{H_0}(z_k, W_0) = (g-3)/2$ for all $z_k \in N(x^*) - z_1$ and for $k > 1$ there exists a unique vertex say $b_k \in W_1$ for which $a_k b_k \in W$. As for each $b \in W_1$, $|N_{(g-3)/2}(b) \cap N(x_1)| \le 1$ (otherwise cycles of length less than g appear) we may denote by v_k the vertex in $N(x_1) - v_1$ such that $d(b_k, v_k) = (g-3)/2$, if any. Thus we obtain

$$d_\Gamma(z_k, N(x_1) \setminus \{v_1, v_k\}) = d(z_k, a_k) + 1 + d(b_k, N(x_1) \setminus \{v_1, v_k\})$$
$$\ge \tfrac{g-3}{2} + 1 + \tfrac{g-1}{2} = g - 1.$$

Let us consider $X_0 = N(x^*) - z_1$ and $X_1 \subseteq N(x_1) - v_1$, with $|X_1| = d-1$. It is clear that $|deg_{\mathcal{B}_\Gamma}(z_k)| \ge d-2 \ge 1$ for all $z_k \in N(x^*) - u_1$ yielding $|E(\mathcal{B}_\Gamma)| \ge (d-2)(d-1) = (d-1)^2 - (d-1)$.

First, suppose that $|deg_{\mathcal{B}_\Gamma}(v)| \ge 1$ for all $v \in N(x_1) - v_1$. From Lemma 7, there is a matching M which covers every vertex in $N(x^*) - z_1$ and every vertex in $N(x_1) - v_1$ if $deg(x_1) = d$. In this case $G^* = (G - \{x^*, x_1\}) \cup M \cup \{z_1 v_1\}$ is a graph with girth $g^* \ge g$ and smaller than G whose vertices have the same degree they had in G; thus G^* is a $(\{d, d+1\}; g^*)$-graph and we are done. Thus suppose that $deg(x_1) = d+1$ and that after adding the matching $M \cup \{z_1 v_1\}$ to $G - \{x^*, x_1\}$ the vertex $v_{d+1} \in (N(x_1) - v_1) \setminus X_1$ remains of degree $d-1$. By Lemma 4 every z_k, $k > 1$, has degree d in G, and we have proved that $d(z_k, N(x_1) \setminus \{v_1, v_k\}) \ge g-1$. Then we add one extra edge $z_k v_{d+1}$ to G^* obtaining a new $(\{d, d+1\}; g^*)$-graph with $g^* \ge g$ and smaller than G, a contradiction to the monotonicity Theorem 1, so we are done.

Therefore we must suppose that there exists $v_2 \in N(x_1) - v_1$ such that $|deg_{\mathcal{B}_\Gamma}(v_2)| = 0$. This implies that $d(v_2, b) = (g-3)/2$ for all $b \in W_1 - b^*$, hence $d(v, W_1 - b^*) = (g-1)/2$ for all $v \in N(x_1) - v_2$. First suppose that $d(v_2, b^*) \ge (g+1)/2$; then $d_\Gamma(z_1, v_2) \ge g-1$, $d_\Gamma(z_k, N(x_1) - v_2) = g-1$ for all $k \ge 2$, thus we consider the set $X_1 \subseteq N(x_1) - v_2$ with $|X_1| = d-1$. It is clear that $|deg_{\mathcal{B}_\Gamma}(w)| \ge d-1$ for all $w \in X_0 \cup X_1$. Using Lemma 7 and reasoning as before we get a contradiction. Therefore we must suppose that $d(v_2, b^*) \le (g-1)/2$. Since $N(x_1) - v_2 \subseteq N_{(g-1)/2}(W_1) \cap V(H_1)$ we have by hypothesis that for all $v \in N(x_1) - v_2$ there exists $\hat{v}_1 \in N(v)$ and $\hat{b}^* \in W_1$ such that $d(\hat{b}^*, \hat{v}_1) \ge (g+1)/2$. As the behavior of any $v \in N(x_1) - v_2$ is the same as vertex x_1, reasoning as before we get a contradiction unless for all $v \in N(x_1) - v_2$ there exists $\hat{v}_2 \in N(v) - \hat{v}_1$ such that $|deg_{\mathcal{B}_{\hat{\Gamma}}}(\hat{v}_2)| = 0$ satisfying $d(\hat{v}_2, b) = (g-3)/2$ for all $b \in W_1 - \hat{b}^*$ and $d(\hat{v}_2, \hat{b}^*) \le (g-1)/2$. Therefore we conclude that every vertex $b \in W_1$ has

$deg_{H_1}(b) = deg(x_1) \in \{d, d+1\}$. Now considering the same graph as in Subcase (a.3.1) we get a contradiction.

Case (b): $\mu_0 = \mu_1 = (g-1)/2$.

Let $x_0 \in V(H_0)$ and $x_1 \in V(H_1)$ satisfy $d(x_i, W_i) = (g-1)/2$, $i = 0, 1$.

First of all note that there must exist a vertex in $N(x_0)$ of degree d, otherwise $G - x_0$ would be either a $\{d, d+1\}$-graph or a d-regular graph. In the former case we get a contradiction because $G - x_0$ is smaller than G and has girth at least g. And in the latter case we consider the graph $(G - x_0) \cup \{u_i x_1\}$ with $u_i \in N(x_0)$, which gives again a contradiction. Similarly, note that there must exist a vertex in $N(x_1)$ of degree d.

Suppose that $deg(x_0) = deg(x_1) = r$ with $r \in \{d, d+1\}$. Let $X_0 = N(x_0), X_1 = N(x_1)$ and $\Gamma = G - \{x_0, x_1\}$. Define $A = \{u_i v_j : u_i \in X_0, v_j \in X_1, d_\Gamma(u_i, v_j) \leq g-2\}$ and consider $\mathcal{B}_\Gamma = K_{|X_0|,|X_1|} - A$. Note that every (u_i, v_j)-path in G goes through an edge of W. Therefore every edge in W gives rise to at most one element in A, otherwise G would contain a cycle of length at most $2(g-3)/2 + 2 = g-1$. Hence $|A| \leq |W| \leq d$ and $|E(\mathcal{B}_\Gamma)| = |K_{r,r}| - |A| \geq r^2 - d$.

If $r = d+1$ then $|E(\mathcal{B}_\Gamma)| \geq (d+1)^2 - d = d^2 + d + 1$ and by Lemma 6, the graph $\mathcal{B}_\Gamma$ contains a perfect matching M. Therefore the graph $G' = G - \{x_0, x_1\} \cup M$ has fewer vertices than G and girth at least g producing a contradiction unless G' is regular of degree d. In this case we consider the graph $G'' = G' \cup \{uv\}$ where $u \in N(x_0)$ is such that $d(u, W_0) = (g-1)/2$ (such a vertex must exist because $deg(x_0) = d+1$ and $|W_1| \leq d$) and $v \in N(x_1)$ such that $uv \notin M$. As G'' is a $(\{d, d+1\}; g)$-graph with fewer vertices than G and girth g a contradiction is again obtained.

Suppose $r = d$. If $deg_{\mathcal{B}_\Gamma}(z) \geq 1$ for all $z \in \mathcal{B}_\Gamma$, then by Lemma 7 there exists a perfect matching M between X_0 and X_1; reasoning as before we obtain again a contradiction. Hence, we may assume that $deg_{\mathcal{B}_\Gamma}(u_1) = 0$ for some $u_1 \in X_0$. This implies that $d_\Gamma(u_1, v_j) = g-2$ for all $v_j \in N(x_1)$, or equivalently $d_\Gamma(v_j, W_1) = (g-3)/2$ for all $v_j \in N(x_1)$. From this, and because $g \geq 5$, we get $|W_1| \geq |N(x_1)| = d$, yielding $|W_1| = d$ (since $d = |W| \geq |W_1|$), and also $N_{(g-3)/2}(v_j) \cap W_1 = \{b_j\}$ for all $v_j \in N(x_1)$. That is, $|N(b_j) \cap W_0| = 1$ for every $b_j \in W_1$. Also we have $N_{(g-1)/2}(u_1) \cap W_1 = W_1$, hence $N_{(g-3)/2}(u_1) \cap W_0 = W_0$ and thus $d(u_i, W_0) = (g-1)/2$ for $i \geq 2$.

Let $u_k \in N(x_0)$, $k \geq 2$, define $\Gamma_k = G - \{u_k, x_1\}$ and consider the sets

$$X_k = \begin{cases} N(u_k) & \text{if } deg(u_k) = d; \\ N(u_k) - x_0 & \text{if } deg(u_k) = d+1; \end{cases}$$

$$X_1 = N(x_1);$$

$$A_k = \{z_i v_j : z_i \in X_k, v_j \in X_1, d_{\Gamma_k}(z_i, v_j) \leq g-2\}.$$

Let $\mathcal{B}_{\Gamma_k} = K_{|X_k|,|X_1|} - A_k$.

If $deg_{\mathcal{B}_{\Gamma_k}}(z) \geq 1$ for all $z \in X_k$, we get a perfect matching M between X_k and $N(x_1)$ by Lemma 7; if $deg(u_k) = d$ the graph $\Gamma_k \cup M$ yields a contradiction; if $deg(u_k) = d+1$ the graph $\Gamma_k \cup M \cup \{x_0 v_j\}$, where v_j is a vertex of $N(x_1)$ with degree d, yields again a contradiction. Therefore we can suppose that for every $u_k \in N(x_0) - u_1$ there exists $\hat{z}_k \in N(u_k)$ such that $d_{\Gamma_k}(\hat{z}_k, v_j) = g-2$ for all $v_j \in N(x_1)$. Hence, $N_{(g-3)/2}(\hat{z}_k) \cap W_0 = W_0$, that is $d_{\Gamma_k}(\hat{z}_k, a_j) = (g-3)/2$ for each $a_j \in W_0$. Therefore $deg_{H_0}(a_j) = d$, $deg(a_j) = d+1$ and $[W_0, W_1]$ is a matching (recall that $|N(b_j) \cap W_0| = 1$ for every $b_j \in W_1$). We can now use the same graph $\hat{G} = (G - \{x_0\} - W) \cup \{b_1 u_1, \ldots, b_d u_d\}$ as used in Case (a.3.2), arriving again at a contradiction.

The only remaining case occurs when x_0 and x_1 have different degrees. Let us suppose $deg(x_0) = d$ and $deg(x_1) = d+1$. As $deg(x_1) = d+1 > |W_1|$, there exists, say $v_{d+1} \in N(x_1)$, such that $d(v_{d+1}, W_1) = (g-1)/2$. We proceed as before, with the sets $X_0 = N(x_0)$ and $X_1 = N(x_1) - v_{d+1}$, finding a graph G' with fewer vertices and the same girth and degrees as G, except for the vertex v_{d+1}. Recall that there must exist a vertex $y \in N(x_0)$ such that $deg(y) = d$. Then we construct the graph $G^* = G' \cup \{y v_{d+1}\}$, which is a new $\{d, d+1\}$-graph with girth g, arriving at a contradiction. This ends the proof of the theorem. $\square$

Acknowledgement

This research was supported by the Ministry of Education and Science, Spain, and the European Regional Development Fund (ERDF) under project MTM2008-06620-C03-02/MTM; also by Catalonian government 2009 SGR 1298. The author J. Salas has been supported by CONACYT.

References

[1] M.C. Balbuena, A. Carmona, J. Fàbrega, and M.A. Fiol. On the order and size of s-geodetic digraphs with given connectivity. *Discrete Math.* 174:19–27, 1997.

[2] C. Balbuena, P. García-Vázquez, and X. Marcote. Sufficient conditions for λ'-optimality in graphs with girth g. *J. Graph Theory* 52:73–86, 2006.

[3] C. Balbuena, D. González-Moreno, and X. Marcote. On the connectivity of semiregular cages. *Networks* 56(8):81–88, 2010.

[4] C. Balbuena and X. Marcote. Monotonicity of the order of $(D;g)$-cages. Submitted.

[5] C. Balbuena and X. Marcote. Diameter and connectivity of $(D;g)$-cages. *International Journal of Computer Math.*, accepted.

[6] F.T. Boesch. Synthesis of reliable networks–A survey. *IEEE Trans Reliability*, 35:240–246, 1986.

[7] F.T. Boesch and R. Tindell. Circulants and their connectivities. *J. Graph Theory* 8:487–499, 1984.

[8] G. Chartrand, R.J. Gould, and S.F. Kapoor. Graphs with prescribed degree set and girth. *Period. Math Hungar.* 6:261–266, 1981.

[9] G. Chartrand and L. Lesniak. *Graphs and digraphs*, 3rd ed. Chapman and Hall, London, 1996.

[10] P. Erdös and H. Sachs. Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe*, 12:251–257, 1963.

[11] H. Fu, K. Huang, and C. Rodger. Connectivity of cages. *J. Graph Theory*, 24:187–191, 1997.

[12] P. Hall. On representatives of subsets. *J. London Math. Soc.*, 10:26–30, 1935.

[13] M. Imase, T. Soneoka, and K. Okada. Connectivity of regular directed graphs with small diameter. *IEEE Trans Comput.*, C-34:267–273, 1985.

[14] T. Jiang. Short even cycles in cages with odd girth. *Ars Combin.*, 59:165–169, 2001.

[15] T. Jiang and D. Mubayi. Connectivity and separating sets of cages. *J. Graph Theory*, 29:35–44, 1998.

[16] Y. Lin, C. Balbuena, X. Marcote, and M. Miller. On the connectivity of (k, g)-cages of even girth. *Discrete Math.*, 308:3249–3256, 2008.

[17] Y. Lin, M. Miller, and C. Balbuena. Improved lower bound for the vertex connectivity of $(\delta; g)$-cages. *Discrete Math.*, 299:162–171, 2005.

[18] Y. Lin, M. Miller, C. Balbuena, and X. Marcote. All $(k; g)$-cages are edge-superconnected. *Networks*, 47:102–110, 2006.

[19] X. Marcote and C. Balbuena. Edge-superconnectivity of cages. *Networks*, 43:54–59, 2004.

[20] X. Marcote, C. Balbuena, I. Pelayo, and J. Fbrega. (δ, g)-cages with $g \geq 10$ are 4-connected. *Discrete Math.*, 301:124–136, 2005.

M-Matrix Inverse problem for distance-regular graphs

Enrique Bendito, Ángeles Carmona, Andrés M. Encinas and Maragarida Mitjana
Universitat Politècnica de Catalunya
Barcelona

Abstract

We analyze when the Moore–Penrose inverse of the combinatorial Laplacian of a distance–regular graph is a M–matrix; that is, it has non–positive off–diagonal elements or, equivalently when the Moore-Penrose inverse of the combinatorial Laplacian of a distance–regular graph is also the combinatorial Laplacian of another network. When this occurs we say that the distance–regular graph has the M–property. We prove that only distance–regular graphs with diameter up to three can have the M–property and we give a characterization of the graphs that satisfy the M-property in terms of their intersection array. Moreover we exhaustively analyze the strongly regular graphs having the M-property and we give some families of distance regular graphs with diameter three that satisfy the M-property.

1 Introduction

Very often problems in biological, physical and social sciences can be reduced to problems involving matrices which have some special structure. One of the most common situation is when the matrix has non–positive off–diagonal and non–negative diagonal entries; that is $L = kI - A$, $k > 0$ and $A \geq 0$, where the diagonal entries of A are less or equal than k. These matrices appear in relation to systems of equations or eigenvalue problems in a broad variety of areas including finite difference methods for solving partial differential equations, input–output production and growth models in economics or Markov processes in probability and statistics. Of course, the combinatorial community can recognize within this type of matrices, the combinatorial Laplacian of a k–regular graph where A is its adjacency matrix.

If k is at least the spectral radius of A, then L is called a *M–matrix.* We remark that M–matrices arise naturally in some discretizations of differential operators, particularly those with a minimum/maximum principle, such as the Laplacian, and as such are well–studied in scientific computing. In fact M–matrices satisfy monotonicity properties that are the discrete counterpart of the minimum principle, and it makes them suitable for the resolution of large sparse systems of linear equations by iterative methods.

As well as a symmetric, irreducible and non–singular M–matrix appears as the discrete counterpart of a Dirichlet problem for a self–adjoint elliptic operator, its inverse corresponds with the Green operator associated with the boundary value problem. On the other hand, when the M–matrix is singular, it can be seen as a discrete analogue of the Poisson equation for a self–adjoint elliptic operator on a manifold without boundary and then, its Moore–Penrose inverse corresponds with the Green operator too. A well–known property of an irreducible non–singular M–matrix is that its inverse is non–negative, [3]. However, the scenario changes dramatically when the matrix is an irreducible and singular M–matrix. In this case, it is known that the matrix has a generalized inverse which is non–negative, but this is not always true for any generalized inverse. For instance, it may happen that the Moore–Penrose inverse has some negative entries. We focus here in studying when the Moore–Penrose inverse of a symmetric, singular and irreducible M–matrix is itself an M–matrix. In particular, we study the case of distance–regular graphs and more specifically strongly regular graphs.

2 Preliminaries

The triple $\Gamma = (V, E, c)$ denotes a finite network; that is, a finite connected graph without loops nor multiple edges, with vertex set V, whose cardinality equals n, and edge set E, in which each edge $\{x, y\}$ has been assigned a *conductance* $c(x, y) > 0$. So, the conductance can be considered as a symmetric function $c\colon V \times V \longrightarrow [0, +\infty)$ such that $c(x, x) = 0$ for any $x \in V$ and moreover, $x \sim y$, that is vertex x is adjacent to vertex y, iff $c(x,y) > 0$. We define the *degree function* k as $k(x) = \sum\limits_{y\in V} c(x,y)$, for each $x \in V$. The usual distance from vertex x to vertex y is denoted by $d(x,y)$ and $D = \max\{d(x,y) : x,y \in V\}$ stands for the *diameter* of Γ. We denote as $\Gamma_i(x)$ the set of vertices at distance i from vertex x, $\Gamma_i(x) = \{y : d(x,y) = i\}$ $0 \le i \le D$. The *complement* of Γ is defined as the graph $\overline{\Gamma}$ on the same vertices such that two vertices are adjacent iff they are not adjacent in Γ; that is $x \sim y$ in $\overline{\Gamma}$ iff $c(x,y) = 0$.

The set of real–valued functions on V is denoted by $\mathcal{C}(V)$. When necessary, we identify the functions in $\mathcal{C}(V)$ with vectors in $\mathbb{R}^{|V|}$ and the endomorphisms of $\mathcal{C}(V)$ with $|V|$–order square matrices.

The *combinatorial Laplacian* or simply the *Laplacian* of the network Γ is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}(u)(x) = \sum_{y\in V} c(x,y)\Big(u(x) - u(y)\Big) = k(x)u(x) - \sum_{y\in V} c(x,y)\,u(y), \quad x \in V.$$

It is well–known that $\mathcal{L}$ is a positive semi–definite self–adjoint operator and has 0 as its lowest eigenvalue whose associated eigenfunctions are constant. So, $\mathcal{L}$ can be interpreted as an irreducible, symmetric, diagonally dominant and singular M–matrix, L. Therefore, the Poisson equation $\mathcal{L}(u) = f$ on V has solution iff $\sum\limits_{x\in V} f(x) = 0$ and, when this happens, there exists a unique solution $u \in \mathcal{C}(V)$ such that $\sum\limits_{x\in V} u(x) = 0$, see [1].

The *Green operator* is the linear operator $\mathcal{G} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ that assigns to any $f \in \mathcal{C}(V)$ the unique solution of the Poisson equation with data $f - \frac{1}{n}\sum\limits_{x\in V} f(x)$ such that $\sum\limits_{x\in V} u(x) = 0$. It is easy to prove that $\mathcal{G}$ is a positive semi–definite self–adjoint operator and has 0 as its lowest eigenvalue whose associated eigenfunctions are constant. Moreover, if $\mathcal{P}$ denotes the projection on the subspace of constant functions then,

$$\mathcal{L} \circ \mathcal{G} = \mathcal{G} \circ \mathcal{L} = \mathcal{I} - \mathcal{P}.$$

In addition, we define the *Green function* as $G : V \times V \longrightarrow \mathbb{R}$ given by $G(x,y) = \mathcal{G}(\varepsilon_y)(x)$, where ε_y stands for the Dirac function at y. Therefore, interpreting $\mathcal{G}$, or G, as a matrix it is nothing else but $\mathsf{L}^\dagger$ the Moore–Penrose inverse of L, the matrix associated with $\mathcal{L}$. In consequence, $\mathsf{L}^\dagger$ is a M–matrix iff $G(x,y) \le 0$ for any $x,y \in V$ with $x \ne y$ and then $\mathcal{G}$ can be identified with the combinatorial Laplacian of a new connected network with the same vertex set, that we denote by $\Gamma^\dagger$.

From now on we will say that a network Γ has the *M–property* iff $\mathsf{L}^\dagger$ is a M–matrix.

Next we obtain a necessary and sufficient condition for a network to have the M–property. In [1] it was proved that for any $x \in V$, there exists a unique $\nu^x \in \mathcal{C}(V)$ such that $\nu^x(x) = 0$, $\nu^x(y) > 0$ for any $y \neq x$ and verifying

$$\mathcal{L}(\nu^x) = 1 - n\varepsilon_x \quad \text{on } V. \tag{1}$$

We call ν^x the *equilibrium measure of* $V \setminus \{x\}$ and then we define *capacity* as the function $\mathsf{cap} \in \mathcal{C}(V)$ given by $\mathsf{cap}(x) = \sum\limits_{y \in V} \nu^x(y)$.

Theorem 1 *The network Γ has the M–property iff for any $y \in V$*

$$\mathsf{cap}(y) \leq n\nu^y(x) \qquad \textit{for any } x \sim y.$$

In this case, $\overline{\Gamma}$ is a subgraph of the subjacent graph of $\Gamma^\dagger$.

Proof: The Green function is given by

$$G(x, y) = \frac{1}{n^2}\big(\mathsf{cap}(y) - n\,\nu^y(x)\big),$$

see [1]. Therefore, $\mathsf{L}^\dagger$ is a M–matrix iff

$$\mathsf{cap}(y) \leq n \min_{x \in V\setminus\{y\}} \big\{\nu^y(x)\big\}.$$

The results follow by keeping in mind that $\min\limits_{x \in V\setminus\{y\}} \big\{\nu^y(x)\big\} = \min\limits_{x\sim y} \big\{\nu^y(x)\big\}$, since if the minimum is attained at $z \not\sim y$, then

$$1 = \mathcal{L}(\nu^y)(z) = \sum_{x \in V} c(z, x)\big(\nu^y(z) - \nu^y(x)\big) \leq 0,$$

which is a contradiction. $\square$

3 Distance–regular graphs with the M–property

We aim here at characterizing when the Moore–Penrose inverse of the combinatorial Laplacian matrix of a distance–regular graph is a M–matrix.

A connected graph Γ is called *distance–regular* if there are integers b_i, c_i, $i = 0, \dots, D$ such that for any two vertices $x, y \in \Gamma$ at distance $i = d(x, y)$, there are exactly c_i neighbours of y in $\Gamma_{i-1}(x)$ and b_i neighbours of y in $\Gamma_{i+1}(x)$, where for any vertex $x \in \Gamma$ the set of vertices at distance i from it is denoted by $\Gamma_i(x)$. Moreover, $|\Gamma_i(x)|$ will be denoted by k_i. In particular, Γ is regular of degree $k = b_0$. The sequence

$$\iota(\Gamma) = \{b_0, b_1, \dots, b_{D-1}; c_1, \dots, c_D\},$$

is called the *intersection array* of Γ. In addition, $a_i = k - c_i - b_i$ is the number of neighbours of y in $\Gamma_i(x)$, for $d(x, y) = i$. Clearly, $b_D = c_0 = 0$, $c_1 = 1$ and the diameter of Γ is D. Usually, the parameters a_1 and c_2 are denoted by λ and μ, respectively. For all the properties related with distance–regular graphs we refer the reader to [4, 7].

The parameters of a distance–regular graph satisfy many relations, among them we will make an extensive use of the following:

(i) $k_0 = 1$ and $k_i = \dfrac{b_0 \cdots b_{i-1}}{c_1 \cdots c_i}$, $\quad i = 1, \dots, D$.

(ii) $n = 1 + k + k_2 + \cdots + k_D$.

(iii) $k > b_1 \geq \cdots \geq b_{D-1} \geq 1$.

(iv) $1 \leq c_2 \leq \cdots \leq c_D \leq k$.

(v) If $i + j \leq D$, then $c_i \leq b_j$ and $k_i \leq k_j$ when, in addition, $i \leq j$.

Additional relations between the parameters give more information about the structure of distance–regular graphs. For instance, Γ is *bipartite* iff $a_i = 0$, $i = 1, \dots, D$, whereas Γ is *antipodal* iff $b_i = c_{D-i}$, $i = 0, \dots, D$, $i \neq \lfloor \frac{D}{2} \rfloor$ and then $b_{\lfloor \frac{D}{2} \rfloor} = t c_{\lceil \frac{D}{2} \rceil}$, $t \geq 1$ and Γ is an antipodal $(t+1)$–cover of its folded graph, see [7, Prop. 4.2.2].

The following lemma shows that the equilibrium measures for a distance–regular graph, and hence the capacity function, can be expressed in terms of the parameters of its intersection array, see [1, Prop. 4.1] for the details.

Lemma 2 *Let Γ be a distance–regular graph. Then, for all $x, y \in V$*

$$\nu^x(y) = \sum_{j=0}^{d(x,y)-1} \frac{1}{k_j b_j}\Big(\sum_{i=j+1}^{D} k_i\Big) \quad \textit{and} \quad \mathsf{cap}(x) = \sum_{j=0}^{D-1} \frac{1}{k_j b_j}\Big(\sum_{i=j+1}^{D} k_i\Big)^2.$$

Proposition 3 *A distance–regular graph Γ has the M–property iff*

$$\sum_{j=1}^{D-1} \frac{1}{k_j b_j}\Big(\sum_{i=j+1}^{D} k_i\Big)^2 \leq \frac{n-1}{k}.$$

Moreover, the subjacent graph of $\Gamma^\dagger$ is K_n when the above inequality is strict and $\overline{\Gamma}$ otherwise.

Proof: From Theorem 1 and Lemma 2, the Moore–Penrose inverse of L is a M–matrix iff

$$\sum_{j=0}^{D-1} \frac{1}{k_j b_j}\Big(\sum_{i=j+1}^{D} k_i\Big)^2 \leq \frac{n(n-1)}{k}$$

that is, iff

$$\frac{(n-1)^2}{k} + \sum_{j=1}^{D-1} \frac{1}{k_j b_j}\Big(\sum_{i=j+1}^{D} k_i\Big)^2 \leq \frac{n(n-1)}{k}.$$

Finally, the above inequality is an equality iff $G(x,y) = 0$ when $d(x,y) = 1$, since for any $y \in V$, $\nu^y(x)$ is constant on $\Gamma_1(y)$. Therefore, the subjacent graph of $\Gamma^\dagger$ is $\overline{\Gamma}$. $\square$

A distance–regular graph of order n has diameter $D = 1$ iff it is the complete graph K_n. In this case, the above inequality holds since the left side term vanishes. Therefore, any complete graph has the M–property. In fact, $\mathsf{L}^\dagger = \frac{1}{n^2}\,\mathsf{L}$, see [2], and hence, $\Gamma^\dagger$ is also a complete network.

Corollary 4 *If Γ has the M–property and $D \geq 2$, then*

$$\lambda \leq 3k - \frac{k^2}{n-1} - n.$$

and hence $n < 3k$.

Proof: When $D \geq 2$, from the inequality in Proposition 3 we get that

$$\frac{(n-k-1)^2}{kb_1} \leq \sum_{j=1}^{D-1} \frac{1}{k_j b_j}\Big(\sum_{i=j+1}^{D} k_i\Big)^2 \leq \frac{n-1}{k}.$$

Therefore, $(n-1-k)^2 \leq (n-1)b_1 = (n-1)(k-1-\lambda)$ and the upper bound for λ follows. In addition, this inequality implies that $0 \leq \lambda < 3k-n$ and then $3k > n$. $\square$

The inequality $3k > n$ turns out to be a strong restriction for a distance–regular graph to have the M–property. For instance, if $n \geq 3$, the n–cycle, C_n, is a distance–regular graph with diameter $D = \left\lfloor \frac{n}{2} \right\rfloor$ whose intersection array is

$$\iota(C_n) = \{2, 1, \dots, 1; 1, \dots, 1, c_D\},$$

where $c_D = 1$ when n is odd and $c_D = 2$ when D is even, see [7]. So, if C_n has the M–property, necessarily $6 > n$ and this occurs iff either $D = 1$; that is $n = 3$, or $D = 2$; that is $n = 4, 5$. Moreover for $n = 4, 5$, C_n has the M–property since

$$(\mathsf{L}^{\dagger})_{ij} = \frac{1}{12n}\Big(n^2 - 1 - 6|i-j|(n - |i-j|)\Big), \qquad i, j = 1, \dots, n,$$

see for instance, [2, 10].

In the following result we generalize the above observation, by showing that only distance–regular graph with small diameter can satisfy the M–property.

Proposition 5 *If* Γ *is a distance–regular graph with the* M*–property, then* $D \leq 3$.

Proof: If $D \geq 4$, then from property (v) of the parameters, $k = k_1 \leq k_i$, $i = 2, 3$ and hence,

$$3k < 1 + 3k \leq 1 + k + k_2 + k_3 \leq n,$$

and hence Γ has not the M–property. $\square$

3.1 Strongly regular graphs

A distance–regular graph whose diameter equals 2 is called *strongly regular graph.* This kind of distance–regular graph is usually represented throughout the four parameters (n, k, λ, μ) instead its intersection array, see [7, 9]. Clearly the four parameters of a strongly regular graph are not independent, since

$$(n-1-k)\mu = k(k-1-\lambda). \tag{2}$$

For this reason some authors drop the parameter n in the above array, see for instance [4]. Moreover, Equality (3) implies that $2k-n \le \lambda < k-1$, since $1 \le \mu \le k$ and $D = 2$.

Observe that the only n–cycles satisfying the M–property are precisely C_3, that is the complete graph with 3 vertices, and C_4 and C_5 that are strongly regular graphs.

In the following result we characterize those strongly regular graphs that have the M–property, in terms of their parameters.

Proposition 6 *A strongly regular graph with parameters* (n, k, λ, μ) *has the M–property iff*

$$\mu \ge k - \frac{k^2}{n-1}.$$

Proof: Clearly for $D = 2$ the inequality in Corollary 4 characterizes the strongly regular graphs satisfying the M–property. The result follows taking into account that from Equality (3)

$$\lambda \le 3k - \frac{k^2}{n-1} - n \iff k(n-1-k) \le \mu(n-1). \qquad \square$$

Kirkland et al. in [11, Theorem 2.4] gave another characterization of strongly regular graphs with the M–property in terms of the combinatorial Laplacian eigenvalues.

It is straightforward to verify that Petersen graph does not have the M–property. So, it is natural to ask if there exist many strongly regular graphs satisfying the above inequality. Prior to answer this question, we recall that if Γ is a strongly regular graph with parameters (n, k, λ, μ), then its complement graph is also a strongly regular graph with parameters $(n, n-k-1, n-2-2k+\mu, n-2k+\lambda)$, see for instance [5], which in particular implies that $\mu \geq 2(k+1) - n$. Strongly regular graphs with the same parameters as their complement are called *conference graphs* and then their parameters are $(4m+1, 2m, m-1, m)$ where $m \geq 2$. Moreover it is known that such a graph exits iff $m = p^2 + q(q+1)$, where $p, q \geq 1$, see [9].

Now we are ready to answer the raised question.

Corollary 7 *If Γ is strongly regular graph, then either Γ or $\overline{\Gamma}$ has the M–property. Moreover, both of them have the M–property iff Γ is a conference graph.*

Proof: If we define $\bar{k} = n - k - 1$, $\bar{\lambda} = n - 2 - 2k + \mu$ and $\bar{\mu} = n - 2k + \lambda$, then

$$\bar{k} - \frac{\bar{k}^2}{n-1} = k - \frac{k^2}{n-1}$$

and hence

$$\bar{\mu} \geq \bar{k} - \frac{\bar{k}^2}{n-1} \Longleftrightarrow \lambda \geq 3k - \frac{k^2}{n-1} - n \Longleftrightarrow \mu \leq k - \frac{k^2}{n-1},$$

where the equality in the left side holds iff the equality in the right side holds. Moreover, any of the above inequalities is an equality iff $\bar{\mu} = \mu$ and $\bar{\lambda} = \lambda$; that is iff Γ is a conference graph. The remaining claims follow from Proposition 6. $\square$

Many strongly regular graphs appear associated with the so–called partial geometries. A *Partial Geometry with parameters* $s, t, \alpha \geq 1$, $pg(s, t, \alpha)$, is an incident structure of points and lines such that every line has $s+1$ points, every point is on $t+1$ lines, two distinct lines meet in at most one point and given a line and a point not in it, there are exactly α lines through the point which meet the line. Therefore, the parameters of a partial geometry satisfy the inequalities $1 \leq \alpha \leq \min\{t+1, s+1\}$. We refer the reader to the surveys [6, 8, 9] for the main properties of partial geometries and their relation with strongly regular graphs.

The number of points and lines in $pg(s, t, \alpha)$ are $n = \dfrac{1}{\alpha}(s+1)(st+\alpha)$ and $\ell = \dfrac{1}{\alpha}(t+1)(st+\alpha)$, respectively. The *point graph of* $pg(s, t, \alpha)$ has the points as vertices and two vertices are adjacent iff they are collinear. Therefore, it is a regular graph with degree $k = s(t+1)$. Moreover, when

$\alpha = s+1$, the partial geometry is called *Linear space* and its point graph is the complete graph K_n. When $\alpha \leq s$, the point graph is a strongly regular graph with parameters $\big(n, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\big)$. A strongly regular graph is called *pseudo geometric graph* if its associated parameters are of the former form.

Corollary 8 *A pseudo geometric graph with parameters* $\big(n, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\big)$ *has the M–property iff*

$$\alpha(2ts+t+\alpha) \geq st(s+1).$$

Next we study when the point graphs associated with some well–known families of partial geometries, o more generally when some families of pseudo geometric graphs, verify the M–property.

1. *Dual Linear Spaces:* In this case $\alpha = t+1$ and hence the point graph has the M–property iff $s \leq 2(t+1)$. When, $t=1$ and $s=m-2$ the corresponding pseudo geometric graph are the so–called *triangular graph* T_m whose parameters are $\big(\binom{m}{2}, 2(m-2), m-2, 4\big)$. So, T_m has the M–property iff $m = 4, 5, 6$. Notice, that T_m is also the line graph of the complete graph K_m.

2. *Transversal Designs:* In this case $\alpha = s$ and hence the corresponding pseudo geometric graph is the *complete multipartite graph* $K_{(s+1)\times(t+1)}$ whose parameters are $\big((s+1)(t+1), s(t+1), (s-1)(t+1), s(t+1)\big)$ and it has the M–property. Observe that these graphs are the complement of $s+1$ disjoint copies of K_{t+1} which are characterized as the unique graphs such that $\mu = 0$, see [9, Theorem 1.2]. Therefore, $K_{(s+1)\times(t+1)}$ are the unique strongly regular graphs such that $\mu = k$, that is the only antipodal strongly regular graphs. Finally, note that the graph $K_{(s+1)\times 2}$ is also know as *Cocktail party graph.*

3. *Dual Transversal Designs:* In this case $\alpha = t$, $t > 1$ and hence the corresponding pseudo geometric graph is the *Pseudo–Latin square graph* $PL_r(m)$ whose parameters are $(m^2, r(m-1), r^2-3r+m, r(r-1))$, where $r = t+1$ and $m = s+1$. It has the M–property iff $s \leq 2t$. For $t=2$ it is the line graph of the complete bipartite graph $K_{m,m}$, also called *squared lattice graph.*

4. *Generalized quadrangles:* In this case $\alpha = 1$, $s > 1$ and hence the parameters of the corresponding pseudo geometric graph are $\big((s+$

$1)(st+1), s(t+1), s-1, t+1\big)$. Therefore, it has the M–property iff $ts+t+1 \geq s^2 t$ and hence iff $t=1$ and $s=2$. Note that when $t=1$ these graphs are the so–called *Hamming graph $H(2, s+1)$ or Lattice.* Observe that the complement of $H(2, s+1)$ is the pseudo–latin square graph $PL_s(s+1)$ that satisfies the M–property.

When $1 < \alpha < \min\{t, s\}$, the point graph of $pg(s, t, \alpha)$ is called *Proper pseudo–geometric.* An example of this structure are the so–called *Kneser graphs $K(m, 2)$, where $m \geq 6$ is even*, in which case $s = \dfrac{m}{2} - 1$, $t = m-4$ and $\alpha = \dfrac{m}{2} - 2$. For arbitrary $m \geq 5$, the *Kneser graph* $K(m, 2)$ is the graph whose vertices represent the 2–subsets of $\{1, \ldots, m\}$, and where two vertices are connected if and only if they correspond to disjoint subsets. The parameters of the Kneser graph $K(m, 2)$ are $\big(\binom{m}{2}, \binom{m-2}{2}, \binom{m-4}{2}, \binom{m-3}{2}\big)$, that coincide with the parameters of the complement of T_m. Therefore, it has the M–property iff $m \geq 7$ as expected. In addition, for m odd $K(m, 2)$ is an example of strongly regular graph that is not a pseudo geometric graph, which also implies that the complement of a pseudo geometric graph is not necessarily a pseudo geometric graph.

3.2 Distance–regular graphs with diameter $D = 3$

In this section we characterize those distance–regular graphs with diameter 3 that have the M–property. In this case, the intersection array is

$$\iota(\Gamma) = (k, b_1, b_2; 1, c_2, c_3).$$

Again the parameters are not independent, since

$$(n-1-k)c_2 c_3 = k b_1 (b_2 + c_3). \tag{3}$$

The next result follows straightforwardly from Proposition 3.

Proposition 9 *A distance–regular graph with $D = 3$ has the M–property iff*

$$k^2 b_1 \big(b_2 c_2 + (b_2 + c_3)^2\big) \leq c_2^2 c_3^2 (n-1).$$

A simple example verifying the above condition is the 3*–cube,* Q_3. Next we study when bipartite or antipodal distance–regular graphs have the M–property. Recall that the first ones are the incidence graph of a symmetric 2–design, whereas the second ones are covers of a complete graph.

The intersection array of a bipartite distance–regular graph with diameter $D = 3$ is $\iota(\Gamma) = \big(k, k-1, k-\mu; 1, \mu, k\big)$, where $1 \le \mu \le k-1$. Then, $n - 1 = \dfrac{1}{\mu}(2k(k-1) + \mu)$ and hence Γ has the M–property iff

$$k(k-1)(4k - 5\mu) \le \mu^2.$$

Notice that the inequality holds when $\mu \ge \dfrac{4k}{5}$. For instance, it is true for $\mu = k-1$ when $k \ge 5$, and it is true for $\mu = k-2$ when $k \ge 10$.

The intersection array of an antipodal distance–regular graph with diameter $D = 3$ is $\iota(\Gamma) = \big(k, t\mu, 1; 1, \mu, k\big)$, where $t \ge 1$ and $1 \le m < k$. These graphs are the $(t+1)$–cover of the complete graph K_{k+1}. Then, $n = (t+1)(k+1)$ and hence Γ has the M–property iff

$$t(k+1)^2 \le \mu k(t+1).$$

When $t = 1$, the antipodal distance–regular graphs are known as *Taylor graphs*, $T(k, \mu)$. Then $T(k, \mu)$ has the M–property iff $(k+1)^2 \le 2k\mu$. Moreover, if Γ is a Taylor graph with $1 \le m < k-1$, it is well-known that the graph $\Gamma_{(2)}$ on the same vertices and such that two vertices of $\Gamma_{(2)}$ are adjacent if and only if their distance in Γ is 2 is also a Taylor graph whose intersection array is

$$\iota(\Gamma_{(2)}) = \big\{k, k-1-m, 1; 1, k-1-m, k\big\}.$$

Then, $\Gamma_{(2)}$ has the M–property iff $2km \le (k-2)^2 - 5$.

Corollary 10 *If Γ is the Taylor graph $T(k, m)$, then either Γ or $\Gamma_{(2)}$ has the M–property, except when $m = \dfrac{k}{2} - 2, \dfrac{k}{2} - 1, \dfrac{k}{2}, \dfrac{k}{2} + 1$ when k is even and $m = \left\lceil \dfrac{k}{2} \right\rceil - 2, \left\lceil \dfrac{k}{2} \right\rceil - 1, \left\lceil \dfrac{k}{2} \right\rceil$ when k is odd, in which case none of them has the M–property.*

Finally, the only bipartite and antipodal distance–regular graphs with $D = 3$ have intersection array

$$\iota(\Gamma) = \big\{k, k-1, 1; 1, k-1, k\big\}$$

and they are called *k–crown graphs.* Therefore, they are Taylor graphs with $\mu = k-1$ and hence they have the M–property for any $k \ge 5$.

Acknowledgement

This work has been partly supported by the Comisión Interministerial de Ciencia y Tecnología under projects MTM2007-62551 and MTM2008-06620-C03-01/MTM.

References

[1] E. Bendito, A. Carmona, A.M. Encinas. Solving boundary value problems on networks using equilibrium measures. *J. Funct. Anal.*, 171:155–176, 2000.

[2] E. Bendito, A. Carmona, A.M. Encinas, M. Mitjana. Generalized inverses of symmetric M–matrices. *Linear Algebra Appl.*, 432:2438-2454, 2010.

[3] A. Berman and R.J. Plemons. *Nonnegative matrices in the mathematical sciences.* Classics in Applied Mathematics, vol. 9, SIAM, 1994.

[4] N. Biggs. Algebraic Potential Theory on Graphs. *Bull. London Math. Soc.*, 29:641-682, 1997.

[5] J.A. Bondy and U.S.R. Murty. *Graph Theory.* Graduate Texts in Mathematics, vol 244, Springer, New York, 2008.

[6] R.C. Bose. Strongly Regular Graphs, Partial Geometries and Parcially Balanced Designs, *Pacific J. Math.*, 13:389-419, 1963.

[7] A.E. Brouwer, A.M. Cohen and A. Neumaier. *Distance–regular graphs.* Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 18, Springer-Verlag, Berlin, 1989.

[8] A.E. Brouwer and J.H. van Lint. Strongly Regular Graphs and Partial Geometries, in *Enumeration and Design*, D.M. Jackson and S.A. Vanstone (eds.), Academic Press, Canada Toronto, 1984, 85–122.

[9] P.J. Cameron, Strongly regular graphs, in *Topics in Algebraic Graph Theory*, L.W. Beineke, R.J. Wilson (eds.), Cambridge University Press, 2004, 203–221.

[10] Y. Chen, S.J. Kirkland, M. Neumann. Group generalized inverses of M–matrices associated with periodic and nonperiodic Jacobi matrices. *Linear Multilinear Algebra*, 39:325–340, 1995.

[11] S.J. Kirkland, M. Neumann. Group inverses of M–matrices associated with nonnegative matrices having few eigenvalues. *Linear Algebra Appl.*, 220:181–213, 1998.

Spectral behavior of some graph and digraph compositions

Romain Boulet
Centre national de la recherche scientifique
and Université de Toulouse
Toulouse

Abstract

Let G be a graph of order n the vertices of which are labeled from 1 to n and let $G_1, \cdots, G_n$ be n graphs. The graph composition $G[G_1, \cdots, G_n]$ is the graph obtained by replacing the vertex i of G by the graph G_i and there is an edge between $u \in G_i$ and $v \in G_j$ if and only if there is an edge between i and j in G. We first consider graph composition $G[K_k, \cdots, K_k]$ where G is regular and K_k is a complete graph and we establish some links between the spectral characterisation of G and the spectral characterisation of $G[K_k, \cdots, K_k]$. We then prove that two non isomorphic graphs $G[G_1, \cdots G_n]$ where G_i are complete graphs and G is a strict threshold graph or a star are not Laplacian-cospectral, giving rise to a spectral characterization of these graphs. We also consider directed graphs, especially the vertex-critical tournaments without non-trivial acyclic interval which are tournaments of the shape $t[\overrightarrow{C}_{k_1}, \cdots, \overrightarrow{C}_{k_m}]$, where t is a tournament and $\overrightarrow{C}_{k_i}$ is a circulant tournament. We give conditions to characterise these graphs by their spectrum.

1 Introduction

Some informations about the structure of the graph can be obtained from the spectrum of a matrix associated to the graph. The most used matrices are the adjacency matrix A and the Laplacian matrix $L = D - A$ where D

is the diagonal matrix of degrees. A graph G is *determined by its spectrum* (DS for short) if any other graph having the same spectrum as G is isomorphic to G; we shall specify the matrix only if there is a risk of confusion (we recall that a regular graph is DS with respect to A if and only if it is DS with respect to L). We can focus on a particular family $\mathcal{F}$ of graphs: a graph G is *characterised by its spectrum in* $\mathcal{F}$ if there are no other graphs in $\mathcal{F}$ cospectral non-isomorphic to G.

Let G be a graph of order n the vertices of which are labeled from 1 to n and let $G_1, \cdots, G_n$ be n graphs. The graph composition $G[G_1, \cdots, G_n]$ is the graph obtained by replacing the vertex i of G by the graph G_i and there is an edge between $u \in G_i$ and $v \in G_j$ if and only if there is an edge between i and j in G. If all the G_i's are isomorphic to a graph H then the graph composition $G[H, \cdots, H]$ is the lexicographic product of G and H and will be noted $G[H]$.

The vertex set of $G[H]$ is the cartesian product $V(G) \times V(H)$ and there is an edge between (u, u') and (v, v') if and only if there is an edge between u and v in G or $u = v$ and there is an edge between u' and v' in H. Throughout this paper, a vertex of a lexicographic product $G[H]$ will be denoted by (u, u') where $u \in V(G)$ and $u' \in V(H)$.

We first consider simple graphs and the lexicographic product of a graph with a complete graph (Section 2). Then in Section 3 we develop a specific example of graph composition $G[K_{k_1}, K_{k_2}, \cdots, K_{k_n}]$ with G a strict threshold graph or a star. Finally (Section 4), we deal with digraphs and in particular compositions of tournaments.

To fix notations, $Sp(M)$ denotes the spectrum of a matrix M; for a graph G, $Sp(G)$ denotes the spectrum of its adjacency matrix and $\mu_1^{(m_1)} \in Sp(M)$ means that μ_i is m_i times an eigenvalue of M (the multiplicity of μ_i is at least m_i, we may allow $\mu_i = \mu_j$ for $i \neq j$). The Laplacian spectrum of G is denoted by $Sp_L(G)$. For a vertex v of a graph G, $N(v)$ denotes the set of neighbours of v in G. A complete graph on n vertices is denoted by K_n. The neighbourhood of a vertex v is denoted by $N(v)$ and is the set of vertices adjacent to v.

2 Composition of simple graphs with complete graphs

We consider the graph composition $G[K_k, \cdots, K_k]$ where K_k stands for the complete graph with k vertices. This kind of graph composition $G[H, \cdots H]$, often denoted by $G[H]$ is also called the lexicographic product of G and H and denoted by $G.H$. Moreover we remark that when H is the complete graph then $G[K_k]$ is equal to the strong product of G and H: $G \boxtimes H$.

Proposition 1 *Let λ_i be the eigenvalues of a graph G on n vertices ($1 \leq i \leq n$). Then the nk eigenvalues of $G[K_k]$ are*

$$Sp(G[K_k]) = \{(-1)^{(nk-n)}\} \cup \{k\lambda_i + k - 1, 1 \leq i \leq n\}.$$

Proof: The proof of this proposition is conducted by writing the block matrix of $G[K_k]$; the sketch of the proof is the same as that of Theorem 16. Another to prove this proposition is to remark that the adjacency matrix of $G[K_k]$ can be written as a Kroneker product of matrices: $(A+I) \otimes J - I$, where A is the adjacency matrix of G, I is the identity matrix and J is the all-ones matrix. Then we use classical result of Kronecker products [7]. $\square$

Lemma 2 [8, 9, 10] *Two regular graphs G and G' are isomorphic if and only if the graphs $G[K_k]$ and $G'[K_k]$ are isomorphic.*

The following lemma is a consequence of Proposition 1.

Lemma 3 *Let $\mathcal{C}_r = \{G[K_k], G$ regular $, k \in \mathbb{N}, k \geq 2\}$. If $H = G[K_k]$ is a graph cospectral with $H' = G'[K_k] \in \mathcal{C}_r$ then G is cospectral with G'.*

We can state the following theorem:

Theorem 4 *Let $\mathcal{C}_r = \{G[K_k], G$ regular $, k \in \mathbb{N}, k \geq 2\}$. If the graph $H = G[K_k] \in \mathcal{C}_r$ is characterised by its spectrum in $\mathcal{C}_r$ then G is determined by its spectrum.*

Proof: If G is not determined by its spectrum then there is a graph G' cospectral with G and non isomorphic to G. Then the graphs $G[K_k]$ and $G'[K_k]$ are cospectral (Proposition 1) and not isomorphic (Lemma 2) and therefore the graph H is not characterised by its spectrum in $\mathcal{C}$. $\square$

Corollary 5 *If $G = \tilde{G}[K_k] \in \mathcal{C}_r$ is DS then $\tilde{G}$ is DS.*

The main problem to prove the converse of this theorem is to prove that if $G[K_k]$ (G DS) is cospectral with $G'[K_{k'}]$ then these graphs are isomorphic. Here we consider this problem for some sub-classes of $\mathcal{C}_r$.

Theorem 6 *Let $\mathcal{B}$ be the family of regular bipartite graph and let $\mathcal{C}_r^{\mathcal{B}} = \{G[K_k], G \in \mathcal{B}, k \in \mathbb{N}, k \geq 2\}$. If $G \in \mathcal{B}$ is determined by its spectrum then the graph $H = G[K_k] \in \mathcal{C}_r^{\mathcal{B}}$ is characterised by its spectrum in $\mathcal{C}_r^{\mathcal{B}}$.*

Proof: Let $G \in \mathcal{B}$ be a regular bipartite graph determined by its spectrum and let $H' = G'[K_{k'}] \in \mathcal{C}_r^{\mathcal{B}}$ be a graph cospectral with $H = G[K_k]$; we have to show that H and H' are isomorphic. Let μ (resp. μ') be the spectral radius of G (resp. G'); since G and G' are bipartite, the minimum eigenvalue of G (resp. G') is $-\mu$ (resp. $-\mu'$). The maximal eigenvalue of H is $k(\mu+1)$ and its minimal eigenvalue is $k(-\mu+1)$. The maximal eigenvalue of H' is $k'(\mu'+1)$ and its minimal eigenvalue is $k'(-\mu'+1)$. Since H and H' are cospectral, we have $k(\mu+1) + k(-\mu+1) = k'(\mu'+1) + k'(-\mu'+1)$ that is $k = k'$. Applying Lemma 3 we have that G' is cospectral with G and since G is DS, G' is isomorphic to G and so H' is isomorphic to H. $\square$

Theorem 7 *Let $\mathcal{C}_r^{\mathcal{P}} = \{G[K_k], |G| prime, G \text{ regular }, k \in \mathbb{N}, k \geq 2\}$. If G is a regular DS graph on a prime number of vertices then $\forall k > 1$ the graph $G[K_k] \in \mathcal{C}_r^{\mathcal{P}}$ is characterised by its spectrum in $\mathcal{C}_r^{\mathcal{P}}$.*

Proof: Let G be a regular DS graph on a prime number of vertices determined by its spectrum and let $H' = G'[K_{k'}] \in \mathcal{C}_r^{\mathcal{P}}$ be a graph cospectral with $H = G[K_k]$; we have to show that H and H' are isomorphic. Let $d = \gcd(k, k')$ and let q, q' be such that $k = dq$ and $k' = dq'$ (q and q' are coprime). We have $H = (G[K_q])[K_d]$ cospectral with $H' = (G[K_{q'}])[K_d]$ and applying Lemma 3 we have that $G[K_q]$ is cospectral with $G'[K_{q'}]$. Let n (resp. n') and r (resp. r') be the number of vertices and the degree of G (resp. G'). We have $nq = n'q'$ and $(r+1)q = (r'+1)q'$. So q' divides nq but q and q' are coprime, thus q' divides n and q' is equal to 1 or n (n is prime).

- If $q' = 1$ then $q = 1$ (otherwise n' is not prime) and we have $n = n'$, $k = k'$ and $G'[K_k]$ is cospectral with $G[K_k]$, so (Lemma 3) G' is cospectral with (and therefore isomorphic to) G. So H' is isomorphic to H.

- If $q' = n$ then n divides $r+1$ but $n \geq r+1$ so $n = r+1$ and G is a complete graph. Then H is also a complete graph wich is DS so H' is isomorphic to H. $\square$

To end this section, we compute the Laplacian spectrum of a graph $G[K_{k_1}, K_{k_2}, \cdots, K_{k_n}]$. The proof, using block matrices, is a classical way in this paper to compute eigenvalues of (di)graphs compositions, we describe it in details.

Theorem 8 *The Laplacian spectrum of $G[K_{k_1}, K_{k_2}, \cdots, K_{k_n}]$ is:*

$$\bigcup_{i=1..n} \left\{ \left(k_i + \sum_{j \in N(i)} k_j \right)^{(k_i - 1)} \right\} \cup Sp(-A(G)\hat{D} + \Delta),$$

where the vertices of G are labelled from 1 to n, $A(G)$ is the adjacency matrix of G, $\hat{D}$ is the diagonal matrix of the k_i's and Δ is the diagonal matrix whose i^{th} entry is $\sum_{j \in N(i)} k_j$.

Proof: The adjacency matrix of K_{k_i} will be also denoted by K_{k_i}, the adjacency matrix of $G[K_{k_1}, K_{k_2}, ..., K_{k_n}]$ will be denoted by A, D is the diagonal matrix of degrees of $G[K_{k_1}, ..., K_{k_n}]$ and $L = D - A$ is the Laplacian of $G[K_{k_1}, ..., K_{k_n}]$. The vector $(\underbrace{1, 1, \cdots, 1}_{p \text{ times}})^T$ is denoted by $\mathbf{1}_p$ or by $\mathbf{1}$ if no confusion can be made. Let u be an eigenvector of K_{k_i} associated to the eigenvalue -1, since the multiplicity of the eigenvalue -1 is $k_i - 1$, there is $k_i - 1$ independant eigenvectors u. As $\mathbf{1}$ is an eigenvector of K_{k_i} associated ti the eigenvalue $k_i - 1$, we have $< u, \mathbf{1} >= 0$ (where $<, >$ is the usual scalar product). Let $\tilde{u} = (\underbrace{0, ..., 0}_{k_1 + ... + k_{i-1} \text{ times}}, u^T, \underbrace{0, ..., 0}_{k_{i+1} + ... + k_n \text{ times}})^T$, we have $A\tilde{u} = -\tilde{u}$ and $D\tilde{u} = (k_i - 1 + \sum_{j \in N(i)} k_j)\tilde{u}$. So $L\tilde{u} = (k_i + \sum_{j \in N(i)} k_j)\tilde{u}$. As a result $k_i + \sum_{j \in N(i)} k_j$ is $k_i - 1$ times an eigenvalue of $G[K_{k_1}, K_{k_2}, \cdots, K_{k_n}]$.

There remains n eigenvalues to find (and n eigenvectors). Let $w = \begin{pmatrix} \alpha_1 \mathbf{1}_{k_1} \\ \alpha_2 \mathbf{1}_{k_2} \\ \vdots \\ \alpha_n \mathbf{1}_{k_n} \end{pmatrix}$ and $v = (\alpha_1, \alpha_2, ..., \alpha_n)^T$ where $\alpha_i \in \mathbb{R}$ for $1 \leq i \leq n$, we have:

$$Aw = \begin{pmatrix} (k_1-1)\alpha_1 \mathbf{1}_{k_1} + (\sum_{j\in N(1)} k_j\alpha_j)\mathbf{1}_{k_1} \\ (k_2-1)\alpha_2 \mathbf{1}_{k_2} + (\sum_{j\in N(2)} k_j\alpha_j)\mathbf{1}_{k_2} \\ (k_3-1)\alpha_3 \mathbf{1}_{k_3} + (\sum_{j\in N(3)} k_j\alpha_j)\mathbf{1}_{k_3} \\ \vdots \\ (k_n-1)\alpha_n \mathbf{1}_{k_n} + (\sum_{j\in N(n)} k_j\alpha_j)\mathbf{1}_{k_n} \end{pmatrix},$$

$$Dw = \begin{pmatrix} (k_1-1+\sum_{j\in N(1)} k_j)\alpha_1 \mathbf{1}_{k_1} \\ (k_2-1+\sum_{j\in N(2)} k_j)\alpha_2 \mathbf{1}_{k_2} \\ (k_3-1+\sum_{j\in N(3)} k_j)\alpha_3 \mathbf{1}_{k_3} \\ \vdots \\ (k_n-1+\sum_{j\in N(n)} k_j)\alpha_n \mathbf{1}_{k_n} \end{pmatrix}$$

so

$$Lw = \begin{pmatrix} ((\sum_{j\in N(1)} k_j)\alpha_1 - \sum_{j\in N(1)} k_j\alpha_j)\mathbf{1}_{k_1} \\ ((\sum_{j\in N(2)} k_j)\alpha_2 - \sum_{j\in N(2)} k_j\alpha_j)\mathbf{1}_{k_2} \\ ((\sum_{j\in N(3)} k_j)\alpha_3 - \sum_{j\in N(3)} k_j\alpha_j)\mathbf{1}_{k_3} \\ \vdots \\ ((\sum_{j\in N(n)} k_j)\alpha_n - \sum_{j\in N(n)} k_j\alpha_j)\mathbf{1}_{k_n} \end{pmatrix}.$$

As a consequence w is an eigenvector of L if and only if $\exists\lambda\in\mathbb{R}$, $\forall i = 1,\cdots,n$, $(\sum_{j\in N(i)} k_j)\alpha_i - \sum_{j\in N(i)} k_j\alpha_j = \lambda\alpha_i$, that is if and only if $v = (\alpha_1,\cdots,\alpha_n)^T$ is an eigenvector of $-A(G)\hat{D}+\Delta$ where $\hat{D}$ is the diagonal matrix of the k_i's and Δ is the diagonal matrix whose i^{th} diagonal entry is equal to $\sum_{j\in N(i)} k_j$. As a result $v = (\alpha_1,\cdots,\alpha_n)^T$ is an eigenvector of $-A\hat{D}+\Delta$ associated to the eigenvalue λ if and only if w is an eigenvector of L associated to the eigenvalue λ.
Moreover a vector w is not a linear combination of the vectors $\tilde{u}$ previously defined because the vectors u are orthogonal to $\mathbf{1}$ so there is no linear combination of vectors u equals to $\alpha\mathbf{1}$, $\alpha\in\mathbb{R}^*$. The w are the n missing eigenvectors and the n missing eigenvalues are the eigenvalues of $-A(G)\hat{D}+\Delta$. $\square$

3 Composition of a threshold graph with complete graphs

3.1 Starlike threshold graphs: definition and Laplacian spectrum

In this section we consider a special class of graphs, the characterization of which cannot be done with theorems stated in the previous section: the graph is not regular and the composition is made with complete graphs of various orders. Moreover, showing the characterisation by the spectrum of a special class of graph is quite frequent in spectral graph theory; indeed, more the considered family of graphs is large, more the risk to have a pair of cospectral non-isomorphic graphs within this family is important.

A *threshold graph* [2] is a graph that can be partitioned into a stable subgraph S and a maximal complete subgraph K such that $S = \{i_1, \cdots, i_p\}$ and $N(i_1) \subset N(i_2) \subset \cdots \subset N(i_p)$. If these inclusions are strict then the threshold graph is called *strict threshold graph*. A *(strict) starlike-threshold graph* is a graph $G[K_{k_1}, \cdots, K_{k_n}]$ where G is a (strict) threshold graph. We can give an alternative definition of a (strict) starlike-threshold graph [2]:

Definition 9 A *starlike-threshold graph* is a connected graph where vertices can be partitioned into $C, D_1, D_2, \cdots, D_p$ such that:

- C is a maximal complete subgraph;
- D_i is a complete subgraph and $\forall u, v \in D_i$, $N(u) \cup \{u\} = N(v) \cup \{v\}$;
- $C_1 \subset C_2 \subset \cdots \subset C_p \subset C$ where $C_i = (N(u) \cup \{u\}) \backslash D_i$ with $u \in D_i$.

If the latest inclusions are strict the starlike-threshold graph is called *strict starlike-threshold graph*

Notations: We set $d_i = |D_i|, c = |C|$, $c_i = |C_i|$, $c'_i = |C_i \backslash C_{i-1}|$ with $c'_1 = c_1$, $c' = |C \backslash C_p|$. The number of vertices of a starlike-threshold graph is denoted by n and we set $n_i = n - \sum_{k=1}^{i-1} d_k = c + \sum_{k=i}^{p} d_k$ for $2 \leq i \leq p$ (we have $n_1 = n$ and $n_p = c + d_p$). A starlike-threshold graph is determined by the parameters p, c, $(d_i)_{1 \leq i \leq p}$, $(c_i)_{1 \leq i \leq p}$. By analogy with a star, the parameter p is called the number of branches.

Before dealing with starlike-threshold graphs, we give some general results on the Laplacian spectrum.

Theorem 10 [6, 13] *Let G be a graph on n vertices whose Laplacian spectrum is $\mu_1 \geq \mu_2 \geq ... \geq \mu_{n-1} \geq \mu_n = 0$. Then:*

(i) $\mu_{n-1} \leq \frac{n}{n-1} \min\{d(v), v \in V(G)\}$.

(ii) *If G is not a complete graph then $\mu_{n-1} \leq \min\{d(v), v \in V(G)\}$.*

(iii) $\mu_1 \leq \max\{d(u) + d(v), uv \in E(G)\}$.

(iv) $\mu_1 \leq n$.

(v) $\sum_i \mu_i = 2|E(G)|$.

(vi) $\mu_1 \geq \frac{n}{n-1} \max\{d(v), v \in V(G)\} > \max\{d(v), v \in V(G)\}$.

Theorem 11 [6] *Let G be a non-complete graph, κ_0 its vertex connectivity, κ_1 its edge connectivity, μ_{n-1} its second smallest Laplacian eigenvalue (also called algebraic connectivity), d_m its minimum degree. Then $\mu_{n-1} \leq \kappa_0 \leq \kappa_1 \leq d_m$*

Definition 12 Let G be a simple graph on n vertices, a vertex of degree $n-1$ is an *universal vertex.*

The following lemma uses only basic properties on the Laplacian spectrum [13].

Lemma 13 *Let G be a graph on n vertices with k universal vertices, then $n^{(k)} \in Sp_L(G)$ and the Laplacian spectrum of $G \backslash \{$universal vertices$\}$ is $(Sp_L(G) \backslash \{n^{(k)},\ 0\} - k) \cup \{0\}$.*

Proposition 14 *Let G be a graph with only one non-zero Laplacian-eigenvalue a, then there is $r \in \mathbb{N}^*$, $p \in \mathbb{N}$ such that the Laplacian spectrum of G is $\{a^{(ra-r)}, 0^{(r+p)}\}$ and G is isomorphic to $rK_a \cup pK_1$.*

Proof: Let G be a graph with only one non-zero Laplacian-eigenvalue a and let H be a connected component of G which is not an isolated vertex; the graph H has only one non-zero eigenvalue a. If H is not complete, by Theorem 11 we have $a \leq \min\{d(v), v \in V(H)\}$, but Theorem 10 gives $a > \max\{d(v), v \in V(H)\}$, contradiction. So H is the complete graph K_a with Laplacian spectrum $\{a^{(a-1)}, 0\}$. $\square$

Theorem 15 [1] *Let G be a graph without isolated vertex. If the Laplacian spectrum of G is $\{k_1^{(k_1-1)}, k_2^{(k_2-1)}, ..., k_n^{(k_n-1)}, 0^{(n)}\}$ with $k_i \in \mathbb{N}\backslash\{0,1\}$ then G is a disjoint union of complete graphs of order $k_1, ..., k_n$.*

Theorem 16 *The Laplacian spectrum of a strict starlike-threshold graph with parameters p, c, $(d_i)_{1\leq i\leq p}$, $(c_i)_{1\leq i\leq p}$ is the multiset:*

$$\bigcup_{i=1}^{p}\left\{n_i^{(c'_i)}, (d_i + \sum_{j=1}^{i} c'_j)^{(d_i-1)}, c_i\right\} \cup \{c^{(c'-1)}, 0\}.$$

Proof: The proof is made by induction on p. Induction Hypothesis: the Laplacian spectrum of a threshold graph of completes with p branches is

$$\bigcup_{i=1}^{p}\left\{n_i^{(c'_i)}, (d_i + \sum_{j=1}^{i} c'_j)^{(d_i-1)}, c_i\right\} \cup \{c^{(c'-1)}, 0\}.$$

$p = 1$: Let $n = |G|$ and let $\mu_1 \geq \mu_2 \geq ... \geq \mu_{n-1} \geq \mu_n = 0$ be the Laplacian eigenvalues (counted with multiplicity). Since G has c_1 universal vertices we have (Lemma 13) $n^{(c_1)} \in Sp_L(G)$. The graph $G\backslash C_1$ is the disjoint union of two completes with d_1 and c' vertices so $Sp_L(G\backslash C_1) = \{d_1^{(d_1-1)}, c'^{(c'-1)}, 0^{(2)}\}$. According to Lemma 13,

$$Sp_L(G\backslash\{\text{universal vertices}\}) = (Sp_L(G)\backslash\{n^{(c_1)}, 0\} - c_1) \cup \{0\}$$

so

$$Sp_L(G\backslash C_1) = (Sp_L(G)\backslash\{n^{(c_1)},\ 0\} - c_1) \cup \{0\}$$

and

$$Sp_L(G)\backslash\{n^{(c_1)},\ 0\} = Sp_L(G\backslash C_1) \setminus \{0\} + c_1 = \{(d_1 + c_1)^{(d_1-1)}, c^{(c'-1)}\}.$$

Thus

$$Sp_L(G) = \{n^{(c_1)}, (d_1 + c_1)^{(d_1-1)}, c^{(c'-1)}, 0\}.$$

The induction hypothesis is true for $p = 1$.

Let us assume that the induction hypothesis is true at rank p and let G be a strict threshold graph of completes with $p + 1$ branches and let $\mu_1 \geq \mu_2 \geq ... \geq \mu_{n-1} \geq \mu_n = 0$ be its Laplacian eigenvalues (counted with multiplicity).

According to Lemma 13, the spectrum of $G\backslash C_1$ is $\{\mu_{c_1+1}-c_1,...,\mu_{n-1}-c_1,0\}$. The graph $G\backslash C_1$ has two connected components: the complete graph D_1 and a strict threshold graph of completes denoted by G_1. The Laplacian of $G\backslash C_1$ has twice the eigenvalue 0, so $\exists i$ such that $\mu_i-c_1=0$ *i.e.* $\mu_i=c_1=c'_1$. We also have that $Sp_L(G\backslash C_1)=Sp_L(D_1)\cup Sp_L(G_1)$. Since the spectrum of a complete graph on k vertices is k with multiplicity $k-1$ and 0 with multiplicity 1 we have that d_1 is an eigenvalue of $G\backslash C_1$ with multiplicity d_1-1. The graph G_1 is a strict threshold graph of completes whose the partitioning of vertices is $C\backslash C_1$, D_2,...,D_{p+1}. Moreover, for $u\in D_i$ we have $(N(u)\cup\{u\})\backslash D_i=C_i\backslash C_1$ and so $|(N(u)\cup\{u\})\backslash D_i|=c_i-c_1$. We also have $|G_1|=n-c_1$.

We apply the induction hypothesis to G_1 in order to obtain its spectrum:

$$\bigcup_{i=1}^{p}\left\{(n_{i+1}-c_1)^{(c'_{i+1})},(d_{i+1}+\sum_{j=1}^{i}c'_{j+1})^{(d_{i+1}-1)},c_{i+1}-c_1\right\}\cup\{(c-c_1)^{(c'-1)},0\}$$

i.e.

$$\bigcup_{i=2}^{p+1}\left\{(n_i-c_1)^{(c'_i)},(d_i+\sum_{j=2}^{i}c'_j)^{(d_i-1)},c_i-c_1\right\}\cup\{(c-c_1)^{(c'-1)},0\}$$

so the spectrum of $G\backslash C_1$ is

$$\{d_1^{(d_1-1)},0\}\cup\bigcup_{i=2}^{p+1}\left\{(n_i-c_1)^{(c'_i)},(d_i+\sum_{j=2}^{i}c'_j)^{(d_i-1)},c_i-c_1\right\}\cup\{(c-c_1)^{(c'-1)},0\}$$

As $Sp_L(G)=\{n^{(c_1)},0\}\cup(Sp_L(G\backslash C_1)\setminus\{0\}+c_1)$ we have

$$Sp_L(G)=\bigcup_{i=1}^{p+1}\left\{n_i^{(c'_i)},(d_i+\sum_{j=1}^{i}c'_j)^{(d_i-1)},c_i\right\}\cup\{c^{(c'-1)},0\}.$$

As a conclusion the induction hyptohesis is true for $p+1$. $\square$

3.2 There are no cospectral non-isomorphic strict starlike-threshold graphs.

Lemma 17 *For a threshold graph of completes with parameters p, c, $(d_i)_{1\leq i\leq p}$, $(c_i)_{1\leq i\leq p}$, we have the following inequalities:*

$$n_1 > n_2 > n_3 > ... > n_p$$
$$\forall j \geq i,\ n_i > d_j + \sum_{k=1}^{i} c'_k$$
$$\forall i, j,\ n_i > c_j$$
$$\forall i,\ n_i > c$$
$$c_1 \leq c_2 \leq c_3 \leq ... \leq c_{p-1} < c$$

Lemma 18 *For $p \geq 2$, if $d_1 + c_1 > n_2$ and $c'_2 \neq 0$ then the multiplicity of $d_1 + c_1$ is $d_1 - 1$.*

Proof: We already know that the multiplicity of $d_1 + c_1$ is greater than or equal to $d_1 - 1$; it remains to show that the other eigenvalues are not equal to $d_1 + c_1$. These eigenvalues are $n_i^{(c'_i)}, (d_i + \sum_{k=1}^{i} c'_k)^{(d_i-1)}, c_i, c^{(c'-1)}, 0$ for $i = 1, ..., p$.
• With the first inequalities of the previous lemma and with $d_1 + c_1 > n_2$ we have $d_1 + c_1 > n_i$ for $i \geq 2$. Obviously we have $n_1 > d_1 + c_1$.
• $d_1 + c_1 > n_2 \Rightarrow d_1 + c_1 > d_j + \sum_{k=1}^{i} c'_k$ for $j \geq 2$ (second inequality of the previous lemma).
• $d_1 + c_1 > n_2 \Rightarrow d_1 + c_1 > c_j$ for all j (third inequality of the previous lemma).
• $d_1 + c_1 > n_2 \Rightarrow d_1 + c_1 > c$ (fourth inquality of the previous lemma).
As a result the remaining eigenvalues are not equal to $d_1 + c_1$, thus the multiplicity of $d_1 + c_1$ is $d_1 - 1$. □

Lemma 19 *For $p \geq 2$, if $d_1 + c_1 < n_2$ and $c'_2 \neq 0$ then the multiplicity of n_2 is c'_2.*

Proof: We already know that the multiplicity of n_2 is greater than or equal to c'_2; it remains to show that the other eigenvalues are not equal to n_2. These eigenvalues are $n_i^{(c'_i)}, (d_i + \sum_{k=1}^{i} c'_k)^{(d_i-1)}, c_i, c^{(c'-1)}, 0$ for $i = 1, ..., p$.

• $n_i \neq n_2,\ \forall i \neq 2$ (Lemma 17).
• $n_2 > d_1 + c_1$ by hypothesis.
• $n_2 > d_j + \sum_{k=1}^{i} c'_k$ pour $j \geq 2$ (second inequality of Lemma 17).
• $n_2 > c_j$ pour tout j (third inequality of Lemma 17).
• $n_2 > c$ (fourth inequality of Lemma 17).
As a result the remaining eigenvalues are not equal to n_2, thus the multiplicity of n_2 is c'_2. □

Lemma 20 *If $d_1 + c_1 = n_2$ and $c'_2 \neq 0$ then the multiplicity of n_2 (i.e. the multiplicity of $d_1 + c_1$) is $c'_2 + d_1 - 1$.*

Proof: We already know that the multiplicity of n_2 is greater than or equal to c'_2 and that the multiplicity of d_1+c_1 is greater than or equal to d_1-1, so if $d_1+c_1 = n_2$ then the multiplicity if n_2 (*i.e.* that of d_1+c_1) is greater than or equal to c'_2+d_1-1. It remains to show that the other eigenvalues are not equal to n_2. These eigenvalues are $n_i^{(c'_i)}, (d_i+\sum_{k=1}^{i} c'_k)^{(d_i-1)}, c_i, c^{(c'-1)}, 0$ for $i = 1, ..., p$.
• $n_i \neq n_2,\ \forall i \neq 2$ (Lemma 17).
• $n_2 > d_j + \sum_{k=1}^{i} c'_k$ pour $j \geq 2$ (second inequality of Lemma 17).
•$n_2 > c_j$ pour tout j (third inequality of Lemma 17).
• $n_2 > c$ (fourth inequality of Lemma 17).
As a result the remaining eigenvalues are not equal to n_2, thus the multiplicity of n_2 is $c'_2 + d_1 - 1$. □

Lemma 21 *Let G be a strict starlike threshold graph with $p \geq 2$ and $c'_2 \neq 0$, the spectrum of which is $\mu_1 > \mu_2 > ... > \mu_q > 0$, and let $m_1, m_2, ..., m_q$ be the multiplicities of these eigenvalues. If $m_2 = \mu_2 - \mu_q - 1$ then $d_1 = \mu_2 - \mu_q$ otherwise $d_1 = \mu_1 - \mu_2$.*

Proof: The spectrum of G is

$$\bigcup_{i=1}^{p} \left\{ n_i^{(c'_i)}, (d_i + \sum_{k=1}^{i} c'_k)^{(d_i-1)}, c_i \right\} \cup \{c^{(c'-1)}, 0\}$$

(Theorem 16). According to Lemma 17, the greatest eigenvalue μ_1 is equal to n and the smallest eigenvalue μ_q is equal to c_1. According to Lemma 17, there are two possible values for the second largest eigenvalue μ_2: n_2

or $d_1 + c_1$. Indeed for $j > 2$ we have $n_j < n_2$, $d_j + \sum_{k=1}^{i} c'_k < n_2$ and for $j \leq 1$ we have $c_j < c < n_2$; except n_1 and $d_1 + c_1$ all the eigenvalues are strictly loweer than n_2.
• If $\mu_2 = n_2 = d_1 + c_1$ the we have $d_1 = \mu_2 - \mu_q$ and $d_1 = \mu_1 - \mu_2$. The lemma is true in this case.
• If $\mu_2 = d_1 + c_1 > n_2$ then, by Lemma 18, the multiplicity of $d_1 + c_1$ is $d_1 - 1$ and we have $m_2 = \mu_2 - \mu_q - 1$ and $d_1 = \mu_2 - \mu_q$. The lemma is true in this case.
• If $\mu_2 = n_2 > d_1 + c_1$ then, by Lemma 19, the multiplicity of n_2 is c'_2, we have $c'_2 < n - d_1 - c_1 - 1 = n_2 - \mu_q - 1$ thus $m_2 \neq \mu_2 - \mu_q - 1$ and $d_1 = \mu_1 - \mu_2$. The lemma is true in this case. □

Theorem 22 *Let G be a strict starlike threshold graph cospectral with a strict starlike threshold graph F with $p = 1$. Then G and F are isomorphic.*

Proof: Let F be a strict starlike threshold graph with $p = 1$ and with parameters d_1, c_1, c'. As G is cospectral with F, the spectrum of G is $\{n^{(c_1)}, (d_1 + c_1)^{(d_1-1)}, c_1, c^{(c'-1)}, 0\}$ and G has c_1 universal vertices. Let G_1 be the graph $G_1 = G\backslash\{\text{universal vertices}\}$, the spectrum of G_1 is $\{d_1^{(d_1-1)}, c'^{(c'-1)}, 0, 0\}$. The graph G_1 has two connected components: a complete and a complete or a strict starlike threshold graph.
• If $Sp_L(G_1) = \{0, 0\}$ then G_1 consists in two isolated vertices and G is completely determined.
• If $Sp_L(G_1) = \{d_1^{(d_1-1)}, 0, 0\}$ then G_1 (and consequently G) is completely determined (Proposition 14).
• If $Sp_L(G_1) = \{c'^{(c'-1)}, 0, 0\}$ then G_1 (and consequently G) is completely determined (Proposition 14).
• Let us assume that $Sp_L(G_1) = \{d_1^{(d_1-1)}, c'^{(c'-1)}, 0, 0\}$. If G_1 has an isolated vertex, then the spectrum of a connected component of G_1 is $\{d_1^{(d_1-1)}, c'^{(c'-1)}, 0\}$. This is not the spectrum of a complete nor the spectrum of a strict starlike threshold graph because the greatest eigenvalue is not equal to the number of vertices. As a result G_1 does not have an isolated vertex and (Theorem 15) G_1 is the union of two completes with d_1 and c' vertices. □

Theorem 23 *There are no cospectral non-isomorphic strict starlike threshold graphs.*

Proof: Let G be a strict starlike-threshold graph cospectral with another strict starlike-threshold graph G'; we have to show that G is isomorphic to G'. The proof is made by induction on the number of branches of G' denoted by p; the induction hypothesis is 'If G is a strict starlike threshold graph cospectral with a starlike threshold graph on p branches then these two graphs are isomorphic'.
• $p = 1$. It is Theorem 22.
• Let us assume the hyothesis true at rank $p-1$ and let G be a strict starlike threshold graph cospectral with a strict starlike threshold graph with p branches. We denote by m_i the multiplicity of the eigenvalue μ_i. We have:

- n is given by the number of eigenvalues or by μ_1.
- $c_1 = m_1$.
- d_1 is given by Lemma 21.

The graph $G \backslash C_1$ is the disjoint union of a complete with d_1 vertices and a strict starlike threshold graph G_1. As we know c_1 and n, we know the spectrum of $G \backslash C_1$ (Lemma13); and as we know d_1, we know the spectrum of G_1:

$$\bigcup_{i=2}^{p} \left\{ (n_i - c_1)^{(c'_i)}, (d_i + \sum_{j=2}^{i} c'_j)^{(d_i - 1)}, c_i - c_1 \right\} \cup \{(c - c_1)^{(c'-1)}\} \cup \{0\}.$$

But (Theorem 16) this is the spectrum of a strict starlike threshold graph with $p-1$ branches, $n_2 - c_1$ vertices, so the graph G_1 is a strict starlike threshold cospectral with a strict starlike threshold graph with $p-1$ branches and therefore isomorphic to this graph by the induction hypothesis. As a result G is isomorphic with G' and the induction hypothesis is true at rank p. □

3.3 Star of completes

A *star of completes* is the graph $S_n[K_{k_0}, K_{k_1}, \cdots, K_{k_n}]$ where S_n is a star with $n+1$ vertices labeled from 0 to n such that the vertex with degree greater than 1 is labeled 0. A star can be seen as a particular case of a threshold graph, a star of completes can be seen as a particular case of a starlike threshold graph.

Theorem 24 *The Laplacian spectrum of a star $S_n[K_{k_0}, \cdots, K_{k_n}]$ of completes is*

$$\{(k_0 + \cdots + k_n)^{(k_0)}\} \cup \bigcup_{i=1}^{n} \{(k_0 + k_i)^{(k_i - 1)}\} \cup \{k_0^{(n-1)}\} \cup \{0\}.$$

We can now state two theorems of characterizations of stars of completes:

Theorem 25 *There are no Laplacian-cospectral non-isomorphic stars of completes.*

Proof: Two cospectral stars of completes have the same number of universal vertices, the deletion of which gives a union of complete graphs. Since there are not two disjoint unions of complete graphs cospectral and non-isomorphic, it ensues that there are no Laplacian-cospectral non-isomorphic stars of completes. $\square$

Theorem 26 *Let $\mathcal{H}$ be the set of graphs with minimum degree strictly greater than the minimum non-zero eigenvalue of its Laplacian matrix. A star of completes belonging to $\mathcal{H}$ is characterised by its Laplacian spectrum in $\mathcal{H}$.*

Proof: Let G be a graph on N vertices with spectrum $\{(k_0+...+k_n)^{(k_0)}\} \cup \bigcup_{i=1}^{n}\{(k_0 + k_i)^{(k_i-1)}\} \cup \{k_0^{(n-1)}\} \cup \{0^{(1)}\}$ and such that $d_{\min} > k_0$. We have $N = k_0 + ... + k_n$ and the spectrum of $\overline{G}$ is $\bigcup_{i=1}^{n}\{(N - k_0 + k_i)^{(k_i-1)}\} \cup \{(N - k_0)^{(n-1)}\} \cup \{0^{(k_0+1)}\}$ so $\overline{G}$ has $k_0 + 1$ connected components. One of these components which has the eigenvalue $N - k_0$, has more than $N - k_0 - 1$ vertices (Theorem 10). Then the k_0 other connected components are isolated vertices. Let H be the connected component of size $N - k_0$, The spectrum of H is $\bigcup_{i=1}^{n}\{(N - k_0 + k_i)^{(k_i-1)}\} \cup \{(N - k_0)^{(n-1)}\} \cup \{0^{(1)}\}$ so the spectrum of $\overline{H}$ is $\bigcup_{i=1}^{n}\{(k_i)^{(k_i-1)}\} \cup \{(0)^{(n)}\}$.

The graph $\overline{H}$ have no isolated vertex; indeed the maximum degree of H, denoted by $d_{\max}^H$, is the maximum degree of $\overline{G}$ that is $d_{\max}^H = N - d_{\min} - 1$. The minimum degree of $\overline{H}$ is $|H| - d_{\max}^H - 1 = (N - k_0) - (N - d_{\min} - 1) - 1 = d_{\min} - k_0 > 0$ so $\overline{H}$ does not have isolated vertices.

By Theorem 15, $\overline{H}$ is a union of complete graphs of size $k_1, ..., k_n$, thus H is the complete multipartite graph $K_{k_1,...,k_n}$ and $\overline{G}$ is the disjoint union of the complete multipartite graph $K_{k_1,...,k_n}$ and k_0 isolated vertices. As a result G is a star of completes $S_n[K_{k_0}, ..., K_{k_n}]$. $\square$

Remark 27 For a graph G, let κ_0 be its vertex connectivity and κ_1 be its edge connectivity. We have (Theorem 11) $\mu_{n-1} \leq \kappa_0 \leq \kappa_1 \leq d_{\min}$ (μ_{n-1} is the second smallest Laplacian eigenvalue, that is k_0). Thus we can obtain corollaries of the previous theorem by replacing the condition $d_{\min} > k_0$ by the condition $d_{\min} > \kappa_0$ or $\kappa_0 < \kappa_1$.

Remark 28 There exists non-DS star of completes, for instance the star of completes $S_6[K_{k_0}, K_5, K_2, K_2, K_2, K_2, K_2]$ is Laplacian-cospectral with and non-isomorphic to $S_6[K_{k_0}, P, K_1, K_1, K_1, K_1, K_1]$ where P is the Petersen graph. Of course, owing to Theorem 26, the cospectral mate is such that its minimum non-zero eigenvalue equals its minimum degree.

4 Composition of tournaments

In this section we deal with the adjacency spectrum of tournaments (that is a digraph in which each pair of nodes is joined by an arc). Compared with simple graphs, few is done to characterise digraphs by their spectrum.

A *circulant matrix* [5] is a matrix whose k^{th} column is a circulant shift of the $(k-1)^{\text{th}}$ column. A *circulant tournament* is a tournament whose adjacency matrix is circulant. We denote by $\overrightarrow{C}_k$ (k odd) the circulant tournament, the vertices of which are labeled from 0 to k such that $N(0) = \{1, 2, \cdots, \frac{k-1}{2}\}$ and $N(i) = (N(0) + i) \bmod [k]$.

Proposition 29 [5] *The eigenvalues of a circulant matrix are*

$$\lambda_r = \sum_{j=0}^{n-1} a_j e^{\frac{2i\pi j}{n} r}, \quad r = 0, ..., n-1,$$

where $(a_0, a_1, ...a_{n-1})'$ is the first column of the matrix. In particular, the eigenvalues of a circulant tournament the vertices of which are labeled from 0 to n are

$$\lambda_r = \sum_{j \in N(0)} e^{\frac{2i\pi j}{n} r}, \quad r = 0, ..., n-1$$

and the eigenvalues of $\overrightarrow{C}_k$ (k odd) are

$$\lambda_r = \sum_{j=1}^{\frac{k-1}{2}} e^{\frac{2i\pi j}{k} r}, \quad r = 0, ..., k-1.$$

This following well-known and straightfoward result is useful to characterise circulant tournament.

Proposition 30 *A tournament is a circulant tournament if and only if its automorphism group contains a full-length cycle.*

This section is motivated by obtaining an algebraic characterization (mainly spectral characterization) of *vertex-critical tournament without non-trivial acyclic interval* (see the definition hereafter). Culus and Jouve [3] recently found a characterization of these graphs through a combinatorial and graph-theoretic approach: these graphs are compositions $t[\overrightarrow{C}_{k_1}, \cdots, \overrightarrow{C}_{k_m}]$.

Definition 31 A subset X of a tournament T is an *interval* (also called convex subset) if for all v in $V(T) \setminus V(X)$ then for all $x \in X$ there is a link from v to x or for all $x \in X$ there is a link from x to v. An acyclic interval is an interval without any cycle, that is a transitive interval. A *non-trivial acyclic interval* is an acyclic interval with at least two vertices.

A *vertex-critical tournament without non-trivial acyclic interval* is a tournament T such that T is without non-trivial acyclic intervals and, for every vertex u of T, $T \setminus u$ has a non-trivial acyclic interval.

Theorem 32 [3] *Every vertex-critical tournament without non-trivial acyclic interval is isomorphic to $t[\overrightarrow{C}_{k_1}, \cdots, \overrightarrow{C}_{k_m}]$ where t is a tournament of order m and where $k_i \in \mathbb{N} \setminus \{0, 1, 2\}$.*

Proposition 33 [5, 11] *The eigenvalues of $\overrightarrow{C}_k$ are:*

$$\lambda_j = \sum_{s=1}^{\frac{k-1}{2}} e^{\frac{2sj\pi}{k} i} = \begin{cases} -\frac{1}{2} + \frac{i}{2} \cot\left(\frac{j\pi}{2k}\right) & \text{if } j \text{ odd} \\ -\frac{1}{2} + \frac{i}{2} \cot\left(\frac{(k+j)\pi}{2k}\right) & \text{if } j \text{ even} \end{cases}, j < k \text{ and } \lambda_k = \frac{k-1}{2}.$$

Theorem 34 *The spectrum of $T = t[\overrightarrow{C}_{k_1}, ..., \overrightarrow{C}_{k_n}]$ is the multiset*

$$\bigcup_{j=1}^{n} \left(Sp(\overrightarrow{C}_{k_j}) \setminus \left\{ \frac{k_j - 1}{2} \right\} \right) \cup Sp(A\hat{D} + \Delta),$$

where A is the adjacency matrix of t, $\hat{D}$ is the diagonal matrix of the k_j's and Δ is the diagonal matrix whose j^{th} entry is $\frac{k_j-1}{2}$.

Proof: The proof is conducted in the same manner as that of Theorem 16. □

We define $\mathcal{T}_r = \{t[\overrightarrow{C}_k]$, t regular$, k \geq 3\}$; it is a subset of the regular tournaments of $\mathcal{T}$ and a subset of the vertex-critical tournaments without non-trivial acyclic interval.

Theorem 35 *The tournament $T = t[\overrightarrow{C}_k] \in \mathcal{T}_r$ is characterised by its spectrum in $\mathcal{T}_r$ if and only if t is determined by its spectrum.*

Proof: We show the first implication ($\Rightarrow$). Let t be a r-regular tournament such that $t[\overrightarrow{C}_k]$ is characterised by its spectrum in $\mathcal{T}_r$ and let t' a tournament cospectral with t. Then tournament t' is r-regular and $t'[\overrightarrow{C}_k]$ is cospectral with $t[\overrightarrow{C}_k]$ and so there is an isomorphism $\varphi : t[\overrightarrow{C}_k] \to t'[\overrightarrow{C}_k]$. Now let assume that u_1 and u_2 are two distinct vertices of t such that there exists a vertex u' of t' and a, a', b' vertices of $\overrightarrow{C}_k$ with $\varphi(u_1, a) = (u', a')$ and $\varphi(u_2, a) = (u', b')$. Since φ is an isomorphism, we have that

$$\{\varphi(v, x), v \in N(u_1), x \in V(\overrightarrow{C}_k)\} \subset N(\varphi(u_1, a)) = N(u', a')$$

and

$$\{\varphi(v, x), v \in N(u_2), x \in V(\overrightarrow{C}_k)\} \subset N(\varphi(u_2, a)) = N(u', b').$$

Consequently:

$$\{\varphi(v, x), v \in N(u_1) \cup N(u_2), x \in V(\overrightarrow{C}_k)\} \subset N(u', a') \cup N(u', b')$$

Since $u_1 \neq u_2$ we have $|N(u_1) \cup N(u_2)| \geq r + 1$ and $|N(u', a') \cup N(u', b')| \geq (r + 1)k$ But $N(u', a') \cup N(u', b') = \{(v', x), v' \in N(u'), x \in V(\overrightarrow{C}_k)\} \cup \{(u', y'), y' \in N(a') \cup N(b')\}$ and $|N(u', a') \cup N(u', b')| \leq rk + 2\frac{k-1}{2} < (r+1)k$ involving a contradiction. As a result if $\varphi(u_1, a) = (u', a')$ and $\varphi(u_2, a) = (u', b')$ then $u_1 = u_2$. If we define the following surjective homomorphism $\pi : t'[\overrightarrow{C}_k] \to t', \pi(u', x) = u'$, we have that $u_1 \neq u_2$ implies $\pi(\varphi(u_1, a)) \neq \pi(\varphi(u_2, a))$. Now, for a given $a \in V(\overrightarrow{C}_k)$ we define the injection $i_a : t \to t[\overrightarrow{C}_k], i(u) = (u, a)$ and $\psi_a = \pi \circ \varphi \circ i_a$ is an isomorphism from t to t'. (it is easy to see that $u_1 \sim u_2 \Rightarrow \psi(u_1) \sim \psi(u2))$. As a result t is isomorphic to t' and t is DS; the first implication of the theorem is proved.

Now we show the converse ($\Leftarrow$): we assume t DS. Let $T = t[\overrightarrow{C}_k]$ cospectral with $T' = t'[\overrightarrow{C}_{k'}]$, that is :

$$\begin{aligned} & \underbrace{n\left(\mathrm{Sp}(\overrightarrow{C}_k) \setminus \left\{\frac{k-1}{2}\right\}\right) \cup \mathrm{Sp}\left(kA_t + \frac{k-1}{2}I\right)}_{\mathrm{Sp}(T)} \\ = \; & \underbrace{n'\left(\mathrm{Sp}(\overrightarrow{C}_{k'}) \setminus \left\{\frac{k'-1}{2}\right\}\right) \cup \mathrm{Sp}\left(kA_{t'} + \frac{k'-1}{2}I\right)}_{\mathrm{Sp}(T')} \end{aligned}$$

If $n = |t| > |t'| = n'$ then $k < k'$ (because $nk = n'k'$).
Let $\lambda \in \mathrm{Sp}(\overrightarrow{C}_k) \setminus \left\{\frac{k-1}{2}\right\}$ such that $\lambda \notin \mathrm{Sp}(\overrightarrow{C}_{k'}) \setminus \left\{\frac{k'-1}{2}\right\}$, so $\lambda \in \mathrm{Sp}\left(kA_{t'} + \frac{k'-1}{2}I\right)$ which is impossible according to the multiplicity of λ in $\mathrm{Sp}(T)$. So:

$$\mathrm{Sp}(\overrightarrow{C}_k) \setminus \left\{\frac{k-1}{2}\right\} \subset \mathrm{Sp}(\overrightarrow{C}_{k'}) \setminus \left\{\frac{k'-1}{2}\right\}$$

As $\mathrm{Sp}(\overrightarrow{C}_r) \setminus \{\frac{r-1}{2}\} = \{-\frac{1}{2} + \frac{i}{2}\cot\left(\pi\frac{2j+1}{2r}\right), j \in \{0...r-1\} \setminus \{\frac{r-1}{2}\}\}$ we have $\exists j' > 0 : \frac{1}{2k} = \frac{2j'+1}{2k'}$ wich implies $k' \geq 3k$ and as a consequence $n \geq 3n'$.

As the eigenvalues of $\mathrm{Sp}(\overrightarrow{C}_{k'})$ are simple, we have that each copy of $\mathrm{Sp}(\overrightarrow{C}_{k'}) \setminus \left\{\frac{k'-1}{2}\right\}$ can only contain one copy of $\mathrm{Sp}(\overrightarrow{C}_k) \setminus \{\frac{k-1}{2}\}$; for that reason $\mathrm{Sp}\left(kA_{t'} + \frac{k'-1}{2}I\right)$ contains $(n-n')$ copies of $\mathrm{Sp}(\overrightarrow{C}_k) \setminus \{\frac{k-1}{2}\}$, so $n' \geq (n-n')(k-1) > n$, contradiction !

As a result $n = n'$, $k = k'$ and therefore $\mathrm{Sp}(t) = \mathrm{Sp}(t')$. Since t is DS, t and t' are isomorpic and $t[\overrightarrow{C}_k]$ is isomorphic to $t'[\overrightarrow{C}_{k'}]$. $\square$

Corollary 37 is an exemple of application of Theorem 35:

Theorem 36 [11] *The circulant tournament $\overrightarrow{C}_p$, p odd, is DS.*

Corollary 37 *There are no graphs in $\mathcal{T}_r$ cospectral non isomorphic to $\overrightarrow{C}_p[\overrightarrow{C}_k]$.*

We end this section by giving an algebraic characterization of some vertex-critical tournament without non-trivial acyclic interval.

Lemma 38 *Let p be a prime number, ζ_p be a primitive root of* 1 *and let* $\lambda = \sum_{j=1}^{\frac{p-1}{2}} e^{\frac{2i\pi j}{p}}$. *Then we have that the field extentions $\mathbb{Q}(\zeta_p)$ and $\mathbb{Q}(\lambda)$ are equal. As a consequence, the degree of the minimal polynomial in $\mathbb{Q}$ of λ is $p-1$ and The conjugates of λ in $\mathbb{Q}$ are $\lambda_r = \sum_{j=1}^{\frac{p-1}{2}} e^{\frac{2i\pi j}{p}r}$, $\quad r = 1, ..., p-1$.*

Proof: We have $\lambda \in \mathbb{Q}(\zeta_p)$ and so $\mathbb{Q}(\lambda) \subset \mathbb{Q}(\zeta_p)$. Let $\zeta_p = e^{\frac{2i\pi}{p}}$ and $k = \frac{p-1}{2}$, then $\lambda = \zeta_p(1 + \zeta_p + ... + \zeta_p^{k-1})$ and

$$\begin{aligned} \bar{\lambda} &= \zeta_p^{p-1}(1 + \zeta_p^{p-1} + ... + \zeta_p^{p-k+1}) \\ &= \zeta_p^{k+1}\zeta_p^{k-1}(1 + \zeta_p^{p-1} + ... + \zeta_p^{p-k+1}) \\ &= \zeta_p^{k+1}(\zeta_p^{k-1} + \zeta_p^{k-2} + ... + 1). \end{aligned}$$

So $\zeta_p^k = \lambda^{-1}\bar{\lambda}$ and $\mathbb{Q}(\zeta_p) \subset \mathbb{Q}(\lambda)$.

Let $\chi(X)$ be the characteristic polynomial of $\overrightarrow{C}_p$, then there is a polynomial $P(X)$ of degree $p-1$ and with coefficients in $\mathbb{Q}$ such that $\chi(X) = (X - \frac{p-1}{2})P(X)$. Since λ is an eigenvalues of $\overrightarrow{C}_p$, then it is a root of P and therefore P is the minimal polynomial of λ. The conjugates of λ in $\mathbb{Q}$ are then the eigenvalues of $\overrightarrow{C}_p$ different from $\frac{p-1}{2}$ and are described in Proposition 29. $\square$

Theorem 39 *Let T be a tournament on n vertices. If the three following conditions are satisfied*
(i) *there is an integer m and prime number $p_1, p_2, ...p_m$ such that $p_1 + .. + p_m = n$ and $m < \min\{p_i\}$;*
(ii) *the automorphism group of T contains the cycles* $(0\ 1\ ...\ p_1-1), (p_1\ p_1+1\ ...\ p_2-1), ..., (p_1+...+p_{m-1}\ \ p_1+...+p_{m-1}+1\ \ ...\ \ p_1+...+p_m-1)$;
(iii) *the adjacency spectrum of T contains the following eigenvalues*

$$\lambda^{(p_s)} = \sum_{j=1}^{\frac{p_s-1}{2}} e^{\frac{2i\pi j}{p_s}}, \quad s = 1, ..., m;$$

then T is a vertex-critical tournament without non-trivial acyclic interval and there exists a tournament t such that $T = t[\overrightarrow{C}_{p_1}, \cdots, \overrightarrow{C}_{p_m}]$.

Proof: Condition (ii) implies that there exists tournaments T_1, ..., T_m with respectively p_1, ..., p_m vertices and a tournament t on m vertices such that $T = t[T_1, T_2, ..., T_m]$. Moreover the automorphism group of T_i contains a full-length cycle, so T_i are circulant tournaments and consequeltly are regular of degree $\frac{p_i-1}{2}$. It remains to show that these tournaments T_i are isomorphic to $\overrightarrow{C}_{p_i}$.

Using block-matrices, as done in Theorem 16, or by analogy with Theorem 34, the spectrum of $T = t[T_1, T_2, ..., T_m]$ is

$$\bigcup_{j=1}^{n} \left(Sp(T_{p_j}) \setminus \left\{ \frac{p_j - 1}{2} \right\} \right) \cup Sp(A\hat{D} + \Delta)$$

where A is the adjacency matrix of t, $\hat{D}$ is the diagonal matrix of the p_j's and Δ is the diagonal matrix whose j^{th} entry is $\frac{p_j-1}{2}$.

Fact 1: an eigenvalue $\lambda^{(p_s)}$ described in condition (iii) cannot be an eigenvalue of $A\hat{D} + \Delta$. Indeed, the characteristic polynomial of $A\hat{D} + \Delta$ has its coefficients in $\mathbb{Z}$ and is of degree m and $\lambda^{(p_s)}$ is complex and its minimal polynomial in $\mathbb{Q}$ is of degree $p_s - 1$ (Lemma 38). But $m < p_s$ by condition (i) so $m = p_s - 1$ and the eigenvalues of $A\hat{D} + \Delta$ are $\lambda^{(p_s)}$ and its conjugates described in Lemma 38. On one hand the trace of $A\hat{D} + \Delta$ is the trace of Δ (which is positive) and on the other hand the trace of $A\hat{D} + \Delta$ is the sum of its eigenvalues that is (according to Proposition 33) $-\frac{1}{2}(p_s - 1) < 0$, involving a contradiction.

Fact 2: an eigenvalue $\lambda^{(p_s)}$ described in condition (iii) cannot be an eigenvalue of T_r with $|T_r| = p_r \neq p_s$. Indeed, if $\lambda^{(p_s)}$ is an eigenvalue of T_r then according to Proposition 29 we have $\lambda^{(p_s)} \in \mathbb{Q}(\zeta_{p_r})$, but $\lambda^{(p_s)} \in \mathbb{Q}(\zeta_{p_s})$, so $\lambda^{(p_s)} \in \mathbb{Q}(\zeta_{p_r}) \cap \mathbb{Q}(\zeta_{p_s}) = \mathbb{Q}$, a contradiction.

According to Facts 1 and 2, $\lambda^{(p_s)}$ is an eigenvalue of T_s with $|T_s| = p_s$. Therefore the p_s eigenvalues of $|T_s|$ are $\lambda_r^{(p_s)} = \sum_{j=1}^{\frac{p_s-1}{2}} e^{\frac{2i\pi j}{p_s} r}$, $r = 1, ..., p_s - 1$ and $\frac{p_s-1}{2}$ (T_s is $\frac{p_s-1}{2}$-regular). The tournament T_s has the same eigenvalues than $\overrightarrow{C}_{p_s}$, so T_s is isomorphic to $\overrightarrow{C}_{p_s}$ (Theorem 36).

This ends the proof of this theorem. $\square$

5 Conclusion

As a conclusion, if we focus on the well-known question (but far from being solved) *Which graphs are determined by their spectrum?*, we realise that the spectrum is not sufficient to (easily) determine graphs. An easier problem, which is often a first step in proving DS graph, consists in showing that some given graphs are *characterised by their spectrum* within a smaller family of graphs. This is what we have done in this paper by giving characterisations of strict starlike threshold graphs or stars of completes. In this paper we also established some links between graphs determined by their spectrum and graph compositions characterised by their spectrum.

Another difficult point is to deal with directed graphs. Indeed the eigenvalues are complex (and as a consequence more difficult to handle) and few digraphs are proved to be DS; moreover if we compare to undirected graph, there are less properties linking the spectrum to the structure for digraphs than for graphs. As we have done with some vertex-critical tournament without non-trivial acyclic interval (which can be written as a digraph composition), another way to extend the problem of finding DS graphs is to consider other algebraic objects related to the graphs: we do not only consider the spectrum but also the automorphism group for instance. The new question arising is *Which (di)graphs are determined by their spectrum and their automorphism group?* in the sense that if we have these two informations (the spectrum and the automorphism group), we wonder if we could associate one and only one graph (up to isomorphism).

References

[1] R. Boulet. Disjoint union of complete graphs characterized by their Laplacian spectrum *Electronic Journal of Linear Algebra*, 18:773–783, 2009

[2] M.R. Cerioli and J.L. Szwarcfiter. Edge clique graphs and some classes of chordal graphs. *Discrete Mathematics*, 242:31–39, 2002.

[3] J.F. Culus and B. Jouve. Convex circuit-free coloration of an oriented graph. *European Journal of Combinatorics*, 30(1):43–52, 2009.

[4] E.R. van Dam and W.H. Haemers. Which graphs are determined by

their spectrum?. *Linear Algebra and its Applications*, 373:241–272, 2003.

[5] P.J. Davis. *Circulant Matrices.* Wiley, New-York, 1979.

[6] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal*, 23(98):298–305, 1973.

[7] R. A. Horn, C. R. Johnson. *Topics in matrix analysis.* Cambridge University Press, 1994.

[8] W. Imrich. Über das lexikographische Produkt von Graphen. *Archiv der Mathematik*, 20(3):228–234, 1969.

[9] L. Lovász. Operations with structures. *Acta Mathematica Academiae Scientiarum Hungaricae*, 18:321–328, 1967.

[10] L. Lovász. On the cancellation law among finite relational structures. *Periodica Mathematica Hungarica*, 1(2):145–156, 1971.

[11] D.A. Gregory, S.J. Kirkland and B.L. Shader. Pick's inequality and tournaments. *Linear algebra and its applications*, 186:15–36, 1993.

[12] S.J. Kirkland and B.L. Shader. Tournament Matrices with Extremal Spectral Properties. *Linear algebra and its applications*, 196:1–17, 1994.

[13] B. Mohar. The Laplacian spectrum of graphs. *Graph Theory, Combinatorics, and Applications*, 2:871–898, 1991.

On the existence of combinatorial configurations

Maria Bras-Amorós and Klara Stokes
Universitat Rovira i Virgili
Tarragona

Abstract

A (v, b, r, k) combinatorial configuration can be defined as a connected, (r, k)-biregular bipartite graph with v vertices on one side and b vertices on the other and with no cycle of length 4. Combinatorial configurations have become very important for some cryptographic applications to sensor networks and to peer-to-peer communities. Configurable tuples are those tuples (v, b, r, k) for which a (v, b, r, k) combinatorial configuration exists. It is proved in this work that the set of configurable tuples with fixed r and k has the structure of a numerical semigroup. The semigroup is completely described whenever $r = 2$ or $r = 3$. For the remaining cases some bounds are given on the multiplicity and the conductor of the numerical semigroup. This leads to some concluding results on the existence of configurable tuples.

1 Introduction

Combinatorial configurations are a particular case of so-called incidence structures which have been recently used for defining peer-to-peer communities for preserving privacy of users in front of search engines [2, 3]. Other applications of configurations related to sensor networks can be found in [12].

A (v, b, r, k)-configuration is a set of v "points" $\mathcal{P} = \{p_1, \ldots, p_v\}$ and a set of b "lines" $\mathcal{L} = \{l_1, \ldots, l_b\}$, such that there are k points on each line, through each point pass r lines and no two points are joined by more than one line. There is a natural bijection between combinatorial configurations and connected bipartite biregular graphs with girth larger than 5. Observe that since these graphs are bipartite, the girth is always even and therefore larger than or equal to 6. In the present work we will treat configurations as such graphs.

One problem when using configurations is the limited number of known configurations, specially for large v and b. We refer the reader to [7, 6] for previously known results on the existence of combinatorial configurations.

In [3] larger configurations are constructed by combining smaller configurations; a (v, b, r, k)-configuration is obtained with parameters of the form $b = b' + b''$ and $v = v' + v''$, from existing configurations with parameters (v', b', r, k) and (v'', b'', r, k). In this article we interpret this result as giving structure to the set of parameters of existing configurations.

A numerical semigroup is a subset of $\mathbb{N}_0$ that contains 0, is closed under addition and has finite complement in $\mathbb{N}_0$.

Fix $r > 1, k > 1$. We will show that the set of all tuples (v, b, r, k) such that there exists a (v, b, r, k) configuration has the structure of a numerical semigroup. This semigroup can be explicitly described if $r = 2$ or $r = 3$. For the general case we give bounds on the multiplicity and the conductor of the numerical semigroup. The new results on the existence of configurable tuples deduced from this work are summarized in Theorem 30.

2 The semigroup of combinatorial configurations

2.1 Previous results on the existence of configurations

Definition 1 An incidence structure is a triple $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, where $\mathcal{P}$ is a set of "points", $\mathcal{L}$ a set of "lines" and $\mathcal{I} \subset (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ is a symmetric incidence relation.

In this article, no geometric meaning is attached to the terms point and line.

Definition 2 A (v, b, r, k)-configuration is an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$, which has

- v points,
- b lines,
- r lines through any point,

- k points on any line,

and in which any two different points are incident with at most one line, or equivalently, any two different lines are incident with at most one point.

Two general references on geometric and combinatorial configurations are [7, 6].

In the following we will suppose that $v \leq b$. This can be done without loss of generality, since if $v > b$ we can take the dual configuration.

Proposition 3 *The conditions $vr = bk$ and $v \geq r(k-1)+1$ are necessary for a non-trivial configuration (v,b,r,k) to exist* [6].

For some values of r and k, more is known.

Theorem 4 *For $k=3$ the necessary conditions of Proposition 3 are sufficient* [5].

There has not been found any example on parameters v,b,r, such that a $(v,b,r,4)-$configuration fails to exist as long as the parameters satisfy the necessary conditions. Regarding symmetric configurations, that is, v_k-configurations, for which $r=k$ and $v=b$, it is known that for $r=k=4$ they all exist for $v \geq 13$.

The necessary conditions are not always sufficient. One example is $k=5$, since there is no configuration 22_5.

For symmetric configurations, existence for some parameters are listed in Table 1, also from [6] and [4]. We see there that also for small values of k and v the existence of v_k-configurations is sometimes unknown, for example it is not known whether or not there exists a 33_6-configuration.

Results on non-symmetric configurations, generalizing the symmetric configurations, are more sparse, at least for large parameters. The state of the art can be found in [7] which actually treats geometric configurations, but also covers results on combinatorial configurations.

One interesting result in our context is the next theorem by Gropp. It guarantees the existence of large configurations and, in fact, the existence of any configuration satisfying the necessary conditions with sufficiently large v (and so b). Its limitation is the restriction on the choice of the parameters r, k.

Theorem 5 *For given k and r with $r = tk$ there is a v_0 depending on k, t such that there is a (v,b,r,k)-configuration for all $v \geq v_0$ satisfying the necessary conditions from Proposition* 3.

$k=5$	21_5	$-$	23_5	24_5	25_5	26_5	27_5	28_5	29_5	30_5	31_5
6	31_6	$-$	?	?	35_6	36_6	37_6	38_6	39_6	40_6	41_6
7	$-$	$-$	45_7	?	?	48_7	49_7	50_7	51_7	52_7	53_7

Table 1: Existence of configurations v_k for $5 \leq k \leq 7$ and $d \leq 10$. v_k means configuration exists, $-$ means configuration does not exist, ? means existence is unknown. The notation v_k-configuration is used to denote a $(v, v, k, k)-$configuration.

2.2 The set of $(r, k)-$configurable tuples

Definition 6 We say that the tuple (v, b, r, k) is *configurable if a* (v, b, r, k)*-configuration exists.*

As we saw in Proposition 3, if (v, b, r, k) is configurable then $vr = bk$ and consequently there exists d such that $v = d\frac{k}{\gcd(r,k)}$ and $b = d\frac{r}{\gcd(r,k)}$. So, to each configurable tuple (v, b, r, k) we can assign an integer d. Two different configurable tuples (v, b, r, k) will have different integers d. Let us call $D_{r,k}$ the set of all possible integers d corresponding to configurable tuples (v, b, r, k). That is,

$$D_{r,k} = \left\{ d \in \mathbb{N}_0 : \left(d\frac{k}{\gcd(r,k)}, d\frac{r}{\gcd(r,k)}, r, k \right) \text{ is configurable} \right\}.$$

Our aim is to study $D_{r,k}$. We will consider the empty graph to be also a configuration and consequently $0 \in D_{r,k}$ for all pair r, k. Obviously $D_{r,k} = D_{k,r}$ and $D_{1,k} = \{0, k\}$. First we will give a complete description of $D_{2,k}$ and a complete description of $D_{3,k}$ and then we will study the general case.

2.3 The case $r = 2$

There is a natural bijection between $(v, b, 2, k)$-configurations and k-regular connected graphs with b vertices and v edges. Two vertices in the graph share an edge if and only if the corresponding nodes in the configuration share a neighbor and viceversa. The following well-known lemma is the key result for describing $D_{2,k}$. We include the proof in order to make the article more self-contained.

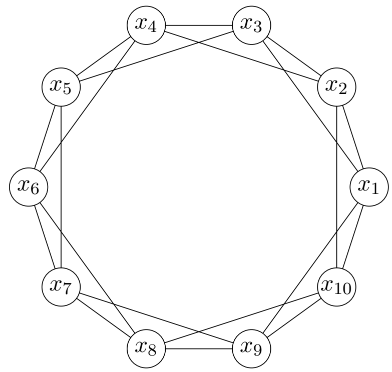


Figure 1: Construction of a connected 4-regular graph with 10 vertices

Lemma 7 *Let k be an even positive integer. A connected k-regular graph with b vertices exists if and only if $b \geq k+1$.*

Proof: By definition, any k-regular graph must have a number of vertices at least $k+1$.

Conversely, suppose $b \geq k+1$. Consider a set of vertices $x_1, \dots, x_b$. Put an edge between x_i and x_j, with $i \leq j$, if $j - i \leq k/2$ or $i + b - j \leq k/2$. This gives a connected k-regular graph with b vertices. □

The construction in this last proof is illustrated in Figure 1.

From the natural bijection between $(v, b, 2, k)$-configurations and k-regular connected graphs with b vertices and v edges, we get the following corollary. We write $\langle a_1, \dots, a_n \rangle$ to denote the numerical semigroup generated by $a_1, \dots, a_n$.

Corollary 8 *If k is an even positive integer then*

$$D_{2,k} = \langle k+1, k+2, \dots, 2k+1 \rangle .$$

Lemma 9 *Let k be an odd positive integer. A connected k-regular graph with b vertices exists if and only if b is even and $b \geq k+1$.*

Proof: By definition, any k-regular graph must have a number of vertices at least $k+1$. Now, since the number of edges is $kb/2$ this means that kb

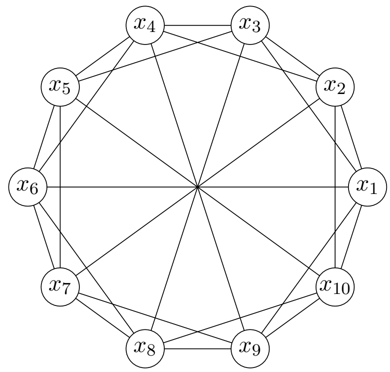


Figure 2: Construction of a connected 5-regular graph with 10 vertices

must be even and since k is odd b must be even. Conversely, suppose b is even and $b \geq k+1$. Consider a set of vertices $x_1, \dots, x_b$. Put an edge between x_i and x_j, with $i \leq j$, if $j - i \leq (k-1)/2$ or $i + b - j \leq (k-1)/2$. Put also edges between x_i and $x_{i+b/2}$ for i from 1 to $b/2$. This gives a connected k-regular graph with b vertices. $\square$

The construction in this last proof is illustrated in Figure 2.

From the natural bijection between $(v, b, 2, k)$-configurations and k-regular connected graphs with b vertices and v edges, we now get the following corollary.

Corollary 10 *If k is an odd positive integer then*

$$D_{2,k} = \left\langle \frac{k+1}{2}, \frac{k+1}{2} + 1, \frac{k+1}{2} + 2, \dots, k \right\rangle.$$

2.4 The case $r = 3$

Because of Theorem 4, the case $r = 3$ is much simpler. It is stated in the next theorem.

Theorem 11 *Suppose $k > 1$ then*

$$D_{3,k} = \begin{cases} \{0, 2k+1, 2k+2, \dots\} & \text{if } k \equiv 0 \bmod 3 \\ \{0, \frac{2k+1}{3}, \frac{2k+1}{3}+1, \frac{2k+1}{3}+2, \dots\} & \text{if } k \equiv 1 \bmod 3 \\ \{0, \frac{2k+2}{3}, \frac{2k+2}{3}+1, \frac{2k+2}{3}+2, \dots\} & \text{if } k \equiv 2 \bmod 3 \end{cases}$$

Proof: By Theorem 4 (by swapping the role of b and v) we know that any tuple $(v, b, 3, k)$ with $b \neq 0$ is configurable if and only if $3v = bk$ and $b \geq k(3-1)+1 = 2k+1$. In particular, the non-zero values b for which there exists a configurable tuple $(v, b, 3, k)$ are exactly those integers $b \geq 2k+1$ such that $\frac{bk}{3}$ is an integer.

If $k \equiv 0 \bmod 3$ then the only condition is $b \geq 2k+1$ which results in

$$d = \frac{b \gcd(3,k)}{3} = \frac{3b}{3} = b \geq 2k+1$$

and this proves the result in this case.

Otherwise, we need $b \geq 2k+1$ and b be a multiple of 3. If $k \equiv 1 \bmod 3$ this is equivalent to $b \in \{2k+1, 2k+4, 2k+7, \dots\}$ and so $d = \frac{b\gcd(3,k)}{3} = \frac{b}{3}$ is in

$$\left\{\frac{2k+1}{3}, \frac{2k+1}{3}+1, \frac{2k+1}{3}+2, \dots\right\}.$$

If $k \equiv 2 \bmod 3$ this is equivalent to $b \in \{2k+2, 2k+5, 2k+8, \dots\}$ and so $d = \frac{b\gcd(3,k)}{3} = \frac{b}{3}$ is in

$$\left\{\frac{2k+2}{3}, \frac{2k+2}{3}+1, \frac{2k+2}{3}+2, \dots\right\}. \quad \square$$

2.5 The general case

We want to prove that $D_{r,k} \subset \mathbb{N}_0$ is a numerical semigroup. The following results on semigroups will be helpful.

Proposition 12 *A set of integers generate a numerical semigroup if and only if they are coprime.*

The proof of this proposition can be found in [13].

Proposition 12 says that in order to prove that a set is a numerical semigroup it is enough to prove that the set is a submonoid of the natural numbers with coprime elements. This means that we need to prove that

- $0 \in D_{r,k}$,
- $D_{r,k}$ is closed under addition,
- at least two elements (and therefore all of the elements) of $D_{r,k}$ are coprime.

The two first conditions ensure that the subset $D_{r,k}$ of the natural numbers is a monoid. The operation of the monoid is addition. The last condition ensures that the monoid is a numerical semigroup. Since the case $r \leq 3$ has been proved earlier, in this section we will suppose that $r, k > 3$.

The set of configurable tuples is a submonoid of the natural numbers

We first observe that since we consider the empty configuration a configuration, $0 \in D_{r,k}$.

We will now prove that the set $D_{r,k}$ is closed under addition.

Lemma 13 *If (v, b, r, k) and (v', b', r, k) are configurable tuples, so is $(v + v', b + b', r, k)$.*

Proof: Suppose we have a (v, b, r, k)-configuration with vertices $\{x_1, \dots, x_v\}$, $\{y_1, \dots, y_b\}$ and a (v', b', r, k)-configuration with vertices $\{x'_1, \dots, x'_{v'}\}$ and $\{y'_1, \dots, y'_{b'}\}$. Consider the graph with vertices $\{x_1, \dots, x_v\} \cup \{x'_1, \dots, x'_{v'}\}$, $\{y_1, \dots, y_b\} \cup \{y'_1, \dots, y'_{b'}\}$ and all the edges in the original configurations. We can assume without loss of generality that the edges x_1y_1, x_vy_b,$x'_1y'_1$, $x'_{v'}y'_{b'}$ belong to the original configurations.

Swap the edges x_vy_b and $x'_1y'_1$ for $x_vy'_1$ and x'_1y_b. This gives a $(v+v', b+b', r, k)$ configuration [3]. An example of this construction is illustrated in Figure 3. □

Since

$$d = v \gcd(r, k)/k = b \gcd(r, k)/r$$

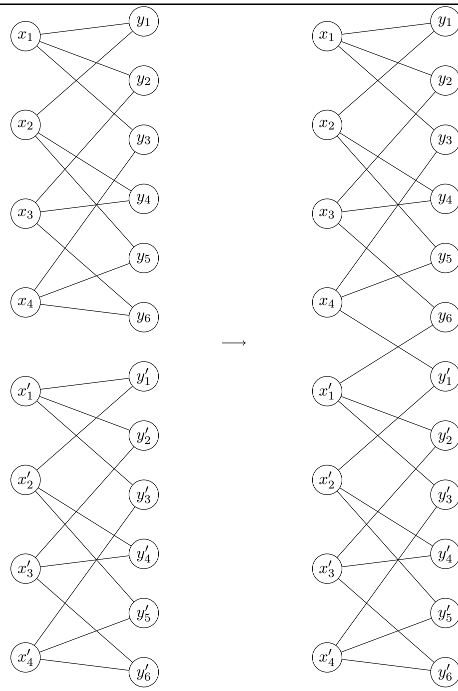


Figure 3: Construction of a $(v+v', b+b', r, k)$ configuration from a (v, b, r, k) configuration and a (v', b', r, k) configuration.

and

$$d' = v' \gcd(r, k)/k = b' \gcd(r, k)/r$$

there exists a

$$d'' = (v + v') \gcd(r, k)/k = (b + b') \gcd(r, k)/r = d + d'.$$

Hence if $d, d' \in D_{r,k}$, then $d + d' \in D_{r,k}$, or in other words $D_{r,k} \subset \mathbb{N}_0$ is closed under addition. Together with the fact that $0 \in D_{r,k}$ we get the result we were looking for.

Proposition 14 $D_{r,k}$ *is a submonoid of the natural numbers.*

The submonoid contains two coprime elements

We start by proving that given any pair of natural numbers (r, k), there exists at least one element in $D_{r,k}$. We do this by constructing a (v, b, r, k)-configuration.

For the construction we need a graph of girth at least 5. In [10] a family of small graphs of girth 5 is constructed, which can be used for our purposes. The existence of this graph is given in the following theorem.

Theorem 15 *Let $q \geq 13$ be an odd prime power and let $n \leq q + 3$. Then there exists a n regular graph with girth 5 and with $2(n-2)(q-1)$ vertices* [10, Theorem 17].

There are other constructions of small graphs for other (larger) girths, such as the ones in [1]. For our purposes taking girth at least 5 is enough.

Now we can construct the connected, r, k biregular graph of girth at least 5 which gives us the (v, b, r, k) configuration we are looking for.

Proposition 16 *For any pair of integers $r > 1, k > 1$, there exists at least one non-zero integer in $D_{r,k}$.*

Proof: The cases in which $r \leq 3$ or $k \leq 3$ have already been proved. We can therefore suppose that $r > 3$ and $k > 3$.

Consider the complete bipartite graph $K_{r,k}$, with edge set E and vertex set V. We consider one spanning tree $T_{r,k}$ of $K_{r,k}$. Then $T_{r,k}$ has vertex set V, but edge set $E' \subset E$ with $|E'| = r + k - 1$.

The number of edges in $K_{r,k}$ outside $T_{r,k}$, that is in $E - E'$, is $n = rk - r - k + 1$. Since $r, k > 3$ we have $n \geq 3$.

From Theorem 15 we know that there exists (at least) a n-regular graph of girth (at least) 5. We take one of these graphs and call it G.

Now we will construct a bipartite $(r, k)-$biregular graph of girth at least 5, using G. Associate to each of the vertices of G a copy of the complete bipartite graph $K_{r,k}$. For all edges ab in G, consider its end vertices a and b and let A and B be the copies of $K_{r,k}$ associated to these vertices. Also let T_A and T_B be the corresponding spanning trees in A and B. Now choose one edge $x_A y_A$ in A, but not in T_A and one edge $x_B y_B$ in B, but not in T_B and swap them so that we instead get two edges $x_A y_B$ and $x_B y_A$. Since G is $n-$regular and n is the number of edges in $K_{r,k}$ that are not in its spanning tree, we can choose different edges $x_A y_A$ and $x_B y_B$ for every edge in G.

In this way we get a bipartite, $(r,k)-$biregular graph of girth at least 5, from a $n-$regular graph of girth at least 5, with $n = rk - r - k + 1$.

The resulting graph may not be connected. If this is the case, we can proceed in two ways.

- We can choose any of the connected subgraphs, and consider that graph to be the incidence graph of the configuration we want to construct. If we choose the smallest connected subgraph, then we minimize the size of the smallest known (v, b, r, k)-configuration proved to exist in this manner;
- We can use the 'addition' law from Lemma 13 to connect all the connected subgraphs.

In any case we get a connected, bipartite, $(r,k)-$biregular graph of girth at least 6, hence the incidence graph of a (v, b, r, k)-configuration. $\square$

We will now construct a second element of $D_{r,k}$, also different from 0, such that the element we already have and the new one are coprime. In order to do so we need the following lemma.

Lemma 17 *Suppose we have a (v, b, r, k)-configuration with $r \geq 3$. There exist three edges in the configuration such that the six ends are all different.*

Proof: It is easy to prove, by the property that no cycle of length 4 exists, that there exists a path with four edges with the five ends being different. Three of these ends will be in one partition of the graph while the other two will be in the other partition. Take the vertex at the end of the path. It must be one of the three in the same partition. Since its degree is at least 3, then it will have one neighbor not in the path. So, by adding the edge from the end of the path to this additional vertex, we obtain a new path with 5 edges with all its vertices being different. By taking the first, third, and fifth edges of this new path we obtain the result. $\square$

This lemma tells us that the vertices $\{x_1, \ldots, x_v\}$, $\{y_1, \ldots, y_b\}$ in a (v, b, r, k)-configuration with $r \geq 3$ can be arranged in a way such that the edges x_1y_1, x_2y_2 and x_vy_b belong to the configuration.

We are now ready to prove the existence of two coprime elements of $D_{r,k}$.

Proposition 18 *$D_{r,k}$ contains two elements $m \neq 0$ and $sm+1$, with $s = rk/\gcd(r,k)$, so that the two are coprime.*

Proof: Because of the results in the previous sections we can assume that r and k are larger than 3. By Proposition 16 and since $D_{r,k} \subseteq \mathbb{N}_0$, there is a minimal non-zero element m in $D_{r,k}$. Let us call

$$v = mk/\gcd(r,k)$$

and

$$b = mr/\gcd(r,k).$$

Select a (v,b,r,k) configuration. Take

$$s = rk/\gcd(r,k)$$

copies of this configuration. Let us call the vertices of the ith copy

$$x_1^{(i)},\ldots,x_v^{(i)},\ y_1^{(i)},\ldots,y_b^{(i)}.$$

By Lemma 17 we can assume that

$$x_1^{(i)}y_1^{(i)},\ x_2^{(i)}y_2^{(i)} \text{ and } x_v^{(i)}y_b^{(i)}$$

belong to the ith copy. Consider $k/\gcd(r,k)$ further vertices

$$x'_1,\ldots,x'_{k/\gcd(r,k)}$$

and $r/\gcd(r,k)$ further vertices

$$y'_1,\ldots,y'_{r/\gcd(r,k)}.$$

Now perform the following changes to the edge set of the graph defined by the union of all parts previously mentioned. It may be clarifying to contemplate Figure 4. In the figure the edges to be removed are dashed, while the edges to add are thick lines.

- For all $2 \leq i \leq s$ replace the edges

  $$x_v^{(i)}y_b^{(i)} \text{ and } x_1^{(i-1)}y_1^{(i-1)}$$

  by

$$x_v^{(i)}y_1^{(i-1)} \text{ and } x_1^{(i-1)}y_b^{(i)}.$$

- Also, remove the edges $x_2^{(i)}y_2^{(i)}$ for all $2 \leq i \leq s$.
- Add the edges

$$\begin{gathered} x'_1y_2^{(1)}, x'_1y_2^{(2)}, \dots, x'_1y_2^{(r)}, \\ x'_2y_2^{(r+1)}, x'_2y_2^{(r+2)}, \dots, x'_2y_2^{(2r)}, \\ \vdots \\ x'_{k/\gcd(r,k)}y_2^{(s-r+1)}, \dots, x'_{k/\gcd(r,k)}y_2^{(s)} \end{gathered}$$

and

$$\begin{gathered} x_2^{(1)}y'_1, x_2^{(2)}y'_1, \dots, x_2^{(k)}y'_1, \\ x_2^{(k+1)}y'_2, x_2^{(k+2)}y'_2, \dots, x_2^{(2k)}y'_2, \\ \vdots \\ x_2^{(s-k+1)}y'_{r/\gcd(r,k)}, \dots, x_2^{(s)}y'_{r/\gcd(r,k)}. \end{gathered}$$

As can be verified, the construction gives a new configuration with parameters

$$\begin{aligned} (v', b', r, k) &= (sv + k/\gcd(r,k),\ sb + r/\gcd(r,k), r, k) \\ &= (smk/\gcd(r,k) + k/\gcd(r,k), \\ &\qquad\qquad smr/\gcd(r,k) + r/\gcd(r,k),\ r,\ k) \\ &= ((sm+1)k/\gcd(r,k),\ (sm+1)r/\gcd(r,k),\ r,\ k) \end{aligned}$$

and so $sm + 1 \in D_{r,k}$. $\square$

From Proposition 18 we deduce that $D_{r,k}$ contains two coprime elements, so that they generate a numerical semigroup and this semigroup is contained in $D_{r,k}$. So the complement of $D_{r,k}$ in $\mathbb{N}_0$ is finite and $D_{r,k}$ is a numerical semigroup.

Theorem 19 *For every pair of integers $r, k \geq 2$, $D_{r,k}$ is a numerical semigroup.*

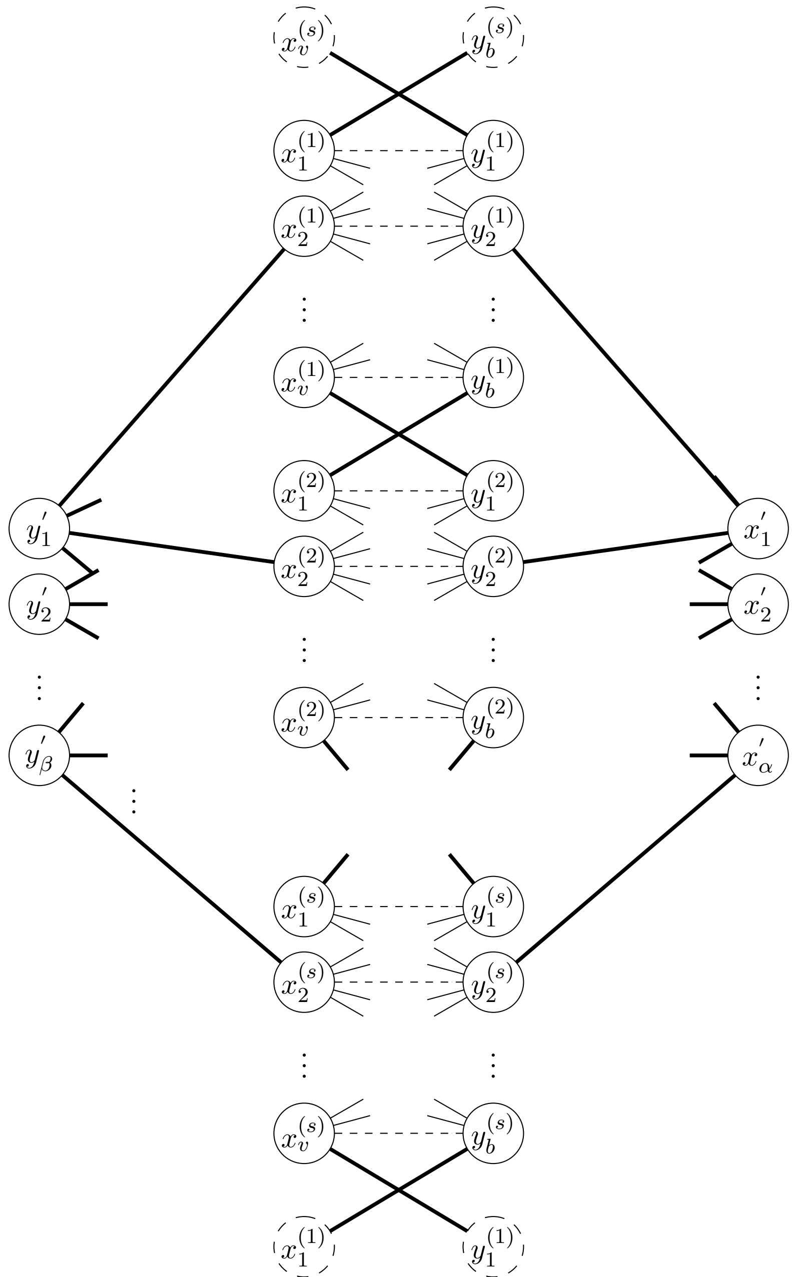


Figure 4: Construction of a $(sv + k/\gcd(r,k), sb + r/\gcd(r,k), r, k)$-configuration from s (v, b, r, k) configurations and $\alpha + \beta = k/\gcd(r,k) + r/\gcd(r,k)$ extra vertices.

We call an element of the complement of a numerical semigroup a gap. Observe that $D_{2,k}$ as well as $D_{3,k}$ for $k > 1$ are ordinary, that is, all their gaps are in a row. However, there are pairs r, k for which $D_{r,k}$ is not ordinary. For example the multiplicity of $D_{5,5}$ is 21, but 22 is a gap, as can be deduced from Table 1. Also, while the multiplicity of $D_{6,6}$ is 31, we have that $33 \notin D_{6,6}$ (see [11]).

3 Bounds on configurable tuples

3.1 A lower bound on the multiplicity of the numerical semigroup $D_{r,k}$

The multiplicity of a numerical semigroup is its smallest non-zero element. Observe that bounds on the multiplicity are bounds regarding the size of the smallest configuration for a given pair of r and k.

For the cases $r = 2$ and $r = 3$, since we know the actual structure of the semigroup we can precise the multiplicity exactly.

Proposition 20 *For $k > 1$ the multiplicity of $D_{2,k}$ is*

$$\begin{cases} k+1 & \text{if } k \text{ is even} \\ \frac{k+1}{2} & \text{if } k \text{ is odd} \end{cases}$$

For $k > 1$ the multiplicity of $D_{3,k}$ is

$$\begin{cases} 2k+1 & \text{if } k \equiv 0 \bmod 3 \\ \frac{2k+1}{3} & \text{if } k \equiv 1 \bmod 3 \\ \frac{2k+2}{3} & \text{if } k \equiv 2 \bmod 3 \end{cases}$$

The proof follows from Corollary 8, Corollary 10, and Theorem 11.

In the next lemma we give a lower bound for the multiplicity of $D_{r,k}$. It is a consequence of Proposition 3.

Lemma 21 *If $d \in D_{r,k}$ and $d \neq 0$ then $d \geq (rk - r + 1)\frac{\gcd(r,k)}{k}$ and, by symmetry, $d \geq (rk - k + 1)\frac{\gcd(r,k)}{r}$.*

For certain $r = k$, the bound is tight. An example is seen in the next proposition.

Proposition 22 *For $r = k = q + 1$ with q a power of a prime, the multiplicity of the numerical semigroup $D_{r,k}$ is $r^2 - r + 1$.*

Proof: There exists a finite projective plane for every power of a prime q. The projective plane is a $(q^2 + q + 1, q^2 + q + 1, q + 1, q + 1)-$configuration. We have that

$$q^2 + q + 1 = (r - 1)^2 + r - 1 + 1 = r^2 - r + 1. \quad \square$$

3.2 An upper bound on the multiplicity of the numerical semigroup $D_{r,k}$

In Proposition 16 we proved that $D_{r,k}$ contains at least one element for every pair (r, k). Counting the points and lines of the configuration constructed in the proof of Proposition 16 we get an upper bound on the multiplicity of $D_{r,k}$.

The graph G from the proof of Proposition 16 has $2(n-2)(q-1)$ vertices, for an odd prime power $q \geq n-3$, $q \geq 13$ and $n = rk-r-k+1$. In the final graph, constructed from G, every vertex of G is replaced by the vertices of the $r, k-$complete graph. Therefore in the final graph, the total number of vertices is $2(n - 2)(q - 1)(r + k)$ and the numbers of points and lines in the corresponding configuration are $2(n - 2)(q - 1)r$ and $2(n - 2)(q - 1)k$ respectively.

A bound on the existence of a prime power

In order to get an exact bound we need a bound on the existence of the prime power q. However, we will not care about prime powers of higher exponents and instead use a famous bound on the existence of primes. The density of prime powers of exponent greater than 1 is small compared with the density of primes.

Proposition 23 *The number of squares, cubes, ... of primes up to x does not exceed*

$$x^{\frac{1}{2}} + x^{\frac{1}{3}} + x^{\frac{1}{4}} + \cdots = O(x^{\frac{1}{2}} \ln x).$$

When it comes to prime numbers, their density is described in the 'Prime number theorem'.

Theorem 24 *Prime number theorem Let $\pi(x)$ be the function counting the number of prime numbers up to x. Then we have*

$$\pi(x) \sim \tfrac{x}{\ln x}$$

The function $x^{\frac{1}{2}} \ln x$ grows much slower than a function containing x like the latter, so that the function counting all prime powers, including those of exponent 1, behaves asymptotically like the prime counting function. For more details on this see [9].

Therefore, when we look for a power of a prime $\geq n-3$, we are more likely to find a prime p than a power of a prime p^m, and it is enough to apply the Bertrand's Postulate in order to get a good bound.

Theorem 25 *Bertrand's Postulate*

If $m > 3$ is an integer, then there always exists at least one prime number p with $m < p < 2m-2$.

Using this, since we want our prime to be greater or equal to $n-3$, we get that there exists at least one prime number in the interval

$$[n-3, 2(n-4)-3] = [n-3, 2n-11].$$

Therefore we get the following upper bound on the multiplicity of $D_{r,k}$.

Proposition 26 *For $r, k > 3$ the multiplicity m of $D_{r,k}$ satisfies*

$$m \leq 2(rk-r-k-1)(2(rk-r-k)-10)\gcd(r,k).$$

Proof: Since $r, k > 3$, we have $n = rk - r - k + 1 > 7$ and so $n - 4 > 3$. Because of the construction of the configuration in Proposition 16 and Bertrand's postulate, we get the following bound on the number of points in the configuration.

$$\begin{aligned}
v &= 2(n-2)(q-1)k \\
&\leq 2(n-2)(2n-11-1)k \\
&= 2(rk-r-k-1)(2(rk-r-k)-10)k
\end{aligned}$$

We have

$$v = d\tfrac{k}{\gcd(r,k)}$$

and therefore

$$d = \tfrac{v\gcd(r,k)}{k}.$$

This means that in this particular configuration

$$\begin{aligned} d &= \tfrac{v\gcd(r,k)}{k} \\ &\leq 2(rk-r-k-1)(2(rk-r-k)-10)\gcd(r,k) \end{aligned}$$

If we had used the bound on the number of lines in the configuration instead, we would have arrived at the same conclusion, simply replacing k by r. □

3.3 An upper bound on the conductor of the numerical semigroup $D_{r,k}$

The largest gap of a numerical semigroup S is called the Frobenius number of S. To Proposition 12 we have associated the following result.

Proposition 27 *The numerical semigroup generated by two coprime positive integers a, b has Frobenius number $(a-1)(b-1)-1$ [15].*

Definition 28 The *conductor* of a numerical semigroup is the smallest element such that all subsequent natural numbers belong to the semigroup.

Hence if the Frobenius number is f, then $c = f + 1$. By bounding the conductor upwards, we get a value from which all subsequent integers give configurable tuples. This is equivalent to giving values v_0 and b_0 such that all tuples (v, b, r, k) with $vr = bk$, $v \geq b_0$ and $b \geq b_0$ are configurable.

As before, for the cases $r = 2$ and $r = 3$ we know exactly the conductor of $D_{r,k}$. Indeed, in these semigroups the conductor is equal to the multiplicity (see Corollary 8, Corollary 10, and Theorem 11) and the multiplicity is given in Proposition 20.

Proposition 29 *Suppose $r, k > 1$ and let $t = rk-r-k-1$. The conductor c of the numerical semigroup $D_{r,k}$ satisfies*

$$c \leq rk((4t^2-16t)^2\gcd(r,k)-4t^2+16t).$$

Proof: By Proposition 26 the multiplicity of $D_{r,k}$ satisfies $m \leq 2(rk - r - k - 1)(2(rk - r - k) - 10)gcd(r,k)$. Now, because of Proposition 18, $sm + 1 \in D_{r,k}$, with $s = rk/gcd(r,k)$, and $sm + 1$ and m are coprime. Therefore Proposition 27 says that $(m-1)sm-1$ is the Frobenius element of a numerical semigroup contained in $D_{r,k}$. We have

$(m-1)sm-1 \leq (2(rk-r-k-1)(2(rk-r-k)-10)\gcd(r,k)-1)2(rk-r-k-1)(2(rk-r-k)-10)rk-1$

and therefore the conductor is bounded by

$c \leq (2(rk-r-k-1)(2(rk-r-k)-10)\gcd(r,k)-1)2(rk-r-k-1)(2(rk-r-k)-10)rk.$

With $t = rk - r - k - 1$ we get

$c \leq (2t(2t-8)\gcd(r,k)-1)2t(2t-8)rk = rk((4t^2-16t)^2\gcd(r,k)-4t^2+16t).$
$\square$

	$k=4$	5	6	7	8
$r=4$	13	17/5	7	25/7	29/2
5		21	13/3	31/7	9/2
6			31	37/7	43/4
7				43	25/4
8					57

Table 2: Lower bounds for the multiplicity of the numerical semigroup $D_{r,k}$

	$k=4$	5	6	7	8
$r=4$	336	240	936	768	4560
5		2800	1008	1584	2288
6			10488	2688	7656
7				28560	5760
8					64672

Table 3: Upper bounds for the multiplicity of the numerical semigroup $D_{r,k}$

	$k=4$	5	6	7	8
$r=4$	450240	1147200	10501920	16493568	166312320
5		39186000	30451680	87761520	209306240
6			659925936	303351552	1406560320
7				5709515280	1857623040
8					33459223296

Table 4: Upper bounds for the conductor of the numerical semigroup $D_{r,k}$

3.4 Results

The upper bounds on the multiplicity and the conductor of $D_{r,k}$ are both huge, while the lower bound on the multiplicity is quite small. In Table 2, Table 3 and Table 4 one can see some examples of the values the bounds take for some r and k. We leave it as an open problem to find better bounds.

3.5 Concluding results

We are ready to collect our results in a final theorem.

Theorem 30 *For any pair of integers r, k, both larger than 1,*

(i) *there exist infinitely many configurable tuples (v, b, r, k);*

(ii) *there exists at least one configurable tuple (v, b, r, k) with*

$$v \leq 2(rk - r - k - 1)(2(rk - r - k) - 10)k$$

and

$$b \leq 2(rk - r - k - 1)(2(rk - r - k) - 10)r;$$

(iii) *all tuples (v, b, r, k) with $vr = bk$,*

- $v \geq d_0 k / \gcd(r, k)$, *and*
- $b \geq d_0 r / \gcd(r, k)$,

are configurable for a certain d_0*;*

(iv) *if* $r=2$ *then*

$$d_0 = \begin{cases} k+1 & \text{if } k \text{ is even} \\ \frac{k+1}{2} & \text{if } k \text{ is odd} \end{cases}$$

if $r=3$ *then*

$$d_0 = \begin{cases} 2k+1 & \text{if } k \equiv 0 \bmod 3 \\ \frac{2k+1}{3} & \text{if } k \equiv 1 \bmod 3 \\ \frac{2k+2}{3} & \text{if } k \equiv 2 \bmod 3 \end{cases}$$

(v) *if* $r,k>3$ *then* $d_0 \geq rk((4t^2-16t)^2 \gcd(r,k) - 4t^2 + 16t)$*, where* $t = rk - r - k - 1$*.*

Proof:

(i) This is a result of the fact that for any (r,k), $D_{r,k}$ is a numerical semigroup,

(ii) This was proven in Proposition 26.

(iii) This is because a numerical semigroup has a conductor d_0, so that every element greater or equal to d_0 pertains to $D_{r,k}$.

(iv) This is a consequence of Proposition 20 and the fact that for the semigroups $D_{2,k}$ and $D_{3,k}$ the multiplicity equals the conductor.

(v) This is the bound on the conductor from Proposition 29. $\square$

Acknowledgement

This work was partly supported by the Spanish Government through projects TIN2009-11689 "RIPUP" and CONSOLIDER INGENIO 2010 CSD2007-00004 "ARES", and by the Government of Catalonia under grant 2009 SGR 1135.

References

[1] G. Araujo-Pardo, C. Balbuena b and T. Héger. Finding small regular graphs of girths 6, 8 and 12 as subgraphs of cages. *Discrete Mathematics*, volume 310, pages 1301-1306, 2010.

[2] J. Domingo-Ferrer and M. Bras-Amorós. Peer-to-peer private information retrieval. In J. Domingo-Ferrer and Y. Saygin, editors, *Privacy in Statistical Databases*, volume 5262 of *Lecture Notes in Computer Science*, pages 315–323. Springer, 2008.

[3] J. Domingo-Ferrer, M. Bras-Amorós, Q. Wu, and J. Manjón. User-private information retrieval based on a peer-to-peer community. *Data and Knowledge Engineering*, 68(11):1237-1252, 2009.

[4] H. Gropp. Nonsymmetric configurations with deficiencies 1 and 2. In *Combinatorics '90 (Gaeta, 1990)*, volume 52 of *Ann. Discrete Math.*, pages 227–239. North-Holland, Amsterdam, 1992.

[5] H. Gropp. Non-symmetric configurations with natural index. *Discrete Mathematics*, 124:87–98, 1994.

[6] H. Gropp. *Handbook Of Combinatorial Designs (Charles J. Colbourn and Jeffrey H. Dinitz ed.)*, chapter Configurations, pages 353–355. Chapman and Hall/CRC, Kenneth H. Rosen, 2007.

[7] B. Grünbaum. *Configurations of Points and Lines*, volume 103 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009.

[8] M. Hall, Jr. *The theory of groups*. The Macmillan Co., New York, N.Y., 1959.

[9] G.H. Hardy. *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work*. 1999.

[10] L. K. Jørgensen. Girth 5 graphs from relative difference sets. *Discrete Mathematics*, 293(1-3):177–184, 2005.

[11] P. Kaski and P. R. J. Östergård. There exists no symmetric configuration with 33 points and line size 6. *Australas. J. Combin.*, 38:273–277, 2007.

[12] J. Lee and D. R. Stinson. A combinatorial approach to key predistribution for distributed sensor networks. In *Wireless Communications and Networking Conference-WCNC 2005*, volume 2, pages 1200–1205, 2005.

[13] J. C. Rosales and P. A. García-Sánchez. *Numerical semigrops*, volume 20 of *Developments in Mathematics*. Springer, New York, 2009.

[14] K. Stokes and M. Bras-Amorós. Optimal configurations for peer-to-peer private information retrieval. Computers and Mathematics with Applications, volume 59 (4), pages 1568 - 1577, 2010.

[15] J.J. Sylvester. On subvariants, i.e. semi-invariants to binary quantics of an unlimited order. *American Journal of Mathematics*, volume 5 (1-4) pages 79-136, 1882.

Symmetric L-graphs

Cristóbal Camarero, Carmen Martínez and Ramón Beivide
Universidad de Cantabria
Santander

Abstract

In this paper we characterize symmetric L-graphs, which are either Kronecker products of two cycles or Gaussian graphs. Vertex symmetric networks have the property that the communication load is uniformly distributed on all the vertices so that there is no point of congestion. A stronger notion of symmetry, edge symmetry, requires that every edge in the graph looks the same. Such property ensures that the communication load is uniformly distributed over all the communication links, so that there is no congestion at any link.

1 Introduction

Many interconnection networks have been based on vertex-transitive (or vertex-symmetric) graphs. This is the case of current parallel computers from Cray, HP and IBM, among others, that are built around torus networks. Tori have displaced meshes that do not use wraparound edges which simplifies planar design at the price of losing vertex-transitivity. Less work has been devoted to edge-transitive (or edge-symmetric) networks. Square tori would be the network of choice as it is symmetric (vertex and edge symmetric). However, for practical reasons such as packaging, modularity, cost and scalability, the number of nodes per dimension might be different. These topologies, are denoted as mixed-radix networks in [3]. Mixed-radix tori have the drawback of being non-edge-transitive which leads to an imbalanced utilization of network links and buffers. For many traffic patterns, the load on the longer dimension is higher than on the shorter one and, hence, links in longer dimension become network bottlenecks, [1].

The present work is devoted to find and characterize the symmetric members of the two-dimensional family of undirected *multidimensional circulant graphs*, defined in [5]. These graphs can be informally denoted as *L-graphs*, since their set of vertices is a subset of a two-dimensional lattice that can be represented as an L-shaped mesh with wrap-around edges. These two-dimensional networks can be considered as a generalization of torus graphs as they have a planar representation when laying them out on a torus surface. L-shaped graphs have been widely considered, for example in [2] and [6], with applications to interconnection networks.

We recall now some definitions and results appeared in [5]. Let $M \in \mathcal{M}_{n\times n}(\mathbb{Z})$ be a $n \times n$ matrix of integers. Given two vectors $a, b \in \mathbb{Z}^n$ we say that a is congruent to b modulo M, which we denote as $a \equiv b \pmod{M}$, if $a - b \in M\mathbb{Z}^n$, where $M\mathbb{Z}^n$ stands for the additive group of column n-vectors with integral coordinates. We will also denote as $\mathbb{Z}^n/M\mathbb{Z}^n$, the group of integral vectors modulo M, which has $|\det(M)|$ elements when M is nonsingular.

Definition 1 Let $M \in \mathcal{M}_{2\times 2}(\mathbb{Z})$ be a non-singular matrix. Let $A = \{\pm a_1, \pm a_2\}$ be a subset of vectors of $\mathbb{Z}^2$ such that $\{a_1, a_2\}$ are linearly independent. The L-graph generated by M and adjacency A, $L(M; A)$, is defined as the graph whose vertex set is formed by the elements of $\mathbb{Z}^2/M\mathbb{Z}^2$ and every vertex u is adjacent to $u + A \pmod{M}$.

In particular, we are interested in those cases whose generating set of jumps is $A = \{e_1, e_2\}$, where $e_1 = (1, 0)^t$ and $e_2 = (0, 1)^t$ form the orthonormal two-dimensional basis. In general we will assume $A = \{e_1, e_2\}$ and therefore we will just denote $L(M; A) = L(M)$.

As stated above, in this paper we study the symmetry of L-graphs. We will proof that the only symmetric L-graphs are either Kronecker products of two cycles or Gaussian graphs. Perfect codes over L-graphs were characterized in [7]. It was shown that L-graphs include Gaussian graphs [8], torus, twisted torus, Kronecker products of two cycles, etc. It is straightforward that any Gaussian graph G_{a+bi} is isomorphic to $L(\begin{pmatrix} a & -b \\ b & a \end{pmatrix})$. Kronecker products of cycles have been proposed for interconnection networks in [11], [10], and codes over them were characterized in [12].

Definition 2 Given a graph $G = (V, E)$, $Aut(G)$ denotes its automorphisms group. G is said to be vertex-transitive if, for any pair of vertices

$v_1, v_2 \in V$ there exists $f \in Aut(G)$ such that $f(v_1) = v_2$. Similarly, G is said to be edge-transitive if for any pair of edges $e_1 = (v_1, v_2), e_2 \in E$ there exists $f \in Aut(G)$ such that $f(e_1) = (f(v_1), f(v_2)) = e_2$. Then, if G is both vertex and edge transitive, then it is called symmetric.

It is easy to see that all L-graphs are vertex-transitive. Therefore, in order to characterize symmetric L-graphs we will study those which are edge-transitive. With this aim, in Section 2 we will consider some properties of isomorphisms between L-graphs. In Section 3, we determine those L-graphs which are symmetric. In Subsection 3.1 we will focus on linear automorphisms (group automorphisms of $\mathbb{Z}^2/M\mathbb{Z}^2$) of L-graphs which act transitively on the set of edges in order to characterize those which are edge-transitive (and therefore symmetric) by means of their generator matrix. Finally, in Subsection 3.2 we analyze some marginal cases of symmetric L-graphs with non-linear automorphisms.

2 Linear Automorphisms of L-graphs

Two multidimensional circulants are *Ádam isomorphic* if there exists an isomorphism between their groups of vertices such that sends the set of jumps of one into the other one. It is clear that any Ádam isomorphic multidimensional circulants are isomorphic, but the opposite it is not true. In [4] it was proved that:

Theorem 3 [4] *Any two finite isomorphic connected undirected Cayley multigraphs of degree 4 coming from abelian groups are Ádam isomorphic, unless they are obtained with the groups and families* $\mathbb{Z}_{4n}$, $(1, -1, 2n+1, 2n-1)$ *and* $\mathbb{Z}_{2n} \times \mathbb{Z}_2$, $((1,0), (-1,0), (1,1), (-1,1))$.

In this Section we obtain a similar result in a different form. We address the method here since it can be extended to higher dimensions, which is considered as future work.

Definition 4 We define the *neighborhood* of a vertex v in the graph $G = (V, E)$ as $N(v) = \{w : (v, w) \in E\}$. Then, the *common neighborhood* of a list of vertices denoted as $v_1, \ldots, v_n$ as $N(v_1, \ldots, v_n) = \bigcap_{i=1}^{n} N(v_i)$.

Theorem 5 *The neighborhood is preserved in graph isomorphisms. That is, if f is a graph isomorphism, then*

$$N(f(v_1), \ldots, f(v_n)) = \{f(w) : w \in N(v_1, \ldots, v_n)\}.$$

Proof: Let f be a graph isomorphism from $G = (V, E)$ into $G' = (V', E')$. We have that $f(w) \in N(f(v_1), \ldots, f(v_n))$ if only if $\forall i, f(w) \in N(f(v_i))$, that is $\forall i, (f(w), f(v_i)) \in E'$. Since f is an isomorphism we have that this is equivalent to $\forall i, (w, v_i) \in E$ so $w \in N(v_1, \ldots, v_n)$. $\square$

Next, we analyze which multidimensional circulant graphs isomorphisms are linear mappings. We need the following:

Definition 6 We say that $a, b, c, d \in A$ form a 4-cycle in $G(M; A)$ if $0 = a + b + c + d$[1]. Then, we say that $G(M; A)$ has not nontrivial 4-cycles if $a, b, c, d \in A$ such that $0 = a+b+c+d$ implies $a = -b$ or $a = -c$ or $a = -d$.

Theorem 7 *If $G(M; A)$ is a multidimensional circulant graph without nontrivial 4-cycles, then for all $a, b \in A$ with $a \neq b$*

$$N(a, b) = \{0, a + b\}.$$

Proof: If $v \in N(a, b)$ then $\exists x, y \in A$ such that $v = a + x = b + y$. Since we have $a - b + x - y = 0$ and $G(M; A)$ has not nontrivial 4-cycles, it must be fulfilled one of the following expressions:

- $a = b$ contradicting the hypothesis,
- $a = -x$ and thus $v = a - a = 0$,
- $a = y$ and thus $v = b + y = a + b$. $\square$

Lemma 8 *Let $G(M; A)$ and $G(M'; A')$ be two isomorphic multidimensional circulant graphs without nontrivial 4-cycles. Then any isomorphism f between $G(M; A)$ and $G(M'; A')$ with $f(0) = 0$ is such that $f(a + b) = f(a) + (b)$ for any $a, b \in A$ with $a \neq b$.*

[1] each of $\{(v, v + a, v + a + b, v + a + b + c, v + a + b + c + d) : v \in V\}$ is a cycle of length 4

Proof: Let $a, b \in A$ with $a \neq b$. From the previous theorem we get that $N(a,b) = \{0, a+b\}$, hence $N(f(a), f(b)) = \{f(0), f(a+b)\} = \{0, f(a) + f(b)\}$. As $f(0) = 0$ we have that $f(a+b) = f(a) + f(b)$. $\square$

Lemma 9 *In multidimensional circulant graphs translations are automorphisms.*

Corollary 10 *If f is an automorphism of $G(M; A)$, then there exists another automorphism f' with $f'(0) = 0$.*

Lemma 11 *Let $f \in Aut\big(G(M;A)\big)$ such that $f(0) = 0$ and $G(M;A)$ has not nontrivial 4-cycles. Then, we have that*

$$\forall t \in G(M;A), \forall a, b \in A, \ f(t+a+b) = f(t+a) + f(t+b) - f(t).$$

Proof: Let $t \in G(M;A)$ and $a, b \in A$. We can define $f_t(v) = f(t+v) - f(t)$, which is an automorphism with $f_t(0) = f(t) - f(t) = 0$. Now if $a \neq b$, by Lemma 8 $\forall t \in G(M;A)$, $f_t(a+b) = f_t(a) + f_t(b)$, which implies $\forall t \in G(M;A)$, $f(t+a+b) - f(t) = f(t+a) - f(t) + f(t+b) - f(t)$.

If $a = b$ and $a \neq -a$ then we have $f(t+a-a) + f(t) = f(t+a) + f(t-a)$ and taking $t = t' + a$, $f(t'+a) + f(t'+a) = f(t'+a+a) + f(t')$.

Finally if $a = b = -a$ then let $c = f_t(a)$ and suppose $c \neq -c$. We have $f_t^{-1}(c) = a$, and because of Lemma 8, $0 = f_t^{-1}(c-c) = f_t^{-1}(c) + f_t^{-1}(-c)$. Hence $f_t^{-1}(c) = -f_t^{-1}(-c) = a = -a = -f_t^{-1}(c)$, so $f_t^{-1}(-c) = f_t^{-1}(c)$ and $c = -c$, which is a contradiction. Hence $f_t(a) = c$ implies $c = -c$ and as consequence we obtain $f_t(a) = -f_t(a)$. From this we get that $f(t+a) - f(t) = -f(t+a) + f(t)$, and taking $t = t' + a$ that $f(t'+a+a) = -f(t'+2a) + f(t'+a) + f(t'+a)$ and by $2a = 0$ that $f(t'+a+a) = f(t'+a) + f(t'+a) - f(t')$. $\square$

Theorem 12 *If the connected multidimensional circulant graph $G(M;A)$ has not nontrivial 4-cycles then any graph automorphism with $f(0) = 0$ is a group automorphism of $\mathbb{Z}^n/M\mathbb{Z}^n$.*

Proof: We need to prove $\forall n_i \in \mathbb{N}, a_i \in A$, $f(\sum_i n_i a_i) = \sum_i n_i f(a_i)$, for which is enough to see that for any $v \in G(M;A)$, $a \in A$, $f(a+v) =$

$f(a) + f(v)$. We proceed by induction in the number of summands of v in terms of elements of A. We have already proved that for 1 summand it holds. Assuming that it holds for N summands: let v be expressed with $N+1$ summands. There exist $b \in A$, $w \in G(M; A)$ such that $v = b + w$ with w expressed with only N summands. Now, because of Lemma 11, $f(a+v) = f(w+a+b) = f(w+a) + f(w+b) - f(w)$. And applying the induction hypothesis we have that $f(a+v) = f(w)+f(a)+f(w+b)-f(w) = f(a) + f(v)$. □

3 Edge-transitivity of L-graphs

In this Section we determine those L-graphs which are edge-transitive. First, in Subsection 3.1 we consider the case where $L(M)$ is an L-graph without nontrivial 4-cycles, that is, we can consider that any automorphism of $L(M)$ is a linear mapping. Later, in Subsection 3.2 we analyze the special cases in which such cycles exist.

3.1 Edge-transitive of L-graphs by Linear Automorphisms

In this Subsection we will consider those L-graphs such that any automorphism is a linear mapping. Some of the following results will be presented not only for L-graphs but for any multidimensional circulant.

Definition 13 A signed permutation is a composition of a permutation with sign changing function.

Definition 14 A signed permutation matrix is a matrix with entries in $\{-1, 0, 1\}$ which has exactly one ± 1 in each row and column.

Note that in $\mathbb{Z}^{n\times n}$ the signed permutation matrices are exactly the unitary matrices, this is, the matrices U such $UU^t = I$. They are related with permutations in the way that for each signed permutation σ we can find a matrix P_σ such that

$$\begin{pmatrix} v_{\sigma(1)} \\ \vdots \\ v_{\sigma(n)} \end{pmatrix} = P_\sigma \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Theorem 15 *Let $G(M)$ be a multidimensional circulant. Then, for each linear automorphism f there exists a signed permutation matrix P such that $\forall\alpha,\ f(\alpha) = P\alpha$.*

Proof: We define P as:

$$P_{i,j} = \begin{cases} 1 & f(e_j) = e_i \\ -1 & f(e_j) = -e_i \\ 0 & \text{otherwise} \end{cases}$$

having $f(e_i) = Pe_i$. Let $\alpha = \sum n_i e_i$.

$$f(\alpha) = \sum_i n_i f(e_i) = \sum_i n_i P e_i = P \sum_i n_i e_i = P\alpha. \quad \square$$

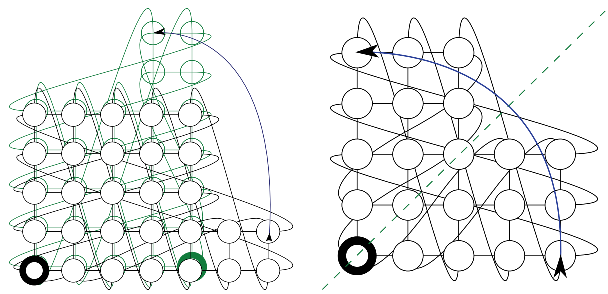
Theorem 16 *Let $G(M)$ be a multidimensional circulant. Then $f(\alpha) = P\alpha$ is a linear automorphism in $G(M)$ if only if there exists a matrix Q such that $PM = MQ$.*

Proof: We prove first the left to right implication. As f must be well-defined, for all i, $0 = P0 \equiv_M PMe_i$. And then exists q_i such that $PMe_i = Mq_i$, gathering all i's together

$$PM = [PMe_1, \dots, PMe_n] = M[q_1, \dots, q_n] = MQ$$

Now, we prove the right to left implication. We begin proving f is well-defined. Let $a \equiv_M b$, there is a γ such that $a - b = M\gamma$, so $Pa - Pb = PM\gamma = MQ\gamma = M\gamma'$, getting $Pa \equiv_M Pb$. Now we prove injectivity. Let $Pa - Pb = M\gamma$ then $a - b = P^{-1}M\gamma = MQ^{-1}\gamma = M\gamma'$. And as $|\det(M)| = |\det(PM)|$ the sizes are equal and so we have bijection. And finally we prove edge preservation. If a and b are connected in $G(M)$ then we have e_i and γ such $a - b = \pm e_i + M\gamma$, and so $Pa - Pb = \pm Pe_i + PM\gamma = \pm e_j + M\gamma'$ thus Pa is connected to Pb in $L(M)$. $\square$

To know if a multidimensional circulant $G(M)$ without nontrivial 4-cycles is edge-transitive we need to look to the multiplicative group of the signed permutation matrices P such $PM = MQ$. It is clear that if a matrix representing a cycle of length n (even if it changes signs) is in the group then by composing it with itself, we can map e_1 to every e_i making the



(a) Rotation (b) Reflexion

Figure 1: Linear Automorphisms of L-graphs

graph edge-transitive. However, there exist groups which have not such cycles but get the mapping, like the fourth alternating group.

In dimension 2 this is simply to see that if for M one of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is in its group.

Theorem 17 *$L(M)$ is edge-transitive if and only if M is right equivalent to one of the following matrices:*

- $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, *for* $P = Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, *for* $P = Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
- $\begin{pmatrix} a & -b \\ a & b \end{pmatrix}$, *for* $P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Proof: The characteristic polynomial of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is $\lambda^2 - 1$, and the one of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is $\lambda^2 + 1$. By Theorem III.2 in [9] it must be the characteristic polynomial of both P and Q in $PM = MQ$. Therefore, we have two cases:

- $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\lambda^2 - 1 = (\lambda+1)(\lambda-1)$, being reducible over $\mathbb{Q}$, Q must be similar to a matrix $Q' = \begin{pmatrix} 1 & p \\ 0 & -1 \end{pmatrix}$, which is similar to either $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \tilde{S} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- $P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, λ^2+1 which is irreducible over $\mathbb{Q}$ and the class number of $\mathbb{Z}[i]$ is 1, so Q must be similar to P (Theorem III.14, in [9]). □

It was proved in [7] that the Kronecker product of two cycles is always an L-graph, as next Theorem shows.

Theorem 18 [7] *Let $a, b \in \mathbb{Z}$. Then, the Kronecker product of the cycles of lengths a and b, denoted as $C_a \times C_b$, is isomorphic to:*

- $L(\begin{pmatrix} \frac{a+b}{2} & \frac{a-b}{2} \\ \frac{a-b}{2} & \frac{a+b}{2} \end{pmatrix})$ *if a and b are odd integers such that $a \geq b$.*

- *2 disjoint copies of* $L(\begin{pmatrix} \frac{a}{2} & \frac{-b}{2} \\ \frac{a}{2} & \frac{b}{2} \end{pmatrix})$ *if a and b are both even integers such that $a \geq b$.*

- $L(\begin{pmatrix} \frac{a}{2} & -b \\ \frac{a}{2} & b \end{pmatrix})$, *if a is an even integer and b is an odd integer.*

Therefore, $L(M)$ in the first and third cases of Theorem 17 is a Kronecker product of two cycles and in the second one it is a Gaussian graph.

3.2 Edge-transitive L-graphs by Nonlinear Automorphisms

In this Subsection we focus on those L-graphs with nontrivial 4-cycles, that is, L-graphs which can be isomorphic but not Ádam isomorphic. Clearly, if there is a nontrivial 4-cycle then there exist $a, b \in A$ which fulfill:

1. $4a \equiv 0 \pmod{M}$
2. $3a + b \equiv 0 \pmod{M}$
3. $2a + 2b \equiv 0 \pmod{M}$

If we consider $ma + nb = 0 \pmod{M}$ it means that there exists $\gamma \in \mathbb{Z}^2$ such that $\kappa = \begin{pmatrix} m \\ n \end{pmatrix} = M\gamma$. Now, let $\gcd \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = gcd(a_1, a_2)$, $\gamma' = \frac{\gamma}{\gcd \gamma}$ and $\kappa' = \frac{\kappa}{\gcd \gamma}$, having $\kappa' = M\gamma'$. As $\gcd \gamma' = 1$ we can build a unit matrix U with γ' as one of its columns, and therefore $M' = MU$ has κ' as a column.

We'll begin with item (3). In this case we obtain the matrix $M = \begin{pmatrix} m & 2 \\ n & 2 \end{pmatrix}$. If $n = 2k$ we have that $\begin{pmatrix} m & 2 \\ 2k & 2 \end{pmatrix}$ is right equivalent to $\begin{pmatrix} m-n & -2 \\ 0 & 2 \end{pmatrix}$. On the other hand, if $n = 2k+1$ then $\begin{pmatrix} m & 2 \\ 2k+1 & 2 \end{pmatrix}$ is right equivalent to $\begin{pmatrix} m-n-1 & -2 \\ 1 & 2 \end{pmatrix}$. Both matrices generate the same graph and the isomorphism is not linear in this case.

If we analyze (1) and (2) in detail, we can obtain an equivalent to Theorem 12 for this case. In fact, we can finally prove that in these cases, all automorphisms are linear.

Finally, there are a few marginal cases which correspond to those matrices M with both columns being nontrivial 4-cycles or with such a column and $2e_i$ in the other, that is, those which can be built by selecting two columns in the set:

$$C = \left\{ \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}.$$

A complete study of the following cases, shows as that most of the combinations are edge-transitive. However, there are cases that lack of a non-linear automorphism, leading to non-edge-transitive graphs.

Up to isomorphism, the L-graphs with 2 different nontrivial solutions to the 4-cycles are:

- With nontrivial 4 cycles but without nonlinear automorphisms.

$$\begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix}$$

- With nonlinear automorphism, which makes them edge-transitive,

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 1 & -1 \end{pmatrix} \simeq \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$$

  with an example in Figure 2.

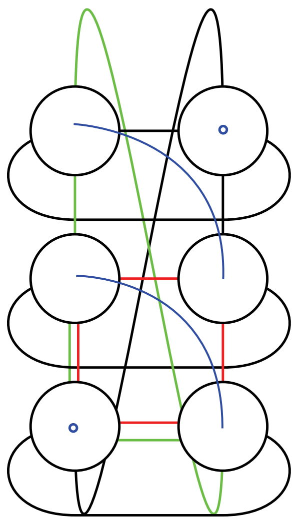


Figure 2: A nonlinear automorphism of $L(M)$, where $M = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$

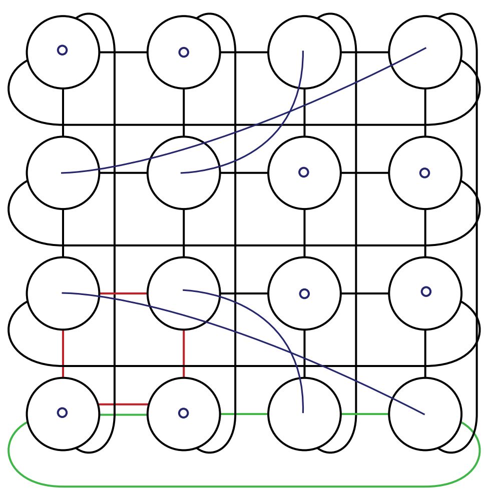


Figure 3: A nonlinear automorphism of the square torus of side 4

- With nonlinear automorphism, but its linear automorphisms already make them edge-transitive,

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$$

with torus as example in Figure 3.

Acknowledgement

This work has been supported by the Spanish Ministry of Education and Science under contract TIN2007-68023-C02-01.

References

[1] Camara, J., Moreto, M., Vallejo, E., Beivide, R., Miguel-Alonso, J., Martinez, C., Navaridas, J. Twisted torus topologies for enhanced interconnection networks. *IEEE Transactions on Parallel and Distributed Systems*, 99 (PrePrints) (2010).

[2] C. K. Wong, Don Coppersmith. A Combinatorial Problem Related to Multimodule Memory Organizations. *Journal of the ACM*, 21(3):392 - 402, July 1974.

[3] Dally, W., Towles, B. Principles and Practices of Interconnection Networks. *Morgan Kaufmann Publishers Inc.*, San Francisco, CA, USA 2003.

[4] C. Delorme, O. Favaron and M. Mahéo. Isomorphism of Cayley Multigraphs of Degree 4 on Finite Abelian Groups. *Europ. J. Combinatorics*, 13:59–61. 1992.

[5] M. A. Fiol. On Congruence in $\mathbb{Z}^n$ and the Dimension of a Multidimensional Circulant". *Discrete Mathematics*, 141:123-134, 1995.

[6] Fiol, M., Yebra, J. L., Alegre, I., and Valero, M. Discrete Optimization Problem in Local Networks and Data Alignment. *IEEE Trans. Comput.*, 36(6):702-713, Jun. 1987.

[7] C. Martínez, C. Camarero, R. Beivide Perfect Graph Codes over Two Dimensional Lattices. Accepted for publication at 2010 *IEEE Internations Symposium on Information Theory.*

[8] C. Martínez, R. Beivide, E. Stafford, M. Moretó, E. M. Gabidulin. Modeling Toroidal Networks with the Gaussian Integers. *IEEE Transactions on Computers*, 57(8):1046–1056, August 2008.

[9] M. Newman. *Integral Matrices.* Pure and Appl. Math. Series, Vol. 45. (Academic Press, New York, 1972.

[10] Barak A. Pearlmutter. Doing the Twist: Diagonal Meshes Are Isomorphic to Twisted Toroidal Meshes. *IEEE Trans. Computers*, 45(6):766-767, 1996.

[11] K. Wendy Tang, Sanjay A. Padubidri. Diagonal and Toroidal Mesh Networks. *IEEE Trans. Computers*, 43(7):815-826, 1994.

[12] J. Zerovnik. Perfect Codes in Direct Product of Cycles - a Complete Characterization. *Advances in Applied Mathematics*, 41:197-205, 2008.

On the vulnerability of some families of graphs

Rocío M. Casablanca, Ana R. Diánez and Pedro García-Vázquez
Universidad de Sevilla
Sevilla

Abstract

The toughness of a noncomplete graph G is defined as $\tau(G) = \min\{|S|/\omega(G-S)\}$, where the minimum is taken over all cutsets S of vertices of G and $\omega(G-S)$ denotes the number of components of the resultant graph $G-S$ by deletion of S. In this paper, we investigate the toughness of the corona of two connected graphs and obtain the exact value for the corona of two graphs belonging to some families as paths, cycles, wheels or complete graphs. We also get an upper and a lower bounds for the toughness of the cartesian product of the complete graph K_2 with a predetermined graph G.

1 Introduction

Throughout this paper, all the graphs are simple, that is, without loops and multiple edges. Notations and terminology not explicitly given here can be found in the book by Chartrand and Lesniak [3].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The graph G is called *connected* if every pair of vertices is joined by a path. A *cutset* in a graph G is a subset $S \subset V(G)$ of vertices of G such that $G-S$ is not connected.
The existence of a cutset is always guaranteed in every graph different from a complete graph K_n. The index of *connectivity* of G, denoted by $\kappa(G)$, is defined as the minimum cardinality over all cutsets of G, if G is a noncomplete graph, or $|V(G)|-1$, otherwise.

There are several measures of vulnerability of a network. The vulnerability parameters one generally encounters are the indices of connectivity and edge-connectivity. These two parameters give the minimum cost to disrupt the network, but they take no account of what remains after the destruction. To measure the vulnerability of networks more properly, some vulnerability parameters have been introduced and studied. Among them are toughness, integrity, scattering number, tenacity and several variants of connectivity and edge-connectivity called conditional connectivity, each of which measures not only the difficulty of breaking down the network but also the damage caused. In general, for most of the aforementioned parameters, the corresponding computational problem is NP-hard. So it is of interest to give the formulae or algorithms for computing these parameters for special classes of graphs. For our purpose, we deal with the notion of *toughness*, introduced by Chvátal [4], which pays special attention to the relationship between the cardinality of the rupture set in the network and the number of components after the rupture. The parameter is defined as

$$\tau(G) = \min\left\{|S|/\omega(G-S) : S \subseteq J(G)\right\},$$

where

$$J(G) = \{S \subset V(G) : S \text{ is a cutset of } G \text{ or } G-S \text{ is an isolated vertex}\},$$

and $\omega(G-S)$ denotes the number of components in the resultant graph $G-S$ by removing S.

Since this parameter was introduced, lots of research has been done, mainly relating toughness conditions to the existence of cycle structures. Historically, most of the research was based on a number of conjectures in [4]. Some of most interesting results are [1, 2, 5]. However, exact values of $\tau(G)$ are known only for a few families of graphs as paths and cycles [4], the cartesian product of two complete graphs [4] and of paths and/or cycles [7], and the composition of two graphs, one of them being a path, a cycle or a complete bipartite graph [7]. In this paper we focus on the toughness of two families of graphs: the *corona* $G \circ H$ of two graphs [6] and the *cartesian product* $K_2 \times G$.

If for each vertex x in a graph G, we introduce a new vertex x' and join x and x' by an edge, the resulting graph is called the *corona* of G. The operation of adding one vertex for each vertex of G and connecting them by an edge can be generalized as follows. The corona of any two graphs G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the ith vertex of G to every vertex in the ith copy of H. Observe that the particular case in which $H = K_1$, the graph $G \circ K_1$ is called the corona of G. The *cartesian product* $K_2 \times G$ of the complete graph K_2 and any graph G is the graph with vertex set $V(K_2) \times V(G)$ in which vertex (i, u), for $i = 1, 2$, is adjacent to vertex (j, v) whenever $i = j$ and $uv \in E(G)$, or $i \neq j$ and $u = v$ [6].

There exists several kinds of interconnection networks whose structure can be modeled in terms of the cartesian product or the corona of two predetermined networks. The cartesian product of graphs seeks to establish parallel connections between identical structures, minimizing the cost of such connections. The corona of two predetermined graphs is often present in electric networks distributed in a big city where each transformer must guarantee the energy supply of its catchment area. In order to optimize resources, the distribution of transformers is made by dividing the city in catchment areas of the same entity. Thus, in terms of Graph Theory, the structure to be analyzed consists of a network transformers, modeled by a graph, G where each transformer is connected with its catchment area, modeled by the graph H. The resultant graph is the corona $G \circ H$ of G and H. In the maintenance of electric networks is relevant to avoid the disruption of the energy supply, but when the failure in some nodes produces the rupture of the network, the greater the number of fragments in which the network has been divided, the greater the cost of reconstruction.

The relationship between the cardinality of a cutset of a graph G and the remaining component after disruption is analyzed by the notion of toughness, defined above. So our aim in this work is to determine the toughness of the corona $G \circ H$ of two connected graphs G and H in terms of known parameters of them. As a consequence, we will deduce the exact value of the corona of some families of graphs involving stars, paths, cycles, wheels or complete graphs. We will also find an upper and a lower bounds for the toughness of $K_2 \times G$, for any arbitrary graph G.

2 The toughness of the corona of two graphs

2.1 Notations and remarks

Let G, H be two connected graphs on m and n vertices, respectively. Let us set $V(G) = \{v_1, \ldots, v_m\}$ and denote by H_i the copy of H that is joined to vertex v_i of G in $G \circ H$. Thus, every cutset S of $G \circ H$ will henceforth expressed as $S = S_0 \cup \bigcup_{i=1}^{m} S_i$, where $S_0 \subseteq V(G)$ and $S_i \subseteq V(H_i)$, for

$i = 1, \ldots, m$. We denote by $\omega_0 = \omega(G - S_0)$, $\omega_i = \omega(H_i - S_i)$, $i = 1, \ldots, m$, that is, the number of component of $G - S_0$ and $H_i - S_i$, $i = 1, \ldots, m$, respectively.

A cutset of $G \circ H$ such that $|S|/\omega(G \circ H - S) = \tau(G \circ H)$ will be called a τ-cut of $G \circ H$. Let us see some remarks on the τ-cut of the corona of two graphs.

Remark 1 If $S = S_0 \cup \bigcup_{i=1}^{m} S_i$ is a cutset of the corona $G \circ H$ of two connected graphs G, H, then $S_0 \neq \emptyset$.

Proof: If $S_0 = \emptyset$ then every vertex of $G \circ H - S$ either is in $V(G)$ or is adjacent to one vertex of G, hence, $G \circ H - S$ is connected, against the fact that S is a cutset of $G \circ H$. $\square$

Remark 2 Let $S = S_0 \cup \bigcup_{i=1}^{m} S_i$ be a τ-cut of the corona $G \circ H$ of two connected graphs G, H. If $v_j \in S_0$ then either $S_j = \emptyset$ or S_j is a cutset of H_j.

Proof: Let $v_j \in S_0$ and suppose by way of contradiction that $S_j \neq \emptyset$ is not a cutset of H_j. Let us consider the set $S^* = S \setminus S_j$. Observe that either $H_j - S_j$ is a component of $G \circ H - S$ or $S_j = V(H_j)$ and H_j is a component of $G \circ H - S^*$. Thus, $\omega(G \circ H - S^*) \geq \omega(G \circ H - S)$ and therefore,

$$\frac{|S^*|}{\omega(G \circ H - S^*)} \leq \frac{|S| - n}{\omega(G \circ H - S)} < \frac{|S|}{\omega(G \circ H - S)} = \tau(G \circ H - S),$$

which contradicts the hypothesis that S is a τ-cut of $G \circ H$. Then either $S_j = \emptyset$ or S_j is a cutset of H_j. $\square$

Remark 3 Let $S = S_0 \cup \bigcup_{i=1}^{m} S_i$ be a τ-cut of the corona $G \circ H$ of two connected graphs G, H. If $v_j \notin S_0$ then $S_j = \emptyset$.

Proof: Let $v_j \notin S_0$ and suppose by way of contradiction that $S_j \neq \emptyset$. Let us consider the set $S^* = S \setminus S_j$. Observe that either $H_j - S_j$ belongs to the component of $G \circ H - S$ that contains vertex v_j or $S_j = V(H_j)$ and H_j belongs to the component of $G \circ H - S^*$ that contains vertex v_j. Thus, $\omega(G \circ H - S^*) = \omega(G \circ H - S)$ and therefore,

$$\frac{|S^*|}{\omega(G \circ H - S^*)} = \frac{|S| - n}{\omega(G \circ H - S)} < \frac{|S|}{\omega(G \circ H - S)} = \tau(G \circ H - S),$$

which is again a contradiction with the fact that S is a τ-cut of $G \circ H$. Then $S_j = \emptyset$. $\square$

Let $S = S_0 \cup \bigcup_{i=1}^{m} S_i$ be a τ-cut of $G \circ H$. From now on, we may assume without loss of generality that the vertices of the set $V(G) = \{v_1, \ldots, v_m\}$ are ordered so that $|S_1| \geq \cdots \geq |S_m|$. Let $k \in \{1, \ldots, m\}$ be the maximum integer such that $S_i \neq \emptyset$ for all $i = 1, \ldots, k$. Then, as an immediate consequence of Remark 1, Remark 2 and Remark 3, it follows that $|S| = |S_0| + \displaystyle\sum_{i=1}^{k} |S_i|$ and $\omega(G \circ H - S) = \omega_0 + \displaystyle\sum_{i=1}^{k} \omega_i + |S_0| - k$.

2.2 Main results

Let G, H be two connected graphs on m and n vertices, respectively. Our purpose is to determine the toughness of the corona $G \circ H$ of G and H. To begin with, given a τ-cut of $G \circ H$, the first question that we must answer is wether every copy of graph H can be disconnected to be disconnected in the same way. The following lemma provides an answer to this question.

Lemma 4 *Let G, H be two connected graphs of order m and n, respectively, and let $S = S_0 \cup \bigcup_{i=1}^{m} S_i$ be a τ-cut of $G \circ H$ of minimum cardinality. If $S_i \neq \emptyset$, $S_j \neq \emptyset$, for $i, j = 1, \ldots, m$ with $i \neq j$, then $|S_i| = |S_j|$ and $\omega_i = \omega_j$.*

Proof: Let us consider the vertex set $V(G) = \{v_1, \ldots, v_m\}$ ordered so that $|S_1| \geq \cdots \geq |S_m|$, and let $k \in \{1, \ldots, m\}$ be the maximum integer such that $S_i \neq \emptyset$ for all $i = 1, \ldots, k$. Thus, $|S| = |S_0| + \sum_{i=1}^{k} |S_i|$. Since S is a τ-cut of $G \circ H$, we have

$$\tau(G \circ H) = \frac{|S_0| + \displaystyle\sum_{i=1}^{k} |S_i|}{\omega_0 + \displaystyle\sum_{i=1}^{k} \omega_i + |S_0| - k} \leq \frac{|S_0| + k|S_\ell|}{\omega_0 + k\omega_\ell + |S_0| - k}, \tag{1}$$

for every $\ell = 1, \ldots, k$,

yielding to

$$\begin{aligned}&\left(|S_0|+\sum_{i=1}^{k}|S_i|\right)k\omega_\ell+(\omega_0+|S_0|-k)\sum_{i=1}^{k}|S_i|\\&\le|S_0|\sum_{i=1}^{k}\omega_i+\left(\omega_0+\sum_{i=1}^{k}\omega_i+|S_0|-k\right)k|S_\ell|,\ \text{for } \ell=1,\dots,k.\end{aligned}\tag{2}$$

By taking summation in (2) we deduce that

$$\begin{aligned}&\left(|S_0|+\sum_{i=1}^{k}|S_i|\right)k\sum_{\ell=1}^{k}\omega_\ell+k(\omega_0+|S_0|-k)\sum_{i=1}^{k}|S_i|\\&\le k|S_0|\sum_{i=1}^{k}\omega_i+\left(\omega_0+\sum_{i=1}^{k}\omega_i+|S_0|-k\right)k\sum_{\ell=1}^{k}|S_\ell|\\&=\left(|S_0|+\sum_{\ell=1}^{k}|S_\ell|\right)k\sum_{i=1}^{k}\omega_i+k(\omega_0+|S_0|-k)\sum_{\ell=1}^{k}|S_\ell|,\end{aligned}$$

which implies that all the inequalities of (2) become equalities, and therefore, all the inequalities of (1) become equalities. Thus,

$$\begin{aligned}&\tau(G\circ H)=\frac{|S_0|+k|S_i|}{\omega_0+k\omega_i+|S_0|-k}=\frac{|S_0|+k|S_j|}{\omega_0+k\omega_j+|S_0|-k},\\&\text{for all } i,j=1,\dots,k,\end{aligned}\tag{3}$$

which means that the set $S^*=S_0\cup\bigcup_{i=1}^{k}S_i^*$, where $S_i^*=S_k$, for all $i=1,\dots,k$, is also a τ-cut. Hence,

$$|S|=|S_0|+\sum_{i=1}^{k}|S_i|\ge|S_0|+k|S_k|=|S^*|,$$

yielding to $|S_1|=\cdots=|S_k|$ because S has minimum cardinality. Moreover, given any two subsets S_i, S_j, with $i,j\in\{1,\dots,k\}$ and $i\neq j$, from (3) it is clear that $\omega_i=\omega_j$. Then the result holds. $\square$

Given a τ-cut $S=S_0\cup\bigcup_{i=1}^{m}S_i$ of $G\circ H$ with minimum cardinality, by Lemma 4 we may assume without loss of generality that for each $i=1,\dots,m$, either $S_i=\emptyset$ or $S_i=S_H$, for some $S_H\subset V(H)$. Furthermore, it follows that either $\omega(H_i - S_i) = 1$ (if $S_i = \emptyset$) or $\omega(H_i - S_i) = \omega(H - S_H)$ (if $S_i = S_H$).

To upper bound the index of toughness of $G \circ H$, it is enough to find a cutset S of $G \circ H$ and compute $|S|/\omega(G \circ H - S)$. There are some alternatives in the choice of such a cutset, as the following proposition shows.

Proposition 5 *Let G, H be two connected graphs of order m and n, respectively. Let $S_H \subset V(H)$ be any cutset of H of cardinality $|S_H| = p$ and denote by $q = \omega(H - S_H)$. Then*

$$\tau(G \circ H) \leq \min\left\{\frac{1}{2}, \frac{\tau(G)}{1+\tau(G)}, \frac{1+p}{1+q}, \frac{1+p}{\frac{1}{\tau(G)}+q}\right\}.$$

Proof: First, let v_j be any vertex of $V(G)$ and let us consider the set $S = \{v_j\}$ in $G \circ H$. Then S is a cutset and $G \circ H - S$ since v_j separates the copy H_j of H from $G \circ H - (\{v_j\} \cup V(H_j))$. Furthermore, $G \circ H - S$ has at least two components, i.e., $\omega(G \circ H - S) = 1 + \omega\left(G \circ H - (\{v_j\} \cup V(H_j))\right) \geq 2$, yielding to $\tau(G \circ H) \leq \frac{|S|}{\omega(G \circ H - S)} \leq \frac{1}{2}$.

Second, let $S \subset V(G)$ be a τ-cut of G. Then S is a cutset of $G \circ H$ and $\omega(G \circ H - S) = \omega(G - S) + |S|$ and therefore,

$$\tau(G \circ H) \leq \frac{|S|}{\omega(G \circ H - S)} \leq \frac{|S|}{\omega(G - S) + |S|} = \frac{\dfrac{|S|}{\omega(G-S)}}{1 + \dfrac{|S|}{\omega(G-S)}} = \frac{\tau(G)}{1+\tau(G)}.$$

Third, let $S_H \subset V(H)$ be any cutset of H of cardinality $|S_H| = p$ and denote by $q = \omega(H - S_H)$. Take any vertex $v_j \in V(G)$ and set $S_j = S_H \subset V(H_j)$. Let us consider the vertex set $S = \{v_j\} \cup S_j$ and observe that S is a cutset of $G \circ H$. Indeed, $\omega(G \circ H - S) = \omega(G - v_j) + \omega(H_j - S_j) \geq 1 + \omega(H_j - S_j)$. Thus, if we denote by $p = |S_j|$ and denote by $q = \omega(H - S_H)$, it follows that

$$\tau(G \circ H) \leq \frac{|S|}{\omega(G \circ H - S)} \leq \frac{1 + |S_j|}{1 + \omega(H_j - S_j)} = \frac{1+p}{1+q}.$$

Finally, take any cutset $S_H \subset V(H)$ of H of cardinality $|S_H| = p$ and denote by $q = \omega(H - S_H)$. Let $S_0 = \{w_1, \ldots, w_{|S_0|}\} \subset V(G)$ be a τ-cut of G and denote by H_i the copy of H joined to vertex w_i in $G \circ H$, for

$i = 1, \ldots, |S_0|$. Let us consider the vertex set $S = S_0 \cup \bigcup_{i=1}^{|S_0|} S_i$, where $S_i = S_H$, for every $i = 1, \ldots, |S_0|$. Clearly S is a cutset of $G \circ H$ and $\omega(G \circ H - S) = \omega(G - S_0) + |S_0|\omega(H - S_H)$. Hence,

$$\begin{aligned}
\tau(G \circ H) \leq \frac{|S|}{\omega(G \circ H - S)} &= \frac{|S_0| + |S_0||S_H|}{\omega(G - S_0) + |S_0|\omega(H - S_H)} \\
&= \frac{|S_0|(1+p)}{\omega(G - S_0) + |S_0|q} \\
&= \frac{\tau(G)(1+p)}{1 + \tau(G)q} \\
&= \frac{1+p}{1/\tau(G) + q}.
\end{aligned}$$

Thus, $\tau(G \circ H) \leq \min \left\{ \frac{1}{2}, \frac{\tau(G)}{1+\tau(G)}, \frac{1+p}{1+q}, \frac{1+p}{\frac{1}{\tau(G)}+q} \right\}$ and the result holds. $\square$

The next result gives a necessary condition for a τ-cut of $G \circ H$ to contain vertices of some copy H_i.

Lemma 6 *Let G, H be two connected graphs of order m and n, respectively, and let $S = S_0 \cup \bigcup_{i=1}^{m} S_i$ be a τ-cut of $G \circ H$ of minimum cardinality. If $S_j \neq \emptyset$ for some $j = 1, \ldots, m$, then $|S_j|/\omega(H_j - S_j) < 1/2$.*

Proof: From Lemma 4 there exists a vertex set $S_H \subset V(H)$ such that either $S_i = \emptyset$ or $S_i = S_H$, for every $i = 1, \ldots, m$. So without loss of generality we may assume that there is an integer $k \in \{1, \ldots, m\}$ such that $S = S_0 \cup \bigcup_{i=1}^{k} S_H$; that is, $S_i = S_H$ if $i \in \{1, \ldots, k\}$ and $S_i = \emptyset$ otherwise. Therefore, it is enough to us to prove that $|S_H|/\omega(H - S_H) < 1/2$. To clarify expressions, denote by $\omega_0 = \omega(G - S_0)$ and $\omega_H = \omega(H - S_H)$. By applying Remark 1, we know that $S_0 \neq \emptyset$, and from Remark 2 and Remark 3 it follows that $k \leq |S_0|$. Thus, $|S| = |S_0| + k|S_H|$ and $\omega(G \circ H - S) = \omega_0 + k\omega_H + |S_0| - k$. By applying Proposition 5 we know that $\tau(G \circ H) \leq 1/2$, which implies that

$$\frac{|S|}{\omega(G \circ H - S)} = \frac{|S_0| + k|S_H|}{\omega_0 + k\omega_H + |S_0| - k} \leq \frac{1}{2},$$

yielding to

$$\frac{|S_H|}{\omega_H} \leq \frac{1}{2} + \frac{\omega_0 - (|S_0| + k)}{2k\omega_H}. \tag{4}$$

Since $S_0 \neq \emptyset$ because of Remark 1, and $k \geq 1$, if S_0 is not a cutset of G then $\omega_0 \leq 1$ (i.e., $\omega_0 = 0$ if $S_0 = V(G)$, and $\omega_0 = 1$ otherwise). Hence, applying inequality $\omega_0 - (|S_0| + k) < 0$ in (4), we have $\frac{|S_H|}{\omega_H} < \frac{1}{2}$. Thus, suppose that $S_0 \subset V(G)$ is a cutset of G.

First assume that $|S_0|/\omega_0 \geq 1$. This means that $\omega_0 - (|S_0| + k) < \omega_0 - |S_0| \leq 0$, yielding in (4) to $\frac{|S_H|}{\omega_H} < \frac{1}{2}$.

Second assume that $|S_0|/\omega_0 < 1$. Since S_0 is a cutset of G then it is also a cutset of $G \circ H$ and $\omega(G \circ H - S_0) = \omega_0 + |S_0|$. Therefore, by using that S is a τ-cut of $G \circ H$, it follows that

$$\frac{|S_0|}{\omega_0 + |S_0|} \geq \tau(G \circ H) = \frac{|S_0| + k|S_H|}{\omega_0 + k\omega_H + |S_0| - k} > \frac{|S_0| + k|S_H|}{\omega_0 + k\omega_H + |S_0|}. \qquad (5)$$

Combining the first and the last members of (5) we deduce that

$$\frac{|S_H|}{\omega_H} < \frac{|S_0|}{\omega_0 + |S_0|} = \frac{\frac{|S_0|}{\omega_0}}{1 + \frac{|S_0|}{\omega_0}} < \frac{1}{2},$$

because $|S_0|/\omega_0 < 1$. This concludes the proof. $\square$

From these previous results it follows the next theorem where the toughness of the corona $G \circ H$ of two connected graphs is determined in terms os some parameter of G and H.

Theorem 7 *Let G, H be two connected graphs of order m and n, respectively. Then the following assertions holds:*

(i) *If $\tau(G) \geq 1$ and $\tau(H) \geq 1/2$, then $\tau(G \circ H) = \dfrac{1}{2}$.*

(ii) *If $\tau(G) < 1$ and $\tau(H) \geq 1/2$, then $\tau(G \circ H) = \dfrac{\tau(G)}{1 + \tau(G)}$.*

(iii) *If $\tau(G) \geq 1$ and $\tau(H) < 1/2$, then*

$$\tau(G \circ H) = \min_{S_H \in J(H)} \left\{ \frac{1 + |S_H|}{1 + \omega(H - S_H)} \right\}.$$

(iv) *If $\tau(G) < 1$ and $\tau(H) < 1/2$, then*

$$\tau(G \circ H) = \min \left\{ \frac{\tau(G)}{1 + \tau(G)}, \min_{S_H \in J(H)} \frac{1 + |S_H|}{\frac{1}{\tau(G)} + \omega(H - S_H)} \right\}.$$

Proof: Let $S = S_0 \cup \bigcup_{i=1}^{m} S_i$ be a τ-cut of $G \circ H$. Without loos of generality we may assume that $V(G) = \{v_1, \dots, v_m\}$, where the vertices are numbered so that $|S_i| \geq |S_{i+1}|$, for all $i = 1, \dots, m-1$. We also may suppose that S has minimum cardinality over all the τ-cuts of $G \circ H$.

First, assume that $\tau(H) \geq 1/2$. Then by applying Lemma 6 we deduce that $S_i = \emptyset$, for all $i = 1, \dots, m$, hence, $S = S_0$. This implies that $\tau(G \circ H) = \frac{|S|}{\omega(G \circ H - S)} = \frac{|S_0|}{\omega_0 + |S_0|}$. Note that $S_0 \neq V(G)$, because otherwise, we have $\omega(G \circ H - S_0) = 0$ and therefore, $\tau(G \circ H) = 1$, which is a contradiction with Proposition 5, thus, $S_0 \subset V(G)$, which means that $\omega(G \circ H - S_0) \geq 1$. If S_0 is not a cutset of G then $\omega(G \circ H - S_0) = 1$ and therefore, $\tau(G \circ H) = \frac{|S_0|}{\omega_0 + |S_0|} = \frac{|S_0|}{1 + |S_0|} \geq \frac{1}{2}$. If S_0 is a cutset of G then $\omega(G \circ H - S_0) \geq 2$ and therefore,

$$\tau(G \circ H) = \frac{|S_0|}{\omega_0 + |S_0|} = \frac{\frac{|S_0|}{\omega_0}}{1 + \frac{|S_0|}{\omega_0}} \geq \frac{\tau(G)}{1 + \tau(G)}.$$

Hence, $\tau(G \circ H) \geq \min\left\{\frac{1}{2}, \frac{\tau(G)}{1+\tau(G)}\right\}$. Moreover, by Proposition 5 we have $\tau(G \circ H) \leq \min\left\{\frac{1}{2}, \frac{\tau(G)}{1+\tau(G)}\right\}$, yielding to

$$\tau(G \circ H) = \min\left\{\frac{1}{2}, \frac{\tau(G)}{1 + \tau(G)}\right\} = \begin{cases} \frac{1}{2}, & \text{if } \tau(G) \geq 1 \\ \frac{\tau(G)}{1+\tau(G)}, & \text{if } \tau(G) < 1 \end{cases},$$

which proves items (i) and (ii).

Second, assume that $\tau(H) < 1/2$. If $S_1 = \emptyset$ then $S_i = \emptyset$ for every $i = 1, \dots, m$, and reasoning as above, we prove that

$$\tau(G \circ H) \geq \min\left\{\frac{1}{2}, \frac{\tau(G)}{1 + \tau(G)}\right\} = \begin{cases} \frac{1}{2}, & \text{if } \tau(G) \geq 1 \\ \frac{\tau(G)}{1+\tau(G)}, & \text{if } \tau(G) < 1 \end{cases} \tag{6}$$

Thus, suppose that $S_1 \neq \emptyset$, then by Lemma 4 we may assume that there exist an integer $k \in \{1, \dots, m\}$ and a nonempty vertex set $S_H \subset V(H)$ such that $S_i = S_H$ if $i \leq k$, and $S_i = \emptyset$ otherwise. Further, from Lemma 6, it follows that $|S_H|/\omega(H - S_H) < 1/2$. Again to clarify expressions, denote by $\omega_H = \omega(H - S_H)$. Notice that $k \leq |S_0|$ because of Remark 2 and Remark 3 and therefore, $\tau(G \circ H) = \frac{|S_0| + k|S_H|}{\omega_0 + k\omega_H + |S_0| - k}$. Since S_0 is also a cutset of $G \circ H$, $|S_0| > |S|$ and S is a τ-cut of $G \circ H$ of minimum cardinality, then

$$\frac{|S_0|}{\omega_0 + |S_0|} > \tau(G \circ H) = \frac{|S_0| + k|S_H|}{\omega_0 + k\omega_H + |S_0| - k},$$

yielding to

$$|S_H|(\omega_0+|S_0|)-|S_0|(\omega_H-1)<0. \tag{7}$$

The function $f(k)=\frac{|S_0|+k|S_H|}{\omega_0+k\omega_H+|S_0|-k}$ has derivate

$$\frac{df}{dk}=\frac{|S_H|(\omega_0+|S_0|)-|S_0|(\omega_H-1)}{(\omega_0+k\omega_H+|S_0|-k)^2},$$

and by (7), we deduce that $f(k)$ is decreasing in k. Hence,

$$\tau(G\circ H)=f(k)\geq f(|S_0|)=\frac{|S_0|\,(1+|S_H|)}{\omega_0+|S_0|\omega_H}. \tag{8}$$

If S_0 is not a cutset of G then $\omega_0\leq 1$ ($\omega_0=0$ if $S_0=V(G)$, and $\omega_0=1$ otherwise), and from (8) we have

$$\begin{aligned}\tau(G\circ H)\geq\frac{|S_0|\,(1+|S_H|)}{1+|S_0|\omega_H} &= \frac{1+|S_H|}{1/|S_0|+\omega_H}\\ &\geq \frac{1+|S_H|}{1+\omega_H}\\ &\geq \min_{S_H\in J(H)}\left\{\frac{1+|S_H|}{1+\omega_H}\right\}.\end{aligned} \tag{9}$$

If S_0 is a cutset of G then $|S_0|/\omega_0\geq\tau(G)$ and therefore, from (8) it follows that

$$\begin{aligned}\tau(G\circ H)\geq\frac{|S_0|\,(1+|S_H|)}{\omega_0+|S_0|\omega_H} &= \frac{1+|S_H|}{\omega_0/|S_0|+\omega_H}\\ &\geq \frac{1+|S_H|}{1/\tau(G)+\omega_H}\\ &\geq \min_{S_H\in J(H)}\left\{\frac{1+|S_H|}{1/\tau(G)+\omega_H}\right\}.\end{aligned} \tag{10}$$

(iii) Suppose that $\tau(G)\geq 1$, then combining (6), (9) and (10), we deduce that

$$\begin{aligned}\tau(G\circ H) &\geq \min\left\{\frac{1}{2},\min_{S_H\in J(H)}\left\{\frac{1+|S_H|}{1+\omega_H}\right\},\min_{S_H\in J(H)}\left\{\frac{1+|S_H|}{1/\tau(G)+\omega_H}\right\}\right\}\\ &= \min\left\{\frac{1}{2},\min_{S_H\in J(H)}\left\{\frac{1+|S_H|}{1+\omega_H}\right\}\right\}.\end{aligned}$$

Since $\tau(H) > 1/2$ then there exists a cutset $S_H \subset V(H)$ such that $|S_H|/\omega_H < 1/2$, which implies that $2|S_H| + 1 \leq \omega_H$. Then

$$\frac{1+|S_H|}{1+\omega_H} \leq \frac{\omega_H - |S_H|}{1+\omega_H} = \frac{1+\omega_H-(1+|S_H|)}{1+\omega_H} = 1 - \frac{1+|S_H|}{1+\omega_H},$$

which means that $\dfrac{1+|S_H|}{1+\omega_H} \leq 1/2$ and therefore,

$$\tau(G \circ H) = \min_{S_H \in J(H)} \left\{ \frac{1+|S_H|}{1+\omega_H} \right\}.$$

(iv) Now suppose that $\tau(G) < 1$, then from (6), (9) and (10) it follows that

$$\tau(G \circ H) \geq \min \left\{ \frac{\tau(G)}{1+\tau(G)}, \min_{S_H \in J(H)} \left\{ \frac{1+|S_H|}{1/\tau(G)+\omega_H} \right\} \right\}. \quad \square$$

$\circ$	S_n	P_n	C_n	$W_{1,n}$	K_n
S_m	$\begin{cases} \frac{1}{m}, & \text{si } m > n-2, \\ \frac{2}{m+n-2}, & \text{si } m \leq n-2. \end{cases}$	$\frac{1}{m}$	$\frac{1}{m}$	$\frac{1}{m}$	$\frac{1}{m}$
P_m	$\begin{cases} \frac{1}{3}, & \text{si } n < 5, \\ \frac{2}{n+1}, & \text{si } n \geq 5. \end{cases}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
C_m	$\frac{2}{n}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$W_{1,m}$	$\frac{2}{n}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
K_m	$\frac{2}{n}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Table 1: The toughness of the corona of some families of graphs.

As a consequence of Theorem 2, the toughness of the corona of some families of graphs can be derived. Let $n \geq 3$ be an integer. Let us denote by P_n and C_n the path and the cycle with n vertices, respectively; by S_n the complete bipartite graph $K_{1,n-1}$; by $W_{1,n}$ the wheel with $n+1$ vertices; and by K_n the complete graph of order n. As a consequence of Theorem 2, the toughness of the corona of two graphs, one of them being a complete graph is deduced. Further, in Table 1 we can find the toughness of the corona of two graphs belonging to some of these families: stars, paths, cycles, wheels and complete graphs.

Corollary 8 *Let $m \geq 3$, $n \geq 3$ be two integers and let G, H be two connected graphs. Then the following assertions hold:*

(i) $$\tau(G \circ K_n) = \begin{cases} \frac{1}{2}, & \text{if } \tau(G) \geq 1, \\ \frac{\tau(G)}{1+\tau(G)}, & \text{if } \tau(G) < 1. \end{cases}$$

(ii) $$\tau(K_m \circ H) = \begin{cases} \frac{1}{2}, & \text{if } \tau(H) \geq 1/2, \\ \min\limits_{S_H \subset J(H)} \left\{ \dfrac{1+|S_H|}{1+\omega(H-S_H)} \right\}, & \text{if } \tau(H) < 1/2. \end{cases}$$

3 The toughness of the cartesian product $\mathbf{K_2} \times \mathbf{G}$

The main result in this section is the following theorem in which the toughness of $K_2 \times G$ is determined in terms of the some invariants of graph G.

Theorem 9 *Let G be a connected graph of minimum degree δ and independence number β. Then*

$$\min\left\{\tau(G), \frac{|V(G)|}{1+\beta}\right\} \leq \tau(K_2 \times G) \leq \min\left\{2\tau(G), \frac{\delta+1}{2}\right\}.$$

Acknowledgement

This research is supported by the Ministry of Education and Science, Spain, and the European Regional Development Fund (ERDF) under project MTM2008-06620-C03-02 and under the Andalusian Government project P06-FQM-01649.

References

[1] D. Bauer, S.L. Hakimi and E. Schmeichel. Recognizing tough graphs is NP-hard. *Discrete Appl. Math.*, 28:191–195, 1990.

[2] H.J. Broersma, E. Engbers and H. Trommel. Various results on the toughness of graphs. *Networks*, 33:233–238, 1999.

[3] Gary Chartrand and Linda Lesniak. *Graphs and Digraphs.* Chapman and Hall/CRC, United States of America, 2005.

[4] V. Chvátal. Tough graphs and Hamiltonian circuits. *Discrete Math.*, 5:215–228, 1973.

[5] R. Faudree, R. Gould, M. Jacobson, L. Lesniak and A. Saito. Toughness, degrees and 2-factors. *Discrete Math.*, 286:245–249, 2004.

[6] R. Frucht and F. Harary. On the corona of two graphs. *Aequationes Math.*, 4:322–325, 1970.

[7] W.D. Goddard and H.C. Swart. On the toughness of a graph. *Quaestiones Math.*, 13:217–232, 1990.

[8] R.E. Pippert. On the toughness of a graph. *Lecture Notes in Math.*, 303:225–233, 1972.

On the diameter of random planar graphs

Guillaume Chapuy
Simon Fraser University
Vancouver

Éric Fusy
École Polytechnique
Palaiseau

Omer Giménez and Marc Noy
Universitat Politècnica de Catalunya
Barcelona

Abstract

We show that the diameter $D(G_n)$ of a random labelled connected planar graph with n vertices is asymptotically almost surely of order $n^{1/4}$, in the sense that there exists a constant $c > 0$ such that

$$P(D(G_n) \in (n^{1/4-\epsilon}, n^{1/4+\epsilon})) \geq 1 - \exp(-n^{c\epsilon})$$

for ϵ small enough and n large enough ($n \geq n_0(\epsilon)$). We prove similar statements for rooted 2-connected and 3-connected maps and planar graphs.

1 Introduction

The diameter of random maps has attracted a lot of attention since the pioneering work by Chassaing and Schaeffer [4] on the radius $r(Q_n)$ of random quadrangulations with n vertices, where they show that $r(Q_n)$ rescaled by $n^{1/4}$ converges as $n \to \infty$ to an explicit (continuous) distribution related to the Brownian snake. This suggests that random maps of size n are to be rescaled by $n^{1/4}$ in order to converge; precise definitions of the convergence can be found in [12, 7], and the (spherical) topology of the limit is studied in [8, 14]; some general statements about the limiting profile and radius are obtained in [11, 13]. At the combinatorial level, the two-point function of random quadrangulations has surprisingly a simple exact expression, a beautiful result found in [3] that allows one to derive easily the limit distibution (rescaled by $n^{1/4}$) of the distance between two randomly chosen vertices in a random quadrangulation. In contrast, little is known about the profile of random *unembedded* connected planar graphs, even if it is strongly believed that the results should be similar as in the embedded case.

We have not been able to show a limit distribution for the profile (or radius, diameter) of a random connected planar graph rescaled by $n^{1/4}$; instead we have obtained large deviation results on the diameter that strongly support the belief that $n^{1/4}$ is the right scaling order. We say that a property A, defined for all values n of a parameter, holds asymptotically almost surely if

$$P(A) \to 1, \qquad \text{as } n \to \infty.$$

In this case we write a.a.s. In this paper we need a certain rate of convergence of the probabilities. Suppose property A depends on a real number $\epsilon > 0$ (usually very small). Then we say that A holds a.a.s. with exponential rate if there is a constant $c > 0$, such that for every ϵ small enough there exist an integer $n_0(\epsilon)$ so that

$$P(\text{not } A) \leq e^{-n^{c\epsilon}} \qquad \text{for all } n \geq n_0(\epsilon). \tag{1}$$

The diameter of a graph (or map) G is denoted by $D(G)$. The main results proved in this paper are the following.

Theorem 1 *The diameter of a random connected labelled planar graph with n vertices is, a.a.s. with exponential rate, in the interval*

$$(n^{1/4-\epsilon}, n^{1/4+\epsilon}).$$

Theorem 2 *Let $1 < \mu < 3$. The diameter of a random connected labelled planar graph with n vertices and $\lfloor \mu n \rfloor$ edges is in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$ a.a.s. with exponential rate.*

This contrasts with so-called "subcritical" graph families, such as trees, outerplanar graphs, series-parallel graphs, where the diameter is in the interval $(n^{1/2-\epsilon}, n^{1/2+\epsilon})$ a.a.s. with exponential rate. (see the remark just before the bibliography).

The basis of our proof is the result for planar maps mentioned above. Then we prove the result for 2-connected maps using the fact that a random map has a large 2-connected component a.a.s. A similar argument allows us to extend the result to 3-connected maps, which proves it also for 3-connected planar graphs, because they have a unique embedding in the sphere. We then reverse the previous arguments and go first to 2-connected and then connected planar graphs, but this is not straightforward. One difficulty is that the largest 3-connected component of a random 2-connected graph does not have the typical ratio between number of edges and number of vertices, and this is why we must study maps with a given ratio between edges and vertices. In addition, we must show that there is a 3-connected component of size $n^{1-\epsilon}$ a.a.s. with exponential rate, and similarly for blocks. Finally, we must show that the height of the tree associated to the decomposition of a 2-connected graph into 3-connected components is at most n^{ϵ}, and similarly for the tree of the decomposition of a connected graph into blocks.

For lack of space, proofs are omitted in this extended abstract.

2 Quadrangulations and maps

We recall here the definitions of maps. A *planar map* (shortly called a map here) is a connected unlabelled graph embedded in the plane up to isotopic deformation. Loops and multiple edges are allowed. A *rooted map* is a map where an edge incident to the outer face is marked so as to have the outer face on its left; the *root-vertex* is the origin of the root. A *quadrangulation* is a map where all faces have degree 4.

We recall Schaeffer's bijection (itself an adaptation of an earlier bijection by Cori and Vauquelin [5]) between labelled trees and quadrangulations. A *rooted plane tree* is a rooted map with a unique face. A *labelled tree* is a rooted plane tree with a integer label $\ell(v) \in \mathbb{Z}$ on each vertex v so that the labels of the extremities of each edge $e = (v, v')$ satisfy $|\ell(v) - \ell(v')| \leq 1$, and such that the root vertex has label 0. A useful observation is that labelled trees are in bijection with rooted plane trees where a subset of the edges is oriented arbitrarily (for the onto mapping, one orients an edge with labels $(i, i+1)$ toward the vertex with label $i+1$ and one leaves an edge of type (i, i) unoriented). Thus the number of labelled trees with n edges is $3^n C_n$ with $C_n := (2n)!/n!/(n+1)!$ the nth Catalan number. A *signed* labelled tree is a pair (τ, σ) where τ is a labelled tree and σ is an element of $\{-1, +1\}$.

Theorem 3 (Schaeffer [15], Chassaing, Schaeffer [4]) *Signed labelled trees with n vertices are in bijection with rooted quadrangulations with n vertices and a secondary pointed vertex v_0. Each vertex v of a labelled tree corresponds to a non-pointed vertex ($\neq v_0$) in the associated quadrangulation Q, and $\ell(v) - \ell_{min} + 1$ gives the distance from v to v_0 in Q, where l_{min} is the minimum label in the tree.*

From this bijection, it is easy to show large deviation results for the diameter of a quadrangulation (the basic idea, originating in [4], is that the typical depth k of a vertex in the tree is $n^{1/2}$, and the typical discrepancy of the labels along a branch is $k^{1/2} = n^{1/4}$). The main result we use, from [6], is the property that (under general conditions) the height of a random tree of size n from a given family has diameter in $(n^{1/2-\epsilon}, n^{1/2+\epsilon})$ a.a.s. with exponential rate.

Lemma 4 (Flajolet et al. Theorem 3.1 in [6]) *Let $\mathcal{T}$ be a family of rooted trees endowed with a weight-function $w(.)$ so that the corresponding weighted series $y(z)$ is admissible (in a precise analytic sense not defined here).*

Let ξ be a height-parameter and let T_n be taken at random in $\mathcal{T}_n$ under the weighted distribution in size n. Then $\xi(T_n) \in (n^{1/2-\epsilon}, n^{1/2+\epsilon})$ a.a.s. with exponential rate.

Proposition 5 *The diameter of a random rooted quadrangulation with n vertices is, a.a.s. with exponential rate, in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$.*

We also need a weighted version of the previous theorem. Recall that a rooted quadrangulation Q has a unique bicoloration of its vertices in black and white such that the origin of the root is black and each edge connects a black with a white vertex. Call it the *canonical bicoloration* of Q. Given $x > 0$, a rooted quadrangulation with v black vertices is weighted with parameter x if we assign to it weight x^v. The next theorem generalizes Proposition 5 to the weighted case. The analytical part of the proof is a little more delicate since the system specifying weighted labelled trees is two-lines, and has to be transformed to a one-line equation in order to apply Lemma 4.

Theorem 6 *Let $0 < a < b$. The diameter of a random quadrangulation weighted by x is, a.a.s. with exponential rate, in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$, uniformly over $x \in [a, b]$.*

We recall the classical bijection between rooted quadrangulations with n faces (and thus $n+2$ vertices) and rooted maps with n edges. Starting from Q endowed with its canonical bicoloration, add in each face a new edge connecting the two (diagonally opposed) black vertices. Return the rooted map M formed by the newly added edges and the black vertices, rooted at the edge corresponding to the root-face of Q, and with same root-vertex as Q. Conversely, to obtain Q from M, add a new white vertex v_f inside each face f of M (even the outer face) and add new edges from v_f to every corner around f; then delete all edges from M, and take as root-edge of Q the one corresponding to the incidence root-vertex/outer-face in M. Clearly, under this bijection, vertices of a map correspond to black vertices of the associated quadrangulation, and faces correspond to white vertices.

Map families are here weighted at their vertices, i.e., for a given parameter $x > 0$, a map with v vertices has weight x^v.

Theorem 7 *Let $0 < a < b$. The diameter of a random rooted map with n edges and weight x at the vertices is in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$, uniformly over $x \in [a, b]$.*

Here it is convenient to include the empty map in the families $\mathcal{M} = \cup_n \mathcal{M}_n$ of rooted maps and $\mathcal{C} = \cup_n \mathcal{C}_n$ of rooted 2-connected maps. As described by Tutte in [16], a rooted map M is obtained by taking a rooted 2-connected map C, called the *core* of M, and then inserting in each corner i of C an arbitrary rooted map M_i. The maps M_i are called the *pieces* of M. Denoting by $M(x, z)$ ($C(x, z)$, resp.) the series of rooted connected (2-connected, resp.) maps according to non-root vertices and edges, this decomposition yields

$$M(x, z) = C(x, H(x, z)), \quad \text{where } H(x, z) = zM(x, z)^2, \tag{2}$$

since a core with k edges has $2k$ corners where to insert rooted maps.

An important property of the composition scheme is to preserve the uniform distribution, as well as the (vertex-)weighted distribution. Precisely, let M be a rooted map with n edges and weight x at the vertices. Let C be the core of M, call k its size, and let $M_1, \ldots, M_{2k}$ be the pieces of M, call $n_1, \ldots, n_{2k}$ their sizes. Then, conditioned to have size k, C is a random rooted 2-connected map with k edges and weight x at vertices, and conditioned to have size n_i the ith piece M_i is a random rooted map with n_i edges and weight x at vertices.

Lemma 8 *Let $0 < a < b$, and let $x \in [a, b]$. Let $\rho^{(x)}$ be the radius of convergence of $z \mapsto M(x, z)$. Following [1], define*

$$\alpha^{(x)} = \frac{H(x, \rho^{(x)})}{\rho^{(x)} H_z(x, \rho^{(x)})}.$$

Let $n \geq 0$, and let M be a random rooted map with n edges and weight x at vertices. Let $X_n = |C|$ be the size of the core of M, and let $M_1, \ldots, M_{2|C|}$ be the pieces of M. Then

$$P\big(X_n = \lfloor \alpha^{(x)} n \rfloor,\ \max(|M_i|) \leq n^{3/4}\big) = \Theta(n^{-2/3})$$

uniformly over $x \in [a, b]$.

In [1] the authors derive the limit distribution of X_n and they show that $P(X_n = \lfloor \alpha^{(x)} n \rfloor) = \Theta(n^{-2/3})$. So Lemma 8 says that the asymptotic order of $P(X_n = \lfloor \alpha^{(x)} n \rfloor)$ is the same under the additional condition that all pieces are of size at most $n^{3/4}$ (one could actually ask $n^{2/3+\delta}$ for any $\delta > 0$). A closely related result proved in [9] is that, for any fixed $\delta > 0$, there is a.a.s. no piece of size larger than $n^{2/3+\delta}$ provided the core has size larger than $n^{2/3+\delta}$.

Theorem 9 *For $0 < a < b$, the diameter of a random rooted 2-connected map with n edges and weight x at vertices is, a.a.s. with exponential rate, in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$, uniformly over $x \in [a, b]$.*

In a similar way as when one goes from connected to 2-connected maps, there is a decomposition of 2-connected maps in terms of 3-connected components that allows to transfer the diameter concentration property from

2-connected to 3-connected maps. In this section it is convenient to exclude the loop-map from the family of 2-connected maps, so all 2-connected maps are loopless.

As shown by Tutte [16], a rooted 2-connected map C is either a series or parallel composition of 2-connected maps, or it is obtained from a rooted 3-connected map T where each non-root edge e is possibly substituted by a rooted 2-connected map C_e (identifying the extremities of e with the extremities of the root of C_e). In that case T is called the *3-connected core* of C and the components C_e are called the *pieces* of C. Call $C(x,z)$ ($\widehat{C}(x,z)$) the series counting rooted 2-connected maps (rooted 2-connected maps with a 3-connected core, resp.) according to vertices not incident to the root (variable x) and edges (variable z). Call $T(x,z)$ the series counting rooted 3-connected maps according to vertices not incident to the root (variable x) and edges (variable z). Then

$$\widehat{C}(x,z) = T(x, C(x,z)). \tag{3}$$

Accordingly, for a random rooted 2-connected map with n edges, weight x at vertices, and conditioned to have a 3-connected core T of size k, T is a random rooted 3-connected map with k edges and weight x at vertices; and each piece C_e conditioned to have a given size n_e is a random rooted 2-connected map with n_e edges and weight x at vertices.

Calling f_e the degree of the root face of C_e, we have

$$D(T) \leq D(C) \leq D(T) \cdot \max_e(f_e) + 2\max_e(D(C_e)). \tag{4}$$

The first inequality is trivial. The second one follows from the fact that a diametral path P in C starts in a piece, ends in a piece, and in between it passes by adjacent vertices $v_1, \ldots, v_k$ of H such that for $1 \leq i < k$, v_i and v_{i+1} are connected in H by an edge e and P travels in the piece C_e to reach v_{i+1} from v_i (since P is geodesic, its length in C_e is bounded by the distance from v_i to v_{i+1}, which is clearly bounded by f_e).

Theorem 10 *Let $0 < a < b$. The diameter of a random 3-connected map with n edges with weight x at the vertices is, a.a.s. with exponential rate, in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$, uniformly over $x \in [a,b]$.*

3 Planar graphs

We need 3-connected graphs labelled at the edges (this is enough to avoid symmetries). The number of edges is now m, and n is reserved for the number of vertices. By Whitney's theorem 3-connected graphs have a unique embedding on the sphere (up to reflexion). Hence from the last theorem on 3-connected maps we obtain directly the following:

Theorem 11 *Let $0 < a < b$. The diameter of a random 3-connected planar graph with m edges with weight x at the vertices is, a.a.s. with exponential rate, in the interval $(m^{1/4-\epsilon}, m^{1/4+\epsilon})$.*

Before handling 2-connected planar graphs we treat the closely related family of (planar) *networks*. A *network* is a connected simple planar graph with two marked vertices called the poles, such that adding an edge between the poles, called the root-edge, makes the graph 2-connected. At first it is convenient to consider the networks as labelled at the edges.

Theorem 12 *Let $0 < a < b$. The diameter of a random network with m edges with weight x at the vertices is, a.a.s. with exponential rate, in the interval*

$$(m^{1/4-\epsilon}, m^{1/4+\epsilon}),$$

uniformly over $x \in [a, b]$.

Lemma 13 *Let $1 < a < b < 3$. For $N_{n,m}$ a network with n vertices and m labelled edges taken uniformly at random, $D(N_{n,m}) \in (n^{1/4-\epsilon}, n^{1/4+\epsilon})$ a.a.s. with exponential rate, uniformly over $m/n \in [a, b]$.*

An important remark is that networks with n vertices and m edges can be labelled either at vertices or at edges, and the uniform distribution in one case corresponds to the uniform distribution in the second case. Hence the result of Lemma 13 holds for random networks with n vertices and m edges and labelled at vertices.

It is proved in [2] that for a random network N_n with n vertices the ratio $r = \#edges/\#vertices$ is concentrated around a certain $\mu \approx 2.2$, implying that for $\delta > 0$ $P(r \notin [\mu - \delta, \mu + \delta])$ is exponentially small. Hence $D(N_n) \in (n^{1/4-\epsilon}, n^{1/4+\epsilon})$ a.a.s. with exponential rate. The same holds for the diameter of a random 2-connected planar graph B_n with n vertices (indeed 2-connected planar graphs are a subset of networks, the ratios of the cardinalities being of order n). We obtain:

Theorem 14 *The diameter of a random 2-connected planar graph with n vertices is, a.a.s. with exponential rate, in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$.*

We prove here from Theorem 14 that a random connected planar graph with n vertices has diameter in $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$ a.a.s. with exponential rate. We use the well known decomposition of a connected planar graph C into 2-connected blocks such that the incidences of the blocks with the vertices form a tree. An important point is that if C is chosen uniformly at random among connected planar graphs with n vertices, then each block B of C is uniformly distributed when conditioned to have a given size. Formulated on pointed graphs, the block-decomposition ensures that a pointed planar graph is obtained as follows: take a collection of 2-connected pointed planar graphs, and merge their pointed vertices into a single vertex; then attach at each non-marked vertex v in these blocks a pointed connected planar graph C_v. Calling $C(z)$ $(B(z))$ the series counting pointed connected (2-connected, resp.) planar graphs, this yields the equation

$$F(z) = z\exp(B'(F(z))), \quad \text{where } F(z) = zC'(z). \tag{5}$$

Note that the inverse of $F(z)$ is the function $\phi(u) = u\exp(-g(u))$, where $g(u) := B'(u)$. Call ρ the radius of convergence of $C(z)$ and R the radius of convergence of $B(u)$.

Lemma 15 *A random connected planar graph with n vertices has a block of size at least $n^{1-\epsilon}$ a.a.s. with exponential rate.*

Lemma 15 directly implies that a random connected planar graph with n vertices has diameter at least $n^{1/4-\epsilon}$. Indeed it has a block of size $k \geq n^{1-\epsilon}$ a.a.s. with exponential rate and since the block is uniformly distributed in size k, it has diameter at least $k^{1/4-\epsilon}$ a.a.s. with exponential rate.

Let us now prove the upper bound, which relies on the following lemma:

Lemma 16 *The block-decomposition tree τ of a random connected planar graph with n vertices has diameter at most n^{ϵ} a.a.s. with exponential rate.*

Lemma 16 easily implies that the diameter of a random connected planar graph C with n vertices is at most $n^{1/4+\epsilon}$ a.a.s. with exponential rate. Indeed, calling τ the block-decomposition tree of C and B_i the blocks of C, one has

$$D(C) \leq D(\tau) \cdot \mathrm{max}_i D(B_i).$$

Lemma 16 ensures that $D(\tau) \leq n^{\epsilon}$ a.a.s. with exponential rate. Moreover Theorem 14 easily implies that a random 2-connected planar graph of size $k \leq n$ has diameter at most $n^{1/4+\epsilon}$ a.a.s. with exponential rate, whatever $k \leq n$ is (proof by splitting in two cases: $k \leq n^{1/4}$ and $n^{1/4} \leq k \leq n$). Hence, since each of the blocks has size at most n, $\max_i D(B_i) \leq n^{1/4+\epsilon}$ a.a.s. with exponential rate. Therefore we have completed the proof of Theorem 1.

Theorem 17 *The diameter of a random connected planar graph with n vertices is, a.a.s. with exponential rate, in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$.*

Similarly one shows that a random planar graph with n vertices has a connected component of size at least $n^{1-\epsilon}$ a.a.s. with exponential rate, which yields Theorem 1.

To show Theorem 2, one needs to extend the statements of Theorem 14 and Lemmas 15, 16 to the case of a random graph of size n with weight $y > 0$ on each edge. Then, one uses the fact (proved in [10]) that for each $\mu \in (1, 3)$ there exists $y > 0$ such that a random planar graph with n edges and weight y on edges has probability $\Theta(n^{-1/2})$ to have $\lfloor \mu n \rfloor$ edges.

We conclude with a remark on so-called "subcritical" graph families, these are the families where the system

$$y = z \exp(B'(y)) =: F(z, y) \tag{6}$$

to specify pointed connected from pointed 2-connected graphs in the family is admissible, i.e., $F(z, y)$ is analytic at (ρ, τ) where ρ is the radius of convergence of $y = y(z)$ and $\tau = y(\rho)$.

Define the *block-distance* of a vertex v in a vertex-pointed connected graph G as the minimal number of blocks one can use to travel from the pointed vertex to v; and define the *block-height* of G as the maximum of the block-distance over all vertices of G. With the terminology of Lemma 4, one easily checks that the block-height is a height-parameter for the system (6). Hence by Lemma 4, the block-height h of a random pointed connected graph G with n vertices from a subcritical family is in $[n^{1/2-\epsilon}, n^{1/2+\epsilon}]$ a.a.s. with exponential rate. Clearly $D(G) \geq h - 1$ since the distance between two vertices is at least the block-distance minus 1. Hence $D(G) \geq n^{1/2-\epsilon}$ a.a.s. with exponential rate. For the upper bound, note that $D(G) \leq h \cdot \max_i(|B_i|)]$, where the B_i's are the blocks of G. Because of the subcritical condition one easily shows that $\max_i(|B_i|) \leq n^{\epsilon}$ a.a.s. with exponential rate. This implies that $D(G) \leq n^{1/2+\epsilon}$ a.a.s. with exponential rate.

References

[1] C. Banderier, P. Flajolet, G. Schaeffer, and M. Soria. Random maps, coalescing saddles, singularity analysis, and Airy phenomena. *Random Structures Algorithms*, 19(3/4):194–246, 2001.

[2] E. Bender, Z. Gao, and N. Wormald. The number of labeled 2-connected planar graphs. *Electron. J. Combin.*, 9:1–13, 2002.

[3] J. Bouttier, P. Di Francesco, and E. Guitter. Geodesic distance in planar graphs. *Nucl. Phys.*, B663:535–567, 2003.

[4] P. Chassaing and G. Schaeffer. Random planar lattices and integrated superBrownian excursion. *Probab. Theory Related Fields*, 128(2):161–212, 2004.

[5] R. Cori and B. Vauquelin. Planar maps are well labeled trees. *Canad. J. Math.*, 33(5):1023–1042, 1981.

[6] P. Flajolet, Z. Gao, A. M. Odlyzko, and L. B. Richmond. The distribution of heights of binary trees and other simple trees. *Combin. Probab. Comput.*, 2(145-156), 1993.

[7] J. F. Le Gall. The topological structure of scaling limits of large planar maps. *Invent. Math.*, 169(3):621–670, 2007.

[8] J. F. Le Gall and F. Paulin. Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere. *Geom. Funct. Anal.*, 18:893–918, 2008.

[9] J. Gao and N. Wormald. The size of the largest components in random planar maps. *SIAM J. Discrete Math.*, 12(2):217–228, 1999.

[10] O. Giménez and M. Noy. Asymptotic enumeration and limit laws of planar graphs. *J. Amer. Math. Soc.*, 22:309–329, 2009.

[11] J. F. Marckert and G. Miermont. Invariance principles for random bipartite planar maps. *Ann. Probab.*, 35(5):1642–1705, 2007.

[12] J. F. Marckert and A. Mokkadem. Limit of normalized random quadrangulations: The brownian map. *Ann. Probab.*, 34(6):2144–2202, 2006.

[13] G. Miermont. An invariance principle for random planar maps. In *Fourth Colloquium in Mathematics and Computer Sciences CMCS'06, DMTCS Proceedings AG*, pages 39–58, 2006.

[14] G. Miermont. On the sphericity of scaling limits of random planar quadrangulations. *Elect. Comm. Probab.*, 13:248–257, 2008.

[15] G. Schaeffer. *Conjugaison d'arbres et cartes combinatoires aléatoires.* PhD thesis, Université Bordeaux I, 1998.

[16] W. T. Tutte. A census of planar maps. *Canad. J. Math.*, 15:249–271, 1963.

Dual Concepts of Almost Distance-Regularity and the Spectral Excess Theorem

Cristina Dalfó, Miquel Àngel Fiol and Ernest Garriga
Universitat Politècnica de Catalunya
Barcelona

Edwin R. van Dam
Tilburg University
Tilburg

Abstract

Generally speaking, 'almost distance-regular' graphs are graphs that share some, but not necessarily all, regularity properties that characterize distance-regular graphs. In this paper we first propose two dual concepts of almost distance-regularity. In some cases, they coincide with concepts introduced before by other authors, such as partially distance-regular graphs. Our study focuses on finding out when almost distance-regularity leads to distance-regularity. In particular, some 'economic' (in the sense of minimizing the number of conditions) old and new characterizations of distance-regularity are discussed. Moreover, other characterizations based on the average intersection numbers and the recurrence coefficients are obtained. In some cases, our results can also be seen as a generalization of the so-called spectral excess theorem for distance-regular graphs.

1 Preliminaries

Almost distance-regular graphs, recently studied in the literature, are graphs which share some, but not necessarily all, regularity properties that characterize distance-regular graphs. Two examples of these are the partially distance-regular graphs [17] and the m-walk-regular graphs [8].

In this paper we propose two dual concepts of almost distance-regularity, and study some cases when distance-regularity is attained. As in the theory of distance-regular graphs [2, 4], the two proposed concepts lead to several duality results.

To some extent, this paper can be considered as a follow-up of our paper [7], where other concepts of almost distance-regularity, such as distance-polynomial graphs and partially walk-regular graphs, were addressed.

Here we are specially interested in the case when almost distance-regularity leads to distance-regularity. In particular, some 'economic' (in the sense of minimizing the number of conditions) old and new characterizations of distance-regularity are discussed. Moreover, other characterizations based on the preintersection parameters and the average intersection numbers are obtained. In some cases, our results can be also seen as a generalization of the so-called spectral excess theorem for distance-regular graphs (see [12]; for short proofs, see [9, 13]). This theorem characterizes distance-regular graphs by the eigenvalues and the average number of vertices at extremal distance. A dual version of this theorem is also derived.

The rest of this section is mainly devoted to introduce the notation used throughout the paper and discuss some basic results. Let us begin with some notation for graphs and their spectra.

1.1 Graphs and their spectra

Throughout this paper, $G = (V, E)$ denotes a simple, connected, δ-regular graph, with order $n = |V|$ and adjacency matrix $\boldsymbol{A}$. The *distance* between two vertices u and v is denoted by $\partial(u, v)$, so that the *eccentricity* of a vertex u is $\mathrm{ecc}(u) = \max_{v\in V}\partial(u, v)$ and the *diameter* of the graph is $D = \max_{u\in V}\mathrm{ecc}(u)$. The set of vertices at distance i from a given vertex $u \in V$ is denoted by $\Gamma_i(u)$, for $i = 0, 1, \dots, D$. The *degree* of a vertex u is denoted by $\delta(u) = |\Gamma_1(u)|$. The *distance-$i$ graph* G_i is the graph with vertex set V and where two vertices u and v are adjacent if and only if $\partial(u, v) = i$ in G. Its adjacency matrix $\boldsymbol{A}_i$ is usually referred to as the *distance-i matrix* of G. The spectrum of G is denoted by

$$\mathrm{sp}\, G = \mathrm{sp}\, \boldsymbol{A} = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\},$$

where the different eigenvalues of G are in decreasing order, $\lambda_0 > \lambda_1 > \cdots > \lambda_d$, and the superscripts stand for their multiplicities $m_i = m(\lambda_i)$. In particular, note that $\lambda_0 = \delta$, $m_0 = 1$ (since G is δ-regular and connected) and $m_0 + m_1 + \cdots + m_d = n$.

For a given ordering of the vertices of G, the vector space of linear combinations (with real coefficients) of the vertices is identified with $\mathbb{R}^n$, with canonical basis $\{\boldsymbol{e}_u : u \in V\}$. Let $Z = \prod_{i=0}^{d}(x - \lambda_i)$ be the minimal polynomial of $\boldsymbol{A}$. The vector space $\mathbb{R}_d[x]$ of real polynomials of degree at most d is isomorphic to $\mathbb{R}[x]/(Z)$. For every $i = 0, 1, \ldots, d$, the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}_i = \operatorname{Ker}(\boldsymbol{A} - \lambda_i \boldsymbol{I})$ is given by the Lagrange interpolating polynomial

$$\lambda_i^* = \frac{1}{\phi_i} \prod_{\substack{j=0 \\ j \neq i}}^{d} (x - \lambda_j) = \frac{(-1)^i}{\pi_i} \prod_{\substack{j=0 \\ j \neq i}}^{d} (x - \lambda_j)$$

of degree d, where $\phi_i = \prod_{j=0, j\neq i}^{d}(\lambda_i - \lambda_j)$ and $\pi_i = |\phi_i|$. These polynomials satisfy $\lambda_i^*(\lambda_j) = \delta_{ij}$. The matrices $\boldsymbol{E}_i = \lambda_i^*(\boldsymbol{A})$, corresponding to these orthogonal projections, are the (*principal*) *idempotents* of $\boldsymbol{A}$, and satisfy the known properties: $\boldsymbol{E}_i \boldsymbol{E}_j = \delta_{ij} \boldsymbol{E}_i$; $\boldsymbol{A}\boldsymbol{E}_i = \lambda_i \boldsymbol{E}_i$; and $p(\boldsymbol{A}) = \sum_{i=0}^{d} p(\lambda_i) \boldsymbol{E}_i$, for any polynomial $p \in \mathbb{R}[x]$ (see, for example, [16, p. 28]). The (u-)*local multiplicities* of the eigenvalue λ_i are defined as

$$m_u(\lambda_i) = \|\boldsymbol{E}_i \boldsymbol{e}_u\|^2 = \langle \boldsymbol{E}_i \boldsymbol{e}_u, \boldsymbol{e}_u \rangle = (\boldsymbol{E}_i)_{uu}, \qquad u \in V,\ i = 0, 1, \ldots, d,$$

and satisfy $\sum_{i=0}^{d} m_u(\lambda_i) = 1$, $u \in V$, and $\sum_{u \in V} m_u(\lambda_i) = m_i$, $i = 0, 1, \ldots, d$ (see [12]).

1.2 The predistance and preidempotent polynomials

From the spectrum of a given (arbitrary) graph, $\operatorname{sp} G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, one can generalize the distance polynomials of a distance-regular graph by considering the following scalar product in $\mathbb{R}_d[x]$:

$$\langle f, g \rangle_\triangle = \frac{1}{n} \operatorname{tr}\left(f(\boldsymbol{A}) g(\boldsymbol{A})\right) = \frac{1}{n} \sum_{i=0}^{d} m_i f(\lambda_i) g(\lambda_i). \tag{1}$$

Then, by using the Gram-Schmidt method and normalizing appropriately, it is immediate to prove the existence and uniqueness of an orthogonal system of so-called *predistance polynomials* $\{p_i\}_{0 \le i \le d}$ satisfying $\deg p_i = i$ and $\langle p_i, p_j \rangle_\triangle = \delta_{ij} p_i(\lambda_0)$, for any $i, j = 0, 1, \ldots, d$. For details, see [12, 11].

As every sequence of orthogonal polynomials, the predistance polynomials satisfy a three-term recurrence of the form

$$xp_i = \beta_{i-1}p_{i-1} + \alpha_i p_i + \gamma_{i+1}p_{i+1}, \qquad i = 0, 1, \ldots, d, \tag{2}$$

where the constants β_{i-1}, α_i, and γ_{i+1} are the Fourier coefficients of xp_i in terms of p_{i-1}, p_i, and p_{i+1}, respectively (and $\beta_{-1} = \gamma_{d+1} = 0$), initiated with $p_0 = 1$. Some basic properties of these coefficients, such as $\alpha_i+\beta_i+\gamma_i = \lambda_0$ for $i = 0, 1, \ldots, d$, and $\beta_i n_i = \gamma_{i+1}n_{i+1} \neq 0$ for $i = 0, 1, \ldots, d-1$, where $n_i = \|p_i\|^2_\triangle = p_i(\lambda_0)$, can be found in [5].

For any graph the sum of all the predistance polynomials gives the *Hoffman polynomial*

$$H = \sum_{i=0}^{d} p_i = \frac{n}{\pi_0}\prod_{i=1}^{d}(x-\lambda_i) = n\,\lambda_0^*, \tag{3}$$

which characterizes regular graphs by the condition $H(\boldsymbol{A}) = \boldsymbol{J}$, the all-1 matrix [17]. Note that (3) implies that the leading coefficient ω_d of H (and also of p_d) is $\omega_d = n/\pi_0$.

From the predistance polynomials, we define the so-called *preidempotent polynomials* q_j, $j = 0, 1, \ldots, d$, by:

$$q_j(\lambda_i) = \frac{m_j}{n_i}p_i(\lambda_j), \qquad i = 0, 1, \ldots, d,$$

and they are orthogonal with respect to the scalar product

$$\langle f, g\rangle_\blacktriangle = \frac{1}{n}\mathrm{tr}\,(f\{\boldsymbol{A}\}g\{\boldsymbol{A}\}) = \frac{1}{n}\sum_{i=0}^{d} n_i f(\lambda_i)g(\lambda_i), \tag{4}$$

where $f\{\boldsymbol{A}\} = \frac{1}{\sqrt{n}}\sum_{i=0}^{d} f(\lambda_i)p_i(\boldsymbol{A})$. From this, it can be proved that the preidempotent polynomials satisfy $\mathrm{sch}\, q_j = j$ (that is, the number of sign-changes of the sequence $q_j(\lambda_0)$, $q_j(\lambda_1)$, …, $q_j(\lambda_d)$ equals j) and $\langle q_i, q_j\rangle_\blacktriangle = \delta_{ij}q_i(\lambda_0)$ for any $i, j = 0, 1, \ldots d$ (see [10, 16]). Moreover, the values of each preidempotent polynomial q_j at the points $\lambda_0, \lambda_1, \ldots, \lambda_d$ satisfy the three-term recurrence

$$\lambda_i q_j(\lambda_i) = \gamma_i q_j(\lambda_{i-1}) + \alpha_i q_j(\lambda_i) + \beta_i q_j(\lambda_{i+1}), \qquad i = 0, 1, \ldots, d, \tag{5}$$

started with $q_j(\lambda_0) = m_j$. Note that, since $q_j(\lambda_0) = m_j$, the duality between the two scalar products (1) and (4) and their associated polynomials is made apparent by writing

$$\langle p_i, p_j \rangle_\triangle = \frac{1}{n}\sum_{l=0}^{d} m_l p_i(\lambda_l) p_j(\lambda_l) = \delta_{ij} n_i, \qquad i,j = 0,1,\ldots,d, \quad (6)$$

$$\langle q_i, q_j \rangle_\blacktriangle = \frac{1}{n}\sum_{l=0}^{d} n_l q_i(\lambda_l) q_j(\lambda_l) = \delta_{ij} m_i, \qquad i,j = 0,1,\ldots,d. \quad (7)$$

1.3 Vector spaces, algebras and bases

Let G be a graph with diameter D, adjacency matrix $\boldsymbol{A}$ and $d+1$ distinct eigenvalues. We consider the vector spaces $\mathcal{A} = \mathbb{R}_d[\boldsymbol{A}] = \operatorname{span}\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^d\}$ and $\mathcal{D} = \operatorname{span}\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_D\}$, with dimensions $d+1$ and $D+1$, respectively. Then, $\mathcal{A}$ is an algebra with the ordinary product of matrices, known as the *adjacency* or *Bose-Mesner algebra*, with possible bases $A_a = \{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^d\}$, $A_p = \{p_0(\boldsymbol{A}), p_1(\boldsymbol{A}), p_2(\boldsymbol{A}), \ldots, p_d(\boldsymbol{A})\}$, and $A_\lambda = \{\lambda_0^*(\boldsymbol{A}), \lambda_1^*(\boldsymbol{A}), \ldots, \lambda_d^*(\boldsymbol{A})\} = \{\boldsymbol{E}_0, \boldsymbol{E}_1, \ldots, \boldsymbol{E}_d\}$. Notice that A_p and A_λ are orthogonal bases. From the properties of the idempotents, the change-of-basis matrix $\boldsymbol{P}$ from A_λ to A_p has entries $P_{ij} = p_j(\lambda_i)$ and inverse $\boldsymbol{P}^{-1} = \frac{1}{n}\boldsymbol{Q}$, where $Q_{ji} = q_i(\lambda_j)$. This gives the respective transformations

$$p_i(\boldsymbol{A}) = \textstyle\sum_{j=0}^{d} p_i(\lambda_j)\boldsymbol{E}_j, \qquad i = 0,1,\ldots,d, \quad (8)$$

$$\boldsymbol{E}_j = \tfrac{1}{n}\textstyle\sum_{i=0}^{d} q_j(\lambda_i) p_i(\boldsymbol{A}), \qquad j = 0,1,\ldots,d. \quad (9)$$

Besides, since $\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^D$ are linearly independent, we have that $\dim \mathcal{A} = d+1 \geq D+1$ and, therefore, we always have $D \leq d$ [2]. It is natural to study the case when equality is attained, $D = d$. In this case, we say that the graph G has *spectrally maximum* diameter. Moreover, $\mathcal{D}$ forms an algebra with the entrywise or Hadamard product of matrices, defined by $(\boldsymbol{X} \circ \boldsymbol{Y})_{uv} = \boldsymbol{X}_{uv}\boldsymbol{Y}_{uv}$. We call $\mathcal{D}$ the *distance* $\circ$*-algebra*, which has orthogonal basis $D_\lambda = \{\lambda_0^*[\boldsymbol{A}], \lambda_1^*[\boldsymbol{A}], \lambda_2^*[\boldsymbol{A}], \ldots, \lambda_d^*[\boldsymbol{A}]\} = \frac{1}{\sqrt{n}}\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_d\}$, where $f[\boldsymbol{A}] = \frac{1}{\sqrt{n}}\sum_{i=0}^{d} f(\lambda_i)\boldsymbol{A}_i$.

From now on, we work with the vector space $\mathcal{T} = \mathcal{A} + \mathcal{D}$, and relate the distance-i matrices $\boldsymbol{A}_i \in \mathcal{D}$ with the matrices $p_i(\boldsymbol{A}) \in \mathcal{A}$. Note that $\boldsymbol{I}$, $\boldsymbol{A}$, and $\boldsymbol{J}$ are matrices in $\mathcal{A} \cap \mathcal{D}$ since $\boldsymbol{J} = H(\boldsymbol{A}) \in \mathcal{A}$. Thus, $\dim(\mathcal{A} \cap \mathcal{D}) \geq 3$, if G is not a complete graph (in this exceptional case, $\boldsymbol{J} = \boldsymbol{I} + \boldsymbol{A}$). Recall that $\mathcal{A} = \mathcal{D}$ if and only if G is distance-regular (see [2, 4]), which is therefore equivalent to $\dim(\mathcal{A} \cap \mathcal{D}) = d + 1$.

Note that for any pair of (symmetric) matrices $\boldsymbol{R}, \boldsymbol{S} \in \mathcal{T}$, we have

$$\operatorname{tr}(\boldsymbol{RS}) = \sum_{u\in V} (\boldsymbol{RS})_{uu} = \sum_{u\in V}\sum_{v\in V} \boldsymbol{R}_{uv}\boldsymbol{S}_{vu} = \operatorname{sum}(\boldsymbol{R} \circ \boldsymbol{S}).$$

Thus, we can define a scalar product in $\mathcal{T}$ in two equivalent forms:

$$\langle \boldsymbol{R}, \boldsymbol{S}\rangle = \frac{1}{n}\operatorname{tr}(\boldsymbol{RS}) = \frac{1}{n}\operatorname{sum}(\boldsymbol{R} \circ \boldsymbol{S}). \tag{10}$$

Observe that the factor $1/n$ assures that $\|\boldsymbol{I}\|^2 = 1$, whereas $\|\boldsymbol{J}\|^2 = n$. Note also that $\|\boldsymbol{A}_i\|^2 = \overline{\delta}_i$ (the *average degree* of G_i), and $\|\boldsymbol{E}_j\|^2 = \frac{m_j}{n} = \overline{m}_j$ (the *average multiplicity* of λ_j). According to (1) and (4), this scalar product of matrices satisfies $\langle f(\boldsymbol{A}), g(\boldsymbol{A})\rangle = \langle f, g\rangle_\triangle$ and $\langle f\{\boldsymbol{A}\}, g\{\boldsymbol{A}\}\rangle = \langle f, g\rangle_\blacktriangle$ for $\boldsymbol{A} \in \mathcal{A}$. Moreover, for $\boldsymbol{A} \in \mathcal{D}$, we have that $\langle f[\boldsymbol{A}], g[\boldsymbol{A}]\rangle = \langle f, g\rangle_\bullet = \frac{1}{n}\sum_{l=0}^{d} \overline{\delta}_l f(\lambda_l)g(\lambda_l)$.

1.4 Preintersection numbers

Given any triple of integers $i, j, k = 0, 1, \ldots, d$, the *preintersection number* ξ_{ij}^k is the Fourier coefficient of p_ip_j in terms of p_k, that is:

$$\xi_{ij}^k = \frac{\langle p_ip_j, p_k\rangle_\triangle}{\|p_k\|_\triangle^2} = \frac{1}{nn_k}\sum_{l=0}^{d} m_l p_i(\lambda_l)p_j(\lambda_l)p_k(\lambda_l). \tag{11}$$

With this notation, notice that the recurrence coefficients in (2) correspond to the preintersection numbers as follows: $\alpha_i = \xi_{1,i}^i$, $\beta_i = \xi_{1,i+1}^i$, and $\gamma_i = \xi_{1,i-1}^i$.

As expected, when G is distance-regular (which implies $D = d$ and $n_i = \overline{\delta}_i$ for $i = 0, 1, \ldots, d$), the predistance polynomials and the preintersection numbers become the *distance polynomials* and the *intersection numbers* p_{ij}^k satisfying:

$$\boldsymbol{A}_i = p_i(\boldsymbol{A}), \qquad \boldsymbol{A}_i\boldsymbol{A}_j = \sum_{k=0}^{d} p_{ij}^k \boldsymbol{A}_k, \quad i, j = 0, 1, \ldots, d.$$

We also define the *average intersection numbers* $\overline{p}_{ij}^k$ as the average of the numbers $|\Gamma_i(u) \cap \Gamma_j(v)|$ over all vertices at distance $\partial(u, v) = k$. Notice that

$$\overline{p}_{ij}^{k} = \frac{\langle \boldsymbol{A}_i\boldsymbol{A}_j, \boldsymbol{A}_k\rangle}{\|\boldsymbol{A}_k\|^2} = \frac{1}{n\overline{\delta}_k}\mathrm{sum}\,(\boldsymbol{A}_i\boldsymbol{A}_j \circ \boldsymbol{A}_k), \tag{12}$$

to be compared with the expression (11) for the preintersection numbers. In particular, let $\overline{a}_i = \overline{p}_{1,i}^{i}$, $\overline{b}_i = \overline{p}_{1,i+1}^{i}$, and $\overline{c}_i = \overline{p}_{1,i-1}^{i}$.

For an arbitrary graph, we say that the intersection number p_{ij}^k, with $i,j,k = 0,1,\dots,D$, is *well defined* if $p_{ij}^k(u,v) = (\boldsymbol{A}_i\boldsymbol{A}_j)_{uv} = |\Gamma_i(u)\cap\Gamma_j(v)|$ is a constant over all vertices u,v at distance $\partial(u,v)=k$, and let $a_i = p_{1,i}^i$, $b_i = p_{1,i+1}^i$, and $c_i = p_{1,i-1}^i$. The matrices

$$\boldsymbol{R}_{ij} = \boldsymbol{A}_i\boldsymbol{A}_j - \sum_{k=0}^{D}\overline{p}_{ij}^{k}\boldsymbol{A}_k, \tag{13}$$

defined for $i,j = 0,1,\dots,D$, allow us to give the following characterization.

Proposition 1 *The intersection number p_{ij}^k is well defined if and only if $\boldsymbol{R}_{ij}\circ\boldsymbol{A}_k = \boldsymbol{O}$, and then $p_{ij}^k = \overline{p}_{ij}^k$.*

Proof: Note first, from (13), the orthogonal decomposition of the matrix $\boldsymbol{A}_i\boldsymbol{A}_j$ with respect to the subspace $\mathcal{D}\subset\mathcal{T}$ is:

$$\boldsymbol{A}_i\boldsymbol{A}_j = \sum_{k=0}^{D}\frac{\langle \boldsymbol{A}_i\boldsymbol{A}_j, \boldsymbol{A}_k\rangle}{\|\boldsymbol{A}_k\|^2}\boldsymbol{A}_k + \boldsymbol{R}_{ij} = \sum_{k=0}^{D}\overline{p}_{ij}^{k}\boldsymbol{A}_k + \boldsymbol{R}_{ij}, \tag{14}$$

where $\boldsymbol{R}_{ij}\in\mathcal{D}^{\perp}$ and $i,j=0,1,\dots,D$. From this, assume first that $\boldsymbol{R}_{ij}\circ\boldsymbol{A}_k = \boldsymbol{O}$. Then, for every pair of vertices u,v at distance $\partial(u,v)=k$,

$$|\Gamma_i(u)\cap\Gamma_j(v)| = (\boldsymbol{A}_i\boldsymbol{A}_j)_{uv} = (\boldsymbol{A}_i\boldsymbol{A}_j\circ\boldsymbol{A}_k)_{uv} = \overline{p}_{ij}^{k}(\boldsymbol{A}_k)_{uv} = \overline{p}_{ij}^{k}\,,$$

and p_{ij}^k is well defined and coincides with $\overline{p}_{ij}^k$. Conversely, if p_{ij}^k is well defined, then $\overline{p}_{ij}^k = p_{ij}^k$ and $\boldsymbol{A}_i\boldsymbol{A}_j\circ\boldsymbol{A}_k = p_{ij}^k\boldsymbol{A}_k$. Thus, $\boldsymbol{R}_{ij}\circ\boldsymbol{A}_k = \boldsymbol{O}$, as claimed. □

Notice that, in particular, (14) yields

$$\boldsymbol{A}\boldsymbol{A}_i = \overline{b}_{i-1}\boldsymbol{A}_{i-1} + \overline{a}_i\boldsymbol{A}_i + \overline{c}_{i+1}\boldsymbol{A}_{i+1} + \boldsymbol{R}_{1i}. \tag{15}$$

Thus, as a by-product, adding up the equalities in (15) for $i=0,1,\dots,D$, and taking into account that $\overline{a}_i+\overline{b}_i+\overline{c}_i = \lambda_0$, we have that $\sum_{i=0}^{D}\boldsymbol{R}_{1i} = \boldsymbol{O}$.

2 Two dual approaches to almost distance-regularity

In the context of spectrally maximum diameter $D = d$, two known characterizations of distance-regularity involving the distance matrices $\boldsymbol{A}_i$, $0 \leq i \leq D$, and the idempotents $\boldsymbol{E}_j$, $0 \leq j \leq d$, are the following:

(**C1**) G is distance-regular if and only if there exist constants p_{ji} such that

$$\boldsymbol{A}_i \boldsymbol{E}_j = p_{ji} \boldsymbol{E}_j \tag{16}$$

for every $0 \leq i \leq D$ and $0 \leq j \leq d$.

(**C2**) G is distance-regular if and only if there exist constants q_{ij} such that

$$\boldsymbol{E}_j \circ \boldsymbol{A}_i = q_{ij} \boldsymbol{A}_i \tag{17}$$

for every $0 \leq j \leq d$ and $0 \leq i \leq D$.

Here it is worth noting that, for general graphs with $D \leq d$, the conditions (16) are a characterization of the so-called *distance-polynomial graphs*, introduced in [19] (see also [3, 5]). This is equivalent to $\mathcal{D} \subset \mathcal{A}$ (but not necessarily $\mathcal{D} = \mathcal{A}$), that is, every distance matrix $\boldsymbol{A}_i$ is a polynomial in $\boldsymbol{A}$. On the other hand, the conditions (17) are equivalent to $\mathcal{A} \subset \mathcal{D}$ and, hence, to $\mathcal{A} = \mathcal{D}$ (which implies $d = D$) as $\dim \mathcal{A} \geq \dim \mathcal{D}$. Then, in this general setting, (**C2**) is 'stronger' than (**C1**) as a characterization of distance-regularity.

From now on, we limit ourselves to the 'non-degenerate' case $D = d$ and, consequently, we will use indistinctly both symbols depending on what we are referring to. In this context, notice that in (16) and (17), and using standard notation in the theory of distance-regular graphs and association schemes (see, for instance, [4, 16]), we have:

$$p_{ji} = P_{ji} = p_i(\lambda_j) \quad \text{and} \quad q_{ij} = \frac{1}{n} Q_{ij} = \frac{1}{n} q_j(\lambda_i),$$

where $0 \leq i, j \leq d$. (In [15, 7], q_{ij} is also denoted by $m_{ij} = m_{uv}(\lambda_j)$ since it is referred to as the *uv-crossed local multiplicity* of λ_j for every pair of vertices u, v at distance $\partial(u, v) = i$.)

The above suggests the following two definitions of almost distance-regularity:

(**D1**) For a given i, $0 \le i \le D$, a graph G is *i-punctually distance-regular* when

$$\boldsymbol{A}_i\boldsymbol{E}_j = p_{ji}\boldsymbol{E}_j, \qquad j = 0, 1, \dots, d, \tag{18}$$

and G is *m-partially distance-regular* when it is i-punctually distance-regular for all $i \le m$.

(**D2**) For a given j, $0 \le j \le d$, a graph G is *j-punctually eigenspace distance-regular* when

$$\boldsymbol{E}_j \circ \boldsymbol{A}_i = q_{ij}\boldsymbol{A}_i \qquad , \qquad i = 0, 1, \dots, D, \tag{19}$$

and G is *m-partially eigenspace distance-regular* when it is j-punctually eigenspace distance-regular for all $j \le m$.

The following theorem summarizes some of the known characterizations of distance-regularity in terms of the above concepts.

Theorem 2 [7, 10, 11, 12, 14] *A graph G with $d+1$ distinct eigenvalues and diameter $D = d$ is distance-regular if and only if any of the following statements is satisfied:*

($a1$) *G is $(d-1)$-partially distance-regular.*

($a2$) *G is $(d-1)$-partially eigenspace distance-regular.*

($b1$) *G is d-punctually distance-regular.*

($b2$) *G is j-punctually eigenspace distance-regular for $j = 1, d$.*

Here it is worth emphasizing the duality between characterizations ($b1$) and ($b2$) which can be stated as follows: A graph G as above is distance-regular if and only if any of the two following conditions is satisfied:

($b1$) $\boldsymbol{A}_0(=\boldsymbol{I}), \boldsymbol{A}_1(=\boldsymbol{A}), \boldsymbol{A}_D \in \mathcal{A}$;

($b2$) $\boldsymbol{E}_0(=\frac{1}{n}\boldsymbol{J}), \boldsymbol{E}_1, \boldsymbol{E}_d \in \mathcal{D}$.

3 Almost distance-regularity (D1)

In this section we study some characterizations of almost distance-regularity, according to definition (**D1**), and how they give, in the extreme cases, new versions of the spectral excess theorem characterizing distance-regular graphs. First, we remark that some basic characterizations of punctual distance-regularity, in terms of the distance matrices and the idempotents, were given in [7].

Proposition 3 [7] *Let $D = d$. Then, G is i-punctually distance-regular if and only if any of the following conditions is satisfied:*

(*a*) $\boldsymbol{A}_i \in \mathcal{A}$,

(*b*) $p_i(\boldsymbol{A}) \in \mathcal{D}$,

(*c*) $\boldsymbol{A}_i = p_i(\boldsymbol{A})$.

In order to derive some new characterizations, let $P_i(u)$ denote the number of shortest paths from a vertex u to the vertices in $\Gamma_i(u)$. As it is well known, in a distance-regular graph $P_i(u)$ does not depend on u and can be computed from the recurrence coefficients and the number $|\Gamma_i(u)| = p_i(\lambda_0) = n_i$ as:

$$P_i(u) = b_0 b_1 \cdots b_{i-1} = n_i c_i c_{i-1} \cdots c_1.$$

For any (regular) graph G, we consider the average value of $P_i(u)$ over all vertices of G:

$$\overline{P}_i = \frac{1}{n}\sum_{u\in V} P_i(u) = \frac{1}{n}\operatorname{sum}(\boldsymbol{A}^i \circ \boldsymbol{A}_i) = \langle \boldsymbol{A}^i, \boldsymbol{A}_i\rangle, \tag{20}$$

which plays a role in the following result.

Proposition 4 *For any graph G with predistance polynomials p_i having leading coefficients ω_i and recurrence coefficients $\gamma_i, \alpha_i, \beta_i$, $i = 0, 1, \ldots, d$, we have*

$$\overline{P}_i \le \frac{1}{\omega_i}\sqrt{p_i(\lambda_0)\overline{\delta}_i} = \sqrt{\beta_0\beta_1\cdots\beta_{i-1}\overline{\delta}_i\gamma_i\gamma_{i-1}\cdots\gamma_1}, \tag{21}$$

and equality occurs if and only if G is i-punctually distance-regular.

Proof: From (20) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\omega_i\overline{P}_i &= \omega_i\langle \boldsymbol{A}^i, \boldsymbol{A}_i\rangle = \langle p_i(\boldsymbol{A}), \boldsymbol{A}_i\rangle \le \|p_i(\boldsymbol{A})\|\|\boldsymbol{A}_i\| = \sqrt{p_i(\lambda_0)\overline{\delta}_i} \\ &= \sqrt{\frac{\beta_0\beta_1\cdots\beta_{i-1}}{\gamma_1\gamma_2\cdots\gamma_i}\overline{\delta}_i}\,,\end{aligned}$$

and (21) follows since $\omega_i = (\gamma_1\gamma_2\cdots\gamma_i)^{-1}$. Moreover, equality occurs if and only if the matrices $p_i(\boldsymbol{A})$ and $\boldsymbol{A}_i$ are proportional, which is equivalent to G being i-punctually distance-regular by Proposition 3. $\square$

As a corollary, we can prove the following result:

Proposition 5 *For one $i = 0, 1 \dots, d$, let G be a graph having $\overline{a}_i^{(i)}$ as the average number of shortest paths among all pairs of vertices at distance i, and predistance polynomial p_i with leading coefficient ω_i. Then, G is i-punctually distance-regular if and only if*

$$\omega_i\overline{a}_i^{(i)} = 1 \quad \textit{and} \quad \overline{\delta}_i = p_i(\lambda_0). \tag{22}$$

Proof: Just notice that $\overline{a}_i^{(i)} = \frac{1}{n\overline{\delta}_i}\mathrm{sum}\,(\boldsymbol{A}^i \circ \boldsymbol{A}_i) = \frac{\overline{P}_i}{\overline{\delta}_i}$. Hence, from Proposition 4, we have that

$$\omega_i\overline{a}_i^{(i)} \le \sqrt{p_i(\lambda_0)/\overline{\delta}_i},$$

with equality if and only if G is i-punctually distance-regular. Thus, if the conditions (22) hold, G satisfies the claimed property. Conversely, if G is i-punctually distance-regular, both equalities in (22) are simple consequences of $p_i(\boldsymbol{A}) = \boldsymbol{A}_i$. Indeed, the first one comes from considering the uv-entries, with $\partial(u,v) = i$, in the above matrix equation, whereas the second one is obtained by taking square norms. $\square$

In particular, when $i = d$, the first condition in (22) always holds since

$$\overline{a}_d^{(d)} = \frac{1}{\overline{\delta}_d}\langle \boldsymbol{A}^d, \boldsymbol{A}_d\rangle = \frac{1}{\overline{\delta}_d\omega_d}\langle H(\boldsymbol{A}), \boldsymbol{A}_d\rangle = \frac{1}{\overline{\delta}_d\omega_d}\langle \boldsymbol{J}, \boldsymbol{A}_d\rangle = \frac{1}{\overline{\delta}_d\omega_d}\|\boldsymbol{A}_d\|^2 = \frac{1}{\omega_d}.$$

Then, in this case, $\overline{\delta}_d = p_d(\lambda_0)$ suffices for having (d-punctually) distance-regularity (see Theorem 2(b1)).

Now, let us consider the more global concept of partial distance-regularity. In this case, we also have the following new result where, for a given $0 \le i \le d$, $s_i = \sum_{j=0}^{i} p_j$, $t_i = H - s_{i-1} = \sum_{j=i}^{d} p_j$, $\boldsymbol{S}_i = \sum_{j=0}^{i}\boldsymbol{A}_i$, and $\boldsymbol{T}_i = \boldsymbol{J} - \boldsymbol{S}_{i-1} = \sum_{j=i}^{d}\boldsymbol{A}_i$.

Proposition 6 *A graph G is m-partially distance-regular if and only any of the following conditions holds:*

(*a*) *G is i-punctually distance-regular for $i = m, m-1, \ldots, \max\{2, 2m-d\}$.*

(*b*) *G is m-punctually distance-regular and $t_{m+1}(\boldsymbol{A}) \circ \boldsymbol{S}_m = \boldsymbol{O}$.*

(*c*) *$s_i(\boldsymbol{A}) = \boldsymbol{S}_i$ for $i = m, m-1$.*

Proof: In all cases, the necessity is clear since $p_i(\boldsymbol{A}) = \boldsymbol{A}_i$ for every $0 \le i \le m$ (for (b), note that $t_{m+1}(\boldsymbol{A}) = \boldsymbol{J} - s_m(\boldsymbol{A})$). Then, let us prove sufficiency. The result in (a) is basically Proposition 3.7 in [7]. In order to prove (b), we show by (backward) induction that $p_i(\boldsymbol{A}) = \boldsymbol{A}_i$ and $t_{i+1}(\boldsymbol{A}) \circ \boldsymbol{S}_i = \boldsymbol{O}$ for $i = m, m-1, \ldots, 0$. By assumption, these equations are valid for $i = m$. Suppose now that $p_i(\boldsymbol{A}) = \boldsymbol{A}_i$ and $t_{i+1}(\boldsymbol{A}) \circ \boldsymbol{S}_i = \boldsymbol{O}$ for some $i > 0$. Then $t_i(\boldsymbol{A}) \circ \boldsymbol{S}_i = \boldsymbol{A}_i$ and, multiplying both terms by $\boldsymbol{S}_{i-1}$ (Hadamard product), we get $t_i(\boldsymbol{A}) \circ \boldsymbol{S}_{i-1} = \boldsymbol{O}$. So, what remains is to show that $p_{i-1}(\boldsymbol{A}) = \boldsymbol{A}_{i-1}$. To this end, let us consider the following three cases:

(i) For $\partial(u,v) > i-1$, we have $(p_{i-1}(\boldsymbol{A}))_{uv} = 0$.

(ii) For $\partial(u,v) = i-1$, we have $(t_{i+1}(\boldsymbol{A}))_{uv} = 0$, so $(p_{i-1}(\boldsymbol{A}))_{uv} = (s_{i-1}(\boldsymbol{A}))_{uv} = (s_{i-1}(\boldsymbol{A}))_{uv} + (\boldsymbol{A}_i)_{uv} = (s_i(\boldsymbol{A}))_{uv} = 1 - (t_{i+1}(\boldsymbol{A}))_{uv} = 1$.

(iii) For $\partial(u,v) < i-1$, we use the recurrence (2) to write:

$$\begin{aligned} xt_i = \sum_{j=i}^{d} xp_j &= \sum_{j=i}^{d} (\beta_{j-1}p_{j-1} + \alpha_j p_j + \gamma_{j+1}p_{j+1}) \\ &= \beta_{i-1}p_{i-1} - \gamma_i p_i + \sum_{j=i}^{d} (\alpha_j + \beta_j + \gamma_j) p_j \\ &= \beta_{i-1}p_{i-1} - \gamma_i p_i + \delta t_i, \end{aligned}$$

which gives

$$\boldsymbol{A} t_i(\boldsymbol{A}) = \beta_{i-1} p_{i-1}(\boldsymbol{A}) - \gamma_i \boldsymbol{A}_i + \delta t_i(\boldsymbol{A}).$$

Then, since $(t_i(\boldsymbol{A}))_{uv} = (\boldsymbol{A}_i)_{uv} = 0$ and $\beta_{i-1} \neq 0$, we get

$$(p_{i-1}(\boldsymbol{A}))_{uv} = \frac{1}{\beta_{i-1}} (\boldsymbol{A} t_i(\boldsymbol{A}))_{uv} = \frac{1}{\beta_{i-1}} \sum_{w \in \Gamma(u)} (t_i(\boldsymbol{A}))_{wv} = 0,$$

since $\partial(v,w) \le \partial(v,u) + \partial(u,w) \le i-1$ for the relevant w.

From (i), (ii), and (iii), we have that $p_{i-1}(\boldsymbol{A}) = \boldsymbol{A}_{i-1}$, so by induction G is m-partially distance-regular, and the sufficiency of (b) is proven. Finally, the sufficiency of (c) follows from that of (b) because $s_i(\boldsymbol{A}) = \boldsymbol{S}_i$ for $i = m, m-1$ implies that $p_m(\boldsymbol{A}) = (s_m - s_{m-1})(\boldsymbol{A}) = \boldsymbol{S}_m - \boldsymbol{S}_{m-1} = \boldsymbol{A}_m$ and $t_{m+1}(\boldsymbol{A}) \circ \boldsymbol{S}_m = (\boldsymbol{J} - s_m(\boldsymbol{A})) \circ \boldsymbol{S}_m = (\boldsymbol{J} - \boldsymbol{S}_m) \circ \boldsymbol{S}_m = \boldsymbol{O}$. □

In particular, notice that any of the above conditions with $m = d$ implies the known fact that a graph is distance-regular if and only if it is d-punctually distance-regular, that is, $p_d(\boldsymbol{A}) = \boldsymbol{A}_d$ (see Theorems 2(b1) and 3(c)).

As is well known, (punctually) distance-regular graphs are not characterized by the spectrum when $d \ge 3$. However, characterizations of such graphs are possible if some more additional information, such as the average degree $\overline{\delta}_d$ of Γ_d, is available. In this case, we speak of 'quasi-spectral' characterizations. For instance, in our context of having spectrally maximum diameter, Proposition 4.1 in [7] reads as follows:

Proposition 7 [7] *Let $i \le D = d$. Then,*

$$\overline{\delta}_i \le \frac{1}{n} \left(\sum_{j=0}^{d} \frac{\overline{m}_{ij}^2}{m_j} \right)^{-1} \tag{23}$$

with equality if and only if G is i-punctually distance-regular.

In this characterization we have used the *average crossed local multiplicities*, which are

$$\overline{m}_{ij} = \frac{1}{n\overline{\delta}_i} \sum_{\partial(u,v)=i} m_{uv}(\lambda_j) = \frac{\langle \boldsymbol{E}_j, \boldsymbol{A}_i \rangle}{\|\boldsymbol{A}_i\|^2}, \tag{24}$$

and where $m_{uv}(\lambda_j) = (\boldsymbol{E}_j)_{uv}$ are the *crossed local multiplicities*. Proposition 7, together with the above mentioned characterization of distance-regularity, yields the following theorem.

Theorem 8 [7] *Let $D = d$. Then, G is distance-regular if and only if*

$$\overline{\delta}_d = \frac{1}{n} \left(\sum_{j=0}^{d} \frac{\overline{m}_{dj}^2}{m_j} \right)^{-1}. \tag{25}$$

In our case of spectrally maximum diameter, $m_{uv}(\lambda_j)$ is a constant over all pairs of vertices u, v at distance d since

$$m_{uv}(\lambda_j) = (\lambda_j^*(\boldsymbol{A}))_{uv} = \frac{(-1)^j}{\pi_j}(\boldsymbol{A}^d)_{uv} = \frac{(-1)^j}{\pi_j}\frac{\pi_0}{n}(H(\boldsymbol{A}))_{uv} = \frac{(-1)^j}{n}\frac{\pi_0}{\pi_j},$$

which therefore also equals $\overline{m}_{dj}$, and Theorem 8 corresponds, in fact, to the spectral excess theorem.

Given some vertex u and an integer m, $0 \le m \le \mathrm{ecc}(u)$, we denote by $N_m(u)$ the *m-neighborhood* of u, which is the set of vertices that are at distance at most m from u. The following theorem was proved in [11] by using results from [12]:

Theorem 9 [11] *Let G be a graph with predistance polynomials p_i, $0 \le i \le d$, and let $s_m = \sum_{i=0}^m p_i$. Then $s_m(\lambda_0)$ is upper bounded by the harmonic average of the numbers $|N_m(u)|$, that is,*

$$s_m(\lambda_0) \le \frac{n}{\sum_{u\in V} |N_m(u)|^{-1}},$$

and equality is attained if and only if $s_m(\boldsymbol{A}) = \boldsymbol{S}_m$.

The following theorem is a direct consequence of Proposition 6 and Theorem 9 and can be seen as a generalization of the spectral excess theorem.

Theorem 10 *A graph G is m-partially distance-regular if and only if*

$$s_i(\lambda_0) = \frac{n}{\sum_{u\in V} |N_i(u)|^{-1}}$$

for $i = m-1, m$.

4 Almost eigenspace distance-regularity (D2)

Following the duality between definitions (**D1**) and (**D2**), it seems natural to conjecture the dual of Proposition 3: a graph G is *j-punctually eigenspace distance-regular* if and only if any of the following conditions is satisfied:

(*a*) $\boldsymbol{E}_j \in \mathcal{D}$,

(*b*) $q_j[\boldsymbol{A}] \in \mathcal{A}$,

(c) $\boldsymbol{E}_j = \frac{1}{\sqrt{n}} q_j[\boldsymbol{A}]$.

However, although (a) is clearly equivalent to definition (**D2**) and (c) $\Rightarrow$ (a), (b), until now we have not been able to prove any of the other equivalences and we leave them as conjectures. In this framework, and in view of the recurrence (5), a partial result could be the following:

Proposition 11 *A graph G is j-punctually eigenspace distance-regular if and only if $\boldsymbol{E}_j = \sum_{i=0}^{d} q_{ij}\boldsymbol{A}_i$ where the constants q_{ij} satisfy the recurrence*

$$\lambda_j q_{ij} = \overline{c}_i q_{i-1,j} + \overline{a}_i q_{ij} + \overline{b}_i q_{i+1,j}, \qquad i = 0, 1, \ldots, d, \tag{26}$$

started with $q_{0j} = m_j/n$.

Proof: We only need to prove necessity. If G is j-punctually eigenspace distance-regular, then $\boldsymbol{E}_j \circ \boldsymbol{A}_k = q_{kj}\boldsymbol{A}_k$ holds for some constants q_{kj}, $k = 0, 1, \ldots, d$, and then

$$\boldsymbol{E}_j = \boldsymbol{E}_j \circ \boldsymbol{J} = \boldsymbol{E}_j \circ \sum_{k=0}^{d} \boldsymbol{A}_k = \sum_{k=0}^{d} q_{kj}\boldsymbol{A}_k. \tag{27}$$

Then, using this and (15) we get

$$\begin{aligned}
\lambda_j \boldsymbol{E}_j &= \boldsymbol{A}\boldsymbol{E}_j = \boldsymbol{A}\sum_{k=0}^{d} q_{kj}\boldsymbol{A}_k \\
&= \sum_{k=0}^{d} q_{kj}(\overline{b}_{k-1}\boldsymbol{A}_{k-1} + \overline{a}_k\boldsymbol{A}_k + \overline{c}_{k+1}\boldsymbol{A}_{k+1}) + \sum_{k=0}^{d} q_{kj}\boldsymbol{R}_{1k} \\
&= \sum_{k=0}^{d} (\overline{c}_k q_{k-1,j} + \overline{a}_k q_{kj} + \overline{b}_k q_{k+1,j})\boldsymbol{A}_k + \sum_{k=0}^{d} q_{kj}\boldsymbol{R}_{1k},
\end{aligned}$$

where $\boldsymbol{R}_{1k} \in \mathcal{D}^{\perp}$. By taking the scalar product with $\boldsymbol{A}_i$, $i = 0, 1, \ldots, d$, we get

$$\lambda_j\langle \boldsymbol{E}_j, \boldsymbol{A}_i\rangle = \lambda_j \sum_{k=0}^{d} q_{kj}\langle \boldsymbol{A}_k, \boldsymbol{A}_i\rangle = \lambda_j q_{ij}\overline{\delta}_i = (\overline{c}_i q_{i-1,j} + \overline{a}_i q_{ij} + \overline{b}_i q_{i+1,j})\overline{\delta}_i,$$

proving (26). Finally, taking traces in (27), we have

$$m_j = \operatorname{tr}\boldsymbol{E}_j = \textstyle\sum_{k=0}^{d} q_{kj}\operatorname{tr}\boldsymbol{A}_k = nq_{0j},$$

and q_{0j} is as required. $\square$

Notice that the recurrences (26) and (5) are the same, provided that $\alpha_i = \overline{a}_i$, $\beta_i = \overline{b}_i$ and $\gamma_i = \overline{c}_i$. Then, in this case, we would have $q_{ij} = \frac{1}{n} q_j(\lambda_i)$ and, from (27), $(a) \Rightarrow (c)$, and hence (**D2**), (a) and (c) would be equivalent.

The following result can be seen as the dual of Proposition 7.

Proposition 12 *Let $j \le d$. Then,*

$$m_j \ge n \sum_{i=0}^{D} \overline{\delta}_i \overline{m}_{ij}^2 \tag{28}$$

with equality if and only if G is j-punctually eigenspace distance-regular.

Proof: From (24), we find that the orthogonal projection of $\boldsymbol{E}_j$ on $\mathcal{D}$ is $\widehat{\boldsymbol{E}_j} = \sum_{i=0}^{D} \overline{m}_{ij} \boldsymbol{A}_i$. Now, the inequality (28) comes from $\|\widehat{\boldsymbol{E}_j}\|^2 \le \|\boldsymbol{E}_j\|^2 = \frac{1}{n} m_j$ and, in case of equality, definition (**C2**) applies with $q_{ij} = \overline{m}_{ij}$. $\square$

To emphasize the duality between this result and Proposition 7, notice that $\overline{m}_j = \frac{m_j}{n}$ is the average of the local multiplicities $m_{0j} = m_u(\lambda_j)$ over the n vertices of the graph. Then, using this, (23) and (28) become

$$\frac{1}{\overline{\overline{\delta}}_i} \ge \sum_{j=0}^{d} \frac{\overline{m}_{ij}^2}{\overline{m}_j} \qquad \text{and} \qquad \overline{m}_j \ge \sum_{i=0}^{D} \overline{\delta}_i \overline{m}_{ij}^2. \tag{29}$$

By using Theorem 2$(b2)$ and Proposition 12, we have the following characterization of distance-regularity.

Theorem 13 *Let $D = d$. Then, G is distance-regular if and only if*

$$\overline{m}_1 = \sum_{i=0}^{D} \overline{\delta}_i \overline{m}_{i1}^2 \quad \textit{and} \quad \overline{m}_d = \sum_{i=0}^{D} \overline{\delta}_i \overline{m}_{id}^2. \tag{30}$$

This result can be seen as the dual of the spectral excess theorem (Theorem 8) with the condition written as

$$\frac{1}{\overline{\overline{\delta}}_d} = \sum_{j=0}^{d} \frac{\overline{m}_{dj}^2}{\overline{m}_j}.$$

References

[1] N. Biggs. *Algebraic Graph Theory.* Cambridge University Press, Cambridge, 1974, second edition, 1993.

[2] R.A. Beezer. Distance polynomial graphs. In *Proceedings of the Sixth Caribbean Conference on Combinatorics and Computing, Trinidad*, 51–73, 1991.

[3] A.E. Brouwer, A.M. Cohen and A. Neumaier. *Distance-Regular Graphs.* Springer-Verlag, Berlin-New York, 1989.

[4] M. Cámara, J. Fàbrega, M.A. Fiol and E. Garriga. Some families of orthogonal polynomials of a discrete variable and their applications to graphs and codes. *Electron. J. Combin.*, 16(1):#R83, 2009.

[5] C. Dalfó, E.R. van Dam, M.A. Fiol, E. Garriga and B.L. Gorissen. On almost distance-regular graphs. Submitted.

[6] C. Dalfó, M.A. Fiol and E. Garriga. On k-walk-regular graphs. *Electron. J. Combin.*, 16(1):#R47, 2009.

[7] E.R. van Dam. The spectral excess theorem for distance-regular graphs: a global (over)view. *Electron. J. Combin.*, 15(1):#R129, 2008.

[8] M.A. Fiol. On pseudo-distance-regularity. *Linear Algebra Appl.*, 323:145–165, 2001.

[9] M.A. Fiol. Algebraic characterizations of distance-regular graphs. *Discrete Math.*, 246:111–129, 2002.

[10] M.A. Fiol and E. Garriga. From local adjacency polynomials to locally pseudo-distance-regular graphs. *J. Combin. Theory Ser. B*, 71:162–183, 1997.

[11] M.A. Fiol and E. Garriga. On the algebraic theory of pseudo-distance-regularity around a set. *Linear Algebra Appl.*, 298:115–141, 1999.

[12] M.A. Fiol, S. Gago and E. Garriga. A simple proof of the spectral excess theorem for distance-regular graphs. *Linear Algebra Appl.*, 432:2418–2422, 2010.

[13] M.A. Fiol, E. Garriga and J.L.A. Yebra. Locally pseudo-distance-regular graphs. *J. Combin. Theory Ser. B*, 68:179–205, 1996.

[14] M.A. Fiol, E. Garriga and J.L.A. Yebra. Boundary graphs: The limit case of a spectral property. *Discrete Math.*, 226:155–173, 2001.

[15] C.D. Godsil. *Algebraic Combinatorics.* Chapman and Hall, NewYork, 1993.

[16] A.J. Hoffman. On the polynomial of a graph. *Amer. Math. Monthly*, 70:30–36, 1963.

[17] D.L. Powers. Partially distance-regular graphs. In *Graph Theory, Combinatorics, and Applications, Vol. 2. Proc. Sixth Quadrennial Int. Conf. on the Theory and Appl. of Graphs, Western Michigan University, Kalamazoo, 1988* (Y. Alavi et al., eds.), Wiley, New York, 991–1000, 1991.

[18] P.M. Weichsel. On distance-regularity in graphs. *J. Combin. Theory Ser. B*, 32:156–161, 1982.

Large Edge-non-vulnerable Graphs

Charles Delorme
Université Paris-Sud
Orsay

Abstract

In this paper we study the graphs such that the deletion of any edge does not increase the diameter. We give some upper bounds for the order of such a graph with given maximum degree and diameter. On the other hand construction of graphs provide lower bounds. As usual, for this kind of problems, there is often a gap between these two bounds.

1 Introduction

A A graph is said *edge-non-vulnerable* if its diameter is unchanged after deletion of any one of its edges.

Such graphs do exist. For example the graph on 4 vertices on the left of figure 1 has diameter 2, and the removal of an edge gives a graph isomorphic to one of the other graphs in the picture, both have diameter 2.

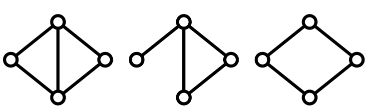


Figure 1: A (toy) edge-non-vulnerable graph

An obvious upper bound for these edge-non-vulnerable graphs with given maximum degree Δ and diameter D is the classical Moore bound, namely $1 + \Delta \sum_{k=0}^{D-1} (\Delta - 1)^k$. But this can be easily improved, since the condition imposes that between any pair of distinct vertices at least two paths of length $\leq D$ exist, and this implies the upper bound

$$n \leq n(\Delta, D) = 1 + \frac{1}{2}\Delta \sum_{k=0}^{D-1} (\Delta - 1)^k$$

Clearly, no graph of diameter 1 (in other words no complete graph) is edge-non-vulnerable, since the removal of the edge betwen x and y either disconnects the graph (if its order is 2) or increases its diameter to 2 (if the order is larger than 2.

2 Diameter 2, upper bound

For diameter 2, we have $n(\Delta, D) = 1 + \Delta^2/2$. This bound obviously cannot be attained if Δ is odd! So what about $(\Delta^2 + 1)/2$? This number is odd, therefore it is not compatible with a regular graph of degree Δ. Moreover, if some vertex has degree $< \Delta$, counting paths from that vertex decreases the bound to $1 + (\Delta - 1)^2/2 \leq (\Delta^2 - 1)/2$. So, what about $(\Delta^2 - 1)/2$? The toy graph of figure 1 shows that this bound $(\Delta^2 - 1)/2$ can be obtained for $\Delta = 3$. For the next odd degree $\Delta = 5$, the cartesian sum of K_3 and K_4 has the wanted property and order, namely $12 = (5^2 - 1)/2$.

For even degrees, the bound is attained only if a *distance-regular graph* (see [1]) with intersection matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ \Delta & 1 & 2 \\ 0 & \Delta - 2 & \Delta - 2 \end{bmatrix}$$

The techniques of distance-regular graphs lead to the computation of the eigenvalues: they are Δ and the two roots of $X^2 + X - +2 - \Delta$. Such an eigenvalue $\lambda \neq \Delta$ has then multiplicity

$$\frac{1 + \Delta^2/2)}{1 + \frac{\lambda^2}{\Delta} + \frac{(\lambda+1)^2}{\Delta(\Delta-2)/2)}}$$

that is also

$$\frac{(2 + \Delta^2)\Delta(\Delta - 2)}{4(\Delta - 1)^2 + (4 - \Delta)\lambda}$$

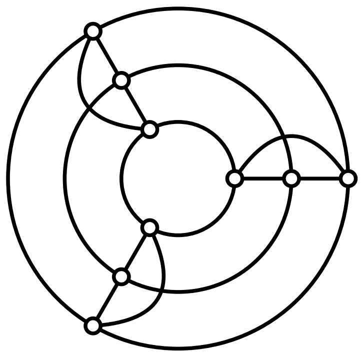


Figure 2: The optimal graph for degree 4 and diameter 2

Since this should be an integer, we should have either $\Delta = 4$ or λ integer, and in this case, $2(2\lambda + 1$ has to divide $(\lambda^2 + 2\lambda + 3)(\lambda^2 + \lambda + 2)\lambda$ and therefore $2(2\lambda + 1$ has to divide 63. This allows only the values 2, 4, 14, 22, 112, 994 for Δ.

The case $\Delta = 2$ is not interesting, the case $\Delta = 4$ gives a graph shown in figure 2 The case $\Delta = 22$, $n = 243$ is known: it is the Berlekamp, van Lint and Seidel graph, a Cayley graph on the group $(\mathbb{Z}/3\mathbb{Z})^5$ ([1, p. 360]).

The case $\Delta - 14$, $n = 99$ is unsolved, according to G. Exoo's list of unknown strongly regular graphs ([4])

3 Diameter 3, upper bound

The condition of edge-non-vulnerability is then: each edge lies in some cycle of length at most 4, each path of length 2 not already in a 4-cycle should be in a 5-cycle, unless each of its edges is in a 3-cycle, and at last, each path of length 3 should be in a cycle of length at most 6.

This provides the bound $n = 1 + \Delta + \Delta_2 + \Delta_3$, where $\Delta_2 \leq \Delta(\Delta - 1) - \lceil \Delta/2 \rceil$ and $\Delta_3 \leq \lfloor \Delta_2(\Delta - 2)/2 \rfloor$.

For $\Delta = 3$, this improved bound is 10, and the cartesian sum of a 5-cycle and K_2 is convenient.: figure 3

For $\Delta = 4$, the bound is 25. However, to attain this value it is necessary that the edge set is partitioned into 4-cycles, this is clearly not possible in a graph with $25 \cdot 4/2 = 50$ edges. The same obstruction occurs for all

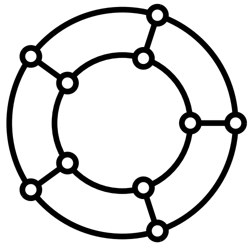


Figure 3: Optimal graph for $\Delta = 3$, and $D = 3$

degrees multiple of 4. For $\Delta \equiv 7$ or 9 (mod 8), the hand-shaking lemma also indicates that is bound is still too high!

4 Higher diameters, upper bound

It happens that the computation of improved upper bounds becomes more and more complicated. Just an example: for $D = 4$ and $\Delta = 3$, one has between vertices at distance 2 from a vertex v at least two edges. Thus at most 8 edges connect the sphere at distance 2 to the one at distance 3. Since the paths of length 2 that are not already in a 5-cycle have to be in a 6-cycle, the 7 sphere has at most 7 vertices, and the sphere at diatance 4 from v has at most 3. Thus a bound is 20. But the graph should then have two pentagons through each vertex, this makes at least 8 pentagons. If a vertex is on 3 pentagons, the bound becomes 19, and even 18 owing to the hand-shaking lemma. There are 30 edges. If an edge belong to 3 pentagons, its endvertices do. Otherwise there are 10 edges belonging to 2 pentagons, some pentagon has at least two such edges: if these edges are adjacent, their common vertex in on 3 pentagons, if the 10 edges are not adjacent they form a matching. The last vertex of a pentagon that has already 2 edges belonging to 2 pentagons is on 3 pentagons.

Thus the bound is now 18. It is easy to check that if the graph contains a cycle of length 3 or 4, the bound is only 16. On the other hand one can build convenient graphs on 16 vertices: figure 4.

Thus the optimal graph has 16 or 18 vertices.

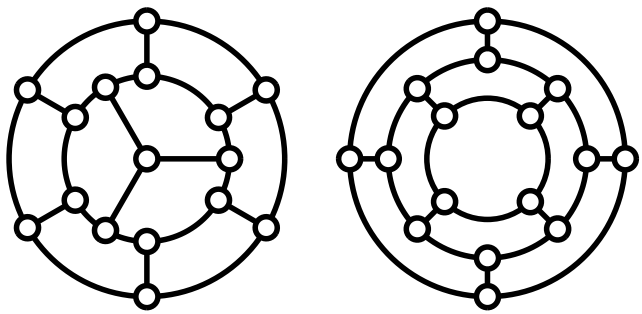


Figure 4: Graphs for degree 3 and diameter 4

5 Cartesian sums and categorical products

A first general construction is the *cartesian sum* of graphs,: the vertex set of $G_1 \square G_2$ has vertex set the product of the vertex sets, the edges of $GG_1 \square G_2$ are the pairs $\{(x,y),(x',y)\}$ where $\{x,x'\}$ is an edge of G_1 and y avertex of G_2 and the pairs $\{(x,y),(x,y')\}$ where x is a vertex of G_1 and $\{y,y'\}$ an edge of G_2.

The cartesian sum of G_1 (diameter D_1 , maximum degree Δ_1) and G_2 (diameter D_2 , maximum degree Δ_2) has maximum degree $\Delta_1 + \Delta_2$ and diameter $D_1 + D_2$, and is edge-non-vulnerable provided if $D_1 + D_2 > 2$,

For example, the cartesian sum of K_2 and Petersen graph has $n = 20$, $D = 3$, $\Delta = 4$, no so far from the (unaccessible) 25.

The *categorical product* of graphs (that may have loops) G_1 and G_2 has vertex set the product of the vertex sets, the edges of $G_1 \times G_2$ are the pairs (or loops) $\{(x,x'),(y,y')\}$ wher $\{x,x'\}$ and $\{y,y'\}$ are edges or loops of G_1 and G_2.

The maximum degree of $G_1 \times G_2$ is the product of the maximum degrees in G_1 and in G_2. The distance between (x,y) and (x',y') is the minimum between the lengths of paths (elementary or not) *of same parity* connecting x,x' and y,y'.

Figure 5 shows the product of the graph K_2^+ made from an edge with a loop at each end and first a 5-cycle with 3 loops and then with a triangle (this gives a graph isomorphic to the octahedron).

The graph of figure 2 is the categorical product of two triangles, that is $K_3 \times K_3$ or $K_3^{\times 2}$. It is also the cartesian sum of two triangles. The distance

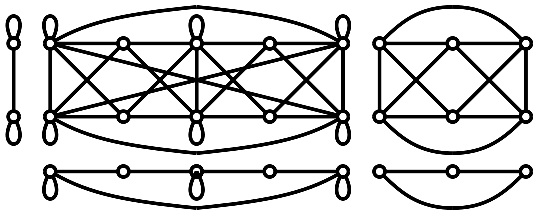


Figure 5: Examples od products with K_2^+

between (x, y) and (x, y') is 2 because there is a non-elementary path of length 2 from x to itself and a path of length 2 from y to y'; the distance between (x, y) and (x', y') is 1 because there is a path of length 1 from x to x' and a path of length 1 from y to y'; and so on.

6 Biplanes

A biplane is a bipartite distance-regular graph with intersection array $(d, d-1, d-2; 1, 2, d)$, thus of order $2n = d^2 - d + 2$ The Bruck-Ryser-Chowla theorem allows the existence of such graphs only if either n is even and $d-2$ is a square or n is odd and $x^2 = (d-2)y^2 + (-1)^{(n-1)/2}2z^2$ has integer non null solutions ([1, p. 698]).

If a biplane has a polarity, the quotient has degree d (with loops counting for 1), order $(d^2 + -d + 2)/2$ and diameter at most 2, and each edge either has a loop at its two endpoints, or lies in a triangle.

For $d = 2$, we have (with a bit cheating) the 4-cycle and its quotient K_2^+.

For $d = 3$,we have the usual cube, and the quotients K_4, $K_{2,1,1}$ (the toy graph of Figure 1 with loops on the vertices of degree 2, and C_4 with a loop at each vertex.

For degree 4, we have a graph with polarities, its quotient (that has always 4 loops) is shown in Figure 6.

For degree 5 the quotient also has edges with two ends occupied by loops (there are always 5 loops)

For degree 6, several quotients are possible, with 0 loops ($K_4 \square K_4$ or Shrikhande graph) or 16 loops (Clebsch graph), among other less symmetric graphs; Eigenvalue considerations impose that the number of loops is a multiple of 4.

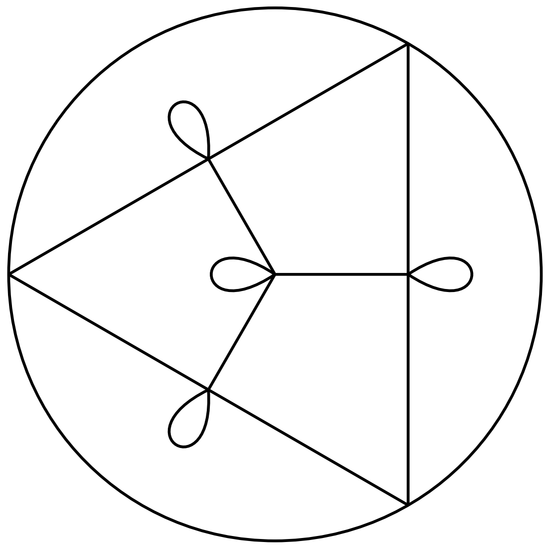


Figure 6: A quotient of a biplane of degree 4

For degree 9, one has a quotient with 37 vertices labeled with the elements of the field $\mathbb{Z}/37\mathbb{Z}$, and x, y are adjacent if their sum is one of the non-null 4-th powers in the field, that is 1, 7, 9, 10, 12, 16, 26, 33, 34. Here the vertices with loops are never adjacent.

For degree 11, there are several biplanes, with qoutients of order 56. One of them has 56 loops: the Gewirtz graph. Other quotients, have a number of loops congruent to 2 modulo 6.

The categorical products of these quotients with Erd***H***os-Rényi graphs are convenient and for some degrees and diameter 2.

7 Diameter 2: lower bounds

The product of K_2^+ with the Erd***H***os Rényi-graphs of degree d and order d^2-d+1 (with their loops) has diameter 2, degree $2d$, order $2(d^2-d+1)$, that is close to the upper bound $\sim d^2/2$.

For some degrees we have special constructions

- degree 6, some quotients of a biplane
- degree 8 the categorical product $K_3 \times K_3 \times K_3$

• degree 9, some quotients of a biplane.
Let us summarize our results for small degrees in table 1.

Table 1: Some results for diameter 2

Δ	3	4	5	6
n	$\underline{4}$	$\underline{9}$	$\underline{12}$	16
	$K_{1,1,2}$	$K_3 \times K_3$	$K_3 \square K_4$	$K_4 \square K_4$
Δ	7	8	9	10
n	20	27	37	42
	$K_5 \square K_4$	$(K_3)^{\times 3}$	quot. bipl.	$K_2^+ \times$ER(5)
Δ	12	14	16	
n	63	84	117	
	$K_3^{\times 2} \times$ ER(3)	$K_4 \times$ ER(5)	$K_3^{\times 2} \times$ ER(4)	
Δ	18	20	22	
n	146	189	$\underline{243}$	
	$K_2^+ \times$ER(9)	$(K_3)^{\times 2} \times$ ER(5)	BvLS	

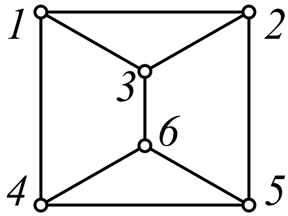
8 Twisted products

Since the cartesian sum clearly spills some edges with an excessive number of 4-cycles, we may improve things here and there.

• product $G \ltimes C_5$. The vertex set is the product of the vertex sets of G and C_5, the edges are the pairs $\{(g, a), g(, a')\}$ with g vertex of G and $\{a, a'\}$ an edge of C_5, and then G is endowed with an orientation, and C_5 with a permutation π exchanging the edges and non-edges of C_5, and we add the edges $\{(g, a), (g', \pi(a))\}$ (in other words, each edge of G is replaced by a Petersen graph). This gives a graph with diameter $D(G) + 1$, maximum degree $\Delta(G) + 2$, that is edge-non-vulnerable provided that $D(G) \geq 3$ and vertices at distance D in G are connected by two internally disjoint paths of length D.

• product $G \ltimes P(4t + 1)$, where $P(4t + 1)$ is the Paley graph on $4t + 1$ vertices. The diameter is $D(G)+1$, and the maximum degree $\Delta(G)+2t$, and the graph is edge-non-vulnerable provided that $D(G) \geq 3$ and vertices at distance D in G are connected by two internally disjoint paths of length D.

• product $G \times C_{13}$, a similar construction, $\Delta(G \times C_{13}) = \Delta(G) + 2$, and



The edges of the graph and their images by the involutive vertex permutation (1)(2)(36)(45) are all the edges of K_6.

Figure 7: The graph A_6 to be used in twisted products

$D(G \times C_{13}) = D(G) + 2$ under the same condition. Endowing C_{13} with the labels in $\mathbb{Z}/13\mathbb{Z}$ so that edges are labeled $\{a, a+1\}$, the permutation π sends the vertex i to the vertex $5i$ (so that π^2 is an isomorphism of C_{13}.

In the same vein, $G \ltimes A_6$, a similar construction, $\Delta(G \times A_6) = \Delta(G)+2$, and $D(G \times A_6) = D(G) + 2$ under the same condition. Here A_6 and its permutation π are represented in figure 7.

9 Line graphs

The line-graph L of a bipartite graph of degree $d \geq 3$, order n and diameter D has diameter D, degree $2d - 2$ and order $dn/2$; each edge of L is in a triangle, and each pair of vertices of L at distance D is connected by 2 paths of length D. Thus the graph is non-edge-vulnerable.

The well-known large cubic bipartite graphs give for diameters 2, 3, 4 and 6 graphs of order 9 (the one we have already seen, from $K_{3,3}$), 21 (from Heawood graph), 45 (from Tutte's 8-cage), 189 (from Tutte's 12-cage). Besides the cubic bipartite graph of diameter 5 and order 56 described by Bond and Delorme [2] provides an edge-non-vulnerable graph on 84 vertices. having degree 4 and diameter 5. Some of these graphs are represented on figure 8. For diameters 3, 4 and 6, the line graphs of bipartite Moore graphs give some results.

10 A small census

We collect some results in the table 2.

This graph on 56 vertices of Figure 10 is the graph 56.2 in the list of M. Conder. Hea denotes Heawood graph, and TC Tutte-Coxeter graph; the O_k's are the so-called odd graphs

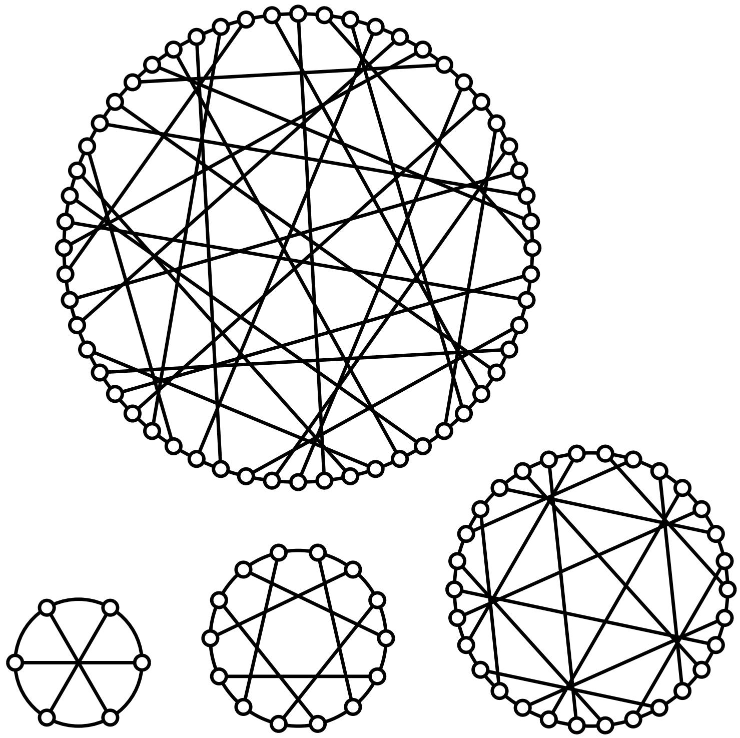


Figure 8: Large bipartite cubic graphs of diameters 2, 3, 4, 5

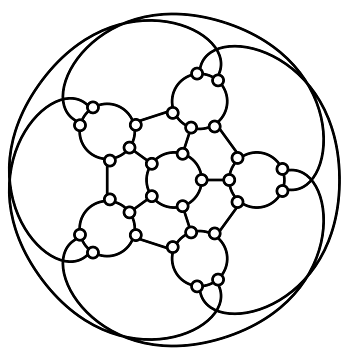


Figure 9: An edge- non-vulnerable graph on 30 vertices

Table 2: Some lower bounds

$\Delta\backslash D$	3	4	5	6
3	10	16	30	56
	$C_5 \,\square\, K_2$	fig. 4	fig.9	fig. 10
4	21	45	84	189
	LG	LG	LG	LG
5	30	70	182	390
	Pet.$\square K_3$	Hea. $\ltimes C_5$	Hea. $\ltimes C_{13}$	TC. $\ltimes C_{13}$.
6	52	175	462	1456
	LG	$O_4 \ltimes C_5$	O_6	LG
7	72	210	630	1716
	$24 \,\square\, K_3$	$O_4 \ltimes A_6$	$O_5 \ltimes C_5$	O_7
8	105	425	756	6825
	LG	LG	$O_5 \ltimes A_6$	LG

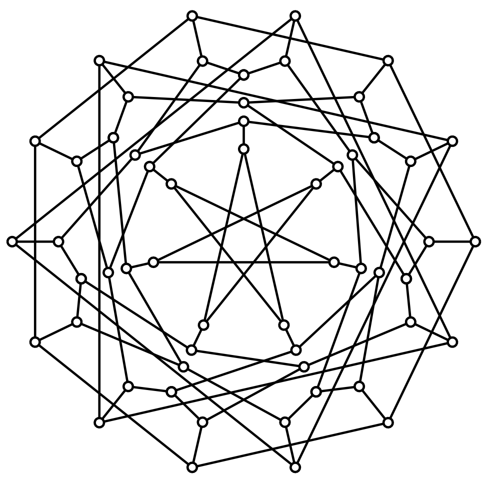


Figure 10: An edge- non-vulnerable graph on 56 vertices

11 Conclusion

We have given some indications on the large graphs with maximum degree and diameter whose diameter is unchanged after deletion of an edge. In the related problem with vertex deletion, the sufficient (but not necessary: see the line graphs of cubic graphs) condition that every path of length $\ell \geq 1$ should be in a cycle of length at most $\ell + D$ is replaced by the slightly weaker: every path of length $\ell \geq 2$ should be in a cycle of length at most $\ell + D$. Thus some of our graphs also provide solutions for the vertex-non-vulnerability, although not always as large as possible. See for example the survey paper by Fàbraga, Gómez and Yebra [5]

References

[1] A. E. Brouwer,, A. Cohen, A. Neumaier. *Distance Regular Graphs.* Springer-Verlag, Ergebnisse der Mathematik undihrer Grenzgebiete 3. Folge, Band 18, 1989.

[2] J. Bond, C. Delorme. New large bipartite graphs with given degree and diameter. *Ars Combinatoria*, 25 C:123–132, 1988.

[3] M. Conder. Census of trivalent symmetric graphs up to 2048 vertices. `http://www.math.auckland.ac.nz/~conder/symmcubic2048list.txt`

[4] G. Exoo. List of unknown strongly regular graphs up to 196 vertices. `http://isu.indstate.edu/ge/COMBIN/SRG/index.html`

[5] J. Gómez, J. Fàbrega and J. L. A. Yebra. On large $(\Delta, D, D, 1)$-graphs. *Networks*, to appear.

A mathematical model for dynamic memory networks

Josep Fàbrega, Miquel A. Fiol, Oriol Serra and José L.A. Yebra
Universitat Politècnica de Catalunya
Barcelona

Abstract

The aim of this paper is to bring together the work done several years ago by M.A. Fiol and the other authors to formulate a quite general mathematical model for a kind of permutation networks known as dynamic memories. A dynamic memory is constituted by an array of cells, each storing one datum, and an interconnection network between the cells that allows the constant circulation of the stored data. The objective is to design the interconnection network in order to have short access time and a simple memory control. We review how most of the proposals of dynamic memories that have appeared in the literature fit in this general model, and how it can be used to design new structures with good access properties. Moreover, using the idea of projecting a digraph onto a de Bruijn digraph, we propose new structures for dynamic memories with vectorial capabilities. Some of these new proposals are based on iterated line digraphs, which have been widely and successfully used by M.A. Fiol and his coauthors to solve many different problems in graph theory.

1 Introduction

A *dynamic memory* is constituted by an array of cells, each storing one datum, and an interconnection network between the cells that allows the movement of the data from a cell to a neighboring one at time-unit intervals. The result of all these data transfers during such an interval is called a *memory transformation* and it is a permutation of the contents of the memory. Because of the network topology (each cell is connected to a reduced number of neighboring ones), only a small number of these memory transformations are available at each time-unit interval.

Therefore, in a dynamic memory we must distinguish between the *physical address* of a datum, which is its current physical location, and its *logical address* which can be thought of as its initial location. To access a requested datum, it must be sent (by a suitable sequence of memory transformations) to a specific cell, called the *read/write (r/w) cell*, where data can be read or written. Hence, the *control problem* in a dynamic memory consists of the following two steps:

(a) To find the physical address of the requested datum from its logical address and some additional information about the memory transformations that have been applied to the memory.

(b) To determine an optimal sequence of memory transformations that route the datum to the r/w cell.

Stone [20, 21] was the first author to propose a general model for dynamic memories. Since then, there have been many different proposals: Aho and Ullman [1], Iyer and Sinclair [11, 12], Kluge [13], Lenfant [15], Morris, Valiere III and Wisniewski [16], Wong and Tang [23], and the authors [5, 6, 7, 8, 9, 18, 19, 24].

In our formulation, the interconnection network of the dynamic memory is modeled by a δ-regular strongly connected digraph D in which the vertices represent the storing cells and the arcs the links between them. We refer to [3] for the standard concepts on digraphs. The memory transformations correspond to a decomposition into permutations of D in the following sense. A *decomposition into permutations* [5] of a δ-regular digraph D is a set $\{\gamma_i;\ 0 \le i \le \delta-1\}$ of δ permutations of the vertices of D that satisfies:

$$\gamma_i(x) \text{ is a vertex adjacent from } x; \quad \gamma_i(x) \neq \gamma_j(x) \text{ for } i \neq j. \tag{1}$$

It is easily shown that any δ-regular digraph can be decomposed (usually in several ways) into permutations. Note that such a decomposition associates a permutation to every arc of D: $(x, y) \mapsto \gamma_i$ if $\gamma_i(x) = y$, which is actually an (*arc-*)*coloring* of D, since different permutations (colors) are associated to the δ arcs to and from any vertex. Conversely, given a set V together with a set $\{\gamma_i;\ 0 \le i \le \delta - 1\}$ of permutations of V that satisfy the second condition in (1), we can consider the δ-regular digraph $(V, \{\gamma_i\})$ that has V as set of vertices and where, in view of the first condition in (1), each vertex x is adjacent to the vertices $\gamma_i(x)$, $0 \le i \le \delta - 1$. As it is seen in the next section, a useful way to obtain a digraph decomposed into permutations consists in identifying its vertex set V with the set of elements of a group G.

2 Group of permutations of a decomposition

We recall that given two groups G and H together with an homomorphism of H into the set of automorphisms of G, $\Pi\colon H \longrightarrow Aut\ G$, $\Pi(h) = \pi_h$, the (*external*) *semidirect product* $G \rtimes H$ is the group with set of elements $\{(g, h);\ g \in G, h \in H\}$ and composition rule

$$(g_1, h_1) \star (g_2, h_2) = (g_1 \pi_{h_1}(g_2), h_1 h_2). \tag{2}$$

It coincides with the direct product $G \times H$ if and only if $\Pi = 0$, that is, when $\Pi(h)$ is always the identity. In this paper H will already be a subgroup of $Aut\ G$ so that, with Π the canonical embedding, (2) becomes $(g_1, \pi_1) \star (g_2, \pi_2) = (g_1 \pi_1(g_2), \pi_1 \pi_2)$. See for instance [17] for the standard concepts on group theory used in this work.

In order to control a dynamic memory we need to know the structure of the set of its different states. This is the group generated by the memory transformations. A useful characterization of this group can be obtained by considering V as the set of elements of a group G, that is, each vertex of V stands for an element of G. Then, given $g_i \in G$ and $\pi_i \in Aut\ G$ for $0 \le i \le \delta - 1$, we can define the permutations

$$\gamma_i(x) = g_i \pi_i(x) \quad \forall x \in V, \quad 0 \le i \le \delta - 1, \tag{3}$$

and consider the digraph $D = (G, \{\gamma_i\})$ that has the decomposition into permutations $\{\gamma_i;\ 0 \le i \le \delta - 1\}$. (The elements g_i and the automorphisms π_i should be chosen in such a way that D is strongly connected.) Let $\Sigma = \langle \gamma_0, \gamma_1, \ldots, \gamma_{\delta-1} \rangle$ be the permutation group generated by $\gamma_0, \gamma_1, \ldots, \gamma_{\delta-1}$.

To characterize Σ, let $H = \langle \pi_0, \pi_1, \ldots, \pi_{\delta-1} \rangle$. We have the following result, see [8, 9].

Theorem 1 *The permutation group Σ is isomorphic to a subgroup of the semidirect product $G \rtimes H$.*

We can identify Σ and its image in $G \rtimes H$, that is, we can identify every permutation $\sigma \in \Sigma$ with the pair $(g, \pi) \in G \rtimes H$ given by $\sigma(x) = g\pi(x)$. Since the digraph D is supposed to be strongly connected, the group Σ acts transitively. Hence, its order satisfies $|\Sigma| = N\ell$, where $N = |G|$ and ℓ is, for any $x \in G$, the order of the stabilizer $\mathrm{St}_\Sigma(x)$ of x in Σ. For $x = e$, the identity of G, the subgroup $\mathrm{St}_\Sigma(e)$ consists of the elements of Σ of the form (e, π), since $(g, \pi)(e) = e \iff g = e$. It follows that the number of right (or left) cosets of $\mathrm{St}_\Sigma(e)$ in Σ equals N.

3 Control of the memory

At every instant, the state of the memory is completely determined by a permutation or *memory address map* $\sigma \in \Sigma$, such that $\sigma(x)$ is the physical address of the datum with logical address x. Since $\sigma(x) = g\pi(x)$, the memory address map σ is determined in turn by the pair $g \in G$, $\pi \in H$. To obtain π we need a simulator of the group H, but g can then be deduced from the knowledge of the logical address s of the datum which is presently at the r/w cell w, for $\sigma(s) = g\pi(s) = w$ leads to $g = w(\pi(s))^{-1} = w\pi(s^{-1})$. In other words, the memory acts as a simulator of the group G. Now the physical address of a datum with logical address x is given by

$$\sigma(x) = g\pi(x) = w\pi(s^{-1})\pi(x) = w\pi(s^{-1}x). \tag{4}$$

Reciprocally, with $y = \sigma(x)$, the *memory configuration* σ^{-1} which gives the logical address of the datum in any cell y is

$$\sigma^{-1}(y) = s\pi^{-1}(w^{-1}y). \tag{5}$$

Once the physical address $y = \sigma(x)$ of the requested datum x is known, the next problem is how to transfer it to the r/w cell w. From the considerations above there are ℓ different permutations $\tau = (g_\tau, \pi_\tau) \in \Sigma$ such that $(g_\tau, \pi_\tau)(y) = g_\tau \pi_\tau(y) = w$. This means $g_\tau = w(\pi_\tau(y))^{-1} = w\pi_\tau(y^{-1})$, so that they all have the form

$$(w\pi(y^{-1}), \pi) \in \Sigma \tag{6}$$

for ℓ appropriate choices of $\pi \in H$. In the digraph $(G, \{\gamma_i\})$ that models the dynamic memory, the required path from y to w corresponds to any factorization of one of these ℓ permutations as a product of the γ_i. Of course, in order to reduce the access time, the number of factors should be minimized both by an adequate choice of π in (6) and when factorizing.

Besides sending the datum at y to w, it is often required that the memory attains a given state. For instance, when accessing a block of data its ordering should not be modified. When G is the cyclic group $\mathbb{Z}_N$, this can be accomplished whenever the choice $\pi = \iota$, the identity of H, is allowed in (6), for then the permutation applied to the memory is, using additive notation, $x \mapsto g + x$, with $g = w - y$.

In the next section we review how most of the proposals of dynamic memories that have appeared in the literature fit in our general model, and how this model can be used to design structures with optimal access time or with good properties for sequential access. In Sections 5 and 6, using the idea of projecting a digraph onto a de Bruijn one, we propose new structures for dynamic memories with vectorial capabilities. Some of these new proposals are based on iterated line digraphs.

4 Applications of the model

To illustrate the generality of the model, we now show how several known proposals of dynamic memories fit into it.

When G is a finite group generated by $\Delta = \{a_0, a_1, \ldots, a_{\delta-1}\}$ and we choose $g_i = a_i$, $\pi_i = \iota$, $0 \le i \le \delta - 1$, so that $\gamma_i(x) = a_i x$, $0 \le i \le \delta - 1$, the corresponding digraph, $D = (G, \{\gamma_i\})$, which is δ-regular and strongly connected, is the *Cayley digraph* $D(G, \Delta)$. Since $H = \{\iota\}$, we obtain $\Sigma \simeq G$ and so $\ell = 1$. Therefore, the corresponding dynamic memory has the least possible number of states $N = |G|$, and it can be controlled with just the information of the contents of the r/w cell w, i.e., (4) and (5) give

$$\sigma(x) = ws^{-1}x \quad \text{and} \quad \sigma^{-1}(y) = sw^{-1}y. \tag{7}$$

A generalization of this simple model is the following. The well-known de Bruijn digraphs $B(\delta, n)$ are set up on the set of $N = \delta^n$ vertices $V = \{x = x_0x_1 \ldots x_{n-1};\ x_i \in S\}$, where S is an alphabet on δ symbols, and vertex x is adjacent to vertex y if and only if $y = x_1x_2 \ldots x_{n-1}x_n$, $x_n \in S$. Given a finite group Γ of order δ generated by $\Delta = \Gamma$, we can associate to Γ a de Bruijn digraph $B(\delta, n)$, with a decomposition into permutations $\{\gamma_i\}$, defining $V(B(\delta, n)) = \{x = x_0x_1 \dots x_{n-1},\ x_i \in \Gamma\}$ and each vertex x being adjacent to

$$\gamma_i(x) = x_1 \dots x_{n-1}x_n,\ x_n = a_ix_0, \tag{8}$$

through the arc $[x, \gamma_i(x)]$ for all $a_i \in \Gamma$. For $n = 1$, $B(\delta, 1)$ is just the Cayley digraph of Γ when generated by all its elements. Thus, $B(\delta, n)$ with the decomposition into permutations (8) may be called the *n-dimensional Cayley digraph* of Γ, see [7].

It is noteworthy that the number of different states $|\Sigma|$ of a dynamic memory modelled by these digraphs is at least $n\delta^n$. Indeed, because of the loop at vertex $00 \dots 0$, there always exists a permutation γ that fixes it: $\gamma(00 \dots 0) = 00 \dots 0$. Then, the n permutations γ, γ^2, ..., γ^n also fix this vertex and are necessarily different since $\gamma^p(00 \dots 01) \neq \gamma^q(00 \dots 01)$ for $p \neq q$, $1 \leq p, q \leq n$. Thus $\ell \geq n$ and $|\Sigma| \geq n\delta^n$ for any decomposition into permutations of $B(\delta, n)$. We next describe a structure that attains this bound.

In our context we can consider V as the set of elements of the direct product $G = \mathbb{Z}_\delta \times \cdots \times \mathbb{Z}_\delta$ with n terms. If we now choose for all automorphisms π_i the *perfect shuffle* permutation $\pi_i = S$, $0 \leq i \leq \delta - 1$, $S(x_0x_1 \dots x_{n-1}) = x_1 \dots x_{n-1}x_0$, and $g_i = 00 \dots 01$, $0 \leq i \leq \delta - 1$, we obtain the decomposition into permutations of the digraph $(V, \{\gamma_i\}) = B(\delta, n)$ given in (8). It follows that $H = \langle S \rangle = C_n$ is the cyclic group with elements S, S^2, ..., $S_n = \iota$, so that $|G \rtimes H| = n\delta^n$, and then $|\Sigma| \leq n\delta^n$ because $\Sigma \cong \Sigma' \subset G \rtimes H$. Being a decomposition into permutations of $B(\delta, n)$, we necessarily have $|\Sigma| = n\delta^n$ as stated. This corresponds to $\Sigma \cong G \rtimes H$.

The control of the memory can then be achieved with the knowledge of the logical address s of the datum in the r/w cell w and the number p (mod n) of shuffle permutations that have been applied to the memory, see also [6]. Since each permutation γ_i includes a shuffle permutation, p can be obtained from a cyclic register that counts modulo n the number of permutations applied to the memory. Then (4), with $\pi = S^p$, gives $y = \sigma(x) = w + S^p(x - s)$, that is (with additive notation),

$$\begin{aligned} &y_0y_1 \dots y_{n-1} = \sigma(x_0x_1 \dots x_{n-1}) = \\ &\qquad w_0 + x_p - s_p\ w_1 + x_{p+1} - s_{p+1} \dots w_{n-1} + x_{p+n-1} - s_{p+n-1} \end{aligned} \tag{9}$$

is the present position of the datum with logical address x. Once we know

y, we can use the standard shortest path routing algorithm in $B(\delta, n)$ to send x to the r/w cell in at most n time-unit intervals.

For $\delta = 2$ we can write $\sigma(x) = w \oplus S^p(s \oplus x)$, where $\oplus$ stands for componentwise addition modulo 2. This is the structure proposed by Morris, Valiere III and Wisniewski in [16] for a memory of $N = 2^n$ cells, which is equivalent, except for the numeration of the cells, to the one proposed by Stone in [20, 22]. More precisely, Stone makes use of the perfect shuffle (γ_0) and the *exchange shuffle* (γ_1) permutations, which correspond in our formulation to $\pi_0 = \pi_1 = S$, $g_0 = 00\ldots0$ and $g_1 = 00\ldots010$. The general case ($\delta \neq 2$) described above has been studied by the authors in [6, 8].

To achieve sequential access to a block of data, Aho and Ullman proposed in [1] an architecture for a dynamic memory of $N = \delta^n - 1$ cells, using a pair of transformations, that can bring any datum to the r/w cell in $\mathcal{O}(\log N)$ steps. Moreover, once two consecutive data have been accessed, the rest of the block can be accessed in one step per datum. This structure corresponds in our formulation to $G = \mathbb{Z}_N$, $N = \delta^n - 1$, $\delta \geq 2$, and permutations $\gamma_0(x) = \delta x$ ($\pi_0(x) = \delta x, g_0 = 0$) and $\gamma_1(x) = x - 1$ ($\pi_1 = \iota, g_1 = -1$). Then $H = \mathbb{Z}_n$ is the cyclic group with elements $\pi_0, \pi_0^2, \ldots, \pi_0^n = \iota$, and $\Sigma \simeq G \rtimes H$. Therefore, the control of the memory requires, besides the knowledge of the logical address of the datum in the r/w cell, the number p (mod n) of permutations π_0 applied to the memory. Then, (4) and (5) give respectively

$$\sigma(x) = w + \pi_0^p(x - s) = w + \delta^p(x - s); \tag{10}$$

$$\sigma^{-1}(y) = s + \pi_0^{n-p}(y - w) = s + \delta^{n-p}(y - w). \tag{11}$$

To transfer the datum at y to w without altering the ordering of the memory, write $y - w$ in base δ as $y - w = \sum_{k=0}^{n-1} r_k \delta^k = \sum_{k=0}^{n-1} \gamma_0^k(r_k)$. Therefore, the required permutation given by (6) with $\pi = \iota$, $(\omega - y, \iota) = (-(y - w), \pi_0^n)$, can be obtained as $\gamma_1^{r_0}\gamma_0\gamma_1^{r_1}\gamma_0 \cdots \gamma_1^{r_{n-1}}\gamma_0$. As pointed out before, if only cyclic permutations of the form (g, ι) are performed on the memory, the memory address map is given by $\sigma(x) = g + x$, and the cyclic structure is always preserved.

This structure has been slightly improved by Stone in [22] and Wong and Tang in [23] who replace π_0 by π_0^{n-c} for different values of c.

In [8, 9] it is shown how the mathematical model can also be used to describe and study other dynamic memories based on de Bruijn digraphs [5], [6], [16], [20], [22], the variants of the memory of Aho and Ullman

[22], [23], the memory of Lenfant [15], together with its improved version studied in [18], the Odd-sized memories proposed by Morris, Valiere III and Wisniewski in [16], an optimal size memory proposed by the authors, and other proposals of the authors suitable for sequential access.

5 Dynamic memories based on iterated line digraphs

As it is well known, de Bruijn digraphs can be obtained by line digraph iterations. In the iterated line digraph $L^n(D)$ of a digraph D, each vertex x represents a sequence $x_0x_1\dots x_n$ of vertices of D such that each x_j is adjacent to x_{j+1} in D, and each vertex x is adjacent to the vertices of the form $x_1\dots x_nx_{n+1}$. It is shown in [10] that, if D is a δ-regular digraph on N vertices with diameter k, its iterated line digraph $L^n(D)$ is also δ-regular on δ^nN vertices and has diameter $k+n$. The de Bruijn digraph $B(\delta,n)$ is just $L^{n-1}(F_\delta)$, where F_δ is the complete symmetric digraph on δ vertices. Another well-know family of iterated line digraphs is the family of Kautz digraphs $K(\delta,n)=L^{n-1}(F^*_\delta)$, where F^*_δ is now the complete symmetric digraph on $\delta+1$ vertices without loops. The order of $K(\delta,n)$ is $\delta^n+\delta^{n-1}$. In [7] a proposal for a dynamic memory based on $K(\delta,n)$ is presented. See also [2] for a mathematical description of the group of permutations generated by a decomposition into permutations of a Kautz digraph.

Any iterated line digraph $L^n(D)$ of a δ-regular digraph D decomposed into permutations can be adequately mapped onto the de Bruijn digraph $B(\delta,n)$. Notice first that, if $\{\tau_i;\ 0\le i\le\delta-1\}$ is a decomposition into permutations of $F_\delta=B(\delta,1)$, the permutations

$$\phi_i(j_0j_1\dots j_{n-1})=j_1\dots j_{n-1}\tau_i(j_0),\ 0\le i\le\delta-1, \tag{12}$$

where $0\le j_k\le\delta-1$, form a decomposition of $B(\delta,n)=L^{n-1}(F_\delta)$. Let D be a strongly connected δ-regular digraph G' with a decomposition into permutations $\{\beta_i;\ 0\le i\le d-1\}$, and consider its iterated line digraph $L^n(D)$. Every vertex x of $L^n(D)$ corresponds to a sequence of successively adjacent vertices of D, $x_0x_1\dots x_n$. For $0\le i\le n-1$ let β_{j_i} be the permutation belonging to $\{\beta_i\}$ such that $\beta_{j_i}(x_i)=x_{i+1}$. Thus, an alternative way of representing x is $x_0;j_0j_1\dots j_{n-1}$ since $\beta_{j_0}(x_0)=x_1$, $\beta_{j_1}(x_1)=x_2$ and so on. Now vertex x is adjacent to the vertices $y=\beta_{j_0}(x_0);j_1\dots j_{n-1}j_n$,

$0 \leq j_n \leq \delta - 1$, and it is easily verified that the permutations

$$\psi_i(x_0; j_0 j_1 \dots j_{n-1}) = \beta_{j_0}(x_0); j_1 \dots j_{n-1} \tau_i(j_0) \tag{13}$$

for $0 \leq i \leq \delta - 1$, form a decomposition into permutations of $L^n(D)$. Moreover, we obtain as a consequence the following result:

Proposition 2 *Let Φ be the mapping of the vertex set of $L^n(D)$ onto the vertex set of $B(\delta, n)$ defined by $\Phi(x_0; j_0 j_1 \dots j_{n-1}) = j_0 j_1 \dots j_{n-1}$. Then, Φ is a digraph homomorphism such that $\Phi(\psi_i) = \phi_i$ for all i.*

Proof: If vertex x of $L^n(D)$ is adjacent to vertex $y = \psi_i(x)$, then vertex $\Phi(x)$ is adjacent to vertex $\Phi(y) = \Phi(\psi_i(x)) = \phi_i(\Phi(x))$. $\square$

The case $D = B(\delta, 1)$, that is when the mapping Φ is an homomorphism from $B(\delta, n+1)$ onto $B(\delta, n)$, has been studied in the context of the design of feedback shift registers by Lempel [14].

To use $L^n(D)$ as a model for a dynamic memory with vectorial capabilities, select a permutation τ_i of the decomposition of F_δ that fixes an element, that is, such that $\tau_i(j) = j$ for some j, $0 \leq j \leq \delta - 1$, and consider a digraph D of order N that has an N-length cycle. In the decomposition into permutations of D we choose the permutation β_j as the one associated to this cycle. Then, the permutation ψ_i of (13) is such that

$$\psi_i(x_0; jj \dots j) = \beta_j(x_0); j \dots j \tau_i(j) = x_1; jj \dots j. \tag{14}$$

Let the memory be organized in such a way that it consists of δ^n blocks or vectors of N words, each stored at the N cells that are mapped by the homomorphism Φ onto a common vertex of $B(\delta, n)$. If the r/w cell w is at $x_0; jj \dots j$, with the routing algorithm of $B(\delta, n)$ it is possible to send one of the data of any block to w and then, by (14), the application of ψ_i allows the rest of the block to visit the r/w cell at one time-unit interval per word. Notice also that, according to (13), the application of any permutation ψ alters the ordering of the N words of each block as it does the permutation β_{j_0}. Therefore, if D has a multiple ring structure with a permutation β associated to each ring, the initial cyclic ordering of the blocks will be preserved. Alternatively, if N read/write cells are available and they are placed at the N cells $*; jj \dots j$, the N words of each block can be accessed simultaneously.

6 Vectorial dynamic memories

In this last section we will make better use of the idea of projecting a digraph onto a de Bruijn one in order to obtain a dynamic memory with vectorial capabilities. Let $D = (V, A)$ be a strongly connected δ-regular digraph with a decomposition into permutations $\{\gamma_i;\ 0 \le i \le \delta - 1\}$ that generate a group Σ, and consider the digraph $D^* = (V^*, A^*)$ where $V^* = \mathbb{Z}_m \times V$ and each vertex (a, x) is adjacent to the vertices $(a+1, y)$ for $[x, y] \in A$. Then, the decomposition into permutations of D leads naturally to the decomposition into permutations of D^* defined by

$$\gamma_i^*((a, x)) = (a + 1, \gamma_i(x)),\ 0 \le i \le \delta - 1. \tag{15}$$

A consequence of this definition is that the projection

$$\begin{aligned} \Phi :\ D^* &\longrightarrow D \\ \Phi((a, x)) &= x \end{aligned} \tag{16}$$

is a digraph homomorphism that preserves the coloring, that is, it transforms the permutation γ_i^* of D^* into the permutation γ_i of D.

Now, let $\Sigma^* = \langle \gamma_0^*, \gamma_1^*, \ldots, \gamma_{d-1}^* \rangle$ and suppose that D^* is strongly connected. Then, $|\Sigma^*| = |V^*|\ell^* = mN\ell^*$ where $\ell^* = |\mathrm{St}_{\Sigma^*}((a, x))|$ and $N = |V|$. The following result restricts the possible values of ℓ^*.

Proposition 3 *ℓ^* divides $\ell = |St_\Sigma(x)|$.*

Proof: The mapping $\psi : \Sigma \to \Sigma^*$ defined by $\alpha = \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_n} \mapsto \alpha^* = \gamma_{i_1}^*\gamma_{i_2}^*\cdots\gamma_{i_n}^*$ is an isomorphism between the subgroup of $\mathrm{St}_\Sigma(x)$ formed by the permutations that can be expressed as product of n (multiple of m) permutations γ_i and $\mathrm{St}_{\Sigma^*}((a, x))$. Therefore, $\mathrm{St}_{\Sigma^*}((a, x))$ can be seen as a subgroup of $\mathrm{St}_\Sigma(x)$ and its order ℓ^* must divide the order ℓ of $\mathrm{St}_\Sigma(x)$. □

For $D = B(\delta, n)$ with the decomposition into permutations given by $\gamma_i(x_0 x_1 \ldots x_{n-1}) = x_1 \ldots x_{n-1}\, x_0 + i$, the above construction results in the digraph D^* with vertex set $V^* = \mathbb{Z}_m \times (\mathbb{Z}_\delta)^n$ and the decomposition into permutations $\{\gamma_i^*,\ 0 \le i \le \delta - 1\}$ defined by:

$$\gamma_i^*(a, x_0 x_1 \ldots x_{n-1}) = (a + 1, x_1 \ldots x_{n-1}\, x_0 + i). \tag{17}$$

We call these digraphs $D_s(m,\delta,n)$ since they are the directed version of the graph $C_s(m,\delta,n)$ considered by Delorme and Fahri in [4]. The projection Φ is now

$$\begin{aligned} \Phi:\ D_s(m,\delta,n) &\longrightarrow B(\delta,n) \\ \Phi((a,x_0x_1\dots x_{n-1})) &= x_0x_1\dots x_{n-1}. \end{aligned} \tag{18}$$

The digraphs $D_s(m,\delta,n)$ can model a dynamic memory to store δ^n vectors of size m. Just assign to the component v_x^j, $0\le j\le m-1$, of vector v_x, $0\le x\le \delta^m-1$, the logical address $(j,x_0x_1\dots x_{n-1})$ if x equals $x_0x_1\dots x_{n-1}$ in base δ. Using the projection Φ we can consider that each vector v_x is stored in a cell of a n-dimensional Cayley digraph $B(\delta,n)$. Therefore, the control of $D_s(m,\delta,n)$ for vectors is equivalent to the control of $B(\delta,n)$ described in Section 4. The use of the permutations γ_i^* do not modify the cyclic order of the components of each vector.

Notice now that when $x_0=x_1=\cdots=x_{n-1}$, all the arcs of the m-cycle $(0,x)\to(1,x)\to\cdots\to(m-1,x)\to(0,x)$ correspond to γ_0^*. As a consequence, if the r/w cell is at one of these vertices, once we have attained a component of the vector, the permutation γ_0^* brings at that cell the m components of the vector in cyclic order. The maximum access time is $n+m-1$, that is, n time-unit intervals to access a component of the vector, and $m-1$ time-unit intervals to access the requested component in the worst case.

To control the memory, suppose that $(b,s_0s_1\dots s_{n-1})$ is the logical address of the datum at the r/w cell, at say $w=(a,00\dots0)$, after q permutations have been applied. Then:

(a) The knowledge of $s_0s_1\dots s_{n-1}$ and p ($\equiv q \bmod n$) gives the position of each vector—control in $B(\delta,n)$;

(b) $a-b$ ($\equiv q \bmod m$) gives the position of its components.

Therefore, the amount of information required to control the memory is $[m,n]\,\delta^n$ where $[m,n]=\operatorname{lcm}(m,n)$. Since $[m,n]\delta^n=m\delta^n(n/(m,n))$ (where $(m,n)=\gcd(m,n)$), besides the knowledge of the logical address of the datum in the r/w cell, we need to count modulo $\ell^*=n/(m,n)$ the number of permutations applied to the memory. Two interesting particular cases are:

(1) m divides n. The memory has only $|\Sigma^*|=m\delta^n$ states and it can be controlled with just the knowledge of the logical address of the datum in the r/w cell.

(2) $m = \delta = 2$. This case corresponds to a memory of 2^{n+1} cells and worst-case access time $n+1$ as in $B(2, n+1)$, but with a smaller average distance and requiring less amount of information to be controlled ($[2, n]2^n$ instead of $(n+1)2^{n+1}$).

Acknowledgement

This work has been supported by the Spanish Research Council under project MTM2008-06620-C03-01 and by the Catalan Research Council under project 2009SGR01387.

References

[1] A.V. Aho and J.D. Ullman. Dynamic memories with rapid random and sequential access. *IEEE Trans. Comput.*, C-23:272–276, 1974.

[2] J.M. Brunat. Kautz digraphs as group action graphs and Cayley digraphs. *Proc. IWONT 2010*, Barcelona, Spain, June 2010.

[3] G. Chartrand and L. Lesniak. *Graphs and Digraphs.* Wadsworth and Brooks, 1986.

[4] C. Delorme and G. Fahri. Large graphs with given degree and diameter —Part I. *IEEE Trans. Comput.*, C-33:857–860, 1984.

[5] J. Fàbrega, M.A. Fiol and J.L.A. Yebra. Permutation decomposition of dynamic memory networks. *Proc. ISMM Int. Conf. MIMI'84*, Bari, Italy, June 1984, 186–189.

[6] J. Fàbrega, M.A. Fiol and J.L.A. Yebra. Dynamic memory networks for random and block access. *Proc. ISMM Int. Conf. MIMI'85*, Sant Feliu de Guixols, Spain, June 1985, 204–207.

[7] M.A. Fiol, J. Fàbrega and J.L.A. Yebra. Cayley and Kautz digraphs as models for dynamic memory networks. *Proc. IV IAESTED Int. Symp. AI'86*, Insbruck, Austria, Feb. 1986, 62–65.

[8] M.A. Fiol, J. Fàbrega and J.L.A. Yebra. The design and control of dynamic memory networks. *Proc. 1988 IEEE Int. Symp. on Circuits and Systems*, Espoo, Finland, June 1988, 175–179.

[9] M.A. Fiol, J. Fàbrega, O. Serra and J.L.A. Yebra. A unified approach to the design and control of dynamic memory networks. *Parallel Process. Lett.*, 3:445–446, 1993.

[10] M.A. Fiol, J.L.A. Yebra and I. Alegre. Line digraph iterations and the (d, k) digraph problem. *IEEE Trans. Comput.*, C-33:400–403, 1984.

[11] B.R. Iyer and J.B. Sinclair. Dynamic memory interconnection for rapid access. *IEEE Trans. Comput.*, C-33:923–927, 1984.

[12] B.R. Iyer and J.B. Sinclair. Comparison of dynamic memory organizations: a tutorial. *J. VLSI Comput. Systems*, 1:217–241, 1985.

[13] W. Kluge. A dynamic memory with fast random access and page transfer properties. *Digital Processes* 1:279–293, 1975..

[14] A. Lempel. On a homomorphism of the de Bruijn graph and its application to the design of feedback shift registers. *IEEE Trans. Comput.*, C-19:1204–1209, 1970.

[15] J. Lenfant. Fast random and sequential access to dynamic memories of any size. *IEEE Trans. Comput.*, C-26:847–855, 1977.

[16] S.B. Morris, A. Valiere III and R.A. Wisniewski. Processes for random and sequential accessing in dynamic memories. *IEEE Trans. Comput.*, C-28:225–237, 1979.

[17] D.J. Robinson. *A Course in the Theory of Groups*. Springer, 1980.

[18] O. Serra, A.S. Lladó and M.A. Fiol. Fast cyclic shift register for dynamic memories of any size. *Proc. V IAESTED Int. Symp. AI'87*, Grindelwald, Switzerland, Feb. 1987, 43–46.

[19] O. Serra, M.A. Fiol. An optimal size network for dynamic memories. *Proc. ISMM Int. Conf. MIMI'88*, Sant Feliu de Guixols, Spain, June 1988, 287–291.

[20] H.S. Stone. Dynamic memories with enhanced data access. *IEEE Trans. Comput.*, C-21:359–366, 1972.

[21] H.S. Stone. *Discrete Mathematical Structures and their Applications*. Science Research Associates, 1973.

[22] H.S. Stone. Dynamic memories with fast random and sequential access. *IEEE Trans. Comput.*, C-24:1167–1174, 1975.

[23] C.K. Wong and D.T. Tang. Dynamic memories with faster random and sequential access. *IBM J. Res. Devel.*, 21: 281–288, 1977.

[24] J.L.A. Yebra, J. Fàbrega and M.A. Fiol. Interconnection networks for dynamic memories: Tours in de Bruijn digraphs. *3rme Colloque Int. Theorie des Graphes et Combinatoire*, Marseille, France, Jun. 1986.

Algebraic characterizations of bipartite distance-regular graphs

Miquel Àngel Fiol (invited speaker)
Universitat Politècnica de Catalunya
Barcelona

Abstract

Bipartite graphs are combinatorial objects bearing some interesting symmetries. Thus, their spectra—eigenvalues of its adjacency matrix—are symmetric about zero, as the corresponding eigenvectors come into pairs. Moreover, vertices in the same (respectively, different) independent set are always at even (respectively, odd) distance. Both properties have well-known consequences in most properties and parameters of such graphs. Roughly speaking, we could say that the conditions for a given property to hold in a general graph can be somehow relaxed to guaranty the same property for a bipartite graph. In this paper we comment upon this phenomenon in the framework of distance-regular graphs for which several characterizations, both of combinatorial or algebraic nature, are known. Thus, the presented characterizations of bipartite distance-regular graphs involve such parameters as the numbers of walks between vertices (entries of the powers of the adjacency matrix $\boldsymbol{A}$), the crossed local multiplicities (entries of the idempotents $\boldsymbol{E}_i$ or eigenprojectors), the predistance polynomials, etc. For instance, it is known that a graph G, with eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_d$ and diameter $D = d$, is distance-regular if and only if its idempotents $\boldsymbol{E}_1$ and $\boldsymbol{E}_d$ belong to the vector space $\mathcal{D}$ spanned by its distance matrices $\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}_2, \ldots \boldsymbol{A}_d$. In contrast with this, for the same result to be true in the case of bipartite graphs, only $\boldsymbol{E}_1 \in \mathcal{D}$ need to be required.

1 Preliminaries

Let $G=(V,A)$ be a (simple and connected) graph with adjacency matrix $\boldsymbol{A}$, and spectrum

$$\operatorname{sp} G=\operatorname{sp}\boldsymbol{A}=\{\lambda_0^{m_0},\lambda_1^{m_1},\dots,\lambda_d^{m_d}\}, \tag{1}$$

where the different eigenvalues of G are in decreasing order, $\lambda_0>\lambda_1>\cdots>\lambda_d$, and the superscripts stand for their multiplicities $m_i=m(\lambda_i)$. In particular, note that when G is δ-regular, the largest eigenvalue is $\lambda_0=\delta$ and has multiplicity $m_0=1$ (as G is connected). Moreover, all the multiplicities add up to $n=|V|$, the number of vertices of G.

Recall also that G is bipartite if and only if it does not contain odd cycles. Then, its adjacency matrix is of the form

$$\boldsymbol{A}=\begin{pmatrix}\boldsymbol{O} & \boldsymbol{B}\\ \boldsymbol{B}^\top & \boldsymbol{O}\end{pmatrix}.$$

(Here and hereafter, it is assumed that the block matrices have the appropriate dimensions.) Moreover, for any polynomial $p\in\mathbb{R}_d[x]$ with even and odd parts p_0 and p_1, we have

$$p(\boldsymbol{A})=p_0(\boldsymbol{A})+p_1(\boldsymbol{A})=\begin{pmatrix}\boldsymbol{C} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{D}\end{pmatrix}+\begin{pmatrix}\boldsymbol{O} & \boldsymbol{M}\\ \boldsymbol{M}^\top & \boldsymbol{O}\end{pmatrix}. \tag{2}$$

Also, the spectrum of G is symmetric about zero: $\lambda_i=-\lambda_{d-i}$ and $m_i=m_{d-i}$, $i=0,1,\dots,d$. (In fact, a well-known result states that a connected graph G is bipartite if and only if $\lambda_0=-\lambda_d$; see, for instance, Cvetković et al. [6].) This is due to the fact that, if $(u|v)$ is an eigenvector with eigenvalue λ_i, then $(u|-v)$ is an eigenvector for the eigenvalue $-\lambda_i$. As shown below, a similar symmetry also applies to the entries of the (*principal*) *idempotents* $\boldsymbol{E}_i$ representing the projections onto the eigenspaces $\mathcal{E}_i$, $i=0,1,\dots,d$. To see this, first recall that, for any graph with eigenvalue λ_i having multiplicity m_i, its corresponding idempotent can be computed as $\boldsymbol{E}_i=\boldsymbol{U}_i\boldsymbol{U}_i^\top$, where $\boldsymbol{U}_i$ is the $n\times m_i$ matrix whose columns form an orthonormal basis of $\mathcal{E}_i$. For instance, when G is δ-regular and has n vertices, its largest eigenvalue $\lambda_0=\delta$ has eigenvector $\boldsymbol{j}$, the all-1 vector, and corresponding idempotent $\boldsymbol{E}_0=\frac{1}{n}\boldsymbol{j}\boldsymbol{j}^\top=\frac{1}{n}\boldsymbol{J}$, where $\boldsymbol{J}$ is the all-1 matrix. Alternatively, we can also compute the idempotents as $\boldsymbol{E}_i=\lambda_i^*(\boldsymbol{A})$ where λ_i^* is the Lagrange interpolating polynomial of degree d satisfying $\lambda_i^*(\lambda_j)=\delta_{ij}$. That is,

$$\lambda_i^* = \frac{1}{\phi_i}\prod_{\substack{j=0\\ j\neq i}}^{d}(x-\lambda_j) = \frac{(-1)^i}{\pi_i}\prod_{\substack{j=0\\ j\neq i}}^{d}(x-\lambda_j)$$

where $\phi_i = \prod_{j=0,j\neq i}^{d}(\lambda_i-\lambda_j)$ and $\pi_i = |\phi_i|$. Then, the idempotents of $\boldsymbol{A}$ satisfy the known properties: $\boldsymbol{E}_i\boldsymbol{E}_j = \delta_{ij}\boldsymbol{E}_i$; $\boldsymbol{A}\boldsymbol{E}_i = \lambda_i\boldsymbol{E}_i$; and $p(\boldsymbol{A}) = \sum_{j=0}^{d} p(\lambda_j)\boldsymbol{E}_j$, for any polynomial $p\in\mathbb{R}[x]$ (see, for example, Godsil [16, p. 28]). In particular, taking $p=1$ we obtain, $\sum_{j=0}^{d}\boldsymbol{E}_j = \boldsymbol{I}$ (as expected), and for $p=x$ we have the *spectral decomposition theorem* $\boldsymbol{A} = \sum_{j=0}^{d}\lambda_j\boldsymbol{E}_j$. The entries of the idempotents $m_{uv}(\lambda_i) = (\boldsymbol{E}_i)_{uv}$ has been recently called *crossed uv-local multiplicities* and satisfy

$$a_{uv}^{(j)} = (\boldsymbol{A}^j)_{uv} = \sum_{i=0}^{d} m_{uv}(\lambda_i)\lambda_i^j. \tag{3}$$

(See [15, 8, 7]). In particular, when $u=v$, $m_u(\lambda_i) = m_{uu}(\lambda_i)$ are the so-called *local multiplicities* of vertex u, satisfying $\sum_{i=0}^{d} m_u(\lambda_i) = 1$, $u\in V$, and $\sum_{u\in V} m_u(\lambda_i) = m_i$, $i=0,1,\ldots,d$ (see [12]).

From any of the above expressions of $\boldsymbol{E}_i$ we deduce that, when G is bipartite, such parameters satisfy:

- $m_{uv}(\lambda_i) = m_{uv}(\lambda_{d-i})$, $\quad i=0,1,\ldots,d,\quad$ if $\mathrm{dist}(u,v)$ is even.
- $m_{uv}(\lambda_i) = -m_{uv}(\lambda_{d-i})$, $i=0,1,\ldots,d,\quad$ if $\mathrm{dist}(u,v)$ is odd.

In particular, the local multiplicities bear the same symmetry as the standard multiplicities: $m_u(\lambda_i) = m_u(\lambda_{d-i})$ for any vertex $u\in V$ and eigenvalue λ_i, $i=0,1,\ldots,d$.

Form the above, notice that, when G is regular and bipartite, we have $\boldsymbol{E}_0 = \frac{1}{n}\boldsymbol{J}$ (as mentioned before) and

$$\boldsymbol{E}_d = \frac{1}{n}\begin{pmatrix} \boldsymbol{J} & -\boldsymbol{J}\\ -\boldsymbol{J} & \boldsymbol{J}\end{pmatrix}. \tag{4}$$

2 Polynomials and regularity

The *predistance polynomials* $p_0,p_1,\ldots,p_d$, $\deg p_i = i$, associated to a given graph G with spectrum $\mathrm{sp}\,G$ as in (1), are a sequence of orthogonal polynomials with respect to the scalar product

$$\langle f, g\rangle = \frac{1}{n}\,\mathrm{tr}(f(\boldsymbol{A})g(\boldsymbol{A})) = \frac{1}{n}\sum_{i=0}^{d} m_i f(\lambda_i)g(\lambda_i),$$

normalized in such a way that $\|p_i\|^2 = p_i(\lambda_0)$ (this makes sense as it is known that always $p_i(\lambda_0) > 0$). Notice that, in particular, $p_0 = 1$ and, if G is δ-regular, $p_1 = x$. Indeed,

- $\langle 1, x\rangle = \frac{1}{n}\sum_{i=0}^{d} m_i\lambda_i = 0$.
- $\|1\|^2 = \frac{1}{n}\sum_{i=0}^{d} m_i = 1$.
- $\|x\|^2 = \frac{1}{n}\sum_{i=0}^{d} m_i\lambda_i^2 = \delta = \lambda_0$.

Moreover, if G is bipartite, the symmetry of such a scalar product yields that p_i is even (respectively, odd) for even (respectively, odd) degree i.

In terms of the predistance polynomials, the *preHoffman polynomial* is $H = p_0 + p_1 + \cdots + p_d$, and satisfies $H(\lambda_0) = n$ (the order of the graph) and $H(\lambda_i) = 0$ for $i = 1, 2, \dots, d$ (see Cámara et al. [5]). In [17], Hoffman proved that a (connected) graph G is regular if and only if $H(\boldsymbol{A}) = \boldsymbol{J}$, in which case H becomes the *Hoffman polynomial.* (In fact, H is the unique polynomial of degree at most d satisfying this property.) Furthermore, when G is regular and bipartite, the even and odd parts of H, H_0 and H_1, satisfy, by (2):

$$H_0(\boldsymbol{A}) = \begin{pmatrix} \boldsymbol{J} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{J} \end{pmatrix} \quad \text{and} \quad H_1(\boldsymbol{A}) = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{J} & \boldsymbol{O} \end{pmatrix}. \tag{5}$$

As far as we know, the following proposition is new and can be seen as the biregular counterpart of Hoffman's result. Recall that a bipartite graph $G = (V_1 \cup V_2, E)$ is called (δ_1, δ_2)-*biregular* when all the n_1 vertices of V_1 has degree δ_1, and the n_2 vertices of V_2 has degree δ_2. So, counting in two ways the number of edges $m = |E|$ we have that $n_1\delta_1 = n_2\delta_2$.

Proposition 1 *Let G be a bipartite graph with $n = n_1+n_2$ vertices, predistance polynomials $p_0, p_1, \dots, p_d$, and consider the odd part of its preHoffman polynomial; that is, $H_1 = \sum_{i \text{ odd}} p_i$. Then, G is biregular if and only if*

$$H_1(\boldsymbol{A}) = \alpha \begin{pmatrix} \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{J} & \boldsymbol{O} \end{pmatrix} \tag{6}$$

with $\alpha = \dfrac{n_1+n_2}{2\sqrt{n_1 n_2}}$.

Proof: Assume first that G is biregular with degrees, say, δ_1 and δ_2. Then, $\lambda_0 = -\lambda_d = \sqrt{\delta_1\delta_2}$ with respective (column) eigenvectors $\boldsymbol{u} = (\sqrt{\boldsymbol{\delta_1}}\boldsymbol{j}|\sqrt{\boldsymbol{\delta_2}}\boldsymbol{j})$ and $\boldsymbol{v} = (\sqrt{\boldsymbol{\delta_1}}\boldsymbol{j}| - \sqrt{\boldsymbol{\delta_2}}\boldsymbol{j})$, with the $\boldsymbol{j}$'s being all-1 vectors with appropriate lengths. Therefore, the respective idempotents are

$$\begin{aligned}
\boldsymbol{E}_0 &= \frac{1}{\|u\|^2}uu^\top = \frac{1}{n_1\delta_1+n_2\delta_2}\begin{pmatrix} \boldsymbol{\delta_1 J} & \sqrt{\boldsymbol{\delta_1\delta_2}}\boldsymbol{J} \\ \sqrt{\boldsymbol{\delta_1\delta_2}}\boldsymbol{J} & \boldsymbol{\delta_2 J} \end{pmatrix}, \\
\boldsymbol{E}_d &= \frac{1}{\|v\|^2}vv^\top = \frac{1}{n_1\delta_1+n_2\delta_2}\begin{pmatrix} \boldsymbol{\delta_1 J} & -\sqrt{\boldsymbol{\delta_1\delta_2}}\boldsymbol{J} \\ -\sqrt{\boldsymbol{\delta_1\delta_2}}\boldsymbol{J} & \boldsymbol{\delta_2 J} \end{pmatrix}.
\end{aligned}$$

As $H_1(x) = \frac{1}{2}[H(x) - H(-x)]$ and $H(\lambda_i) = n\delta_{0i}$, we have that $H_1(\lambda_0) = n/2$, $H_1(\lambda_i) = 0$ for $i \neq 0, d$, and $H_1(\lambda_d) = -n/2$. Hence, using the properties and the above expressions of the idempotents,

$$\begin{aligned}
H_1(\boldsymbol{A}) &= \sum_{i=0}^{d} H_1(\lambda_i)\boldsymbol{E}_i = H_1(\lambda_0)\boldsymbol{E}_0 + H_1(\lambda_d)\boldsymbol{E}_d \\
&= \frac{n}{2}(\boldsymbol{E}_0 - \boldsymbol{E}_d) = \frac{n\sqrt{\delta_1\delta_2}}{n_1\delta_1+n_2\delta_2}\begin{pmatrix} \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{J} & \boldsymbol{O} \end{pmatrix}.
\end{aligned}$$

Thus, the result follows since $n_1\delta_1 = n_2\delta_2$. Conversely, if (6) holds, and $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{B} \\ \boldsymbol{B}^\top & \boldsymbol{O} \end{pmatrix}$, the equality $\boldsymbol{A}H_1(\boldsymbol{A}) = H_1(\boldsymbol{A})\boldsymbol{A}$ yields

$$\begin{pmatrix} \boldsymbol{BJ} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{B}^\top\boldsymbol{J} \end{pmatrix} = \begin{pmatrix} \boldsymbol{JB}^\top & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{JB} \end{pmatrix}.$$

Thus, $(\boldsymbol{BJ})_{uv} = (\boldsymbol{JB}^\top)_{uv}$ implies that $\delta(u) = \delta(v)$ for any two vertices $u, v \in V_1$, whereas $(\boldsymbol{B}^\top\boldsymbol{J})_{wz} = (\boldsymbol{JB})_{wz}$ means that $\delta(w) = \delta(z)$ for any two vertices $w, z \in V_2$. Thus, G is biregular and the proof is complete. □

Notice that the constant α is the ratio between the arithmetic and geometric means of the numbers n_1, n_2. Hence, (6) holds with $\alpha = 1$ if and only if $n_1 = n_2$ or, equivalently, G is regular.

In fact, the above result could be reformulated (and proved) by saying that a (general) bipartite graph is connected and biregular if and only if there exists a polynomial satisfying (6).

3 Distance-regular graphs

Let G be a graph with diameter D, adjacency matrix $\boldsymbol{A}$ and $d+1$ distinct eigenvalues. Let $\boldsymbol{A}_i$, $i=0,1,\ldots,D$, be the distance-i matrix of G, with entries $(\boldsymbol{A}_i)_{uv}=1$ if $\operatorname{dist}(u,v)=i$ and $(\boldsymbol{A}_i)_{uv}=0$ otherwise. Then,

$$\mathcal{A}=\mathbb{R}_d[\boldsymbol{A}]=\operatorname{span}\{\boldsymbol{I},\boldsymbol{A},\boldsymbol{A}^2,\ldots,\boldsymbol{A}^d\}$$

is an algebra, with the ordinary product of matrices and orthogonal basis $\{\boldsymbol{E}_0,\boldsymbol{E}_1,\ldots,\boldsymbol{E}_d\}$ and $\{p_0(\boldsymbol{A}),p_1(\boldsymbol{A}),\ldots,p_d(\boldsymbol{A})\}$, called the *adjacency algebra*, whereas

$$\mathcal{D}=\operatorname{span}\{\boldsymbol{I},\boldsymbol{A},\boldsymbol{A}_2,\ldots,\boldsymbol{A}_D\}$$

forms an algebra with the entrywise or Hadamard product of matrices, defined by $(\boldsymbol{X}\circ\boldsymbol{Y})_{uv}=\boldsymbol{X}_{uv}\boldsymbol{Y}_{uv}$. We call $\mathcal{D}$ the *distance $\circ$-algebra*. Note that, when G is regular, $\boldsymbol{I},\boldsymbol{A},\boldsymbol{J}\in\mathcal{A}\cap\mathcal{D}$ since $\boldsymbol{J}=H(\boldsymbol{A})=\sum_{i=0}^{D}\boldsymbol{A}_i$. Thus, $\dim(\mathcal{A}\cap\mathcal{D})\geq 3$, if G is not a complete graph (in this exceptional case, $\boldsymbol{J}=\boldsymbol{I}+\boldsymbol{A}$). In this algebraic context, an important result is that G is distance-regular if and only if $\mathcal{A}=\mathcal{D}$, which is therefore equivalent to $\dim(\mathcal{A}\cap\mathcal{D})=d+1$ (and hence $d=D$); see, for instance, Biggs [2] or Brower et al. [4]. This leads to the following definitions of distance-regularity where, for types (a) and (b), p_{ji} and q_{ij}, $i,j=0,1,\ldots,d$, are constants, p_i, $i=0,1,\ldots,d$, are the predistance polynomials, and q_j, $j=0,1,\ldots,d$, are the polynomials defined by $q_j(\lambda_i)=m_j\frac{p_i(\lambda_j)}{p_i(\lambda_0)}$, $i,j=0,1,\ldots,d$:

(a) G distance-regular
$$\begin{aligned}
&\iff \boldsymbol{A}_i\boldsymbol{E}_j=p_{ji}\boldsymbol{E}_j, \qquad i,j=0,1,\ldots,d(=D),\\
&\iff \boldsymbol{A}_i=\sum_{j=0}^{d}p_{ji}\boldsymbol{E}_j, \qquad i=0,1,\ldots,d(=D),\\
&\iff \boldsymbol{A}_i=\sum_{j=0}^{d}p_i(\lambda_j)\boldsymbol{E}_j, \quad i=0,1,\ldots,d(=D),\\
&\iff \boldsymbol{A}_i\in\mathcal{A}, \quad i=0,1,\ldots,d(=D).
\end{aligned}$$

$$\begin{aligned}
(b)\ G \text{ distance-regular} \quad &\iff \quad \boldsymbol{E}_j \circ \boldsymbol{A}_i = q_{ij}\boldsymbol{A}_i, \qquad i,j = 0,1,\ldots,d,\\
&\iff \quad \boldsymbol{E}_j = \sum_{j=0}^{d} q_{ij}\boldsymbol{A}_i, \qquad j = 0,1,\ldots,d,\\
&\iff \quad \boldsymbol{E}_j = \frac{1}{n}\sum_{i=0}^{d} q_j(\lambda_i)\boldsymbol{A}_i, \quad j = 0,1,\ldots,d,\\
&\iff \quad \boldsymbol{E}_j \in \mathcal{D}, \quad j = 0,1,\ldots,d.
\end{aligned}$$

In fact, for general graphs with $D \leq d$, the conditions of type (a) are a characterization of the so-called *distance-polynomial graphs*, introduced by Weichsel [19] (see also Beezer [3] and Dalfó et al. [7]). This is equivalent to $\mathcal{D} \subset \mathcal{A}$ (but not necessarily $\mathcal{D} = \mathcal{A}$); that is, every distance matrix $\boldsymbol{A}_i$ is a polynomial in $\boldsymbol{A}$. In contrast with that, the conditions of type (b) are equivalent to $\mathcal{A} \subset \mathcal{D}$ and, hence, to $\mathcal{A} = \mathcal{D}$ (which implies $d = D$) as $\dim \mathcal{A} \geq \dim \mathcal{D}$.

Note also that in (a) (respectively, in (b)) the second implication is obtained from the first one by using that $\sum_{i=0}^{d} \boldsymbol{A}_i = \boldsymbol{J}$ (respectively, $\sum_{j=0}^{d} \boldsymbol{E}_j = \boldsymbol{I}$).

Moreover, with the $a_i^{(j)}$, $i,j = 0,1,\ldots,d$, being constants, we also have:

$$\begin{aligned}
(c)\ G \text{ distance-regular} \quad &\iff \quad \boldsymbol{A}^j \circ \boldsymbol{A}_i = a_i^{(j)}\boldsymbol{A}_i, \qquad i,j = 0,1,\ldots,d,\\
&\iff \quad \boldsymbol{A}^j = \sum_{i=0}^{d} a_i^{(j)}\boldsymbol{A}_i, \qquad j = 0,1,\ldots,d,\\
&\iff \quad \boldsymbol{A}^j = \frac{1}{n}\sum_{i=0}^{d}\sum_{l=0}^{d} q_{il}\lambda_l^j\boldsymbol{A}_i, \quad j = 0,1,\ldots,d,\\
&\iff \quad \boldsymbol{A}^j \in \mathcal{D}, \quad j = 0,1,\ldots,d,
\end{aligned}$$

where we have used (3) with $a_{uv}(j) = a_i^{(j)}$ and $m_{uv}(\lambda_l) = q_{il}$ for vertices u, v at distance $\text{dist}(u,v) = i$.

4 Characterizing bipartite distance-regular graphs

A general phenomenon is that the above conditions for being distance-regular can be relaxed giving more 'economic' characterizations (see [11]). Thus, the purpose of the following three theorems is twofold: First to show

how, for general graphs, such conditions can be relaxed if we assume some extra natural hypothesis (such as regularity) and, second, to study what happens in the case of bipartite graphs.

Theorem 2 (i) *A graph G with predistance polynomials $p_0, p_1, \ldots, p_d$ is distance-regular if and only if any of the following conditions holds:*

$(a1)$ $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ *for* $i = 2, 3, \ldots, d$.

$(a2)$ *G is regular and* $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ *for* $i = 2, 3, \ldots, d-1$.

$(a3)$ *G is regular and* $\boldsymbol{A}_d = p_d(\boldsymbol{A})$.

$(a4)$ *G is regular and* $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ *for* $i = d-2, d-1$.

(ii) *A bipartite graph G with predistance polynomials $p_0, p_1, \ldots, p_d$ is distance-regular if and only if*

$(a5)$ *G is regular and* $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ *for* $i = 3, 4, \ldots, d-2$.

Proof: Statement $(a1)$ with $i = 0, 1, \ldots, d$ is a well-known result; see, for example, Bannai and Ito [1]. For our case, just notice that always $p_0(\boldsymbol{A}) = \boldsymbol{A}_0 = \boldsymbol{I}$ and, as $\boldsymbol{I} + \boldsymbol{A} + \sum_{i=2}^{d} p_i(\boldsymbol{A}) = \boldsymbol{J}$, G is regular and hence $p_1(\boldsymbol{A}) = \boldsymbol{A}_1 = \boldsymbol{A}$; Condition $(a2)$ is a consequence of $(a1)$ taking into account that, under the hypotheses, $\boldsymbol{A}_d = \boldsymbol{J} - \sum_{i=0}^{d-1} \boldsymbol{A}_i = H(\boldsymbol{A}) - \sum_{i=0}^{d-1} p_i(\boldsymbol{A}) = p_d(\boldsymbol{A})$ (see Dalfó et al. [7]); $(a3)$ was first proved by Fiol et al. in [14] (see also van Dam [9] or Fiol et al. [13] for short proofs); and $(a4)$ is a consequence of a more general result in [7] characterizing m-partially distance-regularity (G is called m-partially distance-regular if $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ for any $i = 0, 1, \ldots, m$). Thus, we only need to prove $(a5)$. This is a consequence of $(a2)$ since, if G is δ-regular, $\boldsymbol{A}_2 = p_2(\boldsymbol{A}) = \boldsymbol{A}^2 - \delta \boldsymbol{I}$. Moreover, from (5) and assuming first that d is even,

$$\boldsymbol{A}_{d-1} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{J} & \boldsymbol{O} \end{pmatrix} - \sum_{\substack{i=1 \\ i \text{ odd}}}^{d-3} \boldsymbol{A}_i = H_1(\boldsymbol{A}) - \sum_{\substack{i=1 \\ i \text{ odd}}}^{d-3} p_i(\boldsymbol{A}) = p_{d-1}(\boldsymbol{A})$$

whereas, if d is odd,

$$\boldsymbol{A}_{d-1} = \begin{pmatrix} \boldsymbol{J} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{J} \end{pmatrix} - \sum_{\substack{i=0 \\ i \text{ even}}}^{d-3} \boldsymbol{A}_i = H_0(\boldsymbol{A}) - \sum_{\substack{i=0 \\ i \text{ even}}}^{d-3} p_i(\boldsymbol{A}) = p_{d-1}(\boldsymbol{A}),$$

and the proof is complete. □

The above results suggest the following question:

Problem 3 Prove or disprove: A regular bipartite graph G with predistance polynomial p_{d-1} is distance-regular if and only if $\boldsymbol{A}_{d-1} = p_{d-1}(\boldsymbol{A})$.

With respect to the characterizations of type (b), we can state the following result:

Theorem 4 (i) *A graph G with idempotents $\boldsymbol{E}_0, \boldsymbol{E}_1, \ldots, \boldsymbol{E}_d$ is distance-regular if and only if any of the following conditions holds:*

$(b1)$ $\boldsymbol{E}_j \in \mathcal{D}$ *for* $j = 0, 1, \ldots, d$.

$(b2)$ $\boldsymbol{E}_j \in \mathcal{D}$ *for* $j = 0, 1, \ldots, d-1$.

$(b3)$ *G is regular and* $\boldsymbol{E}_j \in \mathcal{D}$ *for* $j = 1, 2, \ldots, d-1$.

$(b4)$ *G is regular and* $\boldsymbol{E}_j \in \mathcal{D}$ *for* $j = 1, d$.

(ii) *A bipartite graph G with idempotent $\boldsymbol{E}_1$ is distance-regular if and only if*

$(b5)$ *G is regular and* $\boldsymbol{E}_1 \in \mathcal{D}$.

Proof: Statement $(b1)$ (see also (b) in Section 3) is also well-known and comes from the fact that G is distance-regular if and only if $\mathcal{A} = \mathcal{D}$; Condition $(b2)$ is a consequence of $(b1)$ since, under the hypotheses, $\boldsymbol{E}_d = \boldsymbol{I} - \sum_{j=0}^{d-1} \boldsymbol{E}_j \in \mathcal{D}$; $(b3)$ comes from $(b2)$ since, if G is regular, then $\boldsymbol{E}_0 = \frac{1}{n}\boldsymbol{J} = \frac{1}{n}H(\boldsymbol{A}) \in \mathcal{D}$; $(b4)$ was proved by the author in [10] (see also [11]). Finally, $(b5)$ can be seen as a consequence of $(b4)$ since, under the hypotheses, (4) yields

$$\boldsymbol{E}_d = \sum_{\substack{i=0 \\ i \text{ even}}}^{d} \boldsymbol{A}_i - \sum_{\substack{i=0 \\ i \text{ odd}}}^{d} \boldsymbol{A}_i \in \mathcal{D}$$

and the proof is complete. □

Now let us go to the characterizations of type (c) which are given in terms of the numbers $a_{uv}^{(j)} = (\boldsymbol{A}^j)_{uv}$ of walks of length $j \geq 0$ between vertices u, v at distance $\mathrm{dist}(u, v) = i$, $i = 0, 1, \ldots, D$. When such numbers do not depend on u, v but only on i and j, we write $a_{uv}^{(j)} = a_i^{(j)}$. In particular, notice that always $a_0^{(0)} = a_1^{(1)} = 1$ and G is δ-regular if and only if $a_2^{(2)} = \delta$.

Theorem 5 (i) *A graph G, with diameter D and $d+1$ distinct eigenvalues, is distance-regular if and only if, for any two vertices u, v at distance* $\mathrm{dist}(i,j) = i$, *any of the following conditions holds:*

($c1$) $a_{uv}^{(j)} = a_i^{(j)}$ *for $i = 0, 1, \ldots, D$ and $j \geq i$.*

($c2$) $a_{uv}^{(j)} = a_i^{(j)}$ *for $i = 0, 1, \ldots, D$ and $j = i, i+1, \ldots, d$.*

($c3$) *$D = d$, and $a_{uv}^{(j)} = a_i^{(j)}$ for $i = 0, 1, \ldots, D$ and $j = i, i+1, \ldots, d-1$.*

($c4$) *G is regular, $D = d$, and $a_{uv}^{(j)} = a_i^{(j)}$ for $i = 0, 1, \ldots, D-1$ and $j = i, i+1$.*

(ii) *A bipartite graph G is distance-regular if and only if*

($c5$) *G is regular, $D = d$, and $a_{uv}^{(j)} = a_i^{(j)}$ for $i = j = 2, 3, \ldots, D-2$.*

Proof: Characterization ($c1$) was first proved by Rowlinson [18]; Statement ($c2$) is a straightforward consequence of ($b1$) since $\mathcal{A} = \mathrm{span}\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^d\}$; ($c3$) comes from ($c2$) since, if G is regular and $D = d$, the number of d-walks between any two vertices u, v at distance d, is a constant:

$$a_{uv}^{(d)} = (\boldsymbol{A}^d)_{uv} = \frac{\pi_0}{n}[H(\boldsymbol{A})]_{uv} = \frac{\pi_0}{n}(\boldsymbol{J})_{uv} = \frac{\pi_0}{n} = a_d^{(d)};$$

($c4$) derives from a similar result in [10] (not requiring $D = d$) and the above reasoning on $a_{uv}^{(d)}$. Finally, ($c5$) is a consequence of ($c4$) since, when G is bipartite, there are no walks of length $j = i+1$ between vertices at distance i and, thus, $a_i^{(i+1)} = 0$. Moreover, if G is δ-regular and $D = d$, $a_{d-1}^{(d-1)} = \frac{1}{\delta} a_d^{(d)} = \frac{\pi_0}{n\delta}$. □

Problem 6 Give similar characterizations of types (a), (b) and (c) for distance biregular graphs.

Acknowledgement

Research supported by the Ministerio de Educación y Ciencia (Spain) and the European Regional Development Fund under project MTM2008-06620-C03-01, and by the Catalan Research Council under project 2009SGR1387.

References

[1] E. Bannai and T. Ito. *Algebraic Combinatorics I: Association Schemes.* Benjamin/Cummings, London, 1974, 1984.

[2] N. Biggs. *Algebraic Graph Theory.* Cambridge University Press, Cambridge, 1974, second edition, 1993.

[3] R.A. Beezer. Distance polynomial graphs, in *Proceedings of the Sixth Caribbean Conference on Combinatorics and Computing, Trinidad, 1991*, 51–73.

[4] A.E. Brouwer, A.M. Cohen, and A. Neumaier. *Distance-Regular Graphs.* Springer-Verlag, Berlin-New York, 1989.

[5] M. Cámara, J. Fàbrega, M.A. Fiol, and E. Garriga. Some families of orthogonal polynomials of a discrete variable and their applications to graphs and codes *Electron. J. Combin.* 16(1):#R83, 2009.

[6] D. M. Cvetković, M. Doob and H. Sachs. Spectra of Graphs, Theory and Application. VEB Deutscher Verlag der Wissenschaften, Berlin, second edition, 1982.

[7] C. Dalfó, E.R. van Dam, M.A. Fiol, E. Garriga, and B.L. Gorissen. On almost distance-regular graphs. *J. Combin. Theory Ser. A*, to appear.

[8] C. Dalfó, M.A. Fiol, and E. Garriga. On k-walk-regular graphs. *Electron. J. Combin.* 16(1):#R47, 2009.

[9] E.R. van Dam. The spectral excess theorem for distance-regular graphs: a global (over)view. *Electron. J. Combin.* 15(1):#R129, 2008.

[10] M.A. Fiol. On pseudo-distance-regularity. *Linear Algebra Appl.*, 323:145–165, 2001.

[11] M.A. Fiol. Algebraic characterizations of distance-regular graphs. *Discrete Math.*, 246:111–129, 2002.

[12] M.A. Fiol and E. Garriga. From local adjacency polynomials to locally pseudo-distance-regular graphs. *J. Combin. Theory Ser. B*, 71:162–183, 1997.

[13] M.A. Fiol, S. Gago, and E. Garriga. A simple proof of the spectral excess theorem for distance-regular graphs, *Linear Algebra Appl.*, 432:2418–2422, 2010.

[14] M.A. Fiol, E. Garriga, and J.L.A. Yebra. Locally pseudo-distance-regular graphs. *J. Combin. Theory Ser. B*, 68:179–205, 1996.

[15] M.A. Fiol, E. Garriga, and J.L.A. Yebra. Boundary graphs: The limit case of a spectral property. *Discrete Math.*, 226:155–173, 2001.

[16] C.D. Godsil. *Algebraic Combinatorics*. Chapman and Hall, NewYork, 1993.

[17] A.J. Hoffman. On the polynomial of a graph, *Amer. Math. Monthly*, 70:30–36, 1963.

[18] P. Rowlinson, Linear algebra, in *Graph Connections* (L.W. Beineke and R.J. Wilson, eds.), Oxford Lecture Ser. Math. Appl., Vol. 5, 86–99, Oxford Univ. Press, New York, 1997.

[19] P.M. Weichsel. On distance-regularity in graphs. *J. Combin. Theory Ser. B*, 32:156–161, 1982.

Topology of Cayley graphs applied to inverse additive problems

Yahya Ould Hamidoune (invited speaker)
Université Pierre and Marie Curie, Paris 06
Paris

Dedicated to M. A. Fiol

Abstract

We present the basic isopermetric structure theory, obtaining some new simplified proofs. Let $1 \le r \le k$ be integers. As an application, we obtain simple descriptions for the subsets S of an abelian group with $|kS| \le k|S|-k+1$ or $|kS-rS|-(k+r)|S|$, where where ℓS denotes as usual the Minkowski sum of ℓ copies of S. These results may be applied to several questions in Combinatorics and Additive Combinatorics including the Frobenius Problem, Waring's problem in finite fields and the structure of abelian Cayley graphs with a big diameter.

1 Introduction

The connectivity of a graph is just the smallest number of vertices disconnecting the graph. In order investigate more sophisticated properties of graphs, several authors proposed generalizations of connectivity. The reader may find details on this investigation in the chapter [2]. Investigating the isoperimetric connectivity in Cayley graphs is just one of the many facets of Additive Combinatorics. It is also one of the many facets of Network topology. For space limitation, we concentrate on Additive Combinatorics, but the reader may find details and a bibliography in the recent paper [15] concerning the other aspect.

Let $\Gamma = (V, E)$ be a reflexive graph. The minimum of the objective function $|\Gamma(X)|-|X|$, restricted to subsets X with $|X| \ge k$ and $|V \setminus \Gamma(X)| \ge$

k, the k-isoperimetric connectivity. Subsets achieving the above minimum are called k-fragments. k-fragments with smallest cardinality are called k-atoms. It was proved by the author in [7], that distinct k-atoms of Γ intersect in at most $k-1$ elements, if the size of the k-atom of Γ is not greater than the size of the k-atom of Γ^{-1}. Let $1 \in S$ be a finite generating subset of a group G such that the cardinality 1-atom of the Cayley graph defined by S is not greater than the cardinality 1-atom of the Cayley graph defined by S^{-1}. Then a 1-atom H containing 1 is a subgroup. The last result applied to a group with a prime order is just the Cauchy-Davenport Theorem. It has several other implications and leads to few lines proof for result having very tedious proof using the classical transformations. In particular, it was applied recently by the author [14] to a problem of Tao [19].

In the abelian case, things are much easier. Assume that G is abelian and let $1 \in H$ be a k-atom of the Cayley graph defined by S. If $k = 1$, then H a subgroup (the condition involving S^{-1} is automatically verified). In particular, there is a subgroup which is a 1-fragment. A maximal such a group is called an hyper-atom. Assuming now that $k = 2$ and that $\kappa_2 \leq |S|-1$. It was proved in [8] that either $|H| = 2$ or H is a subgroup. It was proved also in [8] that either S is an arithmetic progression or there is a non-zero subgroup which is a 1-fragment, if $|S| \leq (|G|+1)/2$. Let Q be a hyper-atom of S and let $\phi : G \mapsto G/Q$ denotes the canonical morphism. The author proved in [12] that $\phi(S)$ is either an arithmetic progression or satisfies the sharp Vosper property (to be defined later) if $|S| \leq (|G|+1)/2$.

Let G be an abelian group and let A, B be finite non-empty subsets of G, with $|A+B| = |A|+|B|-1-\mu$. Kneser's Theorem states that $\pi(A+B) \neq \{0\}$, where $\pi(A+B) = \{x : x+A+B = A+B\}$. The hard Kemperman Theorem, which needs around half a page to be formulated, describes recursively the subsets A and B if $\mu = 1$. Its classical proof requires around 30 pages. It was applied by Lev [18] to propose a dual description, that looks easier to implement than Kemperman's description.

The above structure isoperimetric results were used in [12, 13] to explain the topological nature of Kemperman Theory and to give a shorter proof of it. Our method involve few technical steps and use some duality arguments and the strong isoperimetric property. We suspect that it could be drastically simplified. In this paper, we shall verify this hypothesis for Minkowski sums of the form $rS - sS$, obtaining very simple proofs and tight descriptions. This case covers almost all the known applications. Also, Modern Additive Combinatorics deals almost exclusively with $rS - sS$, c.f. [20].

The organization of the paper is the following:

Section 2 presents the isoperimetric tools, with complete proofs in order to make the paper self-contained. In particular, this section contains a proof of the fundamental property of k-atoms. In Section 3, we start by showing the structure of 1-atoms of arbitrary Cayley graphs. We then restrict ourselves to the abelian case. We give in this section an new simplified proof for the structure theorem of 2-atoms. We deduce from it the structure of hyper-atoms. In Section 4, we give easy properties of the decomposition modulo a subgroup which is a fragment. Easy proofs of the Kneser's theorem and a Kemperman type result for kA are then presented.

In Section 5, we investigate universal periods for kS introducing a new object: the sub-atom. It follows from a result by Balandraud [1] that $|TS| \leq |T| + |S| - 2$ implies that $T + S + K = T + S$, where K is the final kernel of S (a subgroup contained in the atom of S described in [1]). We shall prove that the $kS + M = kS$, if $|kS| \leq k|S| - k$, where M is the sub-atom. Clearly $K \subset M$. The case $rS - sS$, where $r \geq s \geq 1$, is solved easily in Section 5, by showing that one of the following holds

- S is an arithmetic progression,
- $|sS - rS| \geq \min(|G| - 1, (r + s)|S|)$,
- $|H| \geq 2$ and $sS - rS + H = sS - rS$, where H is an hyper-atom of S.

Readers familiar with Kemperman Theory could appreciate the simplicity of this result. In Section 6, we obtain the following description:

Let $k \geq 3$ be an integer and let $0 \in S$ be a finite generating subset of an abelian group G such that S is not an arithmetic progression, kS is aperiodic and $|kS| = k|S| - k + 1$. Let H be a hyper-atom of S and let S_0 be a smaller H-component of S. Then $(S \setminus S_0) + H = (S \setminus S_0)$ and $|kS_0| = k|S_0| - k + 1$. Moreover $\phi(S)$ is an arithmetic progression, where $\phi : G \mapsto G/H$ denotes the canonical morphism.

Necessarily $|H| \geq 2$, since S is not an arithmetic progression.

2 Basic notions

Recall a well known fact:

Lemma 1 *(folklore) Let a, b be elements of a group G and let H be a finite subgroup of G. Let A, B be subsets of G such that $A \subset aH$ and $B \subset Hb$. If $|A| + |B| > |H|$, then $AB = aHb$.*

Let H be a subgroup of an abelian group G. Recall that a H–*coset* is a set of the $a + H$ for some $a \in G$. The family $\{a + H; a \in G\}$ induces a partition of G. The trace of this partition on a subset A will be called an H–*decomposition* of A.

By a graph, we shall mean a directed graph, identified with its underlying relation. Undirected graphs are identified with symmetric graphs. We recall the definitions in this context.

An ordered pair $\Gamma = (V, E)$, where V is a set and $E \subset V \times V$, will be called a *graph* or a *relation* on V. Let $\Gamma = (V, E)$ be a graph and let $X \subset V$. The *reverse* graph of Γ is the graph $\Gamma^- = (V, E^-)$, where $E^- = \{(x, y) : (y, x) \in E\}$. The *degree* (called also outdegree) of a vertex x is

$$d(x) = |\Gamma(x)|.$$

The graph Γ will be called *locally-finite* if for all $x \in V$, $|\Gamma(x)|$ and $|\Gamma^-(x)|$ are finite. The graph Γ is said to be r*-regular* if $|\Gamma(x)| = r$, for every $x \in V$. The graph Γ is said to be r*-reverse-regular* if $|\Gamma^-(x)| = r$, for every $x \in V$. The graph Γ is said to be r*-bi-regular* if it is r*-regular* and r*-reverse-regular*.

- The minimal degree of Γ is defined as $\delta(\Gamma) = \min\{|\Gamma(x)| : x \in V\}$.
- We write $\delta_{\Gamma^-} = \delta_-(\Gamma)$.
- The boundary of X is defined as $\partial_\Gamma(X) = \Gamma(X) \setminus X$.
- The exterior of X is defined as $\nabla_\Gamma(X) = V \setminus \Gamma(X)$.
- We shall write $\partial^-_\Gamma = \partial_{\Gamma^-}$. This subset will be called the *reverse-boundary* of X.
- We shall write $\nabla^-_\Gamma = \nabla_{\Gamma^-}$.

In our approach, $\Gamma(v)$ is just the image of v by the relation Γ and $\Gamma^-(v)$ requires no definition since Γ^- is defined in Set Theory as the reverse of Γ.

An automorphism of a graph $\Gamma = (V, E)$ is a permutation f of V such that $f(\Gamma(v)) = \Gamma(f(v))$, for any vertex v. A graph $\Gamma = (V, E)$ is said to be *vertex-transitive* if for any ordered pair of vertices there is an automorphism mapping the first one to the second.

Let A, B be subsets of a group G. The *Minkowski product* of A with B is defined as

$$AB = \{xy \ : \ x \in A \text{ and } y \in B\}.$$

Let S be a subset of G. The subgroup generated by S will be denoted by $\langle S\rangle$. The graph (G, E), where $E = \{(x, y) : x^{-1}y \ \in S\}$ is called a *Cayley graph*. It will be denoted by $\mathrm{Cay}(G, S)$. Put $\Gamma = \mathrm{Cay}(G, S)$ and let $F \subset G$. Clearly $\Gamma(F) = FS$. One may check easily that left-translations are automorphisms of Cayley graphs. In particular, Cayley graphs are vertex-transitive.

Let $\Gamma = (V, E)$ be a reflexive graph. We shall investigate the *boundary operator* $\partial_\Gamma : 2^V \to 2^V$. When the context is clear, the reference to Γ will be omitted. Since Γ is reflexive, we have in this case $|\partial(X)| = |\Gamma(X)| - |X|$.

Let $\mathcal{A} \subset 2^V$ be a family of finite subsets of V. We define the *connectivity* of $\mathcal{A}$ as

$$\kappa(\mathcal{A}) = \min\{|\partial(X)| \ : \ X \in \mathcal{A}\}.$$

An $X \in \mathcal{A}$ with $\kappa(\mathcal{A}) = |\partial(X)|$ will be called a *fragment*.

A fragment with a minimal cardinality will be called an *atom*.

Put

$$\mathcal{S}_k(\Gamma) = \{X : k \leq |X| < \infty \text{ and } |\Gamma(X)| \leq |V| - k \ \}.$$

We shall say that Γ is *k-separable* if $\mathcal{S}_k(\Gamma) \neq \emptyset$. In this case, we write

$$\kappa_k(\Gamma) = \kappa(\mathcal{S}_k).$$

By a *k-fragment* (resp. *k-atom*), we shall mean a fragment (resp. atom) of $\mathcal{S}_k$. A k-fragment of Γ^{-1} is sometimes called a *k-negative*) fragment. This notion was introduced by the author in [7]. A relation Γ will be called *k-faithful* if $|A| \leq |V \setminus \Gamma(A)|$, where A is a k-atom of Γ. By a *fragment* (resp. *atom*), we shall mean a 1-fragment (resp. 1- atom).

The following lemma is immediate from the definitions:

Lemma 2 [7] *Let $k \geq 2$ be an integer. A reflexive locally finite k-separable graph $\Gamma = (V, E)$ is a $k-1$-separable graph, and moreover $\kappa_{k-1} \leq \kappa_k$.*

Recall the following easy fact:

Lemma 3 [7] *Let $\Gamma = (V, E)$ be a locally-finite k-separable graph and let A be a k-atom with $|A| > k$. Then $\Gamma^-(x) \cap A \neq \{x\}$, for every $x \in A$.*

Proof: We can not have $\Gamma^-(x) \cap A = \{x\}$, otherwise $A \setminus \{x\}$ would be a k-fragment. $\square$

The next lemma contains useful duality relations:

Lemma 4 [9] *Let X and Y be k-fragments of a reflexive locally finite k-separable graph $\Gamma = (V, E)$. Then*

$$\partial^-(\nabla(X)) = \partial(X), \tag{1}$$
$$\nabla^-(\nabla(X)) = X, \tag{2}$$

Proof: Clearly, $\partial(X) \subset \partial^-(\nabla(X))$

We must have $\partial(X) = \partial^-(\nabla(X))$, since otherwise there is a $y \in \partial^-(\nabla(X)) \setminus \partial(X)$. It follows that $|\partial(X \cup \{y\})| \leq |\partial(X)| - 1$, contradicting the definition of κ_k. This proves (1).

Thus $\Gamma^-(\nabla(X)) = \nabla(X) \cup \partial^-(\nabla(X)) = \nabla(X) \cup \partial(X) = V \setminus X$, and hence (2) holds.

Let $\Gamma = (V, E)$ be a reflexive graph. We shall say that Γ is a *Cauchy graph* if Γ is non-1-separable or if $\kappa_1(\Gamma) = \delta - 1$. h We shall say that Γ is a *reverse-Cauchy graph* if Γ^- is a Cauchy graph.

Clearly, Γ is a Cauchy graph if and only if for every $X \subset V$ with $|X| \geq 1$,

$$|\Gamma(X)| \geq \min\Big(|V|, |X| + \delta - 1\Big). \quad \square$$

Lemma 5 [7] *Let $\Gamma = (V, E)$ be a reflexive finite k-separable graph and let X be a subset of V. Then*

$$\kappa_k = \kappa_{-k}. \tag{3}$$

Moreover,

(i) *X is a k-fragment if and only if $\nabla(X)$ is a k-reverse-fragment,*

(ii) *Γ is a Cauchy graph if and only if it is a reverse-Cauchy graph.*

Proof: Observe that a finite graph is k-separable if and only if its reverse is k-separable. Take a k-fragment X of Γ. We have clearly $\partial_-(\nabla(X)) \subset \partial(X)$. Therefore

$$\kappa_k(\Gamma) \geq |\partial(X)| \geq |\partial^-(\nabla(X))| \geq \kappa_{-k}.$$

The reverse inequality of (3) follows similarly or by duality.

Suppose that X is a k-fragment. By (1) and (3), $|\partial_-(\nabla(X))| = |\partial(X)| = \kappa_k = \kappa_{-k}$, and hence $\nabla(X)$ is a revere k-fragment. The other implication of (i) follows easily. Now (ii) follows directly from the definitions. $\square$

Theorem 6 [7] *Let $\Gamma = (V, E)$ be a reflexive locally-finite k-faithful k-separable graph. Then the intersection of two distinct k-atoms X and Y has a cardinality less than k. Moreover, any locally-finite k-separable graph is either k-faithful or reverse k-faithful.*

Proof:

$\cap$	Y	$\partial(Y)$	$\nabla(Y)$
X	R_{11}	R_{12}	R_{13}
$\partial(X)$	R_{21}	R_{22}	R_{23}
$\nabla(X)$	R_{31}	R_{32}	R_{33}

Assume that $|X \cap Y| \geq k$. By the definition of κ_k,

$$\begin{aligned} |R_{21}| + |R_{22}| + |R_{23}| &= \kappa_k \\ &\leq |\partial(X \cap Y)| \\ &= |R_{12}| + |R_{22}| + |R_{21}|, \end{aligned}$$

and hence

$$|R_{23}| \leq |R_{12}|. \tag{4}$$

Thus,

$$\begin{aligned} |\nabla(X) \cap \nabla(Y)| &= |\nabla(Y)| - |R_{13}| - |R_{23}| \\ &\geq |Y| - |R_{13}| - |R_{12}| \\ &= |X| - |R_{13}| - |R_{12}| = |R_{11}| \geq k. \end{aligned}$$

Thus,

$$\begin{aligned}|R_{12}|+|R_{22}|+|R_{32}| &= \kappa_k\\ &\leq |\partial(X\cup Y)|\\ &\leq |R_{22}|+|R_{23}|+|R_{32}|,\end{aligned}$$

and hence $|R_{12}| \leq |R_{23}|$, showing that $|R_{12}| = |R_{23}|$.

It follows that

$$\kappa_k \leq |\partial(X\cap Y)| \leq |R_{12}|+|R_{22}|+|R_{21}| \leq |R_{12}| \leq |R_{23}|+|R_{22}|+|R_{21}| = \kappa_k,$$

showing that $X \cap Y$ is a k-fragment, a contradiction.

The fact that a locally-finite k-separable graph is either k-faithful or reverse k-faithful follows by Lemma 5. □

3 A structure Theory for atoms

In the sequel, we identify $\mathrm{Cay}(\langle S\rangle, S)$ with S, if $0 \in S$. We shall even work with subsets not containing 1. By $\kappa_k(S)$ we shall mean $\kappa_k(S-a) = \kappa_k(\mathrm{Cay}(\langle S-\rangle, S-a))$, for some $a \in S$. As an exercise, the reader may check that this notion does not depend on a particular choice of $a \in S$.

Theorem 7 *[6] Let $1 \in S$ be a finite proper generating subset of a group G. Let $1 \in H$ be a 1-atom of S.*

(i) *If S is 1–faithful, then H is a subgroup. Moreover $|H|$ divides $\kappa_1(S)$.*

(ii) *If G is abelian and if S is k-separable, then S is k-faithful.*

(iii) *If G is abelian, then H is a subgroup.*

Proof: Take an element $x \in H$. Clearly xH is a 1-atom. Since $(xH)\cap H \neq \emptyset$, we have by Theorem 6, $xH = H$. Since H is finite, H is a subgroup. Now $\kappa_1(S) = |HS| - |H|$, showing the last part of (i).

If G is abelian, then $\mathrm{Cay}(G,S)$ is isomorphic to $\mathrm{Cay}(G,-S)$, and hence S is k-faithful if S is k-separable. Now (iii) follows by combining (i) and (ii). □

Theorem 8 [8] *Let S be a finite generating 2–separable subset of an abelian group G with $0 \in S$ and $\kappa_2(S) \leq |S|-1$. If $0 \in H$ is a 2–atom with $|H| \geq 3$, then H is a subgroup.*

Proof: The proof is by induction. Assume first that $H+Q = H$, where Q is a non-zero subgroup. For every, $x \in H$, we have $|(H+x)\cap H| \geq |x+Q| \geq 2$. By Theorems 7 and 6, $H+x = H$. It follows that H is a subgroup. Assume now that H is aperiodic. Let us first show that $\kappa_1(H) = |H|-1$. Suppose the contrary and take a 1-atom L of H with $0 \in L$. By Theorem 7, L is a subgroup and $|L| \leq \kappa_1(H)$. Take a nonzero element $y \in L$. We have $|H \cup (y+H)| \leq |L+H| = |L| + \kappa_1(H) \leq 2\kappa_1(H) \leq 2|H|-4$. Thus, $|H \cap (y+H)| \geq 2$, and hence $y+H = H$, by Theorem 6.

Take an N-decomposition $S = \bigcup\limits_{1\leq i\leq s} S_i$, with $|S_1+H| \leq \cdots \leq |S_s+H|$. Without loss of generality, we may take $0 \in S_1$. We have necessarily $s \geq 2$. We must have $|S_i| = |N|$, for all $i \geq 2$. Suppose the contrary. By the definition of κ_1, we have $|S_1+H| \geq |S_1|+\kappa_1(H) = |S_1|+|H|-1$. We have also, since H generates N, $|S_i+H| \geq |S_1|+1$. Thus, $|S+H| \geq |S|+|H|-1+1 \geq |S|+|H|$, a contradiction. Now we have $|X+S| = |S\setminus S_1|+|X+S_1$, for any subset X of N. In particular, H is a 2-atom of S_1. If $|S_1| < |S|$, the result holds by Induction. It remains to consider the case $s = 1$.

The relation $|H+S|-|H| \leq |S|-1$ implies that $\kappa_2(H) \leq |H|-1$. By Lemma 3, for every $x \in H$, there $s_x \in S\setminus\{0\}$, with $x - s_x \in H$. We must have

$$|H| \leq |S|-1,$$

otherwise there are distinct elements $x, y \in H$ and an element $s \in S\setminus\{0\}$ such that $x-s, y-s \in H$. It follows that $|(H+s)\cap H| \geq 2$. By Theorem 6, $H+s = H$, a contradiction.

Let $0 \in M$ be a 2-atom of H. Take a non-zero element $a \in M$. Since $\kappa_2(H) = |M+H|-|M|$, $|M|$ divides $\kappa_2(H)$ if M is a subgroup. Thus, the Induction hypothesis implies that $|M| \leq |H|-1$. Since $|M+H| \leq |M|+\kappa_2(H) \leq 2|H|-2$, we have $|H\cap(H+a)| \geq 2$. By Theorem 6, $H+a = H$, a contradiction. □

Theorem 9 ([8],Theorem 4.6) *Let S be a 2–separable finite subset of an abelian group G such that $0 \in S$, $|S| \leq (|G|+1)/2$ and $\kappa_2(S) \leq |S|-1$. If S is not an arithmetic progression, then there is a subgroup which is a 2–fragment of S.*

Proof: Suppose that S is not an arithmetic progression.

Let H be a 2–atom such that $0 \in H$. If $\kappa_2 \leq |S| - 2$, then clearly $\kappa_2 = \kappa_1$ and H is also a 1–atom. By Theorem 7, H is a subgroup. Then we may assume

$$\kappa_2(S) = |S| - 1.$$

By Theorem 8, it would be enough to consider the case $|H| = 2$, say $H = \{0, x\}$. Put $N = \langle x \rangle$.

Decompose $S = S_0 \cup \cdots \cup S_j$ modulo N, where $|S_0 + H| \leq |S_1 + H| \leq \cdots \leq |S_j + H|$. We have $|S| + 1 = |S + H| = \sum\limits_{0 \leq i \leq j} |S_i + \{0, x\}|$.

Then $|S_i| = |N|$, for all $i \geq 1$. We have $j \geq 1$, since otherwise S would be an arithmetic progression. In particular, N is finite. We have $|N + S| < |G|$, since otherwise $|S| \geq |G| - |N| + 1 \geq \frac{|G|+2}{2}$, a contradiction.

Now

$$\begin{aligned} |N| + |S| - 1 &= |N| + \kappa_2(S) \\ &\leq |S + N| = (j+1)|N| \\ &\leq |S| + |N| - 1, \end{aligned}$$

and hence N is a 2-fragment. $\square$

Theorem 9 was used to solve Lewin's Conjecture on the Frobenius number [10].

A H–decomposition $A = \bigcup\limits_{i \in I} A_i$ will be called a *H–modular-progression* if it is an arithmetic progression modulo H.

Recall that S is a Vosper subset if and only if S is non 2–separable or if $\kappa_2(S) \geq |S|$.

Theorem 10 [12] *Let S be a finite generating subset of an abelian group G such that $0 \in S$, $|S| \leq (|G|+1)/2$ and $\kappa_2(S) \leq |S| - 1$. Let H be a hyperatom of S. Then $\phi(S)$ is either an arithmetic progression or a Vosper subset, where ϕ is the canonical morphism from G onto G/H.*

Proof: Let us show that

$$2|\phi(S)| - 1 \leq \frac{|G|}{|H|}. \tag{5}$$

Clearly we may assume that G is finite.

Observe that $2|S+H|-2|H| \leq 2|S|-2 < |G|$. It follows, since$|S+H|$ is a multiple of $|H|$, that $2|S+H| \leq |G|+|H|$, and hence (5) holds.

Suppose now that $\phi(S)$ is not a Vosper subset. By the definition of a Vosper subset, $\phi(S)$ is 2–separable and $\kappa_2(\phi(S)) \leq |\phi(S)|-1$.

Let us show that $\phi(S)$ has no 1–fragment M which is a non-zero subgroup. Assuming the contrary. We have $|\phi(\phi^{-1}(M)+S)| = |M+\phi(S)| \leq |M|+|\phi(S)|-1$. Thus, $|\phi^{-1}(M)+S| \leq |\phi^{-1}(M)|+|H|(|\phi(S)|-1) = |\phi^{-1}(M)|+\kappa_1(S)$. It follows that $\phi^{-1}(M)$ is a 1-fragment. By the maximality of H, we have $|M|=1$, a contradiction. By (5) and Theorem 9, $\phi(S)$ is an arithmetic progression. □

4 Decomposition modulo a fragment

Let H be a subgroup of an abelian group G. Recall that a H*–coset* is a set of the $a+H$ for some $a \in G$. The family $\{a+H; a \in G\}$ induces a partition of G. A non-empty set of the form $A \cap (x+H)$ will be called a H*-component* of A. The partition of A into its H-components will be called a H*-decomposition* of A. By a *smaller* component, we shall mean a component with a smallest cardinality.

Assume now that H is 1-fragment and take a H-decomposition $S = S_0 \cup \cdots \cup S_u$, with $|S_0| \leq \cdots \leq |S_u|$.

We have $|S|-1 \geq \kappa(S) = |H+S|-|H|$.

It follows that for $i \geq 1$, we have

$$2|H|-|S_0|-|S_i| \leq |H+S|-|S| \leq |H|-1,$$

and hence $|S_0|+|S_i| \geq |H|+1$. In particular,

for all $(i,j) \neq (0,0)$, $|S_i|+|S_j| \geq |H|+1$, hence

$$S_i+S_j+H = S_i+S_j,$$

by Lemma 1.

Thus

$$(S \setminus S_0)+S = (S \setminus S_0)+H+S.$$

Similarly

$$((S \setminus S_0))-S = (S \setminus S_0)+H-S.$$

Since $S_0 - S_0 \subset S_1 - S_1 = H$, we have

$$S - S + H = S - S.$$

In particular, $(kS \setminus kS_0) + H = kS \setminus kS_0$.

Proposition 11 *Let S_0 denotes a smaller H-component of S, where H is a non-zero subgroup fragment. We have $S - S + H = S - S$. Let $2 \leq k$ be an integer. Then $(S \setminus S_0) + (k-1)S$ is H-periodic subset with cardinality at least* $\min(|G|, k|S+H| - k|H|)$. *If $kS + H \neq kS$, then $|S_1| > |H|/2 \geq |S_0|$, and $|kS| \geq k|S+H| - k|H| + |kS_0|$. Moreover kS_0 is aperiodic if kS is aperiodic.*

Proof: The first part was proved above. By the definition of κ, we have $|(S \setminus S_0) + (k-1)S| = |(S \setminus S_0)H + (k-1)S| \geq u|H| + (k-1)\kappa = k|S+H| - k|H|$.

Assume now that $kS + H \neq kS$. we have $kS_0 \neq kS_0 + H$, and hence $2S_0 \neq 2S_0 + H$, since $(S \setminus S_0) + (k-1)S$ is H-periodic. By Lemma 1, $|H|/2 \geq |S_0|$. We have now $|S_1| \geq |H| + 1 - |S_0| \geq |H|/2 + 1$. We must also have $kS_0 \cap ((S \setminus S_0) + (k-1)S) = \emptyset$. Thus, $|kS| \geq |(S \setminus S_0) + (k-1)S| + |kS_0| \geq k|S+H| - k|H| + |kS_0|$.

Assume now that kS is aperiodic. Since $(S \setminus S_0) + (k-1)S$ is H-periodic and since the period of kS_0 is a subgroup of H, necessarily kS_0 is aperiodic.
$\square$

Corollary 12 (Kneser, [17]) *Let k be a non-negative integer and let S be a finite subset of an abelian group G. If kS is aperiodic, then $|kS| \geq k|S| - k + 1$*

Proof: Let H be a 1-atom containing 0. By Theorem 7, H is subgroup. Let S_0 denotes a smaller H-component of S. Without loss of generality we may assume that $0 \in S_0$. We may assume $\kappa(S) \leq |S| - 2$, since otherwise $|kS| \geq |S| + (k-1)\kappa(S) = k|S| - k + 1$, and the result holds.

By Proposition 11, kS_0 is aperiodic. By the Induction hypothesis and Proposition 11, $|kS| = |kS_0| + (k-1)(|S+H| - |H|) \geq k|S_0| - k + 1 + (k-1)(|S+H| - |H|) \geq k|S| - k + 1$. $\square$

We shall now complete Proposition 11 in order to deal with the critical pair Theory.

Proposition 13 *Let $2 \leq k$ be an integer. Let S_0 denotes a smaller H-component of S, where H is a non-zero subgroup fragment $kS + H \neq kS$. Assume moreover that kS is aperiodic and $|kS| = k|S| - k + 1$. Then*

(i) *kS_0 is aperiodic,*

(ii) $|kS_0| = k|S_0| - k + 1$,

(iii) $(S \setminus S_0) + H = S \setminus S_0$ *and*

(iv) $|k(S + H)| = k|S + H| - k|H| + |H|$.

Proof: (i) follows by Proposition 11. By Kneser Theorem and Proposition 11,

$$\begin{aligned} k|S| - k + 1 = |kS| &\geq |kS_0| + |(k-1)S + (S \setminus S_0)| \\ &\geq |kS_0| + k|S + H| - k|H| \\ &\geq k|S_0| - k + 1 + k|S + H| - k|H| \geq k|S| - k + 1. \end{aligned}$$

In particular, the inequalities used are equalities and hence (ii) holds and $|S| = |S + H| - |H| + |S_0|$, proving (iii). Also, it follows that $|kS + H| = |(k-1)S + (S \setminus S_0)| + |H| = k|S + H| - k|H| + |H|$, proving (iii). □

We can deduce now a Kemperman type result for kS.

Corollary 14 *Let $k \geq 2$ be an integer and let S be a finite subset of an abelian group G such that kS is aperiodic and $|kS| = k|S| - k + 1$. There is a non-zero subgroup H such $(S \setminus S_0) + H = (S \setminus S_0)$, where S_0 is an H-component of S. Also, $|kS_0| = k|S_0| - k + 1$ and $|k\phi(S)| = k|\phi(S)| - k + 1$, where $\phi : G \mapsto G/H$ denotes the canonical morphism. Moreover one of the following holds:*

- *S_0 is an arithmetic progression,*
- *$k = 2$ and $S_0 = x - ((S_0 + H) \setminus S_0)$, for some x.*

Proof: Take a non-zero subgroup H with minimal cardinality such $k(S + H) = k|S + H| - k|H| + |H|$ and $(S \setminus S_0) + H = (S \setminus S_0)$, where S_0 is an

H-component of S. Notice that G is such a group. Since the period of kS_0 is a subgroup of H, kS_0 is aperiodic and hence

$$|kS_0| = k|S_0| - k + 1,$$

using the relation $|kS| = k|S| - k + 1$.

Observe that S_0 can not have a fragment non-zero subgroup Q. Otherwise we have by Proposition 11, $k(S_0 + Q) = k|S_0 + Q| - k|Q| + |Q|$ and $(S_0 \setminus T_0) + Q = (S_0 \setminus T_0)$, where T_0 is a Q-component of S_0. It would follow that $k(S + Q) = k|S + Q| - k|Q| + |Q|$ and $(S \setminus T_0) + Q = (S \setminus T_0)$, a contradiction. Let H_0 be the subgroup generated by $S_0 - S_0$. By Theorem 9, either (i) holds or one of the following holds:

- S_0 is non 2-separable. We have necessarily $|2S_0| = |H_0| - 1$. Take $a \in S_0$ and put $\{b - a\} = H_0 \setminus (2(S_0 - a))$. Necessarily $b - (S_0 - a) = H_0 \setminus (S_0 - a)$, and thus $b - S_0 = H_0 + a \setminus (S_0) = (S_0 + H_0) \setminus S_0$.

- S_0 is a 2-separable Vopser subset. We must have $k = 2$, otherwise The condition $|2S_0| \geq \min(|H_0| - 1, 2|S_0|)$. But $|H_0| \geq |kS_0| \geq 2|S_0| + |S_0| - 1 \geq 2|S_0| + 1$, observing that S_0 is not an arithmetic progression. By Kneser's Theorem, $|kS_0| \geq k|S_0| - k + 2$, a contradiction. Since $|2S_0| = 2|S_0| - 1$ and since S_0 is a Vosper subset, we have necessarily $|2S_0| = |H_0| - 1$. Take $a \in S_0$ and put $\{b - a\} = H_0 \setminus (2(S_0 - a))$. Thus, $b - (S_0 - a) = H_0 \setminus (S_0 - a)$, and hence

$$b + a - S_0 = (H_0 + a) \setminus S_0 = (S_0 + H_0) \setminus S_0. \quad \square$$

In the above result, the structure of S is completely determined by the structure of S_0 and by the structure of $\phi(S)$. Unfortunately $k\phi(S)$ is sometimes periodic. In order transform the last result, we investigate the S, where kS is periodic and where one element has a unique expressible element. The methods of Kemperman solve very easily this question, as shown in [12].

The hyper-atomic approach avoids the last difficulty and lead to a simpler description, as we shall see later.

5 Universal periods

Let T and S be finite subsets of an abelian group. It follows from a result by Balandraud that $|TS| \leq |T| + |S| - 2$ implies that $T + S$ has a universal period contained in the atom of S. We shall construct a universal period for kS which is bigger in general.

We shall first prove that $S-S$ has a universal period containing the atom if S is not an arithmetic progression and if $|S-S|$ is not very big.

Theorem 15 *Let $r \geq s \geq 1$ be integers and let S be a finite subset of an abelian group G and let H be a hyper-atom of S. One of the following holds:*

(i) *S is an arithmetic progression,*

(ii) $|sS - rS| \geq \min(|G|-1, (r+s)|S|)$,

(iii) *The hyper-atom H is a non-zero-subgroup and $sS-rS+H = sS-rS$.*

Proof: Assume that (i) and (ii) do not hold. It follows that S is 2-separable and non-vosperian. Let H be a hyper-atom of S. By Theorem 9, $|H| \geq 2$. By Proposition 11, $S-S+H = S-S$. Therefore, $sS-rS+H = sS-rS$. □

Proposition 11 suggests a very simple method giving another universal period for kS containing necessarily Balandraud period.

Let H be a subgroup fragment of S. An H-component S_0 of S will be called *desertic* component if $|S_0| \leq |H|/2$. By Proposition 11, the desertic component is unique if it exists. We shall say that S is a *desert* if it has a desertic component.

Given a subset A, with $\kappa(A) \leq |A|-2$. We define a *desert* sequence $A_0, \cdots, A_\ell$, verifying the following conditions:

- $A_0 = A$,
- A_{i+1} is a desert for $0 \leq i \leq \ell - 1$,
- A_ℓ is not a desert.

Such a sequence exists and is unique, since Proposition 11 asserts that A_i is unique for $1 \leq i \leq \ell$. The sequence must end since H_i is a finite group with size $< |H_{i-1}|/2$. The *sub-atom* M of A is defied as $M = H_\ell$ if H_ℓ is non-zero. Otherwise $M = H_{\ell-1}$. In particular, the sub-atom is a non-zero subgroup.

Theorem 16 *Let k be a non-negative integer and let S be a finite subset of an abelian group G. If $|kS| \leq k|S| - k$, then*

$$kS + M = kS,$$

where M is the sub-atom of S.

Proof: We use the last notations. The proof is by induction on ℓ. We have $\kappa_1(S) \leq |S|-2$, and hence $|H_0| \geq 2$. By Proposition 11, $(S \setminus S_0)+(k-1)S$ is H-periodic. We may assume that $kS_0 \cap ((S \setminus S_0)+(k-1)S) = \emptyset$, otherwise kS is H_0-periodic. Proposition 11, $|kS| = |kS_0| + |(S \setminus S_0) + (k-1)S| \geq k|S_0| - k + 1 + ku|H| \geq k|S| - k + 1$. In particular, $|kS_0| \leq k|S_0| - k$. Notice that S and S_0 have the same sub-atom. By the induction hypothesis $kS_0 + M = kS_0$. It follows that $kS + M = kS$. □

6 Hyper-atoms and the critical pair Theory

Applications of hyper-atoms to the critical pair theory where first obtained in [12]. A more delicate notion of hyper-atoms was introduced in [13].

Theorem 17 *Let $k \geq 2$ be an integer and let S be a finite subset of an abelian group G such that S is not an arithmetic progression, kS is aperiodic and $|kS| = k|S| - k + 1$. Let H be a hyper-atom of S and let S_0 be a smaller H-component of S. If $|2S| \neq |G| - 1$, then $|H| \geq 2$. Moreover, $(S \setminus S_0) + H = (S \setminus S_0)$ and $|kS_0| = k|S_0| - k + 1$. Also, either $\phi(S)$ is an arithmetic progression or $k = 2$ and one of the following holds:*

1. *$S = x - (G \setminus S)$, for some x.*
2. *$(\phi(S)-\phi(S_0))\cap(\phi(S_0)-\phi(S)) = \{\phi(0)\}$, where $\phi : G \mapsto G/H$ denotes the canonical morphism.*

Proof: By Kneser's Theorem and since $2S$ is aperiodic, we have $|2S| = 2|S| - 1$. Take an H-decomposition $S = S_0 \cup \cdots \cup S_u$.

Assume first that S is non-2-separable. This forces $|2S| = |G|-1$. Then necessarily $k = 2$, otherwise $3S = G$, by Lemma 1. Put $2S = G \setminus \{x\}$. We have clearly $(x - S) \cap S = \emptyset$. Clearly (1) holds. Assume now that S is 2-separable. By Theorem 9, $|H| \geq 2$.

By Proposition 13, $(S \setminus S_0) + H = (S \setminus S_0)$ and $|kS_0| = k|S_0| - k + 1$.

Assume now that $\phi(S)$ is not an arithmetic progression. By Theorem 10, $\phi(S)$ is a Vosper subset.

Thus, $|\phi(G)| - 1 < 2|\phi(S)| - 1$, otherwise $|\phi((S \setminus S_0) + S)| \geq 2|\phi(S)| - 1$, and hence $|(S \setminus S_0) + S| \geq 2u|H| + |H| \geq 2|S|$, a contradiction. Thus, $|\phi(G)| = 2|\phi(S)| - 1$. In this case, $k = 2$ and $2\phi(S) = \phi(G)$. Necessarily, $2\phi(S_0)$ is uniquely expressible in $2\phi(S)$. In other words $(\phi(S) - \phi(S_0)) \cap (\phi(S_0) - \phi(S)) = \{\phi(0)\}$. $\square$

Corollary 18 *Let $k \geq 3$ be an integer and let S be a finite subset of an abelian group G such that S is not an arithmetic progression, kS is aperiodic and $|kS| = k|S| - k + 1$. Let H be a hyper-atom of S and let S_0 be a smaller H-component of S. Then $(S \setminus S_0) + H = (S \setminus S_0)$ and $|kS_0| = k|S_0| - k + 1$. Moreover $\phi(S)$ is an arithmetic progression, where $\phi : G \mapsto G/H$ denotes the canonical morphism.*

Acknowledgement

The author would like to thank M.A. Fiol for stimulating discussions on connectivity extra-connectivity

References

[1] E. Balandraud. Une variante de la méthode isopérimétrique de Hamidoune, appliquée au théorème de Kneser. *Ann. Inst. Fourier* (Grenoble), 58(3):915–943, 2008.

[2] J. Fàbrega and M. A. Fiol. Futher Topics in Connectivity, in *Handbook of Graph Theory*, (J.L. Gross and Y.Yellen Eds.), CRC Press, Boca Raton, 300–329, 2004.

[3] M. A. Fiol. The superconnectivity of large digraphs and graphs. *Discrete Math.*, 124(1–3):67–78, 1994.

[4] D. Grynkiewicz. A step beyond Kemperman's structure theorem, *Mathematika*, 55(1–2):67–114, 2009.

[5] Y.O. Hamidoune. Sur les atomes d'un graphe orienté. *C.R. Acad. Sc. Paris A*, 284:1253–1256, 1977.

[6] Y.O. Hamidoune. On the connectivity of Cayley digraphs. *Europ. J. Combinatorics*, 5:309-312, 1984.

[7] Y. O. Hamidoune. An isoperimetric method in additive theory. *J. Algebra* 179(2):622–630, 1986.

[8] Y. O. Hamidoune. Subsets with small sums in Abelian groups I: The Vosper property. *European J. Combin.*, 18(5):541–556, 1997.

[9] Y. O. Hamidoune. On small subset product in a group. Structure Theory of set-addition. *Astérisque*, 258:281–308, 1999.

[10] Y. O. Hamidoune. Some results in Additive number Theory I: The critical pair Theory. *Acta Arith.*, 96(2):97–119, 2000.

[11] Y. O. Hamidoune. Some additive applications of the isoperimetric approach. *Annals Institute Fourier* (Grenoble)m, 58(6):2007–2036, 2008.

[12] Y. O. Hamidoune. Hyper-atoms and the critical pair Theory. *Combinatorica*, accepted. arXiv:0805.3522v1.

[13] Y. O. Hamidoune. A Structure Theory for Small Sum Subsets. Preprint, 2009.

[14] Y. O. Hamidoune. Two Inverse results related to a question of Tao. Preprint 2010.

[15] Y. O. Hamidoune, A. Lladó and S. C. López. Vertex-transitive graphs that remain connected after failure of a vertex and its neighbors. *J. Graph Theory*, accepted.

[16] J. H. B. Kemperman. On small sumsets in Abelian groups. *Acta Math.*, 103:66–88, 1960.

[17] M. Kneser. Summenmengen in lokalkompakten abelesche Gruppen, *Math. Zeit.*, 66:88–110, 1956.

[18] V. F. Lev. Critical pairs in abelian groups and Kemperman's structure theorem. *Int. J. Number Theory*, 2(3): 379–396, 2006.

[19] T. Tao. An elementary non-commutative Freiman theorem. `http://terrytao.wordpress.com/2009/11/10/an-elementary-non-commutative-freiman-theorem.`

[20] T. Tao, V.H. Vu. *Additive Combinatorics*, Cambridge Studies in Advanced Mathematics 105, Cambridge University Press, 2006.

[21] G. Vosper. The critical pairs of subsets of a group of prime order. *J. London Math. Soc.*, 31:200–205, 1956.

Graphs with equal domination and 2-domination numbers

Adriana Hansberg
Universitat Politècnica de Catalunya
Barcelona

Abstract

For a graph G a subset D of the vertex set of G is a *k-dominating set* if every vertex not in D has at least k neighbors in D. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality among the k-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$.

Fink and Jacobson showed in 1985 that the inequality $\gamma_k(G) \geq \gamma(G)+k-2$ is valid for every connected graph G. In this paper, we recompile results concerning the case $k = 2$, where γ_k can be equal to γ. In particular, we present the characterization of different graph classes with equal domination and 2-domination numbers as are the cactus graphs, the claw-free graphs and the line graphs.

1 Terminology

We consider finite, undirected, and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n(G)$. The *neighborhood* $N(v) = N_G(v)$ of a vertex v consists of the vertices adjacent to v and $d(v) = d_G(v) = |N(v)|$ is the *degree* of v. By $\delta(G)$ and $\Delta(G)$, we denote the *minimum degree* and the *maximum degree* of the graph G, respectively. For a subset $S \subseteq V$, we define by $G[S]$ the subgraph induced by S. If x and y are vertices of a connected graph G, then we denote with $d_G(x, y)$ the *distance* between x and y in G, i.e. the length of a shortest path between x and y.

With K_n we denote the complete graph on n vertices and with C_n the cycle of length n. We refer to the complete bipartite graph with partition sets of cardinality p and q as the graph $K_{p,q}$. A graph G is a *block-cactus graph* if every block of G is either a complete graph or a cycle. G is a *cactus graph* if every block of G is a cycle or a K_2. If we substitute each edge in a non-trivial tree by two parallel edges and then subdivide each edge, then we speak of a C_4-*cactus*. Let G and H be two graphs. For a vertex $v \in V$, we say that the graph G' arises by *inflating* the vertex v to the graph H if the vertex v is substituted by a set S_v of $n(H)$ new vertices and a set of edges such that $G'[S_v] \cong H$ and every vertex in S_v is connected to every neighbor of v in G by an edge.

The *cartesian product* of two graphs G_1 and G_2 is the graph $G_1 \times G_2$ with vertex set $V(G_1) \times V(G_2)$ and vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. Let u be a vertex of G_1 and v a vertex of G_2. Then the sets of vertices $\{(u, y) \mid y \in V(G_2)\}$ and $\{(x, v) \mid x \in V(G_1)\}$ are called a *row* and, respectively, a *column* of $G_1 \times G_2$. A set of vertices in $V(G_1 \times G_2)$ is called a *transversal* of $G_1 \times G_2$ if it contains exactly one vertex on every row and every column of $G_1 \times G_2$.

A set C of vertices in a graph G is called a *covering* of G if every edge of G is incident with at least one vertex of C. The minimum cardinality of a covering of G is denoted with $\beta(G)$ and is called the *covering number* of G. Let k be a positive integer. A subset $D \subseteq V$ is a k-*dominating set* of the graph G if $|N_G(v) \cap D| \geq k$ for every $v \in V - D$. The k-*domination number* $\gamma_k(G)$ is the minimum cardinality among the k-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$. A k-dominating set of minimum cardinality of a graph G is called a $\gamma_k(G)$-set. For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi, and Slater [12, 13]. More information on k-domination can be found for example in [3, 4, 5, 6, 7, 8].

2 Introduction

Let G be a graph such that $2 \leq k \leq \Delta(G)$ for an integer k and let D be a k-dominating set of G of minimum cardinality. Then $V - D$ is not empty and we can take a vertex $x \in V - D$. If $X \subseteq N_G(x) \cap D$ is a set of neighbors of x in D such that $|X| = k - 1$, then evidently every vertex in $V - ((D - X) \cup \{x\})$ has a neighbor in $(D - X) \cup \{x\}$ and thus the latter is a dominating set of G. This implies the following theorem of Fink and Jacobson, which establishes a relation between the usual domination number $\gamma(G)$ and the k-domination number $\gamma_k(G)$.

Theorem 1 (Fink, Jacobson [6], 1985) *If $k \geq 2$ is an integer and G is a graph with $k \leq \Delta(G)$, then*

$$\gamma_k(G) \geq \gamma(G) + k - 2.$$

The inequality given above is sharp. However, the characterization of the graphs attaining equality is still an open problem. In [9], Hansberg analyzed the extremal graphs for general k and gave several properties for them, among them the next proposition.

Proposition 2 [9] *Let G be a connected graph and k an integer with $\Delta(G) \geq k \geq 2$. If $\gamma_k(G) = \gamma(G) + k - 2$ and D is a minimum k-dominating set, then $\Delta(G[D]) \leq k - 2$.*

In particular, the case $k = 2$ in Theorem 1 is of special interest since it is the only possibility where γ_k can be equal to γ. In this case, Proposition 2 states that if $\gamma(G) = \gamma_2(G)$, then every minimum 2-dominating set is independent. In [11], Hansberg and Volkmann presented some properties on graphs G with $\gamma_2(G) = \gamma(G)$, among them the following one.

Theorem 3 [11] *If G is a connected non-trivial graph with $\gamma_2(G) = \gamma(G)$, then $\delta(G) \geq 2$.*

3 Graphs with $\gamma = \gamma_2$

In [11], Hansberg and Volkmann showed that the graphs with minimum degree at least 2 and equal domination and covering numbers fulfill also that both the domination and the 2-domination numbers are equal.

Theorem 4 [11] *If G is a graph with $\delta(G) \geq 2$ and $\gamma(G) = \beta(G)$, then $\gamma_2(G) = \gamma(G)$.*

Note that it has been shown by Randerath and Volkmann in [15] that the graphs with equal domination and covering numbers have minimum degree $\delta \leq 2$. They also characterized this family of graphs.

In [11], Hansberg and Volkmann characterized the cactus graphs with equal domination and 2-domination numbers.

Theorem 5 [11] *Let G be a cactus graph. Then $\gamma_2(G) = \gamma(G)$ if and only if G is a C_4-cactus.*

In [10], Hansberg, Randerath and Volkmann centered their attention on claw-free graphs and characterized those with equal domination and 2-domination numbers. A *claw-free graph* is a graph which does not contain a $K_{1,3}$ as an induced subgraph.

Following lemma gives two important structural properties of claw-free graphs with equal domination and 2-domination numbers.

Lemma 6 [10] *Let G be a connected nontrivial claw-free graph. If $\gamma(G) = \gamma_2(G)$, then every minimum 2-dominating set D of G fulfills:*

(i) *Every vertex in $V - D$ has exactly two neighbors in D.*

(ii) *Every two vertices $a, b \in D$ are at distance 2 in G.*

Former lemma sets the basis for the characterization of all claw-free graphs with equal domination and 2-domination numbers. Let $\mathcal{H}$ be the family of graphs such that $G \in \mathcal{H}$ if and only if either G arises from a cartesian product $K_p \times K_p$ of two complete graphs of order p for an integer $p \geq 3$ by inflating every vertex but the ones on a transversal (we call it the *diagonal*) to a clique of arbitrary order, or G is a claw-free graph with $\Delta(G) = n(G) - 2$ containing two non-adjacent vertices of maximum degree (see Fig. 1 below).

This family of graphs describes exactly those claw-free graphs with equal domination and 2-domination numbers, as is given in the theorem below.

Theorem 7 [10] *Let G be a connected claw-free graph. Then $\gamma(G) = \gamma_2(G)$ if and only if $G \in \mathcal{H}$.*

If G is a graph, then the *line graph* of G, denoted by $L(G)$, is obtained by associating one vertex to each edge of G, and two vertices of $L(G)$

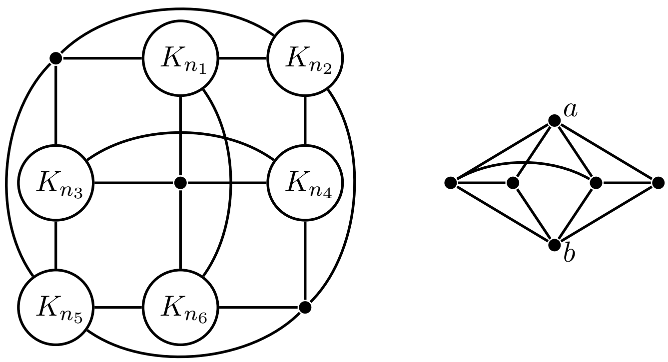


Figure 1: Examples of graphs from the family $\mathcal{H}$
(here, $n_i \in \mathbb{N}$ for $1 \leq i \leq 6$)

being joined by an edge if and only if the corresponding edges in G are incident with each other. If for a graph G there is a graph G' whose line graph is isomorphic to G, then G is called *line graph.* In 1968, Beineke [1] obtained a characterization of line graphs in terms of nine forbidden induced subgraphs. Since the claw is one of those subgraphs, the set of line graphs with $\gamma = \gamma_2$ is contained in $\mathcal{H}$. Using the characterization of line graphs of Krausz [14] and Beineke's forbidden induced subgraphs in line graphs, Hansberg, Randerath and Volkmann were able to characterize the line graphs with equal domination and 2-domination numbers.

Theorem 8 [10] *Let G be a line graph. Then $\gamma_2(G) = \gamma(G)$ if and only if G is either the cartesian product $K_p \times K_p$ of two complete graphs of the same cardinality p or G is isomorphic to the graph J depicted bellow.*

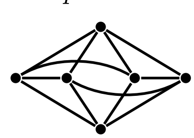


Figure 2: The graph J.

Note that the graphs of the family $\mathcal{H}$, as also the cactus graphs of Theorem 5, contain many induced cycles of length 4. This is not a particular property of these both graph families with equal domination and

2-domination numbers. In fact, this accumulation of induced C_4's can be found in every graph fulfilling equality in Fink and Jacobson's bound. The reason of this particularity lies basically on the assertion of following lemma.

Lemma 9 [9] *Let G be a connected graph with $\gamma_k(G) = \gamma(G) + k - 2$ for an integer k such that $\Delta(G) \geq k \geq 2$. Then, for every vertex $u \in V - D$ and every set $A_u \subseteq N(u) \cap D$ with $|A_u| = k$, there are non-adjacent vertices $x_u, x'_u \in V - D$ such that $D \cap N(x_u) = D \cap N(x'_u) = A_u$.*

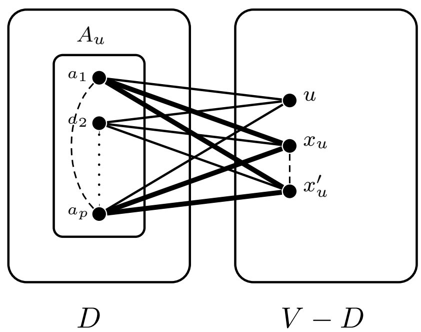


Figure 3: The vertices a_1, x_u, a_p, x'_u induce a cycle of length 4.

Using this lemma, Hansberg was able to prove that for any graph fulfilling Fink and Jacobson's bound, every vertex of G lies on an induced cycle of length four:

Theorem 10 [9] *Let G be a connected graph and k an integer with $\Delta(G) \geq k \geq 2$. If $\gamma_k(G) = \gamma(G) + k - 2$, then every vertex of G lies on an induced cycle of length 4.*

In the same paper, the author presented a sharp lower bound in terms of the domination number for the number of induced cycles of length 4 in this family of graphs.

Theorem 11 [9] *Let G be a connected graph and k an integer with $\Delta(G) \geq k \geq 2$. If $\gamma_k(G) = \gamma(G) + k - 2$, then G contains at least $(\gamma(G) - 1)(k - 1)$ induced cycles of length 4.*

To show that previous bound is tight, let r and k be two positive integers, where $k \geq 2$ and let G be a graph consisting of a complete graph H on $k-1$ vertices and of vertices u_i, v_i, w_i, for $1 \leq i \leq r$, such that every u_i and w_i is adjacent to every vertex of H and to v_i (see Figure 4). It is easy to see that $\gamma_k(G) = k+r-1$, $\gamma(G) = r+1$ and thus $\gamma_k(G) = \gamma(G)+k-2$. Since G contains exactly $r(k-1) = (\gamma(G)-1)(k-1)$ induced cycles of length 4, it follows that the bound in Theorem 10 is sharp.

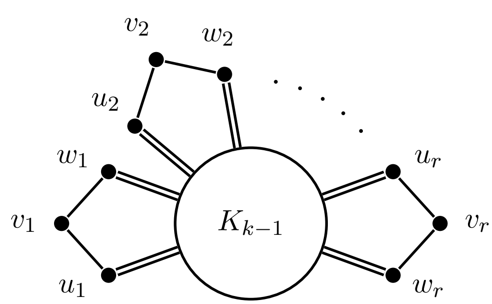


Figure 4: Example of a graph G with $\gamma_k(G) = \gamma(G) + k - 2$ and exactly $(\gamma(G) - 1)(k-1)$ induced cycles of length 4. A double line connecting a vertex u_i or w_i to the complete graph K_{k-1} in the middle means that it is adjacent to all vertices of K_{k-1}.

Using the inequality $\gamma_k(G) \geq \frac{k}{\Delta+k}n$ for any n-vertex graph with maximum degree δ, proved by Fink and Jacobson in [7], the following corollary arises.

Corollary 12 [9] *Let G be a graph and k an integer such that $2 \leq k \leq \Delta(G)$. If $\gamma_k(G) = \gamma(G) + k - 2$, then G contains at least*

$$(\frac{n}{\Delta(G)+1} - 1)(k-1)$$

induced cycles of length 4.

Note that, if G is a graph with $\gamma_k(G) = \gamma(G) + k - 2$ for an integer k with $2 \leq k \leq \Delta(G)$, $\gamma(G)$ is at least 2 and thus $\Delta(G) \leq n(G) - 2$, which implies that the factor $(\frac{n}{\Delta(G)+1} - 1)$ above is always positive.

Reverting the assertion of the theorem, we gain an improvement of Fink and Jacobson's lower bound in Theorem 1 and we obtain, as a corollary, a theorem of Chellali, Favaron, Hansberg and Volkmann.

Corollary 13 [9] *Let G be a graph with $\Delta(G) \leq n(G) - 2$. If G has less than $(\gamma(G) - 1)(k - 1)$ induced cycles of length 4 for an integer k with $\Delta(G) \geq k \geq 2$, then $\gamma_k(G) \geq \gamma(G) + k - 1$.*

Corollary 14 [2] *If G is a graph with at most $k-2$ induced cycles of length 4 for an integer k with $\Delta(G) \geq k \geq 2$, then $\gamma_k(G) \geq \gamma(G) + k - 1$.*

It remains thus as an open problem the characterization of more graph families attaining equality in Fink and Jacobson's bound and specially the interesting case where $\gamma_2 = \gamma$.

Acknowledgement

This research was supported by the Ministry of Science and Innovation, Spain, and the European Regional Development Fund (ERDF) under project MTM2008-066200-C03-02/MTM; also by Catalonian government under project 2009 SGR 1298; and by Future Research postdoc Consolider i-math.

References

[1] L. W. Beineke, *Derived graphs and digraphs, Beiträge zur Graphentheorie*, (Hrsg. H. Sachs, H. Voss, H. Walther), *Teubner*, Leipzig, 17–23, 1968.

[2] M. Chellali, O. Favaron, A. Hansberg and L. Volkmann. On the p-domination, the total domination and the connected domination numbers of graphs. *J. Combin. Math. Combin. Comput.*, 73: 65–75, 2010.

[3] O. Favaron. On a conjecture of Fink and Jacobson concerning k-domination and k-dependence. *J. Combin. Theory Series B*, 39:101–102, 1985.

[4] O. Favaron. k-domination and k-independence in graphs. *Ars Combin.*, 25 C:159–167, 1988.

[5] O. Favaron, A. Hansberg and L. Volkmann. On k-domination and minimum degree in graphs. *J. Graph Theory*, 57:33–40, 2008.

[6] J.F. Fink and M.S. Jacobson. n-Domination in graphs, *Graph Theory with Applications to Algorithms and Computer Science*, *John Wiley and Sons*, New York, 283–300, 1985.

[7] J.F. Fink and M.S. Jacobson. On n-domination, n-dependence and forbidden subgraphs, *Graph Theory with Applications to Algorithms and Computer Science*, *John Wiley and Sons*, New York, 301–311, 1985.

[8] A. Hansberg. *Multiple domination in graphs*, Dissertation, *RWTH Aachen University*, Aachen, 2009, http://darwin.bth.rwth-aachen.de/opus3/volltexte/ 2009/2912.

[9] A. Hansberg. On the k-domination number, the domination number and the cycle of length four. submitted.

[10] A. Hansberg, B. Randerath, L. Volkmann. Claw-free graphs with equal domination and 2-domination numbers. submitted.

[11] A. Hansberg and L. Volkmann. On graphs with equal domination and 2-domination numbers. *Discrete Math.*, 308(11):2277–2281, 2008.

[12] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater. *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[13] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds.) *Domination in Graphs: Advanced Topics. Marcel Dekker*, New York, 1998.

[14] J. Krausz. Démonstration nouvelle d'une théorème de Whitney sur les réseaux (Ungarian). *Mat. Fiz. Lapok*, 50:75–85, 1943.

[15] B. Randerath and L. Volkmann. Characterization of graphs with equal domination and covering number. *Discrete Math.*, 191:159-169, 1998.

Radially Moore graphs of radius three and large odd degree

Nacho López
Universitat de Lleida
Lleida

José Gómez
Universitat Politécnica de Catalunya
Barcelona

Abstract

Extremal graphs which are close related to Moore graphs have been defined in different ways. Radially Moore graphs are one of these examples of extremal graphs. Although it is proved that radially Moore graphs exist for radius two, the general problem remains open. Knor, and independently Exoo, gives some constructions of these extremal graphs for radius three and small degrees. As far as we know, some few examples have been found for other small values of the degree and the radius.

Here, we consider the existence problem of radially Moore graphs of radius three. We use the generalized undirected de Bruijn graphs to give a general construction of radially Moore graphs of radius three and large odd degree.

1 Introduction

Given the values of the maximum degree d and the diameter k of a graph, there is a natural upper bound for its number of vertices n,

$$n \leq M_{d,k} = 1 + d + d(d-1) + \cdots + d(d-1)^{k-1}, \tag{1}$$

where $M_{d,k}$ is known as the *Moore bound*. Graphs attaining such a bound are referred to as *Moore graphs*. In the case of diameter $k = 2$, Hoffman and Singleton [4] proved that Moore graphs exist for $d = 2, 3, 7$ (being unique) and possibly $d = 57$, but for no other degrees. They also showed that for diameter $k = 3$ and degree $d > 2$ Moore graphs do not exist. The enumeration of Moore graphs of diameter $k > 3$ was concluded by Damerell [3], who used the theory of distance-regularity to prove their nonexistence unless $d = 2$, which corresponds to the cycle graph of order $2k + 1$ (an independent proof of it was given by Bannai and Ito [1]).

The fact that there are very few Moore graphs suggested the study of graphs that are in various senses 'close' to being Moore graphs. This 'closeness' has been usually measured as the difference between the (unattainable) Moore bound and the order of the considered graphs. In this sense, the existence of graphs with small 'defect' δ (order $n = M(d,k)-\delta$) has deserved much attention in the literature (see [8]). Another kind of approach considers relaxing some of the constraints implied by the Moore bound. From its definition, all vertices of a Moore graph have the same degree (d) and the same eccentricity (k). We could relax the condition of the degree and admit few vertices with degree $d + \delta$, as Tang, Miller and Lin [10] did for the directed case. Alternatively, we may allow the existence of vertices with eccentricity just on more than the value k they should have. In this context, regular graphs of degree d, radius k, diameter $k + 1$ and order equal to $M_{d,k}$ are referred to as *radially Moore graphs*. Figure 1 shows all (non-isomorphic) cubic radially Moore graphs of radius $k = 2$.

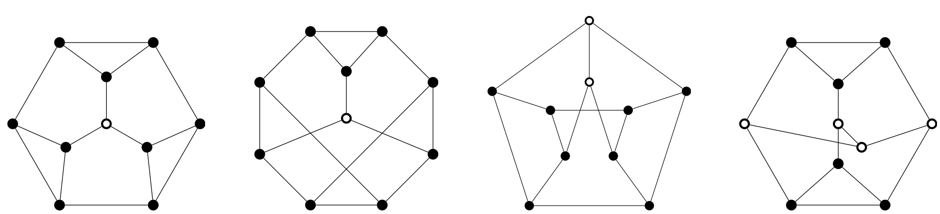


Figure 1: All cubic radially Moore graphs of radius $k = 2$. Vertices with eccentricity k (central vertices) are depicted in white.

It is known that radially Moore graphs of radius $k = 2$ exist for any degree (see [2]). Nevertheless, the situation for $k \geq 3$ seems to be more complicated. Knor [7] (and independently Exoo) found radially Moore graphs of radius $k = 3$ and small degrees. So far, no general construction for radius $k \geq 3$ is known. Besides, Captdevila et al. [2] give the complete enumeration of these extremal graphs for some cases ($k = 2$ and $d = 3, 4$; $k = 3$ and $d = 3$) and rank them according to their 'proximity' to a theoretical Moore graph.

2 The generalized de Bruijn digraphs

The generalized de Bruijn digraphs appear in the context of the optimization problem which tries to minimize the diameter and maximize the connectivity of a digraph with n vertices, each of which has outdegree at most d. The *generalized de Bruijn digraph* $G_B(d, n)$ ($d < n$) is the directed graph with n vertices labeled by the residues modulo n such that an arc from i to j exists if and only if $j \equiv di + k \pmod{n}$, for some $k = 0, \ldots, d-1$. These digraphs were first defined by Imase and Itoh [5] and independently by Reddy et al. [9] as a generalization of the well known de Bruijn digraphs. It is known that each vertex in $G_B(d, n)$ has both indegree and outdegree d and this digraph may contain loops (cycles of length 1) and multiple arcs. The diameter of $G_B(d, n)$ is upper bounded by $\lceil \log_d n \rceil$ (see [5]). Esentially, the generalized de Bruijn digraphs retain all the properties of the de Bruijn digraphs, but have no restriction on the number of vertices. Next, we show the structure of the subdigraph induced by the set of vertices containing either a loop or a digon (cycle of length 2).

Proposition 1 *The digraph* $G_B(d, n)$*, where* $\gcd(d-1, n) = \gcd(d^2-1, n) = 1$*, has* d *loop vertices and* $d^2 - d$ *vertices belonging to a digon.*

Proof: A loop vertex i in $G_B(d, n)$ satisfies the following equation

$$i(d-1) \equiv -k \pmod{n} \tag{2}$$

where $k \in \{0, \ldots, d-1\}$. Since $\gcd(d-1, n) = 1$, there is unique solution of equation 2 for each value of k. As a consequence, there are d loop vertices in $G_B(d, n)$, each of them of the form

$$-k(d-1)^{-1} \pmod{n}, \quad k \in \{0, \ldots, d-1\}.$$

Besides, a vertex i contained in a cycle of length ≤ 2 satisfies,

$$i(d^2-1) \equiv -dk - k' \pmod{n} \tag{3}$$

where $k, k' \in \{0, \ldots, d-1\}$. Since $\gcd(d^2-1, n) = 1$, for every pair (k, k') there is a unique solution of equation 3. Notice that for $k = k'$, equation 3 transforms to equation 2 which it means that there are d solutions of equation 3 corresponding to loop vertices. As a consequence, there are $d^2 - d$ vertices contained in a digon. $\square$

Proposition 2 *The subdigraph of $G_B(d,n)$, where* $\gcd(d-1, n) = \gcd(d^2-1, n) = 1$ *and $n \geq d^3$, induced by the set of vertices contained in either a loop or a digon is isomorphic to the digraph with vertex set $V = \{(i,j) \mid i,j \in \{0, \ldots, d-1\}\}$ and where there is an arc from (i,j) to (j,i), for all $(i,j) \in V$.*

Proof: From the previous proposition, every vertex in the subdigraph of $G_B(d,n)$, induced by the set of vertices contained in either a loop or a digon is of the form

$$(d^2-1)^{-1}(-di-j) \pmod{n}, \quad i,j \in \{0, \cdots, d-1\}.$$

So, every vertex of this subdigraph can be indentified by the pair (i,j), where $(i,j) \in V$. Let us observe that there is an arc from (i,j) to (i',j') in this subdigraph if and only if,

$$(d^2-1)^{-1}(-di'-j') \equiv d(d^2-1)^{-1}(-di-j) + k \pmod{n}$$

for a suitable $k \in \{0, \ldots, d-1\}$. This is equivalent to the following equation:

$$d^2(k-i) + d(i'-j) + (j'-k) \equiv 0 \pmod{n} \tag{4}$$

Since $n \geq d^3$, equation 4 holds if and only if $k - i = i' - j = j' - k = 0$, that is, $k = i$, $i' = j$ and $j' = i$. $\square$

Figure 2 shows a representation of the subdigraph of $G_B(d,n)$ induced by the set of vertices contained in either a loop or a digon, for the particular values $d = 6$ and $n = 1872$. A vertex (i,j) in the picture corresponds to vertex $(d^2-1)^{-1}(-di-j) \pmod{n}$ in $G_B(d,n)$. As an exemple, vertex $(1,2)$ is 1016 in $G_B(d,n)$, this vertex has a unique arc to [from] a vertex belonging to the subdigraph itself. This special vertex is $(2,1)$ (481 in $G_B(d,n)$).

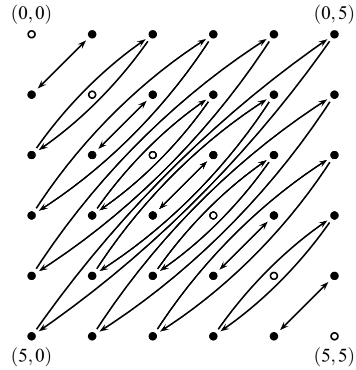


Figure 2: The subdigraph of $G_B(d,n)$ induced by the set of vertices contained in either a loop or a digon, for the particular values $d = 6$ and $n = 1872$. Loop vertices are depicted in white and the corresponding loop-arc has been removed for clarity.

3 The generalized undirected de Bruijn graphs

The *generalized undirected de Bruijn graph*, denoted by $UG_B(d,n)$, is the undirected graph derived from $G_B(d,n)$ by replacing arcs with edges and omitting loops and multiple edges. The diameter of $UG_B(d,n)$ is bounded above by $\lceil \log_d n \rceil$ since for any two distinct vertices u and v in $UG_B(d,n)$, the distance from u to v in the corresponding digraph $G_B(d,n)$ provides an upper bound for the distance between u and v in $UG_B(d,n)$, as it can be seen in [6]. From its own definition, $UG_B(d,n)$ has order n and each vertex has maximum out degree $2d$. More precisely, whenever $\gcd(d-1,n) = \gcd(d^2-1,n) = 1$, $UG_B(d,n)$ has d vertices of degree $2d-2$, d^2-d vertices of degree $2d-1$ and the remaining vertices of degree $2d$. From proposition 2 we derive the following result:

Corollary 3 *The subgraph of $UG_B(d,n)$, where $\gcd(d-1,n) = \gcd(d^2-1,n) = 1$ and $n \geq d^3$, induced by the set of vertices of degree $< 2d$ is isomorphic to the graph with vertex set $V = \{(i,j) \mid i,j \in \{0,\ldots,d-1\}\}$ and where vertex (i,j) is adjacent to (j,i), for all $i > j$.*

Whenever d is even, we can add some extra edges to $UG_B(d,n)$ in order to achieve a $2d$-regular graph.

Proposition 4 *$UG_B(d,n)$, where* $\gcd(d-1,n) = \gcd(d^2-1,n) = 1$, *$n \geq d^3$ and d is even, is a subgraph of a regular graph of degree $2d$ and order n.*

Proof: We add the following adjacency relations to the subgraph of $UG_B(d,n)$ induced by the set of vertices of degree $< 2d$.

$$\begin{cases} (i,j) \sim (i,j+1) & \text{for } j \text{ even;} \\ (i,i) \sim (i+1,i+1) & \text{for } i \text{ even.} \end{cases}$$

The degree of every loop vertex has been increased by two and the degree of every vertex contained in a digon has been increased by one (see figure 3). Hence, the resultant graph is regular of degree $2d$. □

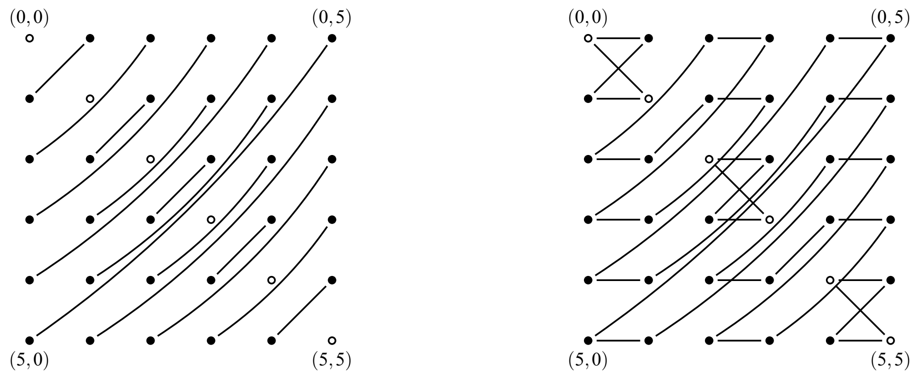


Figure 3: The subdigraph of $UG_B(d,n)$ induced by the set of vertices with degree $< 2d$, for the particular values $d = 6$ and $n = 1872$. This graph is a subgraph of a 2-regular graph, as it shows the right picture. As a consequence, $UG_B(d,n)$ is a subgraph of a $2d$-regular graph.

4 Radially Moore graphs of radius three and large odd degree

Let us start with the tree $T_{d,k}$ given in Fig. 4, which corresponds to the distance preserve spanning tree of a radial Moore graph of degree d and radius k, hanging from a central vertex v (every central vertex in a radial Moore graph of degree d and radius k must reach any other vertex of the graph in at most k steps). In particular, this is the same structure that we observe in a Moore graph hanged from any of its vertices. Let V be the set

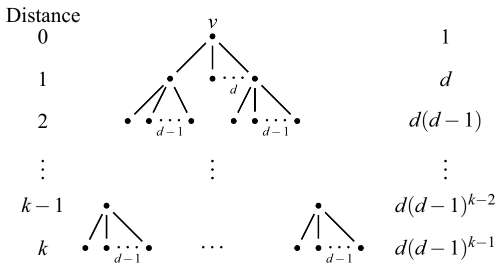


Figure 4: The tree $T_{d,k}$.

of vertices at maximum distance from v in $T_{d,k}$. There are $N = d(d-1)^{k-1}$ of such vertices that we label by the integers modulo $d(d-1)^{k-1}$. Now, for odd d, we build a new graph $GM(d,k)$ taking $T_{d,k}$ as a basis and attaching to V the undirected de Bruijn graph $UG_B(\Delta, N)$, where $\Delta = \frac{d-1}{2}$. That is, the subgraph of $GM(d,k)$ induced by the set of vertices at distance k of v is precisely $UG_B(\Delta, N)$. Obviously $GM(d,k)$ has order $M_{d,k}$ and radius k, since the eccentricity of the 'root' vertex v is k. Next, we prove that for $k = 3$ and bigger enough d, the graph $GM(d,k)$ has diameter $k+1$.

Theorem 5 *$GM(d,3)$ has diameter four for every odd $d \geq 19$.*

Proof: Let v the root of the spanning tree $T = T_{d,3}$ of $G = GM(d,3)$. Since the eccentricity of v is 3, the maximum distance from an adjacent vertex to v is at most 4. Now, we prove the following: Let u_1 and u_2 two vertices at distance three from v, then $\mathrm{dist}(u_1,u_2) \leq 4$ if $d \geq 19$. Since u_1 and u_2 are at distance three from v, we can consider both vertices in $UG_B(\Delta, N)$. Let D be the diameter of $UG_B(\Delta, N)$. Taking into account that $D \leq \log_\Delta N$, and $N = (2\Delta)^2(2\Delta+1)$, then:

$$D \leq \lceil \log_\Delta (2\Delta)^2(2\Delta+1) \rceil \leq \lceil \log_\Delta \Delta^2(8\Delta+4) \rceil \leq \lceil 2 + \log_\Delta (8\Delta+4) \rceil$$

Now, since $8\Delta + 4 \leq \Delta^2$ if $\Delta \geq 9$ then we derive that $D \leq 4$ if $d \geq 19$. That is, $\mathrm{dist}(u_1,u_2) \leq 4$ whenever $d \geq 19$. Now, we prove that when u_1 and u_2 are two vertices of $GM(d,3)$ at least one of them at distance < 3 from v, then $\mathrm{dist}(u_1,u_2) \leq 4$. We can assume that $\mathrm{dist}(u_1,v) = 2$ and

$\mathrm{dist}(u_2, v) = 3$, since otherwise there exist a path from u_1 to u_2 (through v) with length at most 4. Let u_1 be at distance 2 from v, we will see that every vertex at distance three from v is at most at distance 4 from u_1. The set of vertices adjacent to u_1 which are at distance 3 from v is

$$\Gamma_1(u_1) = \{l\Delta + s \mid s = 0, \ldots, 2\Delta - 1\}$$

for some even $0 \leq l \leq 4\Delta^2 + 2\Delta$. That is, $\Gamma_1(u_1)$ is the set of vertices of $UG_B(\Delta, N)$ hanging from u_1. Any vertex of $UG_B(\Delta, N)$ at distance at most three from a vertex in $\Gamma_1(u_1)$ is of the form,

$$\Gamma_4(u_1) = \{l\Delta^4 + \Delta^3 s + \Delta^2 k + \Delta k' + k'' \mid k, k', k'' \in \{0, \ldots, \Delta - 1\}\}$$

In particular, any vertex of $UG_B(\Delta, N)$ belongs to $\Gamma_4(u_1)$ whenever $\Delta \geq 5$, that is, $d \geq 11$. Hence, $u_2 \in \Gamma_4(u_1)$ and, as a consequence, $\mathrm{dist}(u_1, u_2) \leq 4$. Let us observe that the diameter of G cannot be less than four since there is no Moore graph of radius three. $\square$

Note that $GM(d, 3)$ is not a regular graph, since $UG_B(\Delta, N)$ contains vertices with degree $2\Delta - 1$ and $2\Delta - 2$. Nevertheless, we observe that $\gcd(\Delta - 1, N) = \gcd(\Delta^2 - 1, N) = 1$ if and only if $\Delta \equiv 0, 2 \pmod 6$. Hence, in these cases we can apply proposition 4 and derive that $UG_B(\Delta, N)$ is a subgraph of a 2Δ-regular graph. As a consequence, the regularity of $GM(d, 3)$ can be completed.

Theorem 6 *Radial Moore graphs of radius three and degree d do exist for $d = 2\Delta + 1 \geq 19$ and $\Delta \equiv 0, 2 \pmod 6$.*

For other values of Δ it is not clear how to rearrange the subgraph of $UG_B(\Delta, N)$ induced by the set of vertices with degree $< 2\Delta$ in order to complete the regularity. We call *the problem of the regularity completeness* at this special situation.

Problem 7 Solve the problem of regularity completeness for other values of Δ.

Acknowledgement

We would like to thank you Prof. Joan Gimbert for his valuable suggestions in this particular topic.

References

[1] E. Bannai and T. Ito. On finite Moore graphs. *J. Fac. Sci. Univ. Tokyo*, 20:191–208, 1973.

[2] C. Capdevila, J. Conde, G. Exoo, J. Gimbert and N. López. Ranking measures in radially Moore graphs. *Networks*, to appear.

[3] R.M. Damerell. On Moore graphs. *Proc. Camb. Phil. Soc.*, 74:227–236, 1973.

[4] A.J. Hoffman and R.R. Singleton. On Moore graphs with diameter 2 and 3. *IBM Res. Develop.*, 4:497–504, 1960.

[5] M. Imase and M. Itoh. Design to minimize a diameter on a building block network. *IEEE Trans. on Computers*, C-30:439–443, 1981.

[6] J. Kuo and H. Fu. On the diameter of generalized undirected de Bruijn graphs $UG_B(n,m), n^2 < m \leq n^3$. *Networks*, 4:180–182, 2008.

[7] M. Knor. Small radially Moore graphs. *Proceedings of MAGIA*, 59–62, Kocovce 2007.

[8] M. Miller and J. Širáň. Moore graphs and beyond: A survey. *Electronic J. Combin.*, DS14, 2005

[9] S.M. Reddy, D.K. Pradhan and J.G. Kuhl. Directed graphs with minimal diameter and maximal connectivity. *School of Engineering Oakland Univ. Tech. Rep.*, 1980.

[10] J. Tang, M. Miller and Y. Lin. HSAGA and its application for the construction of near-Moore digraphs. *J. Discrete Algorithms*, 6(1):73–84, 2008.

Application Layer Multicast Algorithm

Sergio Machado and Javier Ozón
Universitat Politècnica de Catalunya
Castelldefels

Abstract

This paper presents a multicast algorithm, called MSM-s, for point-to-multipoint transmissions. The algorithm, which has complexity $O(n^2)$ in respect of the number n of nodes, is easy to implement and can actually be applied in other point-to-multipoint systems such as distributed computing. We analyze the algorithm and we provide some upper and lower bounds for the multicast time delay.

1 Introduction

Multicast is a point-to-multipoint means of transmitting data in which multiple nodes can receive the same information from one single source. The applications of multicast include video conferencing, multiplayer networking games and corporate communications. The lack of deployment of IP Multicast has led to considerable interest in alternative approaches at the application layer, using peer-to-peer architectures[7]. In an application layer multicast approach, also called overlay multicast, participating peers organize themselves into an overlay topology for data delivery. In this topology each edge corresponds to an unicast path between two end-systems or peers (also called nodes) in the underlying IP network. All multicast-related functionality is implemented by peers instead of routers, with the goal of depicting an efficient overlay network for multicast data transmission.

In this work we present an algorithm suitable for peer-to-peer multicast transmissions, although the high degree of abstraction of its definition makes it also suitable for its implementation in other layers and in general message-passing systems. The main contribution of this proposal is that the operation of our algorithm is simple, with a complexity of $O(n^2)$, where n is the number of peers, and thus it may adapt dynamically to the characteristics of the source and the network in order to improve the multicast time delay. Algorithm execution may be computed by a single group member, usually the one which multicasts the message, or by all the members after the complete network status has been broadcasted.

2 Single Message Multicast Algorithm

Bar Noi *et al.* introduced in [1] the *Message Passing System Model* $\mathcal{MPS}$ which characterizes systems that use packet switching techniques at the communication network. In this work, we extend the Bar Noi model to $E\mathcal{MPS}(n, \boldsymbol{\lambda}, \boldsymbol{\mu})$, which consists of a set of n full-duplex nodes $\{p_0, \ldots, p_{n-1}\}$ such that each node can simultaneously send and receive a message. The term *message* refers to any atomic piece of data sent by one node to another using the protocols of the underlying layers. For each node p we define the *transmission time* μ_p as the time that requires p to transmit a single message. Moreover, for each pair of nodes p and q in a message-passing system we define the *communication latency* λ_{pq} between p and q as follows. If at time t node p starts to send a single message to node q, then node p sends the message during the time interval $[t, t + \mu_p]$, and node q receives the message during the time interval $[t + \lambda_{pq} - \mu_p, t + \lambda_{pq}]$. Thus λ_{pq} is the transmission time μ_p of node p plus the propagation delay between p and q, as shown in Figure 1. We denote by $\boldsymbol{\mu}$ the vector of all μ_p's and by $\boldsymbol{\lambda}$ the matrix of all λ_{pq}'s in the network. For simplicity sake, we assume that the communication latency is constant, and we consider multicast as a broadcast problem, since we can isolate the receiving nodes of a multicast communication, form with them a complete overlay graph, and then depict a routing table through a broadcast algorithm.

Let p_0 be the source node in $E\mathcal{MPS}(n, \boldsymbol{\lambda}, \boldsymbol{\mu})$ model which has a message to multicast to the set of receiving nodes $R = \{p_1, p_2, \ldots, p_{n-1}\}$, we search for an algorithm that minimizes the *multicast time*, that is, the time at which all nodes in R have received the message. Though the result of $E\mathcal{MPS}$ is a multicast spanning tree, in Figure 2 we show that this problem is different from the well known minimum spanning tree problem.

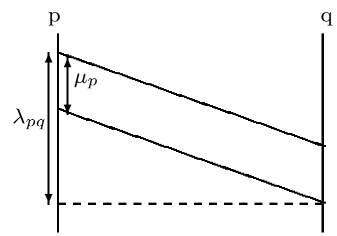


Figure 1: The Postal Model. The latency λ_{pq} is equal to transmission time μ_p plus the propagation delay between p and q.

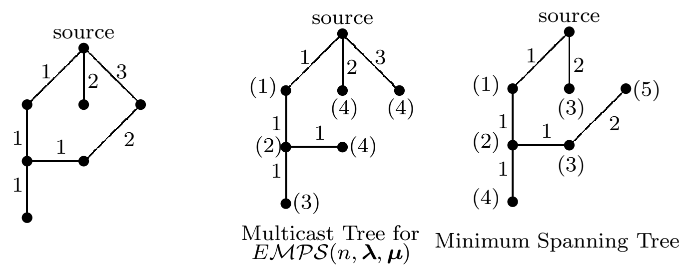


Figure 2: Example of a network where $\mathcal{EMPS}(n, \boldsymbol{\lambda}, \boldsymbol{\mu})$ model does not correspond to the MST problem. We show in parenthesis the multicast delay for each node in the case that transmission time μ is 1 for any node.

The algorithm that we propose, called SMM *Single Message Multicast*, operates as follows: at each step SMM algorithm chooses the node which has not yet received the message and has the lowest cost, that is, the unvisited node that can be reached at minimum time from the nodes which have already received the message. Once the message has been received by this node, the algorithm recalculates the arrival times of the remaining nodes, searches the next node at which the message must be forwarded, and so forth. We assume that when a sending node finishes the retransmission of the message to another node, it begins immediately with another destination node. SMM algorithm is very similar to Dijkstra's shortest path algorithm with the difference that in this case the time delay between two nodes p and q is not constant. Actually, in $\mathcal{EMPS}(n, \boldsymbol{\lambda}, \boldsymbol{\mu})$ this delay is equal to λ_{pq} plus μ_p multiplied by the number of previous retransmissions of node p.

The multicast time achieved by the algorithm SMM is minimum when $\mu_p = 0$ for all the nodes. In this case, the time delay between two nodes p

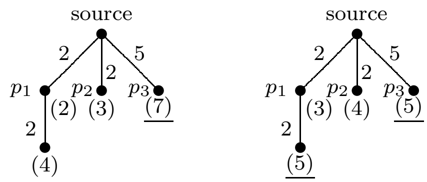


Figure 3: Example of a network where SMM is not optimal. Transmission time is 1 for all the nodes. On the left we apply SMM and on the right another multicast transmission order with a better result.

and q is always the weight λ_{pq} of the edge which joins them, and thus the SMM algorithm corresponds to the optimal algorithm Dijkstra of complexity $O(n^2)$. In a general case, however, the SMM algorithm is not always optimal. Figure 3 shows a network where SMM is not optimal.

Proposition 1 *Algorithm SMM for* $\mathcal{EMPS}(n, \boldsymbol{\lambda}, \boldsymbol{\mu})$ *has complexity* $O(n^2)$.

Proof: At each step SMM searches the node which has not yet received the message with lowest cost. As the maximum number of unvisited nodes is $n-1$ this operation requires at most $n-2$ comparisons. Moreover, the algorithm executes one step for every node which receives the message. Thus, we have $n-1$ steps and at each step we perform at maximum $n-2$ comparisons plus some basic and bounded operations resulting in a complexity of $O(n^2)$. □

2.1 Message Stream Multicast Algorithm

The SMM algorithm has been defined for the multicast of a single message. For a set of messages we can repeat indefinitely the routing table obtained with the SMM algorithm, multicasting each message independently of the others. That means that when one message finally arrives at all the nodes, the message source would proceed to multicast the next message, and so forth. The total delay multicast time of the stream would be in this case the total number of messages M multiplied by the multicast SMM delay for one single message. The main inconvenience of this solution is that the source can not send the next message until the previous one has been received by all the group members and this could slow down the rate of communication.

Next, we consider a new possibility. Before the first message has arrived at all nodes, the source could stop sending it and begin with the second message. With this restriction, the multicast time of the first message will increase, but we will begin to send before the second message. This saving of time between the sending of two consecutive messages will be progressively accumulated, and if the number of messages is large enough it will compensate the increase of the multicast time for one single message. The modified algorithm, that we call MSM-s *Message Stream Multicast*, works as SMM and applies the same multicast scheme for every message with the particularity that it stops the transmission of any message once a node has already sent it s times. Then it will begin to send the next message and so forth. Since the restriction on the number of retransmissions could isolate some nodes of the network, MSM-s should choose a minimum restriction number s to guarantee full-connectivity. As SMM, MSM-s algorithm has complexity $O(n^2)$.

In next sections we prove that under certain conditions it is possible to calculate a minimum number M_σ in such a way that if the number of messages is equal or larger than M_σ then MSM-σ is better than MSM-($\sigma + 1$). Moreover, when restricting the number of transmissions for each node, MSM-s has to take into account the transmission rate of the source. That is, if the source sends at most s times the first message and then, after $s \cdot \mu_r$ time units, stops the transmission of the first message to begin with the second one, we must be able to assume that the source has the second message ready to forward. Otherwise, the source would stop sending the first message before having the second one and would remain unnecessarily idle, with the consequent loss of efficiency.

2.2 Message Stream Multicast Algorithm with Time Restriction

Let p be a node which forwards the message to node q, and let $s_p(s), s_q(s) \leq s$ be the times that p and q forward the message for MSM-s, respectively. In this case the second message will be received at q with a delay of $s_p(s) \cdot \mu_p$ in respect to the first message, since the second message follows the same path but with a source delay of $s_p(s) \cdot \mu_p$, as seen in Figure 4. When the forwarding period $s_q(s) \cdot \mu_q$ of node q is higher than the forwarding period $s_p(s) \cdot \mu_p$ of node p, then successive messages may have higher delays than former messages. In this context, the second message could arrive at node

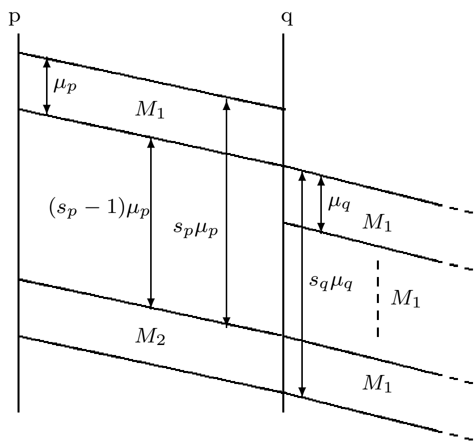


Figure 4: The limit s_q of retransmissions of peer q could be different for every peer q, depending on the transmission times of the peers.

q before it has finished forwarding the first message and then the second message would have to be buffered, with the consequent time delay. This buffering delay would be also accumulated by the third message, and so forth. Nevertheless, this situation may be avoided by limiting the time period $s_q(s) \cdot \mu_q$ at which each node forwards a message, that is, by assuring that the forwarding rate $1/(s_q(s) \cdot \mu_q)$ of any node q is higher than the rate $1/(s_p(s) \cdot \mu_p)$ of any node p which is in the path from the source to node q, including the source. Therefore, the delay of the first message will be always the same as the time delay of any other message, an issue which has great importance in Section 3. Note also that we do not want $1/(s_q(s) \cdot \mu_q) \gg 1/(s_p(s) \cdot \mu_p)$ since in this case q would stop forwarding the first message long before receiving the second one, and then the communication would lose efficiency.

3 Analysis of MSM-s

3.1 Stream Multicast Delay

Let τ_s be the multicast time delay for a single message when the number of transmissions of each node is established up to s, M the number of messages of the stream and μ_r the transmission time of the source, also called root. We assume that s is large enough to arrive at all the nodes of the network. In this case, the total stream multicast delay τ_{Ms} for MSM-s is $\tau_{Ms} = (M-1) \cdot s \cdot \mu_r + \tau_s$. That is, the root sends the first message s times and then, $s \cdot \mu_r$ time units later, it begins with the second and so forth. At moment $(M-1) \cdot s \cdot \mu_r$ the root finishes to send the $(M-1)$th message and it begins with the last message, that will arrive at the last node τ_s time units later. Remember that, as shown in Section 2.2, under certain restrictions, the delay τ_s for the last message is the same as the delay for any other message.

Equation for τ_{Ms} is only valid when the root sends each message s times. When s is large the message may be received by all nodes before the root has sent it s times. Though in this case the node could remain idle and wait until $s \cdot \mu_r$ and then begin to send the second message, this would mean a loss of efficiency. So, for MSM-s, when the message is received for all nodes before the root has sent it s times, we will allow the root to send the second message immediately, without an interval of silence. In this particular case the parameter s should be replaced by the actual number of times $s_r(s) \leq s$ that the root sends each message for MSM-s, and then $\tau_{Ms} = (M-1) \cdot s_r(s) \cdot \mu_r + \tau_s$.

Proposition 2 *Given the algorithm MSM-s for $\mathcal{EMPS}$, the delay of a single message is such that $\tau_{\sigma+\Delta} \leq \tau_\sigma \ \forall \sigma, \Delta > 0$.*

Proof: By construction of the algorithm. When bounding up to $\sigma+\Delta$ the transmissions of each node, MSM-$(\sigma+\Delta)$ will depict a better multicast tree than MSM-$(\sigma+\Delta-1)$ for any message only if there exists a better solution. Otherwise MSM-$(\sigma+\Delta)$ will depict the same multicast tree depicted by MSM-$(\sigma+\Delta-1)$. Thus $\tau_{\sigma+\Delta} \leq \tau_{\sigma+\Delta-1}$. Repeating the argument for $\tau_{\sigma+\Delta-1}$ and $\tau_{\sigma+\Delta-2}$ and so forth, we obtain $\tau_{\sigma+\Delta} \leq \tau_\sigma \ \forall \sigma, \Delta > 0$. $\square$

Theorem 3 *Given the algorithm MSM-s for $\mathcal{EMPS}(n, \boldsymbol{\lambda}, \boldsymbol{\mu})$, we may obtain the conditions such that MSM-σ is faster than MSM-$(\sigma+1)$.*

Proof: First we define, in the case that MSM-σ could be better than MSM-$(\sigma+1)$, the minimum number M_σ of messages from which MSM-σ is better than MSM-$(\sigma+1)$. We begin with $\sigma = 1$. The value of M_1 can be easily obtained once we have computed τ_1, τ_2 and $s_r(2)$ by executing MSM-1 and MSM-2. Remember that $s_r(2)$ is the number of times that the root sends each message in MSM-2 and that, by Proposition 2, $\tau_1 \geq \tau_2$.

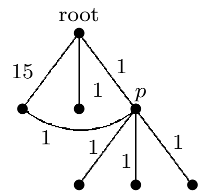


Figure 5: Example of network where $s_r(\sigma) \geq s_r(\sigma+1)$. In particular we have $s_r(3) = 3$ and $s_r(4) = 2$.

Then $\tau_{M1} = (M-1) \cdot \mu_r + \tau_1$ and $\tau_{M2} = (M-1) \cdot s_r(2) \cdot \mu_r + \tau_2$. In this case $s_r(2)$ may be equal to either 1 or 2. In the unusual first case where $s_r(2) = 1$, since $\tau_1 \geq \tau_2$, MSM-2 will be equal or better than MSM-1 for any number of messages. In the more usual case where $s_r(2) = 2$ we establish the restriction $\tau_{M1} \leq \tau_{M2}$ and then $M \geq (\tau_1 - \tau_2)/\mu_r + 1 = M_1$.

For a general case, the number M_σ of messages from which the total stream multicast delay is better for $s = \sigma$ than for $s = \sigma+1$ may be obtained repeating the arguments for M_1. First we obtain by implementing MSM-σ and MSM-$(\sigma+1)$ the following expressions $\tau_{M\sigma} = (M-1) \cdot s_r(\sigma) \cdot \mu_r + \tau_\sigma$ and $\tau_{M\sigma+1} = (M-1) \cdot s_r(\sigma+1) \cdot \mu_r + \tau_{\sigma+1}$. By Proposition 2 we have $\tau_\sigma \geq \tau_{\sigma+1}$. Thus, in the unusual case that $s_r(\sigma) \geq s_r(\sigma+1)$, MSM-$(\sigma+1)$ will be equal or better than MSM-σ for any number of messages. In other case, when $s_r(\sigma) < s_r(\sigma+1)$, we establish $\tau_{M\sigma} \leq \tau_{M(\sigma+1)}$ and then we obtain $M \geq (\tau_\sigma - \tau_{\sigma+1})/((s_r(\sigma+1) - s_r(\sigma)) \cdot \mu_r) + 1 = M_\sigma$. $\square$

Though it is not a usual case, in Figure 5 we depict a network where $s_r(\sigma)$ is greater than $s_r(\sigma+1)$. In particular we have $s_r(3) > s_r(4)$, and thus MSM-4 will be faster than MSM-3 for any number of messages. In order to accomplish the restrictions discussed in section 2.2, we suppose $\mu_r = 2$ and $\mu_p = 1$.

3.2 Analytical Bounds for M_1

In this section, we obtain an analytical bound for M_1, that is, for the number of messages from which the total stream multicast delay is better for $s = 1$ than for $s = 2$. As explained in former section we assume $s_r(2) = 2$. In other case, when $s_r(2) = 1$, MSM-2 will be equal or faster than MSM-1 for any number of messages. Let M be the number of messages; τ_{M1} and

τ_{M2} the multicast delay for MSM-1 and MSM-2 respectively; μ_r the transmission time of the root; and λ_{min} and λ_{max} the minimum and maximum latency between any pair of nodes, respectively. First, we find an upper bound for τ_{M1} and a lower bound for τ_{M2} which we denote respectively by T_1 and t_2. If we force T_1 to be lower or equal than t_2, then MSM-1 will be better than MSM-2:

$$\tau_{M1} \leq T_1 \leq t_2 \leq \tau_{M2} \tag{1}$$

To find T_1 and t_2, we modify slightly the MSM-s performance. First we have:

$$\tau_{M1} \leq (M-1) \cdot \mu_r + (n-1)\lambda_{max} = T_1 \tag{2}$$

Remember that for MSM-1 each node sends each message only once, so MSM-1 depicts a linear tree with $n-1$ links. In this case it is clear that $\tau_{M1} \leq T_1$ since Equation 2 corresponds to the worst case where a message has to cross the $n-1$ links with the maximum latency λ_{max}.

To find a lower bound for τ_{M2} we consider an algorithm with a lower delay than MSM-2. First we assume that the latency for any pair of nodes is the minimum latency λ_{min}. Moreover, in the new algorithm we consider that a node can send the same message simultaneously to two different nodes, that is, that μ_p is equal to 0 for all the nodes. Note that, though this is physically impossible, the new multicast tree will be faster than the tree obtained with MSM-2. Let $N(t), t \in \mathbb{Z}^+$, be the number of nodes that have received the message at step t according to the new algorithm, then $N(t) = 1 + 2 + 4 + \cdots + 2^t = 2^{t+1} - 1$. If we equal $N(t)$ to the number n of nodes we will obtain the number of steps that we need to arrive at all the network $t = \lceil \log_2(n+1) - 1 \rceil$. In this case the new algorithm could send the single message to all the other nodes in $\lceil \log_2(n+1) - 1 \rceil \cdot \lambda_{min}$ time units and then for all the messages we have:

$$\tau_{M2} \geq (M-1) \cdot 2 \cdot \mu_r + \lceil \log_2(n+1) - 1 \rceil \lambda_{min} = t_2 \tag{3}$$

Finally, if according to Equation 1 we force T_1 to be lower or equal than t_2 then τ_{M1} will be also lower or equal than τ_{M2} and MSM-1 will be better than MSM-2. From Equation 2 and Equation 3 it results $M \geq ((n-1)\lambda_{max} - \lceil \log_2(n+1) - 1 \rceil \lambda_{min})/\mu_r + 1$. And since we have considered tighter cases than MSM-1 and MSM-2:

$$M_1 \leq \frac{(n-1)\lambda_{max} - \lceil \log_2(n+1) - 1 \rceil \lambda_{min}}{\mu_r} + 1 \tag{4}$$

Note than when $s_r(2) = 2$ there is always a number of messages from which MSM-1 is better than MSM-2. From Equation 4 we see that this minimum number of messages is linear respect to the number n of nodes. So we can conclude that for the general case that $s_r(2) = 2$, MSM-1 is in general better than MSM-2, provided that the number of messages is usually larger than the number of nodes.

The bound obtained in Equation 4 can be improved by recalculating t_2, that is, by comparing MSM-2 to a tighter algorithm and by using the same lower bound T_1 for MSM-1. We assume that $s_r(2) = 2$. First, we define an algorithm such that, at every step, each node sends the message to one node and such that each node can send the message only twice. We do not consider by the moment time delays. We call $N(t)$ the number of nodes which have received the message at step t. Note that from step $t-1$ to next step t, only the $N(t-1) - N(t-3)$ nodes which have not yet forwarded the message twice can forward it. Thus we have, at step t, the $N(t-1)$ nodes of the last step plus the $N(t-1) - N(t-3)$ nodes that have just received the message one or two iterations before:

$$N(t) = N(t-1) + (N(t-1) - N(t-3)) = 2N(t-1) - N(t-3) \tag{5}$$

In our case we have also $N(0) = 1$, $N(1) = 2$ and $N(2) = 4$. From Table 1 we see that $N(t) = F(t+3) - 1$ where $F(t)$ is the well known Fibonacci serie for $F(0) = 0$ and $F(1) = 1$. Hence, considering $\phi_1 = (1+\sqrt{5})/2)$ and $\phi_2 = (1-\sqrt{5})/2)$ we have:

$$N(t) = F(t+3) - 1 = (\phi_1^{t+3} - \phi_2^{t+3})/\sqrt{5} - 1 \tag{6}$$

As we are determining a lower bound, in order to calculate the number t of steps as a function of the number n of nodes we define $N'(t)$ which is a little faster than $N(t)$ as $N'(t) = (\phi_1^{t+3} + 1)/\sqrt{5} - 1$. Observe that from Equation 6 we have $-1 < \phi_2 < 0$ and then $N'(t) > N(t)$. Hence, if we calculate for $N'(t)$ the number of steps necessary to visit n nodes, we will obtain a lower bound for $N(t)$. For $t \gg 1$ the term ϕ_2^{t+3} is close to 0 and then we have a very accurate bound. If we equal $N'(t)$ to n we obtain $t = \lceil \log_{\phi_1}((n+1)\sqrt{5} - 1) - 3 \rceil$. We can also prove from Figure 6 that at each step t we have a minimum delay of $t \cdot (\lambda_{min} + \mu_{min})/2$ and then:

t				0	1	2	3	4	5	6
N(t)				1	2	4	7	12	20	33
F(t)	0	1	1	2	3	5	8	13	21	34
t	0	1	2	3	4	5	6	7	8	9

Table 1: N(t) vs. Fibonacci Serie.

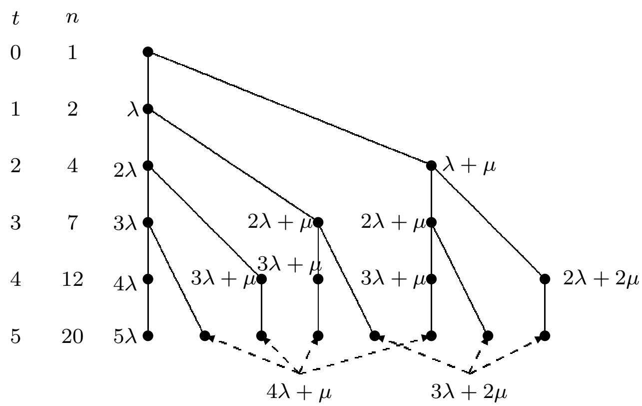


Figure 6: The Fibonacci Tree. Each node forwards the message twice.

$$\tau_{M2} \geq (M-1) \cdot 2 \cdot \mu_r + \left\lceil \log_{\phi_1}((n+1)\sqrt{5}-1) - 3 \right\rceil \frac{\lambda_{min} + \mu_{min}}{2} = t_2$$

Hence, considering the new bound of t_2 with $\phi_1 = (1+\sqrt{5})/2)$ and repeating the arguments from the former section with the same value of T_1, we have:

$$M_1 \leq \frac{(n-1)\lambda_{max} - \left\lceil \log_{\phi_1}((n+1)\sqrt{5}-1) - 3 \right\rceil \frac{\lambda_{min}+\mu_{min}}{2}}{\mu_r} + 1 \qquad (7)$$

This bound is tighter than the bound of Equation 4 depending on the value of μ_{min}. Actually, if μ_{min} is close to λ_{min} the new bound is better than the former, whereas if $\mu_{min} \ll \lambda_{min}$ then we must consider Equation 4. In a practical case we can calculate both bounds and consider the best one.

3.3 An Upper and a Lower Bound for Time Delay in MSM-s

Let τ_{Ms} be the multicast delay for MSM-s and T_s an upper bound of τ_{Ms}. We obtain first a bound for $s = 2$ and then we generalize the result. With this purpose we consider again the algorithm $N(t)$. As we determine an upper bound, in order to calculate the number t of steps as a function of the number n of nodes we define $N''(t)$, which is a little slower than $N(t)$, as $N''(t) = (\phi_1^{t+3} - 1)/\sqrt{5} - 1$. From Equation 6 we have $-1 < \phi_2 < 0$ and then $N''(t) < N(t)$. If we calculate for $N''(t)$ the number of steps that we need to visit n nodes we will obtain an upper bound for $N(t)$. If we equal $N''(t)$ to n we obtain $t = \lceil (\log_{\phi_1}((n+1)\sqrt{5}+1)) - 3 \rceil$. Considering from Figure 6 that at each step t the maximum delay is $t \cdot \lambda_{max}$, we obtain:

$$\tau_2 \leq \lceil (\log_{\phi_1}((n+1)\sqrt{5}+1)) - 3 \rceil \lambda_{max} \tag{8}$$

Note that, if we want to guarantee that $N''(t) < N(t)$, we have to suppose that each node cand send the message twice for MSM-2. Remember also that in section 2.2 we have seen that this is not always possible (in order to avoid congestion). However, we now assume that for MSM-s every node can send the message s times (twice for $s = 2$), even if its rate is lower than the root rate, and then we force the root to send any message with a lower rate than the slowliest node in the graph. In this case, we will not have congestion problems, as referred in section 2.2, and all the messages will have the same delay. Then, since $\tau_s \leq \tau_2 \;\; \forall s \geq 2$ and the root will begin to send a message at most $s \cdot \mu_s$ time units later than the previous one (being μ_s the maximum transmission time of any node in the graph), we obtain:

$$\tau_{Ms} \leq (M-1) \cdot s \cdot \mu_s + \left\lceil \log_{\phi_1}((n+1)\sqrt{5}+1) - 3 \right\rceil \lambda_{max} = T_s \;\; \forall s \geq 2 \tag{9}$$

To find a lower bound t_s for τ_{Ms} we consider, as we did for MSM-2 in section 3.2 but now in a general case, an algorithm such that a node can forward the same message simultaneously to s different nodes. Moreover, we assume that the latency for any pair of nodes is the minimum latency λ_{min}. The new algorithm is therefore better than MSM-s. Let $N(t), t \in \mathbb{Z}^+$, be the number of nodes that have received the message at step t, then $N(0) = 1$ and $N(t) = 1 + s + s^2 + \cdots + s^t = (s^{t+1} - 1)/(s-1)$ for $t > 0$. If we equal $N(t)$ to the number of nodes n we obtain that the number of steps that we need to arrive at all the network for $s \geq 2$ is $t = \lceil (\log_s(n(s-1)+1)) - 1 \rceil$. And thus for $s \geq 2$ we obtain t_s:

$$\begin{aligned} \tau_{Ms} &\geq (M-1) \cdot s_r(s) \cdot \mu_r + \left\lceil \log_s(n(s-1)+1) - 1 \right\rceil \lambda_{min} \\ &\geq (M-1) \cdot \mu_r + \left\lceil \log_s(n(s-1)+1) - 1 \right\rceil \lambda_{min} \end{aligned} \quad (10)$$

For $s = 2$ we obtain the expression in Equation 3 and for $s = 1$ we have $\tau_{M1} \geq (M-1) \cdot \mu_r + (n-1)\lambda_{min}$.

3.4 A General Bound for M_σ

Taking the bounds of the former section and repeating the arguments of section 3.2 for M_1, we obtain a bound for the minimum number M_σ of messages from which MSM-σ is better than MSM-$(\sigma+1)$. As in former sections, this bound has only sense when $s_r(\sigma) < s_r(\sigma+1)$. In other case, MSM-$(\sigma+1)$ is always equal or better than MSM-σ. From Equation 9 we have an upper bound T_σ for $s = \sigma$ and from Equation 10 we obtain a lower bound $t_{\sigma+1}$ for $s = \sigma + 1$. Forcing $T_\sigma \leq t_{\sigma+1}$ we will have $\tau_{M\sigma} \leq T_\sigma \leq t_{\sigma+1} \leq \tau_{M(\sigma+1)}$ and then MSM-σ will be better than MSM-$(\sigma+1)$. This results in the next bound for $\sigma \geq 2$:

$$M \geq \frac{\left\lceil \log_{\phi_1}((n+1)\sqrt{5}+1) - 3 \right\rceil \lambda_{max} - \left\lceil \log_{(\sigma+1)}(n\sigma+1) - 1 \right\rceil \lambda_{min}}{s_r(\sigma+1) \cdot \mu_r - \sigma \cdot \mu_s} + 1$$

Note that we assume $s_r(\sigma+1) \cdot \mu_r \geq \sigma \cdot \mu_s$ and thus $s_r(\sigma+1) = \sigma+1$ (since $\mu_r \leq \mu_s$). Otherwise, we should find tighter values for T_σ and $t_{\sigma+1}$ and then recalculate the bound for M. Finally, since we have a pessimistic case, we obtain for $\sigma \geq 2$:

$$M_\sigma \leq \frac{\left\lceil \log_{\phi_1}((n+1)\sqrt{5}+1) - 3 \right\rceil \lambda_{max} - \left\lceil \log_{(\sigma+1)}(n\sigma+1) - 1 \right\rceil \lambda_{min}}{(\sigma+1) \cdot \mu_r - \sigma \cdot \mu_s} + 1$$

3.5 Robustness of MSM-s

Frequently, real-time applications use unreliable transport-layer protocols such as User Datagram Protocol (UDP). That means that it is not always

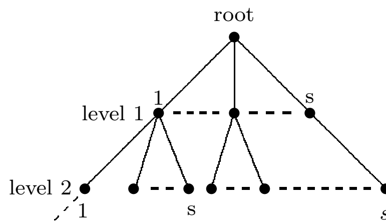


Figure 7: Multicast tree for the calculation of MSM-s robustness.

possible to ensure the ordered and complete arrival of the data at the destination peers. The overlay links of application-layer multicast for real-time applications could therefore provide some degree of reliability. We analyze in this section, under the assumption that there is no message retransmission, the robustness of MSM-s algorithm.

First note that MSM-1 algorithm depicts a linear topology for the multicast tree, that is, a message arrives at a peer which immediately forwards it to another peer and so forth, whereas the MSM-s algorithm depicts in general s divergent paths from each peer of the multicast tree, as shown in Figure 7. In this case, for MSM-1 the probability that a message arrives at l peers is always lower than the probability of arriving at $l-1$ peers. This is because for arriving at the lth peer the message will have to arrive first until the $(l-1)$th peer and then cross successfully the edge between them. This undesirable characteristic does not appears in MSM-s when $s > 1$ due to the different multicast tree that depicts the algorithm, with divergent paths. In MSM-3, for example, the probability of arriving at three peers is much higher than the probability of arriving at only one, an issue which does not happen in MSM-1. Then, as we show in this section, MSM-1 will be less robust than the rest of MSM-s algorithms.

Let $P_c(p,q)$ be the probability that peer q receives correctly a message sent by peer p. For the sake of simplicity we consider that $P_c(p,q) = P_c$ for all the peers. Actually, if we consider $P_c = \max\{P_c(p,q)\}\ \forall p, q \in \mathit{E}\mathcal{MPS}(n, \boldsymbol{\lambda}, \boldsymbol{\mu})$, we will get a lower bound of the robustness of the MSM-s algorithm. We call a peer which has received correctly the message a "visited peer". Note that since the results would be the same, we calculate the robustness only for the transmission of one single message.

We denote by $\bar{n}_s$ the average number of peers that receive the message for MSM-s with a probability P_c for each peer-to-peer communication. To calculate $\bar{n}_s$ we divide the peers into levels, according to Figure 7. We call E_l the average number of peers that receive the message at level l. For the first level we have s peers and then:

$$E_1 = E(r_{11} + r_{21} + \cdots + r_{s1}) = sE(r_{11}) = s(0 \cdot p(0) + 1 \cdot p(1)) = sP_c$$

By definition r_{ij} is 1 if the peer i at level j has received the message and 0 otherwise (we calculate the average number of visited peers when we send only one message). Thus $r_{11} + r_{21} + \cdots + r_{s1}$ is equal to the number of peers that have received the message at the first level. For the s^2 peers of the second level we have:

$$E_2 = E(r_{12} + r_{22} + \cdots + r_{s^2 2}) = s^2 E(r_{12}) = s^2(0 \cdot p(0) + 1 \cdot p(1)) = s^2 P_c^2$$

And in general for the level l:

$$E_l = E(r_{1l} + r_{2l} + \cdots + r_{s^l l}) = s^l E(r_{1l}) = s^l(0 \cdot p(0) + 1 \cdot p(1)) = s^l P_c^l$$

Finally we calculate $\bar{n}_s$ as the sum of the averages of each level, considering that, since the root always has the message, for level 0 this number is 1. We denote by L the number of levels:

$$\begin{aligned} \bar{n}_s &= E_0 + E_1 + \cdots + E_L \\ &= 1 + sP_c + (sP_c)^2 + \cdots + (sP_c)^L \\ &= \frac{(sP_c)^{L+1} - 1}{sP_c - 1} \end{aligned} \tag{11}$$

In this case we assume that the number of peers is $1+s+s^2+\cdots+s^L = (s^{L+1}-1)/(s-1)$ and that we flood the network level by level (this only would happen on a very regular network with little time transmissions). For the MSM-1 algorithm the assumptions of Equation 11 are valid. Since we have $L = n - 1$ it results:

$$\bar{n}_1 = \frac{1 - P_c^n}{1 - P_c} < \frac{1}{1 - P_c}$$

Thus if $1/(1 - P_c)$ is much smaller than the number n of peers, the average number of visited peers with MSM-1 will be also much smaller than n. But if, on the contrary, $1/(1 - P_c)$ is higher than n then the average number of visited peers may be close to n.

n	$P_c = 0.9$	$P_c = 0.99$	$P_c = 0.999$	$P_c = 0.9999$
50	9.95	39.50	48.79	49.88
100	10.00	63.40	95.21	99.51
1000	10.00	100.00	632.30	951.67

Table 2: Average of the number of peers that receive the message for MSM-1, depending on n and P_c.

In Table 2 we see some values of the average for MSM-1 depending on n and P_c. For $P_c = 0.9$ we have $1/(1 - P_c) = 10$ and then the average may not be greater than 10, no matter how high is n. However, for the usual values of $P_c = 0.999$ and $n = 50$ or $n = 100$ we have good averages, close to n. For the other values the average is much smaller than n, which means that the message is not received by a large percentage of peers. In this case the algorithm should automatically change from MSM-1 to MSM-2. For instance, for $n \approx 1000$ and $P_c = 0.999$ we would arrive at only the 63.2% of the peers with MSM-1 whereas for MSM-2 the percentage would be of 99.1%. In this case the percentage for MSM-3 would be only a little higher than for MSM-2: 99.4%. Actually, in a general case the robustness of MSM-2 will be acceptable.

Therefore, when we want to apply MSM-1 to a real network (assuming that MSM-1 can topologically arrive at all the peers with the restriction $s = 1$, that the rate of the source is high enough to provide a new message every μ_r time units, and that for the number of messages that we have the time delay is lower for MSM-1 than for MSM-2), in this case the algorithm itself should estimate the average number of peers that will receive the message for MSM-1 and if it would not be high enough then it should apply MSM-2, consider the new average number and change if necessary to MSM-3, and so forth.

Nevertheless, the assumption that P_c is equal for all the links has major implications on MSM-1 than on MSM-s for $s \geq 2$. In general, in MSM-1 there are a larger percentage of short end-to-end transmissions than in MSM-s for $s \geq 2$. This means that in MSM-1 there will be in general more transmissions than in MSM-s with a probability of success larger than P_c. Thus, the results of Equation 11 can be more pessimistic for $s = 1$ than for $s \geq 2$.

4 Conclusions and Future Work

In this paper we propose an algorithm, called MSM-s, for multicast transmissions in application-layer networks. The fundamental parameter of MSM-s is the value of s, i.e. the maximum number of times that a peer may forward a single message. This parameter is also the maximum degree of each node in the resulting multicast tree. We present a theoretical study of the multicast delay and the robustness of the algorithm.

The practical implementation of the algorithm in real peer-to-peer networks will be part of our future work. Though in this paper we have not considered the possibility of dynamic multicast groups, we plan to define mechanisms for the actualization of the routing tables which take into account the joining and leaving of peers without recalculating the whole table. This could mean a loss of efficiency but would simplify the computation. We consider the possibility of allowing a maximum number of joinings and leavings of peers with only partial changes on routing tables. Once this number is achieved the algorithm would proceed to completely recalculate the tables. This maximum number could be determined by means of theoretical bounds and network measures.

We also plan to study the benefits of the use of the algorithm for two real-time applications: multi-player networking games, which can be considered a static scenario with large restrictions on delay and with multiple points of information, and video-streaming of stored content and live-events, which are dynamic scenarios where the preservation of messages rate is of great importance.

References

[1] A. Bar-Noy and S. Kipnis. Designing broadcasting algorithms in the postal model for message-passing systems. *Proceedings of the Fourth Annual ACM Symposium on Parallel Algorithms and Architectures, SPAA '92*, 13–22, 1992.

[2] M. Castro, P. Druschel, A-M. Kermarrec and A. Rowstron. SCRIBE: A large-scale and decentralised application-level multicast infrastructure. *IEEE Journal on Selected Areas in Communication*, 20(8):85–110, 2002.

[3] Y. Chawathe, S. McCanne and E. A. Brewer. RMX: Reliable Multicast for Heterogeneous Networks. *Proceedings of the Nineteenth Annual Joint Conference of the IEEE Computer and Communications Societies, INFOCOM 2000*, 2:795–804, 2000.

[4] S. E. Deering and D. R. Cheriton. Multicast routing in datagram internetworks and extended LANs. *ACM Trans. Comput. Systems*, 8(2):85–110, 1990.

[5] J. Jannotti, D. K. Gifford, K. L. Johnson, M. F. Kaashoek and J. W. O'Toole. Overcast: Reliable Multicasting with an Overlay Network. *USENIX Symposium on Operating Systems Design and Implementation*, 197–212, 2000.

[6] S. Ratnasamy, P. Francis, M. Handley, R. Karp and S. Shenker. A Scalable Content Addressable Network. *Proceedings of the ACM SIGCOMM 2001 Conference on Applications, Technologies, Architectures, and Protocols for Computer Communications*, 161-172, 2001.

[7] J. H. Saltzer, D. P. Reed and D. D. Clark. End-to-end arguments in system design. *ACM Transactions on Computer Systems*, 2(4):277–278, 1984.

[8] E. W. Zegura, K. L. Calvert and S. Bhattacharjee. How to Model an Internetwork. *Proceedings of the Fifteenth Annual Joint Conference of the IEEE Computer Societies, Networking the Next Generation, INFOCOM '96*, 2:594–602, 1996.

[9] S. Q. Zhuang, B. Y. Zhao, A. D. Joseph, R. H. Katz and J. D. Kubiatowicz. Bayeux: An Architecture for Scalable and Fault-tolerant Wide-area Data Dissemination. *Proceedings of the 11th International Workshop on Network and Operating Systems Support for Digital Audio and Video, NOSSDAV '01*, 11–20, 2001.

On the k-restricted edge-connectivity of matched sum graphs

Xavier Marcote
Universitat Politècnica de Catalunya
Barcelona

Abstract

A *matched sum graph* G_1MG_2 of two graphs G_1 and G_2 of the same order n is obtained by adding to the union (or sum) of G_1 and G_2 a set M of n independent edges which join vertices in $V(G_1)$ to vertices in $V(G_2)$. When G_1 and G_2 are isomorphic, G_1MG_2 is just a *permutation graph.* In this work we derive bounds for the k-restricted edge connectivity $\lambda_{(k)}$ of matched sum graphs G_1MG_2 for $2 \leq k \leq 5$, and present some sufficient conditions for the optimality of $\lambda_{(k)}(G_1MG_2)$.

1 Introduction

Georges and Mauro introduced in [11] the concept of matched sum graphs as follows. Given two graphs G_1, G_2 of the same order $|V(G_1)| = |V(G_2)| = n$ and a set M of n independent edges with one endvertex in $V(G_1)$ and the other one in $V(G_2)$ (a matching between $V(G_1)$ and $V(G_2)$), the *matched sum graph of* G_1 *and* G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup M$. Even though these authors denoted such a graph by $G_1M^+G_2$, we will simplify this writing to G_1MG_2 heretofore for the sake of simplicity. Matched sum graphs are in fact *permutation graphs* —as they were introduced by Chartrand and Harary in [6]— when G_1 and G_2 are isomorphic; hence, matched sum graphs generalize the concept of permutation graphs. Examples of permutation graphs include hypercubes, prisms and some generalized Petersen graphs; see [12, 15, 17, 18] for results on permutation graphs.

This work is devoted to study a particular measure of the connectivity of matched sum graphs, extending (and somehow improving) some other related known results. This measure —which can be seen within the framework of *conditional connectivities*, introduced by Harary in [13]— is the so-called *k-restricted edge connectivity* of a graph G, denoted $\lambda_{(k)}(G)$, which corresponds to the minimum cardinality of a set of edges of G whose deletion results in a disconnected graph with all its components of cardinality at least k. We first derive bounds for the k-restricted edge connectivity of matched sum graphs $G = G_1MG_2$ for $2 \leq k \leq 5$. As a consequence of this, we can present some sufficient conditions to guarantee optimality for $\lambda_{(k)}(G)$, G being a matched sum graph. These new results extend and improve those obtained in [2, 3] in some senses.

From now on, every graph will be assumed to be simple; that is, with neither loops nor multiple edges.

1.1 Notation and terminology

Unless otherwise stated we follow [7] for additional terminology and definitions.

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For every subset X of $V(G)$, $G[X]$ denotes the subgraph of G *induced* by X. For every vertex $x \in V(G)$, the *neighborhood of x* denoted by $N(x) = N_G(x)$ is the set of vertices that are adjacent to x. The *degree* of a vertex x is $d(x) = d_G(x) = |N(x)|$, whereas $\delta = \delta(G)$ is the *minimum degree* over all vertices of G. For every two given proper subsets X, Y of $V(G)$ we denote by $[X, Y]$ the set of edges with one end in X and the other end in Y; when $X = \{x\}$, we write $[x, Y]$ instead of $[\{x\}, Y]$. If X is a proper subset of $V(G)$, let us denote by $w(X) = w_G(X)$ to the set $[X, V(G) \setminus X]$. If the graph G is connected and $1 \leq k \leq |V(G)|$ is an integer, the *minimum k-edge degree of G* is defined as

$$\xi_{(k)}(G) = \min\{|w(X)| : |X| = k,\ G[X] \text{ is connected}\}.$$

Clearly $\xi_{(1)}(G) = \delta(G)$ and $\xi_{(2)}(G) = \min\{d(u)+d(v)-2 : uv \in E(G)\}$, the latter being usually denoted as $\xi(G)$ and called the *minimum edge-degree* of G.

Inspired by the definition of conditional connectivity introduced by Harary [13], Fàbrega and Fiol [9, 10] proposed the concept of k-restricted edge connectivity as follows. For an integer $k \geq 1$ an edge cut W is called a *k-restricted edge cut* if every component of $G - W$ has at least k vertices, where $k \geq 1$ (in the former version due to Fàbrega and Fiol all components obtained by deleting a k-restricted edge cut W from G should have at least $k+1$ vertices, hence $k \geq 0$ was taken; nevertheless, in view of recent related literature we consider in this work cardinality at least k for the components of $G - W$). Assuming that G has k-restricted edge cuts (then G is said to be $\lambda_{(k)}$*-connected*), the *k-restricted edge connectivity* of G, denoted by $\lambda_{(k)}(G)$, is defined as the minimum cardinality over all k-restricted edge cuts of G. From the definition, we immediately have that if $\lambda_{(k)}(G)$ exists, then $\lambda_{(i)}(G)$ exists for any $i < k$ and $\lambda_{(i)}(G) \leq \lambda_{(k)}(G)$. Observe that any edge cut of G is a 1-restricted edge cut and $\lambda_{(1)}(G)$ is just the standard connectivity $\lambda(G)$. Furthermore, the restricted edge connectivity $\lambda'(G)$ defined in [8] is $\lambda'(G) = \lambda_{(2)}(G)$.

As far as the existence of k-restricted edge cuts is concerned, it was shown in [8] that $\lambda_{(2)}(G)$ exists and $\lambda_{(2)}(G) \leq \xi(G)$ if G is not a star and its order is at least 4. For $k = 3$, it was shown [5, 16] that except for a special class of graphs named *flowers*, 3-restricted edge cuts exist and $\lambda_{(3)}(G) \leq \xi_{(3)}(G)$ for any connected graph G with order at least 7. Following Ou [16], a graph F of order $n \geq 2k$ is called a *flower* if it contains a cut-vertex s such that every component of $F - s$ has order at most $k - 1$. The following result was given by Zhang and Yuan in [21].

Theorem 1 [21] *Let G be a connected graph of minimum degree δ and order $n \geq 2(\delta+1)$ that is not isomorphic to any $G^*_{m,\delta}$ (where $G^*_{m,\delta}$ consists of m disjoint copies of K_δ and a new vertex u adjacent to all the vertices in those copies). For all $k \leq \delta+1$, G is $\lambda_{(k)}$-connected with $\lambda_{(k)}(G) \leq \xi_{(k)}(G)$.*

A graph G is said to be $\lambda_{(k)}$*-optimal* if $\lambda_{(k)}(G) = \xi_{(k)}(G)$. For other interesting results on the k-restricted edge connectivity of graphs see [1, 3, 4, 14, 19, 20, 22], among others.

2 Main results

Given a matched sum graph G_1MG_2, it is clear that if $B \subset V(G_i)$ is a set of cardinality k that induces a connected subgraph of G_i then

$$\xi_{(k)}(G_1MG_2) \leq |w_{G_1MG_2}(B)| = |w_{G_i}(B)| + k,$$

which in particular yields to the following remark.

Remark 2 Let $k \geq 1$ and let G_1, G_2 be two graphs of minimum k-edge degrees $\xi_{(k)}(G_1)$, $\xi_{(k)}(G_2)$, respectively. Then for every matched sum graph G_1MG_2 it follows that

$$\xi_{(k)}(G_1MG_2) \leq \min\{\xi_{(k)}(G_1), \xi_{(k)}(G_2)\} + k.$$

A useful result obtained in [3] is recalled next.

Lemma 3 [3] *Let G be a connected graph with minimum degree δ and minimum k-edge-degree $\xi_{(k)}(G)$ with $k \leq \delta + 1$. Then for every $k \geq 2$ and for every $j \in \{0, \dots, k\}$ it follows that*

$$\xi_{(k)}(G) \geq \xi_{(k-j)}(G) + j\delta - 2jk + j(j+1).$$

The following theorem constitutes the main result of this work.

Theorem 4 *Let $2 \leq k \leq 5$ be an integer and let G_1, G_2 be two connected $\lambda_{(k)}$-connected graphs of the same order n and minimum degrees $\delta(G_1) \geq k$, $\delta(G_2) \geq k$, respectively. Then every matched sum graph G_1MG_2 is $\lambda_{(k)}$-connected and*

$$\begin{aligned}
&\min\{n, \lambda_{(k)}(G_1) + \lambda_{(k)}(G_2), \lambda_{(k)}(G_1) + \delta(G_1) - k + 3,\\
&\qquad \lambda_{(k)}(G_2) + \delta(G_2) - k + 3, \xi_{(k)}(G_1MG_2)\}\\
&\leq \lambda_{(k)}(G_1MG_2) \leq \xi_{(k)}(G_1MG_2).
\end{aligned}$$

Proof: Set $\mathcal{M} = G_1MG_2$ from now on. Observe that $n \geq 2k$ because both G_1 and G_2 are $\lambda_{(k)}$-connected. Notice also that $\mathcal{M}$ has no cutvertex, because G_1 and G_2 are connected.

Consider first $G_1 \simeq G_2 \simeq K_n$. In this case, $\mathcal{M}$ is isomorphic to $K_2 \times K_n$, and it is easily seen that this graph is $\lambda_{(k)}$-connected with

$$\lambda_{(k)}(K_2 \times K_n) = n < k(n-k+1) = \xi_{(k)}(K_2 \times K_n).$$

Suppose now that G_1 is a noncomplete graph, then $n = |V(G_1)| \geq \delta(G_1) + 2$. First, when $G_2 \simeq K_n$ we get $\delta(G_2) = n - 1 \geq \delta(G_1) + 1$, hence $\delta(\mathcal{M}) = \delta(G_1) + 1 \leq n - 1$. As a consequence,

$$|V(\mathcal{M})| = 2n \geq 2(\delta(\mathcal{M}) + 1),$$

and $\mathcal{M}$ is $\lambda_{(k)}$-connected with $\lambda_{(k)}(\mathcal{M}) \leq \xi_{(k)}(\mathcal{M})$ following Theorem 1 as $\mathcal{M}$ has no cutvertex. Second, suppose that G_2 is also a noncomplete graph, $n = |V(G_2)| \geq \delta(G_2) + 2$. Then $\delta(\mathcal{M}) = \min\{\delta(G_1), \delta(G_2)\} + 1 \leq n - 1$ and $|V(\mathcal{M})| = 2n \geq 2(\delta(\mathcal{M}) + 1)$ holds. Again from Theorem 1 it follows that $\mathcal{M}$ is $\lambda_{(k)}$-connected with $\lambda_{(k)}(\mathcal{M}) \leq \xi_{(k)}(\mathcal{M})$.

The rest of the proof concerns with the lower bound for $\lambda_{(k)}(\mathcal{M})$. Let $W \subset E(\mathcal{M})$ be a minimum k-restricted edge cut of $\mathcal{M}$, $|W| = \lambda_{(k)}(\mathcal{M})$. Hence $\mathcal{M} - W$ consists of exactly two connected components, H, H^* such that $|V(H)| \geq k$ and $|V(H^*)| \geq k$. Observe that $w(V(H)) = w(V(H^*)) = W = [V(H), V(H^*)]$. If $|V(H)| = k$, then $\lambda_{(k)}(\mathcal{M}) = |W| \geq \xi_{(k)}(\mathcal{M})$ and the result holds. If $W = M$ the result is also true since $\lambda_{(k)}(\mathcal{M}) = |M| = n$. Let us next prove the following claim.

Claim A. *The inequality $\lambda_{(k)}(\mathcal{M}) \geq \xi_{(k)}(\mathcal{M})$ holds provided that any of the following situations occurs:*

(i) *There exist two sets $S_1 \subset V(G_1)$, $S_2 \subset V(G_2)$, $2 \leq |S_1| = k - 2, |S_2| = k - 1$, such that the following conditions hold altogether: $S_1 \cup S_2 \subseteq V(H)$; the subgraphs $\mathcal{M}[S_i]$ are connected, $i = 1, 2$; $\mathcal{M} - W$ contains no edge cd with $c \in S_i$ and $d \in (V(G_i) \setminus S_i) \cap V(H)$, $i = 1, 2$; there exist two vertices $u \in S_1$, $u' \in S_2$ such that $uu' \in E(\mathcal{M} - W)$; $\mathcal{M} - W$ contains no edge $ab' \in M$ where $a \in (V(G_1) \setminus S_1) \cap V(H)$ and $b' \in S_2 - u'$.*

(ii) *There exist two sets $S_1 \subset V(G_1)$, $S_2 \subset V(G_2)$, $|S_1| = |S_2| = k - 1$ for $3 \leq k \leq 4$, and $|S_1| = |S_2| \in \{k - 2, k - 1\}$ for $k = 5$, such that the following conditions hold altogether: $S_1 \cup S_2 = V(H)$; the subgraphs $\mathcal{M}[S_i]$ are connected, $i = 1, 2$; there exist two vertices $u \in S_1$, $u' \in S_2$ such that $uu' \in E(\mathcal{M} - W)$.*

(iii) *$k = 5$ and there exist $S_1 = \{u, w\} \subset V(G_1)$, $S_2 = \{u', v', t'\} \subset V(G_2)$, $S_3 = \{w', z'\} \subset V(G_2)$ ($S_2 \cap S_3 = \emptyset$), $|S_1| = |S_3| = 2$, $|S_2| = 3$, such that the following conditions hold altogether: $S_1 \cup S_2 \cup S_3 \subseteq V(H)$; the subgraphs $\mathcal{M}[S_i]$ are connected, $i = 1, 2, 3$; $\mathcal{M} - W$ contains no edge cd with $c \in S_i$ and $d \in (V(G_i) \setminus S_i) \cap V(H)$, $i = 1, 2, 3$; $uu', ww' \in E(\mathcal{M} - W)$; $\mathcal{M} - W$ contains no edge $ab' \in M$ where $a \in (V(G_1) \setminus S_1) \cap V(H)$ and $b' \in S_2 - u'$.*

(iv) *$k = 5$ and there exist $S_1 = \{u, w\} \subset V(G_1)$, $S_2 = \{u', v', t', z'\} \subset V(G_2)$, $|S_1| = 2$, $|S_2| = 4$, such that the following conditions hold* *altogether:* $S_1 \cup S_2 \subseteq V(H)$*; the subgraphs* $\mathcal{M}[S_i]$ *are connected,* $i = 1,2$*;* $\mathcal{M} - W$ *contains no edge cd with* $c \in S_i$ *and* $d \in (V(G_i) \setminus S_i) \cap V(H)$*,* $i = 1,2$*;* $uu' \in E(\mathcal{M} - W)$*;* $\mathcal{M} - W$ *contains no edge* $ab' \in M$ *where* $a \in (V(G_1) \setminus S_1) \cap V(H)$ *and* $b' \in S_2 - u'$*.*

(v) $k = 5$ *and there exist* $S_1 = \{u, w\} \subset V(G_1)$*,* $S_2 = \{u', v'\} \subset V(G_2)$*,* $S_3 = \{v, t\} \subset V(G_1)$ $(S_1 \cap S_3 = \emptyset)$*,* $|S_1| = |S_2| = |S_3| = 2$*, such that the following conditions hold altogether:* $S_1 \cup S_2 \cup S_3 \subseteq V(H)$*; the subgraphs* $\mathcal{M}[S_i]$ *are connected,* $i = 1,2,3$*;* $\mathcal{M} - W$ *contains no edge cd with* $c \in S_i$*,* $d \in (V(G_i) \setminus S_i) \cap V(H)$*,* $i = 1,2,3$*;* $uu', vv' \in E(\mathcal{M} - W)$*.*

Proof of Claim A. We give the proof for items (i), (ii) and (iii), since (iv) and (v) are proved similarly.

(i) Considering the set $\Omega = \{u\} \cup S_2$ of cardinality k it is clear that the subgraph of $\mathcal{M}$ induced by Ω is connected. Observe that, for every vertex $v \in S_1 - u$, it may exist an edge in $M \setminus W$ which connects v and some vertex in $(V(G_2) \setminus S_2) \cap V(H)$. Then,

$$\begin{aligned}
\lambda_{(k)}(\mathcal{M}) &= |w_{\mathcal{M}}(V(H))| \\
&\geq |w_{\mathcal{M}}(\Omega)| + \textstyle\sum_{v \in S_1 - u}(d_{\mathcal{M}}(v) - 2|[v,\Omega]| - 1) - (|S_1| - 1)(|S_1| - 2) \\
&\geq \xi_{(k)}(\mathcal{M}) + \textstyle\sum_{v \in S_1 - u}(k + 1 - 2 \cdot 2 - 1) - (k-3)(k-4) \\
&\geq \xi_{(k)}(\mathcal{M}) + (k-3)(k-4) - (k-3)(k-4) = \xi_{(k)}(\mathcal{M}),
\end{aligned}$$

after taking into account that $|[v,\Omega]| \leq 2$ for every $v \in S_1 - u$.

(ii) When $|S_1| = |S_2| = k-1$ consider again the set $\Omega = \{u\} \cup S_2$, which induces a connected subgraph of $\mathcal{M}$. It follows that:

$$\begin{aligned}
\lambda_{(k)}(\mathcal{M}) &= |w_{\mathcal{M}}(V(H))| \\
&\geq |w_{\mathcal{M}}(\Omega)| + \textstyle\sum_{v \in S_1 - u}(d_{\mathcal{M}}(v) - 2|[v,\Omega]|) - (|S_1| - 1)(|S_1| - 2) \\
&\geq \xi_{(k)}(\mathcal{M}) + \textstyle\sum_{v \in S_1 - u}(k + 1 - 2 \cdot 2) - (k-2)(k-3) \\
&\geq \xi_{(k)}(\mathcal{M}) + (k-2)(k-3) - (k-2)(k-3) = \xi_{(k)}(\mathcal{M}).
\end{aligned}$$

And when $|S_1| = |S_2| = k-2 = 3$ $(k = 5)$, take the set $L = \{u, w\} \cup S_2$ with $uw \in E(G_1)$, $w \in S_1$. This set has cardinality $k = 5$ and clearly induces a connected subgraph of $\mathcal{M}$. In this case, if $S_1 \setminus \{u, w\} = \{z\}$:

$$\begin{aligned}
\lambda_{(5)}(\mathcal{M}) &= |w_{\mathcal{M}}(V(H))| \geq |w_{\mathcal{M}}(L)| + d_{\mathcal{M}}(z) - 2|[z, L]| \\
&\geq \xi_{(5)}(\mathcal{M}) + (6 - 2 \cdot 3) \geq \xi_{(5)}(\mathcal{M}),
\end{aligned}$$

noticing that $|[z, L]| \leq 3$.

(iii) Take the set of cardinality five $\Omega = S_1 \cup \{u'\} \cup S_3$, which induces a connected subgraph of $\mathcal{M}$. Then:

$$\begin{aligned}\lambda_{(5)}(\mathcal{M}) &= |w_{\mathcal{M}}(V(H))| \\ &\geq |w_{\mathcal{M}}(\Omega)| + d_{\mathcal{M}}(v') + d_{\mathcal{M}}(t') - 2|[\{v',t'\},\Omega]| - 2|[v',t']| - 1 \\ &\geq \xi_{(5)}(\mathcal{M}) + 6 + 6 - 2\cdot 2 - 2 - 1 = \xi_{(5)}(\mathcal{M}) + 5 > \xi_{(5)}(\mathcal{M}),\end{aligned}$$

because vertices v', t' cannot be adjacent in $\mathcal{M}$ to any vertex of S_1 and since it may exist one edge in $M \setminus W$ which connects z' to some vertex in $(V(G_1) \setminus S_1) \cap V(H)$. $\square$

We continue the proof of the theorem by assuming $|V(H)| \geq k+1$, $|V(H^*)| \geq k+1$, $W \neq M$, and that none of the aforementioned five situations (i) to (v) of Claim A (or the corresponding ones obtained by interchanging the roles of either G_1, G_2, or H, H^*) occurs. We write heretofore $W = W_1 \cup W_M \cup W_2$, with $W_1 \subset E(G_1)$, $W_M \subset M$, $W_2 \subset E(G_2)$. Notice that if $W_i \neq \emptyset$ then W_i is an edge cut of G_i due to the minimality of W. The following claim needs to be proved at this point.

Claim B. *If $W_i \neq \emptyset$, every component of $G_i - W_i$ has at least k vertices.*

Proof of Claim B. We use proof by contradiction. Assume that some component of $G_i - W_i$ has at most $k-1$ vertices. Let C be such a component of $(G_1 - W_1) \cup (G_2 - W_2)$ on at most $k-1$ vertices, chosen so that no other component of $(G_1 - W_1) \cup (G_2 - W_2)$ has fewer vertices than C, and (in case two or more components have this minimum order) with the minimum possible number of components of $(G_1 - W_1) \cup (G_2 - W_2)$ to which these components are linked by means of an edge (of M) in $\mathcal{M} - W$. Assume without loss of generality that $W_1 \neq \emptyset$ and that C is a component of $G_1 - W_1$, with $V(C) \subset V(H)$. As $\mathcal{M}$ is $\lambda_{(k)}$-connected it follows that there exist two adjacent vertices $u \in V(C) \subset V(G_1 - W_1) \cap V(H)$ and $u' \in V(G_2 - W_2) \cap V(H)$ such that the edge $uu' \in M$ does not belong to W. Let us prove now the following assertion:

$$\textit{All components of } H - V(C) \textit{ have at least } k \textit{ vertices.} \tag{1}$$

To this end, let C^* be a component of $G_2 - W_2$ to which C is linked by means of an edge of $M \setminus W$, and assume that $|V(C)| \leq |V(C^*)| \leq k-1$ (otherwise the component of $H - V(C)$ containing C^* has cardinality at least k).

Suppose first that $|V(C)| = 1$, $V(C) = \{u\}$. Then $H - u$ is connected as vertex u is only adjacent in H to vertex $u' \in V(C^*)$, and $|V(H - u)| = |V(H)| - 1 \geq k$. Thus, assertion (1) is proved when $k = 2$.

Now, suppose that $2 \leq |V(C)| \in \{k-2, k-1\}$, hence $3 \leq k \leq 5$. Observe that C^* must be linked in $\mathcal{M} - W$ (by means of an edge of $M \setminus W$) to some component $\tilde{C} \neq C$ of $G_1 - W_1$. Indeed, let us see that supposing otherwise that the only component of $G_1 - W_1$ to which C^* is linked is C yields to one of the five situations of Claim A, against our assumptions. When $|V(C^*)| > |V(C)|$ it must be $|V(C^*)| = k - 1$ and $|V(C)| = k - 2$, which corresponds to situation (i) of Claim A; and when $|V(C^*)| = |V(C)|$, it follows that the only component of $G_2 - W_2$ to which C is linked is C^* (by the way C has been chosen), that is to say, $V(H) = V(C) \cup V(C^*)$ and then $|V(C^*)| = |V(C)| = k - 1$ for $3 \leq k \leq 4$ or $|V(C^*)| = |V(C)| \in \{k-2, k-1\} = \{3, 4\}$ for $k = 5$, because $|V(H)| \geq k + 1$; this is situation (ii) of Claim A.

Hence when $2 \leq |V(C)| \in \{k-2, k-1\}$ it follows that C^* is linked in $\mathcal{M} - W$ (by means of an edge of $M \setminus W$) to some component $\tilde{C} \neq C$ of $G_1 - W_1$. In this case, the component of $H - V(C)$ containing C^* has cardinality at least

$$\begin{aligned} &|V(C^*)| + |V(\tilde{C})| \geq 2 \cdot 2 = 4, && \text{if } k = 3, \\ &|V(C^*)| + |V(\tilde{C})| \geq 2(k-2) \geq k, && \text{if } k = 4, 5. \end{aligned}$$

Observe that assertion (1) is then proved when $k = 3, 4$. Hence, to complete the proof of (1) it must be assumed next that $k = 5$ and $|V(C)| = 2$, $V(C) = \{u, w\}$.

First, if C^* is not linked in $\mathcal{M} - W$ (by means of an edge of $M \setminus W$) to any component $\tilde{C} \neq C$ of $G_1 - W_1$ ($H - V(C^*)$ is connected), it turns out that $|V(C^*)| \in \{3, 4\}$; otherwise $|V(C^*)| = 2$ and so $V(H) = V(C) \cup V(C^*)$ according to the way C has been chosen, which is an absurdity because $|V(H)| \geq 6$ by assumption. When $|V(C^*)| = 3$, C is necessarily linked in $\mathcal{M} - W$ (by means of an edge of $M \setminus W$) to some component $\hat{C} \neq C^*$ of $G_2 - W_2$, because $|V(H)| \geq 6$. If $|V(\hat{C})| \geq 3$ then $|V(H) \setminus V(C^*)| \geq |V(C)| + |V(\hat{C})| \geq 5$; hence the set of edges

$$W' = (W \cup \{uu'\}) \setminus w_{G_2}(V(C^*))$$

is a 5-restricted edge cut of $\mathcal{M}$, of cardinality

$$|W'| \leq |W| + 1 - |V(C^*)|(\delta(G_2) - 2) \leq |W| - 8 < |W|,$$

an absurdity. As a consequence $|V(\hat{C})| = 2$, situation (iii) of Claim A. The case $|V(C^*)| = 4$ corresponds to situation (iv) of Claim A.

Second, suppose that C^* is linked in $\mathcal{M} - W$ by means of an edge of $M \setminus W$ to some component $\tilde{C} \neq C$ of $G_1 - W_1$. When $|V(C^*)| \geq 3$ assertion (1) holds, as $|V(C^*)| + |V(\tilde{C})| \geq 3 + 2 = 5$. Hence, consider the case $|V(C^*)| = 2$. Again, if $|V(\tilde{C})| \geq 3$ we are done, then assume $|V(\tilde{C})| = 2$, which corresponds to situation (v) of Claim A. At this point, assertion (1) has been shown to be true for all $2 \leq k \leq 5$.

Once we have seen that every component of $H - V(C)$ has order at least k, it follows that the set of edges

$$W^* = (W \cup \{ww' : w \in V(C), w' \in V(G_2), ww' \in E(H) \setminus W_M\}) \setminus w_{G_1}(V(C))$$

is a k-restricted edge cut of $\mathcal{M}$. But W^* has cardinality

$$|W^*| \leq |W| + |V(C)| - |w_{G_1}(V(C))| \leq |W| - |V(C)| \leq |W| - 1$$

(because $|w_{G_1}(V(C))| \geq 2|V(C)|$ since $\delta(G_1) \geq k$ and $|V(C)| \leq k - 1$), an absurdity. Then the claim has been proved. $\square$

As a consequence of Claim B, if $W_i \neq \emptyset$ then W_i is indeed a k-restricted edge cut of G_i, hence $|W_i| \geq \lambda_{(k)}(G_i)$.

Therefore, when both $W_1, W_2 \neq \emptyset$, then $\lambda_{(k)}(\mathcal{M}) = |W| \geq |W_1| + |W_2| \geq \lambda_{(k)}(G_1) + \lambda_{(k)}(G_2)$, and the theorem holds. Hence we may assume $W_1 \neq \emptyset$ and $W_2 = \emptyset$, and in this case $V(H) \subset V(G_1)$ and $k + 1 \leq |V(H)| = |W_M|$. It follows that

$$\lambda_{(k)}(\mathcal{M}) = |W| = |W_1| + |W_M| = |W_1| + |V(H)|. \tag{2}$$

Set $r = |V(H)| \geq k + 1$. First observe that if $r \geq \delta(G_1) - k + 3$, then from (2) and from the fact that $|W_1| \geq \lambda_{(k)}(G_1)$ (because W_1 is a k-restricted edge cut of G_1) it follows

$$\lambda_{(k)}(\mathcal{M}) \geq \lambda_{(k)}(G_1) + \delta(G_1) - k + 3,$$

and the theorem holds. Therefore we assume $k + 1 \leq r \leq \delta(G_1) - k + 2$. By Lemma 3 we have

$$|W_1| \geq \xi_{(r)}(G_1) \geq \xi_{(k)}(G_1) + (r - k)(\delta(G_1) - r - k + 1). \tag{3}$$

If $r \leq \delta(G_1) - k + 1$, then $(r-k)(\delta(G_1) - r - k + 1) \geq 0$, hence from (2), (3), and from Remark 2 it follows that

$$\lambda_{(k)}(\mathcal{M}) \geq \xi_{(k)}(G_1) + r \geq \xi_{(k)}(G_1) + k + 1 > \xi_{(k)}(\mathcal{M}).$$

Suppose finally that $r = |V(H)| = \delta(G_1) - k + 2$. Taking into account Remark 2 and expressions (2) and (3) yields

$$\lambda_{(k)}(\mathcal{M}) \geq \xi_{(k)}(G_1)+(2k-\delta(G_1)-2)+(\delta(G_1)-k+2) = \xi_{(k)}(G_1)+k \geq \xi_{(k)}(\mathcal{M}).$$

Similarly, under the alternative assumption $W_2 \neq \emptyset$ and $W_1 = \emptyset$ we obtain either

$$\lambda_{(k)}(\mathcal{M}) \geq \xi_{(k)}(\mathcal{M})$$

or

$$\lambda_{(k)}(\mathcal{M}) \geq \lambda_{(k)}(G_2) + \delta(G_2) - k + 3,$$

and the proof of the theorem is now complete. □

A very similar expression to that in Theorem 4 was obtained in [2] for matched sum graphs when $k = 2$. In fact, the only difference lies on the terms $\lambda_{(k)}(G_i)+\delta(G_i)-k+3 = \lambda_{(2)}(G_i)+\delta(G_i)+1$ for $i = 1, 2$ (in the lower bound for $\xi_{(2)}(G_1MG_2)$ in Theorem 4), which are one unit larger than the corresponding terms in the mentioned result in [2]; in this sense, Theorem 4 (slightly) improves the result in [2] for the case $k = 2$. When $k = 3$ and $G_1 \simeq G_2$ (then G_1MG_2 is a permutation graph), Theorem 4 recovers the main result in [3]. Hence the case $k = 3$ of Theorem 4 is a natural generalization for matched sum graphs of the corresponding known result for permutation graphs. As far as we know, cases $k = 4, 5$ of Theorem 4 must be considered as new contributions for the k-restricted edge connectivity of matched sum graphs (thus, also for permutation graphs).

The following results —consequences of Theorem 4— provide conditions on G_1, G_2 to guarantee $\lambda_{(k)}$-optimality for matched sum graphs G_1MG_2 ($\lambda_{(k)}(G_1MG_2) = \xi_{(k)}(G_1MG_2)$) when $2 \leq k \leq 5$.

Corollary 5 *Let $3 \leq k \leq 5$ be an integer and let G_1, G_2 be two connected $\lambda_{(k)}$-connected graphs of minimum degrees $\delta(G_1) \geq 2k-3$, $\delta(G_2) \geq 2k-3$ and order $|V(G_1)| = |V(G_2)| \geq \min\{\xi_{(k)}(G_1), \xi_{(k)}(G_2)\} + k$, and such that $\lambda_{(k)}(G_i) \geq \xi_{(k)}(G_i) - \delta(G_i) + 2k - 3$ for both $i = 1, 2$. Then every matched sum graph G_1MG_2 is $\lambda_{(k)}$-optimal.*

Corollary 6 *Let $3 \leq k \leq 5$ be an integer and let G_1, G_2 be two connected $\lambda_{(k)}$-connected graphs such that $\lambda_{(k)}(G_1) \leq \lambda_{(k)}(G_2)$. Suppose that G_1 and G_2 are $\lambda_{(k)}$-optimal, with minimum degrees $\delta(G_1) \geq 2k-3$, $\delta(G_2) \geq k+2$ and order $|V(G_1)| = |V(G_2)| \geq \xi_{(k)}(G_1) + k$. Then every matched sum graph G_1MG_2 is $\lambda_{(k)}$-optimal.*

References

[1] C. Balbuena, P. García-Vázquez and X. Marcote. Sufficient conditions for λ'-optimality in graphs with girth g. *J. Graph Theory*, 52:73–86, 2006.

[2] C. Balbuena, P. García-Vázquez and X. Marcote. Connectivity measures in matched sum graphs. *Discrete Math.*, 308:1985–1993, 2008.

[3] C. Balbuena, D. González-Moreno and X. Marcote. On the 3-restricted edge connectivity of permutation graphs. *Discrete Appl. Math.*, 157:1586–1591, 2009.

[4] C. Balbuena, X. Marcote and P. García-Vázquez. On restricted connectivities of permutation graphs. *Networks*, 45:113–118, 2005.

[5] P. Bonsma, N. Ueffing and L. Volkmann. Edge-cuts leaving components of order at least three. *Discrete Math.*, 256(1–2):431–439, 2002.

[6] G. Chartrand and F. Harary. Planar permutation graphs. *Ann. Inst. H. Poincaré, Sec. B*, 3:433–438, 1967.

[7] Gary Chartrand and Linda Lesniak. *Graphs and Digraphs.* Chapman and Hall, London, UK, 1996.

[8] A.H. Esfahanian and S.L. Hakimi. On computing a conditional edge-connectivity of a graph. *Information Process. Lett.*, 27:195–199, 1988.

[9] J. Fàbrega and M.A. Fiol. Extraconnectivity of graphs with large girth. *Discrete Math.*, 127:163–170, 1994.

[10] J. Fàbrega and M.A. Fiol. On the extraconnectivity of graphs. *Discrete Math.*, 155:49–57, 1996.

[11] J.P. Georges and D.W. Mauro. On generalized Petersen graphs labeled with a condition at distance two. *Discrete Math.*, 259:311–318, 2002.

[12] W. Goddard, M.E. Raines and P.J. Slater. Distance and connectivity measures in permutation graphs. *Discrete Math.*, 271:61–70, 2003.

[13] F. Harary. Conditional connectivity. *Networks*, 13:347–357, 1983.

[14] A. Hellwig and L. Volkmann. Sufficient conditions for graphs to be λ'-optimal, super-edge-connected, and maximally edge-connected. *J. Graph Theory*, 48:228–246, 2005.

[15] H.-J. Lai. Large survivable nets and the generalized prisms. *Discrete Appl. Math.*, 61:181–185, 1995.

[16] J. Ou. Edge cuts leaving components of order at least m. *Discrete Math.*, 305(1–3):365–371, 2005.

[17] B. Piazza. Edge-connectivity of permutation graphs. *Congr. Numer.*, 65:7–16, 1988.

[18] B.L. Piazza and R.D. Ringeisen. Connectivity of generalized prisms over G. *Discrete Appl. Math.*, 30:229–233, 1991.

[19] S. Wang, J. Yuan and A. Liu. k-Restricted edge connectivity for some interconnection networks. *Applied Math. and Comput.*, 201:587–596, 2008.

[20] Z. Zhang. Sufficient conditions for restricted-edge-connectivity to be optimal. *Discrete Math.*, 307:2891–2899, 2007.

[21] Z. Zhang and J. Yuan. A proof of an inequality concerning k-restricted edge connectivity. *Discrete Math.*, 304:128–134, 2005.

[22] Z. Zhang and J. Yuan. Degree conditions for restricted-edge-connectivity and isoperimetric-edge-connectivity to be optimal. *Discrete Math.*, 307:293–298, 2007.

An Overview of the Degree/Diameter Problem for Directed, Undirected and Mixed Graphs

Mirka Miller (invited speaker)
University of Newcastle, Australia
University of West Bohemia, Pilsen
King's College, London

Abstract

A well-known fundamental problem in extremal graph theory is the *degree/diameter problem*, which is to determine the largest (in terms of the number of vertices) graphs or digraphs or mixed graphs of given maximum degree, respectively, maximum out-degree, respectively, mixed degree; and given diameter. General upper bounds, called Moore bounds, exist for the largest possible order of such graphs, digraphs and mixed graphs of given maximum degree d (respectively, maximum out-degree d, respectively, maximum mixed degree) and diameter k.

In recent years, there have been many interesting new results in all these three versions of the problem, resulting in improvements in both the lower bounds and the upper bounds on the largest possible number of vertices. However, quite a number of questions regarding the degree/diameter problem are still wide open. In this paper we present an overview of the current state of the degree/diameter problem, for undirected, directed and mixed graphs, and we outline several related open problems.

1 Introduction

We are interested in relationships among three graph parameters, namely, maximun degree (respectively, maximum out-degree, respectively, maximum mixed degree), diameter and order (i.e., the number of vertices) of a graph (respectively, digraph, respectively, mixed graph). Fixing the values of two of the parameters, we then wish to maximise or minimise the value of the third parameter. Then there are six possible problems, depending on which parameter we maximise or minimise; however, three of these problems are trivial and so below we formulate only the three non-trivial problems. For undirected graphs the problem statements are then as follows.

- *Degree/diameter problem*: Given natural numbers d and k, find the largest possible number of vertices $n_{d,k}$ in a graph of maximum degree d and diameter $\leq k$.

- *Order/degree problem*: Given natural numbers n and d, find the smallest possible diameter $k_{n,d}$ in a graph of order n and maximum degree d.

- *Order/diameter problem*: Given natural numbers n and k, find the smallest possible maximum degree $d_{n,k}$ in a graph of order n and diameter k.

The statements of the directed version of the problems differ only in that 'degree' is replaced by 'out-degree'. The corresponding statements for the mixed version of the problems use both the (undirected) maximum degree and the maximum out-degree.

The three problems are related but as far as we know they are not equivalent. For both undirected and directed cases, most of the attention has been given to the first problem, some attention has been received by the second problem but the third problem has been largely overlooked so far. The mixed version of all three problems was the last to be formulated and has received only very limitted attention until recently.

In this paper we will consider mainly the degree/diameter problem. For most fixed values of d and k, this problem is still wide open. In the next section we give an overview of the undirected version of the degree/diameter problem. In Section 3 we consider the degree/diameter problem for directed graphs. In Section 4 we present the status of the degree/diameter problem for mixed graphs. The paper concludes with some interesting open problems.

2 Undirected graphs

There is a natural straightforward upper bound on the largest possible order $n_{d,k}$ of a graph G of maximum degree d and diameter k. Trivially, if $d = 1$ then $k = 1$ and $n_{1,1} = 2$; in what follows we therefore assume that $d \geq 2$. Let v be a vertex of the graph G and let n_i, for $0 \leq i \leq k$, be the number of vertices at distance i from v. Then $n_i \leq d(d-1)^{i-1}$, for $1 \leq i \leq k$, and so

$$\begin{aligned} n_{d,k} = \sum_{i=0}^{k} n_i &\leq 1 + d + d(d-1) + \cdots + d(d-1)^{k-1} \\ &= 1 + d(1 + (d-1) + \cdots + (d-1)^{k-1}) \\ &= \begin{cases} 1 + d\,\frac{(d-1)^k - 1}{d-2} & \text{if } d > 2 \\ 2k+1 & \text{if } d = 2 \end{cases} \end{aligned} \tag{1}$$

The right-hand side of (1) is called the *Moore bound* and is denoted by $M_{d,k}$. A graph whose maximum degree is d, diameter k, and order equal to the Moore bound $M_{d,k}$ is called a *Moore graph*; such a graph is necessarily regular of degree d.

Moore graphs do exist: For diameter $k = 1$ and degree $d \geq 1$, they are the complete graphs K_{d+1}. For diameter $k = 2$, Hoffman and Singleton [13] proved that Moore graphs can exist only for $d = 2, 3, 7$ and possibly 57; they are the cycle C_5 for degree $d = 2$, the Petersen graph for degree $k = 3$, and the Hoffman-Singleton graph for degree $k = 7$. The existence of a Moore graph of degree 57 is still an open problem. Damerell [9] proved that there are no Moore graphs (other than cycles K_{2k+2}) of diameter $k \geq 3$. An independent proof of this result was also given by Bannai and Ito [1].

Since Moore graphs exist only in a small number of cases, the study of the existence of large graphs focuses on graphs whose order is 'close' to the Moore bound, that is, graphs of order $M_{d,k} - \delta$, for δ small. The parameter δ is called the *defect*, and the most usual understanding of 'small defect' is that $\delta \leq d$. For convenience, by a (d,k)-*graph* we will understand any graph of maximum degree d and of diameter at most k; if such a graph has order $M^*_{d,k} - \delta$ then it will be referred to as a (d,k)-*graph of defect* δ.

Erdös, Fajtlowitcz and Hoffman [10] proved that, apart from the cycle C_4, there are no graphs of degree d, diameter 2 and defect 1, that is, of order one less than the Moore bound. This was subsequently generalized by Bannai and Ito [2] and also by Kurosawa and Tsujii [15] to all diameters. Hence, for all $d \geq 3$ there are no (d, k)-graphs of defect 1, and for $d = 2$ the only such graphs are the cycles C_{2k}. It follows that for $d \geq 3$ we have $n_{d,k} \leq M_{d,k} - 2$. Only a few values of $n_{d,k}$ are known. Apart from those already mentioned, we have also $n_{4,2} = 15$, $n_{5,2} = 24$, $n_{3,3} = 20$ and $n_{3,4} = 38$. The general frontier in the study of the upper bound of $n_{d,k}$ is defect 2.

Miller, Nguyen and Pineda-Villavicencio [17] found several structural properties of $(d, 2)$-graphs with defect 2, and showed the nonexistence of such graphs for infinitely many values of d. Conde and Gimbert [7] used factorisation of certain polynomials related to the characteristic polynomial of a graph of diameter 2 and defect 2 to prove the nonexistence of $(d, 2)$-graphs with defect 2 for other values of d. Combining these results we obtain that for degree d, $6 \leq d \leq 50$, there are no $(d, 2)$-graphs with defect 2. Moreover, we believe that the following conjecture holds.

Conjecture 1 *For degree $d \geq 6$, there are no $(d, 2)$-graphs with defect 2.*

Little is known about defects larger than two. Jorgensen [14] proved that a graph with maximum degree 3 and diameter $k \geq 4$ cannot have defect two. Taking into account the handshaking lemma when defect is odd, this shows that $n_{3,k} \leq M_{3,k} - 4$ if $k \geq 4$. In 2008, this was improved by Pineda-Villavicencio and Miller [18] to $n_{3,k} \leq M_{3,k} - 6$ if $k \geq 5$. Miller and Simanjuntak [19] proved that for $k \geq 3$, a $(4, k)$-graph cannot have defect 2, showing that $n_{4,k} \leq M_{4,k} - 3$ if $k \geq 3$. Currently, for most values of d and k, the existence or otherwise of (d, k)-graphs with defect 2 remains an open problem.

The lower bounds on $n_{d,k}$ and $n^*_{d,k}$ are obtained from constructions of the corresponding graphs and digraphs. There are many interesting techniques used in these constructions, including algebraic specifications (used to produce de Bruijn and Kautz graphs and digraphs), star product, compounding, and graph lifting - the last three methods all producing large graphs from suitable smaller "seed" or "base" graphs. Additionally, many new largest known graphs have been obtained with the assistance of computers.

In the case of undirected graphs, the gap between the lower bound and the upper bound on $n_{d,k}$ is in most cases wide, providing a good motivation for researchers to race each other for ever larger graphs. Further stimulation is provided by the current table of largest graphs (for degree up to 16 and diameter up to 10), kept up to date by Francesc Comellas on the website

http://maite71.upc.es/grup_de_grafs/grafs/taula_delta_d.html

A larger table (for degree up to 20 and diameter up to 10) is kept by Eyal Loz, Hebert Perez-Roses and Guillermo Pineda-Villavicencio; it is available at

http://combinatoricswiki.org/wiki/The_Degree_Diameter_Problem_for_General_Graphs

3 Directed graphs

As in the case of undirected graphs, there is a natural upper bound on the order, denoted by $n_{d,k}$, of directed graphs (digraphs) of given maximum out-degree d and diameter k. For any given vertex v of a digraph G, we can count the number of vertices at a particular distance from that vertex. Let n_i^*, for $0 \leq i \leq k$, be the number of vertices at distance i from v. Then $n_i^* \leq d^i$, for $0 \leq i \leq k$, and consequently,

$$\begin{aligned} n_{d,k}^* = \sum_{i=0}^{k} n_i^* &\leq 1 + d + d^2 + \cdots + d^k \\ &= \begin{cases} \frac{d^{k+1}-1}{d-1} & \text{if } d > 1 \\ k+1 & \text{if } d = 1 \end{cases} \end{aligned} \tag{2}$$

The right-hand side of (2), denoted by $M_{d,k}^*$, is called the *Moore bound* for digraphs. If the equality sign holds in (2) then the digraph is called a *Moore digraph*.

It is well known that Moore digraphs exist only in the trivial cases when $d = 1$ (directed cycles of length $k+1$, C_{k+1}, for any $k \geq 1$) or $k = 1$ (complete digraphs of order $d+1$, K_{d+1}, for any $d \geq 1$). This was first proved by Plesník and Znám in 1974 [23] and later independently by Bridges and Toueg [6]. In the directed version, the general frontier in the study of the upper bound of $n_{d,k}^*$ is defect 1. For diameter $k = 2$, line digraphs of complete digraphs are examples of $(d, 2)$-digraphs of defect 1, for any $d \geq 2$, showing that $n_{d,2}^* = M_{d,2}^* - 1$. When $d = 2$ there are two other non-isomorphic $(2,2)$-digraphs of defect 1 but for $d \geq 3$ Gimbert [11, 12] proved that line digraphs of complete digraphs are the only $(d,2)$-digraphs of defect 1. Moreover, Conde, Gimbert, Gonzalez, Miret and Moreno [8] proved that there are no $(d,3)$-digraphs with defect 1, for any $d \geq 3$.

On the other hand, focusing on small out-degree instead of diameter, Miller and Fris [16] proved that, for maximum out-degree 2, there are no $(2,k)$-digraphs of defect 1, for any $k \geq 3$. Moreover, Baskoro, Miller, Širáň and Sutton [3] proved, for maximum out-degree 3, that there are no $(3,k)$-digraphs of defect 1, for any diameter greater than or equal to 3. The following conjecture is likely to hold but unlikely to be proved in a simple way.

Conjecture 2 *For maximum out-degree $d \geq 2$ and diameter $k \geq 3$, there are no (d,k)-digraphs with defect 1.*

The study of digraphs of defect two has so far concentrated on digraphs of maximum out-degree $d = 2$. Miller and Širáň [20] proved, for maximum out-degree $d = 2$, that $(2,k)$-digraphs of defect two do not exist, for all $k \geq 3$. For the remaining values of $k \geq 3$ and $d \geq 3$, the question of whether digraphs of defect two exist or not remains completely open.

As in the undirected case, the lower bounds on $n^*_{d,k}$ are obtained from constructions of the corresponding digraphs. The current situation for the best lower bounds in the directed case is much simpler than in the undirected case. In the case of directed graphs, the best known values of $n^*_{d,k}$ are, in almost all cases, given by the corresponding Kautz digraph. One exception is the case of $d = 2$, where the best lower bound for $k \geq 4$ is obtained from Alegre digraph and line digraphs of Alegre digraph.

The difference between lower bound and upper bound on the largest possible order of a digraph of given maximum out-degree and diameter is much smaller than in the undirected case. Correspondingly, it seems much more difficult to find constructions of graphs that would improve the lower bound of $n^*_{d,k}$, and indeed, there has not been any improvement to the lower bound during the last 30 years or so, since the discovery of the Alegre digraph. On the other hand, thanks to the line digraph technique, finding any digraph larger than currently best known would result in much higher "payout" than in the undirected case, giving rise to a whole infinite family of largest known digraphs.

4 Mixed graphs

In many real-world networks, a mixture of both unidirectional and bidirectional connections may exist (e.g., the World Wide Web network, where pages are nodes and hyperlinks describe the connections). For such networks, mixed graphs provide a perfect modeling framework. The idea of "mixed" (or "partially directed") graphs is a generalisation of both undirected and directed graphs.

We start by introducing some definitions which are needed for mixed graphs. Let v be a vertex of a graph G. Denote by $id(v)$ (respectively, $od(v)$) the sum of the number of arcs incident to (respectively, from) v and the number of edges incident with v. Denote by $r(u)$ the number of edges incident with v (i.e., the *undirected degree* of v). A graph G is said to be *regular* of degree d if $od(v) = id(v) = d$, for every vertex v of G. A regular graph G of degree d is said to be *totally regular* with mixed degree d, undirected degree r and *directed degree* $z = d - r$ if for every pair of vertices $\{u, v\}$ of G we have $r(u) = r(v) = r$. Mixed Moore graphs of diameter 2 were first studied by Bosák in [4] and [5] who proved that all mixed Moore graphs are totally regular.

Let G be a mixed graph of diameter k, maximum degree d and maximum out-degree z. Let $r = d - z$. Then the order $n(z, r, k)$ of G is bounded by

$$M_{z,r,k} = 1+(z+r)+z(z+r)+r(z+r-1)+\cdots+z(z+r)^{k-1}+r(z+r-1)^{k-1} \tag{3}$$

We shall call $M_{z,r,k}$ *the mixed Moore bound* for mixed graphs of maximum degree d, maximum out-degree z and diameter k. A mixed graph of maximum degree d, maximum out-degree z, diameter k and order $M_{z,r,k}$ is called a *mixed Moore graph*. Note that $M_{z,r,k} = M_{d,k}$ when $z = 0$ and $M_{z,r,k} = M^*_{d,k}$ when $r = 0$ ($d = r + z$).

A mixed graph G is said to be a *proper* mixed graph if G contains at least one arc and at least one edge.. Most of the known proper mixed Moore graphs of diameter 2, constructed by Bosák, can be considered isomorphic to Kautz digraphs of the same degree and order (with the exception of order $n = 18$). Indeed, they are the Kautz digraphs $Ka(d, 2)$ with all digons (a *digon* is a pair of arcs with the same end points and opposite direction) considered as undirected edges.

Mixed Moore graphs for $k \geq 3$ have been categorised in [22]. Suppose

$d \geq 1$, $k \geq 3$. A finite graph G is a mixed Moore graph of degree d and diameter k if and only if either $d = 1$ and G is Z_{k+1} (the directed cycle on $k+1$ vertices), or $d = 2$ and G is C_{2k+1} (the undirected cycle on $2k+1$ vertices).

It remains to consider Moore graphs of diameter 2. Mixed Moore graphs of diameter 2 were studied by Bosák in [5] using matrix and eigenvalue techniques. Bosák proved that any mixed Moore graph of diameter 2 is totally regular with undirected degree r and directed degree z, where these two parameters r and z must satisfy a tight arithmetic condition obtained by eigenvalue analysis. Thus, apart from the trivial cases $z = 1$ and $r = 0$ (graph Z_3), $z = 0$ and $r = 2$ (graph C_5), there must exist a positive odd integer c such that

$$c \mid (4z-3)(4z+5) \text{ and } r = \frac{1}{4}(c^2+3). \tag{4}$$

Mixed Moore graphs of diameter $k = 2$ and order $n \leq 100$ are summarized in Table 1,where $d = z + r$ and the values of r and z are derived from (4) (see [5]).

5 Conclusion

In this paper we have given an overview of the degree/diameter problem and we pointed out some research directions concerning the three parameters order, diameter and maximum degree for undirected graphs, resp., maximum out-degree for directed graphs, resp., maximum mixed degree for mixed graphs. More specifically, we have been interested in the questions of optimising one of these three parameters (the order) given the values of the other two parameters. We finish by presenting a list of some related open problems in this area.

1. *Does there exist a Moore graph of diameter 2 and degree 57?* This is the best known open problem in this area; it has been open for half a century.

2. *Find graphs (resp. digraphs) which have larger number of vertices than the currently largest known graphs (resp., digraphs).*

n	d	z	r	*existence*	*uniqueness*
3	1	1	0	Z_3	$\surd$
5	2	0	2	C_5	$\surd$
6	2	1	1	$Ka(2,2)$	$\surd$
10	3	0	3	Petersen graph	$\surd$
12	3	2	1	$Ka(3,2)$ (Figure 1)	$\surd$
18	4	1	3	Bosák graph	$\surd$
20	4	3	1	$Ka(4,2)$	$\surd$
30	5	4	1	$Ka(5,2)$	$\surd$
40	6	3	3	unknown	unknown
42	6	5	1	$Ka(6,2)$	$\surd$
50	7	0	7	Hoffman-Singleton graph	$\surd$
54	7	4	3	unknown	unknown
56	7	6	1	$Ka(7,2)$	$\surd$
72	8	7	1	$Ka(8,2)$	$\surd$
84	9	2	7	unknown	unknown
88	9	6	3	unknown	unknown
90	9	8	1	$Ka(9,2)$	$\surd$

Table 1: Mixed Moore graphs of diameter 2 and order ≤ 100.

3. *Prove the diregularity or otherwise of digraphs close to Moore bound for defect greater than one.* Clearly, undirected graphs close to the Moore bound must be regular. It is also easy to see that digraphs close to the directed Moore bound must be out-regular. However, even for quite small defect (as little as 2), there exist digraphs which are in-regular but not out-regular (that is, all vertices have the same in-degree but not the same out-degrees).

4. *Investigate the degree/diameter problem for regular graphs, digraphs and mixed graphs.*

5. *Investigate the existence (and uniqueness) of mixed Moore graphs of diameter* $k = 2$ *and orders* $40, 54, 88, 90$ *and when* $n > 100$.

6. *Find large proper mixed graphs and construct a Table of the largest known proper mixed graphs.*

References

[1] E. Bannai and T. Ito. On finite Moore graphs. *J. Fac. Sci. Tokyo Univ.*, 20:191–208, 1973.

[2] E. Bannai and T. Ito. Regular graphs with excess one. *Discrete Math.*, 37:147–158, 1981.

[3] E.T. Baskoro, M. Miller, J. Širáň and M. Sutton. A complete characterisation of almost Moore digraphs of degree three. *J. Graph Theory*, 48(2):112–126, 2005.

[4] J. Bosák. On the k-index of graphs. *Discrete Math.*, 1(2): 133–146, 1971.

[5] J. Bosák. Partially directed Moore graphs. *Math. Slovaca*, 29: 181–196, 1979.

[6] W.G. Bridges and S. Toueg. On the impossibility of directed Moore graphs. *J. Combinatorial Theory B*, 29: 339–341, 1980.

[7] J. Conde and J. Gimbert. On the existence of graphs of diameter two and defect two. *Discrete Math.*, 309: 3166–3172, 2009.

[8] J. Conde, J. Gimbert, J. Gonzalez, J.M. Miret, R. Moreno. Nonexistence of almost Moore digraphs of diameter three. *Electronic Journal of Combinatorics*, 15, R87, 2008.

[9] R.M. Damerell. On Moore graphs. *Proc. Cambridge Phil. Soc.*, 74: 227–236, 1973.

[10] P. Erdös, S. Fajtlowicz and A.J. Hoffman. Maximum degree in graphs of diameter 2. *Networks*, 10:87–90, 1980.

[11] J. Gimbert. On the existence of (d, k)-digraphs. *16th British Combinatorial Conference (London, 1997)*, *Discrete Math.*, 197/198: 375–391, 1999.

[12] J. Gimbert. Enumeration of almost Moore digraphs of diameter two. *Discrete Math.*, 231: 177–190, 2001.

[13] A.J. Hoffman and R.R. Singleton. On Moore graphs with diameter 2 and 3. *IBM J. Res. Develop.*, 4: 497–504, 1960.

[14] L.K. Jorgensen. Diameters of cubic graphs. *Discrete Applied Math.*, 37/38: 347–351, 1992.

[15] K. Kurosawa and S. Tsujii. Considerations on diameter of communication networks. *Electronics and Communications in Japan*, 64A(4): 37–45, 1981.

[16] M. Miller and I. Fris. Maximum order digraphs for diameter 2 or degree 2. *Pullman volume of Graphs and Matrices*, *Lecture Notes in Pure and Applied Mathematics*, 139: 269–278, 1992.

[17] M. Miller, M.H. Nguyen, G. Pineda-Villavicencio. On the nonexistence of graphs of diameter 2 and defect 2. *JCCMCC*, 71: 5–20, 2009.

[18] M Miller, G. Pineda-Villavicencio. Complete catalogue of graphs of maximum degree 3 and defect at most 4. *Discrete Applied Math.*, 157: 2983–2996, 2009.

[19] M. Miller and R. Simanjuntak. Graphs of order 2 less than the Moore bound. *Discrete Math.*, 308 (13): 2810–1821, 2008.

[20] M. Miller and J. Širáň. Digraphs of degree two which miss the Moore bound by two. *Discrete Math.*, 226: 265–270, 2001.

[21] M.H. Nguyen, M. Miller. Moore bound for mixed graphs. *Discrete Math.*, 308 (23): 5499–5503, 2008.

[22] M.H. Nguyen, M. Miller and J. Gimbert. On mixed Moore graphs. *Discrete Math.*, 307: 964–970, 2007.

[23] J. Plesník and Š. Znám. Strongly geodetic directed graphs. *Acta F. R. N. Univ. Comen. - Mathematica* XXIX:29–34, 1974.

Large graphs of diameter two and given degree

Jozef Širáň
Open University, Milton Keynes
Slovak University of Technology, Bratislava

Jana Šiagiová and Mária Ždímalová
Slovak University of Technology
Bratislava

Abstract

Let $r(d,2)$, $C(d,2)$, and $AC(d,2)$ be the largest order of a regular graph, a Cayley graph, and a Cayley graph of an Abelian group, respectively, of diameter 2 and degree d. The best currently known lower bounds on these parameters are $r(d,2) \geq d^2 - d + 1$ for $d-1$ an odd prime power (with a similar result for powers of two), $C(d,2) \geq (d+1)^2/2$ for degrees $d = 2q-1$ where q is an odd prime power, and $AC(d,2) \geq (3/8)(d^2-4)$ where $d = 4q-2$ for an odd prime power q.

Using a number theory result on distribution of primes we prove, for *all* sufficiently large d, lower bounds on $r(d,2)$, $C(d,2)$, and $AC(d,2)$ of the form $c \cdot d^2 - O(d^{1.525})$ for $c = 1$, $1/2$, and $3/8$, respectively. We also prove results of a similar flavour for vertex-transitive graphs and Cayley graphs of cyclic groups.

1 Introduction

The degree-diameter problem is to find, or at least give good estimates on, the largest order of a graph of given maximum degree and diameter. In a broader scope the problem also includes analysis and classification of the largest graphs of given degree and diameter that have been discovered. History and development of this area of research has been summed up in the relatively recent survey [16]. Despite numerous deep results there remain fundamental problems to be resolved, even for diameter two. We will concentrate on this case and refer the reader interested in higher diameters to [16].

For $d \geq 2$ let $n(d, 2)$ be the largest order of a graph of maximum degree d and diameter 2. The diameter requirement localized at a vertex v of degree d implies that any vertex of such a graph, distinct from v and not adjacent to v, must be one of the at most $d-1$ neighbours of some neighbour of v. This implies the bound $n(d, 2) \leq 1 + d + d(d-1) = d^2 + 1$, known as the *Moore bound* for diameter two. By the landmark result of Hoffman and Singleton [10] who initiated research into the degree-diameter problem, the equality $n(d, 2) = d^2 + 1$ holds if and only if $d = 2$, 3, 7, and possibly 57. The corresponding unique extremal graphs, that is, the Moore graphs of diameter two, are the pentagon, the Petersen graph, and the Hoffman-Singleton graph; the existence of a Moore graph of degree 57 is still in doubt. For all the remaining degrees d we have $n(d, 2) \leq d^2 - 1$ by [7].

The best lower bound on $n(d, 2)$ comes from the graphs constructed by Brown [3] and reads $n(d, 2) \geq d^2 - d + 1$ for all degrees d such that $d-1$ is an odd prime power. It was later observed in [5] and [7] that Brown's graphs can be extended by one vertex if $d-1$ is a power of 2, giving $n(d, 2) \geq d^2 - d + 2$ in this special case. Thus, at least for degrees closely related to prime powers, $n(d, 2)$ is asymptotically d^2.

Since Brown's graphs are not regular while the Moore graphs are, it is of interest to ask about a 'regular' version of the degree-diameter problem. As there are no regular graphs of odd order and odd degree, we also allow, but only for odd d, graphs of odd order with a single vertex of degree $d-1$ and all the remaining vertices of degree d; such graphs will be referred to as *almost regular* of degree d. Let now $r(d, 2)$ denote the largest order of a regular or an almost regular graph of degree $d \geq 2$ and diameter 2. Obviously $n(d, 2) \geq r(d, 2)$ but it is not clear at all whether equality holds, say, for some infinite set of degrees.

Observe that all the known Moore graphs of diameter two are not only regular but vertex-transitive as well. In contrast with this, a result of Higman (presented in [4]) says that if a graph of degree $d = 57$, diameter 2, and order $d^2 + 1 = 3250$ exists, it is not vertex-transitive. For the interest of the reader, the currently known best upper bounds on the order of this hypothetical graph can be found in [13]. This has generated interest in the parameter $vt(d,2)$ defined as the largest order of a vertex-transitive graph of degree $d \geq 2$ and diameter 2. By the above result we have $vt(d,2) = d^2 + 1$ for $d \in \{2,3,7\}$ and $vt(d,2) \leq d^2 - 1$ for all other degrees, including 57. The best available lower bound [15] in this case is $vt(d,2) \geq (8/9)(d+1/2)^2$ for all degrees of the form $d = (3q-1)/2$ where q is a prime power such that $q \equiv 1 \bmod 4$.

A special class of vertex-transitive graphs are Cayley graphs. Given a finite group G with a unit-free generating set S such that $S = S^{-1}$, the *Cayley graph* $Cay(G,S)$ has vertex set G and a pair of vertices $g,h \in G$ are adjacent if $g^{-1}h \in S$. Since this condition is equivalent to $h^{-1}g \in S$ because of $S = S^{-1}$, the Cayley graph $Cay(G,S)$ is undirected. Obviously, the degree of $Cay(G,S)$ is $|S|$, and the diameter of $Cay(G,S)$ is 2 if and only if every non-identity element of $G \backslash S$ is a product of two elements from S. We note that the graphs of [15] giving the lower bound at the end of the previous paragraph are vertex-transitive but not Cayley.

To further refine the analysis, for an arbitrary integer $d \geq 2$ we let $C(d,2)$, $AC(d,2)$, and $CC(d,2)$ denote the largest order of a Cayley graph of a group, an Abelian group, and a cyclic group, respectively, of diameter 2 and degree d. For $d = 2$ the three invariants have value 5, and from results summed up in [16] one can extract the upper bounds $C(d,2) \leq d^2 - 1$ and $CC(d,2) \leq AC(d,2) \leq 1+d+d^2/2$ for all $d \geq 3$. Our interest, however, will be in constructions providing lower bounds which, as it appears, are quite far from the upper bounds. For general Cayley graphs the best lower bound is $C(d,2) \geq (d+1)^2/2$ but we only have it for degrees $d = 2q-1$ where q is an odd prime power [18]. In the Abelian case the best available estimate is $AC(d,2) \geq (3/8)(d^2-4)$ where $d = 4q-2$ for an odd prime power q, and for cyclic groups we have $CC(d,2) \geq (9/25)(d+3)(d-2)$ for $d = 5p-3$ where p is an odd prime such that $p \equiv 2 \bmod 3$; both results have been proved in [14]. By [16] the only known lower bound valid for *all* degrees $d \geq 3$ is the folklore inequality $C(d,2) \geq AC(d,2) \geq CC(d,2) > d^2/4$. The values of $C(d,2)$ for $d \leq 20$ found with the help of computers can be looked up in the tables [19].

Our aim is to extend, at least in the asymptotic sense, the above best bounds on $n(d,2)$, $vt(d,2)$, $C(d,2)$, $AC(d,2)$, and $CC(d,2)$ from the very restricted sets of degrees related to prime powers to *arbitrary* degrees. The basis of our considerations is a number theory result of [2] about gaps between consecutive primes. Details, given in Section 3, are straightforward for Cayley graphs, less obvious for regular graphs, and non-trivial for vertex-transitive non-Cayley graphs.

For completeness we remark that interest in large Cayley graphs and digraphs of given diameter and degree has also been motivated by problems in group theory and theoretical computer science. Group theory connections via the concept of a basis of a group have been outlined in [14] and for links with design of interconnection networks we refer to [16]. Importance of Cayley graphs in computer generation of record large graphs of manageable degree and diameter is well known [16] and emphasized also by the recent results of [12] which led to rewriting the tables of current largest graphs kept in [19].

2 A review of two constructions

To make our explanations self-contained we begin with giving some details about the graphs of Brown [3] and McKay-Miller-Širáň [15].

For any prime power q the graph $B(q)$ of Brown has vertex set all the q^2+q+1 points of a finite projective plane over the Galois field $GF(q)$. In other words, vertices of $B(q)$ are equivalence classes $[a,b,c]$ of triples $(a,b,c) \neq (0,0,0)$ of elements of $GF(q)$, where two triples are equivalent if they are a non-zero multiple of each other. Two distinct vertices $[a,b,c]$ and $[a',b',c']$ are adjacent in $B(q)$ if and only if the triples are orthogonal, that is, if $aa'+bb'+cc'=0$. In the terminology of projective geometry this means that vertices $[a,b,c]$ and $[a',b',c']$ are adjacent if and only if the point $[a,b,c]$ lies on the line with homogeneous coordinates $[a',b',c']$. The vertex set of $B(q)$ may also be identified with one-dimensional subspaces of a three-dimensional linear space over $GF(q)$, with adjacency defined by orthogonality of the subspaces. We remark that Brown's graphs are isomorphic to the graphs of [8] and a polarity version of this construction was given in [1].

The structure of Brown's graphs is known to a great detail and most of the few facts we need for our analysis can be easily verified by elementary linear algebra. Determination of the neighbours of a vertex $[a,b,c]$ of $B(q)$ amounts to classify solutions (x,y,z) of the linear equation $ax+by+cz=0$. Apart from $(0,0,0)$ this equation has q^2-1 non-zero solutions representing $(q^2-1)/(q-1)=q+1$ distinct projective points, which are different from $[a,b,c]$ if and only if $a^2+b^2+c^2 \neq 0$. Thus, a vertex $[a,b,c]$ has q or $q+1$

neighbours according to whether $a^2 + b^2 + c^2$ is equal to zero or not. The number of vertices of degree q is then equal to the number of self-orthogonal projective points (those for which $a^2 + b^2 + c^2 = 0$), which is known to be equal to $q+1$, see e.g. [1]. Further, for any two distinct vertices $u = [a, b, c]$ and $v = [a', b', c']$ the system $ax+by+cz = 0$ and $a'x+b'y+c'z = 0$ consists of two linearly independent equations and hence has a one-dimensional solution space which, apart from the zero solution, represents a projective point, that is, a unique vertex w of $B(q)$. If $w \in \{u, v\}$, then u and v are adjacent and exactly one of them is represented by a self-orthogonal triple (if both u and v were represented by self-orthogonal triples then the system would have two linearly independent solutions (a, b, c) and (a', b', c'), a contradiction), and if $w \notin \{u, v\}$ then w is the unique vertex adjacent to both u and v in $B(q)$.

This discussion shows that the vertex set of $B(q)$ is a disjoint union of two sets for which we now reserve the symbols V and W, where vertices in V and W have degree $q+1$ and q, respectively, with $|V| = q^2$ and $|W| = q+1$. Further, the above arguments imply that the subgraph of $B(q)$ induced by the set W is edgeless, every pair of distinct vertices of W is connected by a unique path of length two, every edge joining two vertices of V lies in a unique triangle, while no edge joining a vertex of V with a vertex of W is contained in any triangle. In particular, $B(q)$ has diameter 2, and since its maximum degree is $q + 1$, we have $n(d, 2) \geq q^2 + q + 1 = d^2 - d + 1$ for $d = q + 1$.

Since $|W| = q + 1$, for any odd q we may extend $B(q)$ by an arbitrary perfect matching M between the (even number of) vertices in W and form thus a new graph $B^*(q)$. Since $B^*(q)$ is regular of degree $q+1$ and still has diameter 2, we have $r(d, 2) \geq d^2 - d + 1$ for $d = q + 1$ where q is an odd prime power. When q is a power of 2 it turns out that all the $q+1$ vertices of degree q in $B(q)$ have the form $[a, b, a+b]$ for $a, b \in GF(q)$ and hence are all joined to the vertex $[1, 1, 1]$ of degree $q+1$. In this case we use a different type of extension of $B(q)$ to a regular graph, pointed out in [5, 7]. Namely, we extend $B(q)$ by a new vertex incident to the $q+1$ vertices of degree q and denote the resulting regular graph $B^*(q)$, again. This way of adding the new vertex does not change the diameter and therefore $r(d, 2) \geq d^2 - d + 2$ for q a power of 2.

Although the extended graphs of Brown lead to very good lower bounds on $r(d, 2)$ when $d - 1$ is a prime power, they cannot be used for bounds on

$vt(d, 2)$ for the same degrees, as we show next.

Proposition 1 *The extended Brown's graphs $B^*(q)$ are not vertex-transitive for any prime power q.*

Proof: The result is straightforward if q is a power of 2, since in this case it can be checked that $B^*(q)$ contains exactly two distinct vertices with identical neighbours.

Now, let q be odd. We will borrow the notation introduced earlier together with the facts about Brown's graphs derived above. We saw there that any pair of vertices in W is connected by exactly one path of length two. Thus, at most $(q+1)q/2$ vertices in V are adjacent to at least one vertex from W (in fact, by a refined argument it can be shown that there are exactly $(q+1)q/2$ such vertices but we will not need this here). This means that there is at least one vertex in V, say, v, not joined to any vertex in W. Since every edge joining two vertices of V lies in a unique triangle, the subgraph of $B(q)$ induced by all the neighbours of v is isomorphic to a matching of $(q+1)/2$ edges. Recall also that no edge incident to a vertex in W is contained in any triangle of $B(q)$. It follows that no matter how $B(q)$ is extended to $B^*(q)$ by a matching M between vertices of W, the subgraph of $B^*(q)$ induced by the set of all neighbours of any vertex $w \in W$ contains just one edge, namely, the edge $w'v'$ where $w' \in W$ is the unique vertex for which $ww' \in M$ and $v' \in V$ is the unique vertex adjacent to both w and w'. We conclude that the graphs $B^*(q)$ are never vertex-transitive. □

The current largest *vertex-transitive* graphs of diameter 2, at least for a certain degrees, are the graphs of McKay-Miller-Širáň [15]; they have also been described in two different but equivalent ways in [17] and [9]. Keeping to the original description, let q be an arbitrary prime power such that $q \equiv 1 \bmod 4$ and let $F = GF(q)$. Let X be the set of all non-zero squares in F. Let $M(q)$ be the graph of order $2q^2$ with vertex set $V_0 \cup V_1$ where, for $r \in \{0, 1\}$, the set V_r consists of triples (i, k, r) with $i, k \in F$. Every vertex (i, k, r) is adjacent to the q vertices $(j, k + (-1)^r ij, 1 - r)$ for all $j \in F$, as well as to the $(q-1)/2$ vertices of the form $(i, k + \xi^r x, r)$ for all $x \in X$, with no other adjacency. It was proved in [15] that the graphs $M(q)$ are vertex-transitive (and non-Cayley) graphs of diameter 2. Since they have degree $d = (3q-1)/2$ and order $2q^2$, for such degrees d we have $vt(d, 2) \geq (8/9)(d + 1/2)^2$.

The full automorphism group of the graphs $M(q)$ was determined in [9]. In what follows we will just need information about certain subgroups of this group, which we collect from [15] and [17]. For any $s, t \in F$ the mapping $f_{s,t}$ given by

$$f_{s,t}(i,k,r) = (i+s, k+t-(-1)^r(is+s^2/2), r)$$

is an automorphism of $M(q)$. It can be checked that the collection of all such mappings forms a group H isomorphic, via the bijection $f_{s,t} \mapsto (s,t)$, to the direct product $F^+ \times F^+$ where F^+ is the additive group of F. Moreover, H acts regularly on both V_0 and V_1.

In order to introduce more automorphisms, let n be the largest integer such that 2^n divides $q-1$ and let $q-1 = 2^n(2\ell+1)$. Letting $\lambda = \xi^{2\ell+1}$ one sees that λ has order 2^n in the multiplicative group F^* of the field F. It follows from [15] that the mapping g given by

$$g(i,k,r) = ((-\xi)^{\ell+r}i, \lambda k, 1-r)$$

is an automorphism of the graph $M(q)$ interchanging V_0 and V_1. The group G generated by g and all the $f_{s,t}$ is a group of automorphisms of $M(q)$, transitive on the vertex set of this graph. We will not need any structural information about G later on.

3 Results

We are now ready to state and prove our main result.

Theorem 2 *We have the following lower bounds:*

(1) $n(d,2) \geq r(d,2) \geq q^2+q+1$ *for all* $d \geq 4$*, with* q *being the largest odd prime power such that* $q \leq d-1$*,*

(2) $vt(d,2) \geq 2q^2$ *for all odd* $d \geq 5$*, where* q *is the largest prime power such that* $q \equiv 5 \bmod 8$ *and* $(d+1)/2 < q < (2d+1)/3$*,*

(3) $C(d,2) \geq 2q^2$ *for all degrees* $d \geq 5$*, where* q *is the largest odd prime power such that* $q \leq (d+1)/2$*,*

(4) $AC(d,2) \geq 6q(q-1)$ *for all* $d \geq 10$*, with* q *being the largest odd prime power such that* $q \leq (d+2)/4$*, and*

(5) $CC(d,2) \geq 9p(p-1)$ *for all degrees* $d \geq 12$*, where* p *is the largest prime such that* $p \equiv 2 \bmod 3$ *and* $d/6 \leq p \leq (d+3)/5$.

Proof: The essence of the method is to use the known results summed up in the previous two sections and extend the corresponding graphs appropriately. We assume throughout the proof that d and q (and p in the last case) satisfy the conditions listed above.

(1) Let q be an odd prime power as in the statement; by Chebyshev's theorem we have $(d-1)/2 < q \leq d-1$. Let $L = B^*(q)$ be an extended Brown's graph of degree $q+1$, diameter 2, and order q^2+q+1. The complement $\overline{L}$ of L has degree $\Delta = q^2-1$. Let $j = \lfloor (d-q-1)/2 \rfloor$, which means that $d = q+1+2j$ if d is even, and $d = q+1+2j+1$ if d is odd. Our strategy will be to extend L by j 2-factors of $\overline{L}$ if d is even, and by j 2-factors and a maximum matching coming from a Hamilton cycle of $\overline{L}$ if d is odd, to obtain a regular and an almost regular graph of degree d and diameter 2, respectively.

Observe that for $q \geq 3$ we have $\Delta > (q^2+q+1)/2$. By Dirac's theorem [6], the graph $\overline{L}$ contains a Hamilton cycle, say, C. The graph $\overline{L} \backslash E(C)$ has even degree and is therefore 2-factorable by the classical result of Petersen; let $F_1, \ldots, F_j$ be a collection of j pairwise edge-disjoint 2-factors of $\overline{L} \backslash E(C)$. If d is even, we let L' be the graph arising from L by putting in all the 2-factors F_i for $1 \leq i \leq j$. By our choice of j the graph L' is regular of degree d. If d is odd, let L' be obtained from L by putting in the 2-factors F_i for $1 \leq i \leq j-1$ together with a matching of $(q^2+q)/2$ edges taken from the Hamilton cycle C. The resulting graph L' is obviously almost d-regular. In both instances, L' has diameter 2 because (as a consequence of the inequality from Chebyshev's theorem) it is not complete and contains L as a spanning subgraph. This shows that $r(d,2) \geq q^2+q+1$ if q is the largest odd prime power not exceeding $d-1$, for any $d \geq 4$.

(2) We refer to the notation regarding the McKay-Miller-Širáň graph $M(q)$ introduced before the statement of this theorem. We will extend the graph $M(q)$ by adding new edges as follows. First, observe that the assumption $q \equiv 5 \bmod 8$ implies that $n = 2$ and $2\ell + 1 = (q-1)/4$, that is, λ is a non-square and $\lambda^2 = -1$. Let $Y \subset F$ be a set of non-squares closed under inverses, that is, $Y = -Y$, such that $|Y| = d - (3q-1)/2$; note that our assumptions imply that this number is positive, even, and smaller than $(q-1)/2$. Let us extend the graphs $M(q)$ by adding, at each vertex (i,k,r), a total of $|Y|$ new edges joining (i,k,r) to the vertices of

the form $(i, k+\lambda^r y, r)$ for all $y \in Y$. The resulting graph, denoted $M_Y(q)$, is regular of degree d and has diameter 2 because it contains $M(q)$ as a subgraph. More importantly, by a direct computation one can verify that the generators $f_{s,t}$ and g of the group G introduced above preserve all the added edges. This shows that G is transitive on vertices of $M_Y(q)$, which completes the proof of (2). As far as the assumption $(d+1)/2 < q < (2d+1)/3$ is concerned, we note that the value of $q = (2d+1)/3$ is taken care of by the McKay-Miller-Širáň graphs for all $q \equiv 1 \bmod 4$, while in the case $q = (d+1)/2$ we obtain the Cayley graphs of [18] which will appear in the next part of the argument.

(3) Let L be a Cayley graph of degree $2q-1$, diameter 2, and order $2q^2$, constructed in [18]. An inspection of the construction shows that the graph L is a Cayley graph $Cay(G,X)$ for a non-Abelian group G of order $2q^2$ containing q^2 involutions, and for an inverse-closed generating set X of size $2q-1$ containing q involutions. It is clearly possible to select additional inverse-closed set $Y \subset G$ disjoint from X such that Y contains $d-2q+1$ generators, including some odd number of involutions if d is even. This yields a Cayley graph $L' = Cay(G, X \cup Y)$ of degree d, diameter 2, and order $2q^2$. Consequently, $C(d,2) \geq 2q^2$ for the largest odd prime power $q \leq (d+1)/2$.

(4) Let L be the Cayley graph $Cay(G,X)$ from [14] of diameter 2 for an Abelian group G of order $6q(q-1)$ that contains precisely 3 elements of order 2, where X is an inverse-closed generating set of size $4q-2$ containing exactly two elements of order 2. Let us select an additional inverse-closed set $Y \subset G$ disjoint from X such that Y contains $d-4q+2$ generators, including an involution if d is odd. Clearly, the Cayley graph $L' = Cay(G, X \cup Y)$ has degree d, diameter 2, and order $6q(q-1)$.

(5) Let L be a Cayley graph $Cay(G,X)$ for a cyclic group G of order $9p(p-1)$, degree $5p-3$ and diameter 2 with the single element of order 2 being outside the generating set X, as given in [14]. Choose an additional inverse-closed set $Y \subset G$ disjoint from X such that Y contains $d-5p+3$ generators, including the involution if d is odd. Clearly, the Cayley graph $L' = Cay(G, X \cup Y)$ has degree d, diameter 2, and order $9p(p-1)$. We only need to apply this process for primes p such that $p \geq d/6$ as otherwise we have a better bound $CC(d,2) > d^2/4$ from [16]. $\square$

Using a highly non-trivial number theory result on the distribution of primes it is possible to eliminate q from some of the bounds appearing in

Theorem 2. The roots of the result are in a still unresolved conjecture of Legendre on the existence of a prime between n^2 and $(n+1)^2$ for any positive integer n. Replacing $(n+1)^2$ with x and allowing any real $x \geq 2$ leads to a stronger form of the conjecture, stipulating that for any real $x \geq 2$ there is a prime p such that $x - 2\sqrt{x} + 1 < p \leq x$. The first result in this direction was given by Hoheisel [11] of which we just need a consequence saying that there exists a real number $\theta < 1$ such that the interval $[x - x^{\theta}, x]$ contains a prime number for any sufficiently large x. Clearly, the stronger form of Legendre's conjecture would follow, at least for sufficiently large x, if one could prove that $\theta < 1/2$ in Hoheisel's result. This has generated research towards making the exponent θ as small as possible. The current record is $\theta = 0.525$, established by Baker, Harman and Pintz [2]. That is, by [2], for any sufficiently large x there is a prime p such that $x - x^{0.525} \leq p \leq x$.

We will use this result of [2] to prove lower bounds on the order of the largest graphs of given degree and diameter 2 for all sufficiently large degrees.

Corollary 3 *For all sufficiently large degrees d we have:*

(a) $n(d, 2) \geq r(d, 2) > d^2 - 2d^{1.525}$,

(b) $C(d, 2) > (1/2)d^2 - 1.39d^{1.525}$,

(c) $AC(d, 2) > (3/8)d^2 - 1.45d^{1.525}$.

Proof: Let d be sufficiently large and let q be the largest odd prime power such that $q \leq D$ where D is equal to $d-1$, $(d+1)/2$, and $(d+2)/4$ according as we are in the case (a), (b), and (c). The result of [2] gives

$$q \geq D - D^{0.525} \tag{1}$$

and we will use this estimate in all three instances. In the case (a) when $D = d - 1$ we invoke the first part of Theorem 2 combined with (1) which, for all sufficiently large d, gives

$$r(d, 2) \geq q^2 + q + 1 > (D - D^{0.525})^2 + D - D^{0.525} > d^2 - 2d^{1.525} .$$

If $D = (d + 1)/2$, by the third part of Theorem 2 together with (1) we obtain

$$C(d, 2) \geq 2q^2 \geq 2(D-D^{0.525})^2 > (1/2)d^2-2^{0.475}d^{1.525} > (1/2)d^2-1.39d^{1.525}$$

for all sufficiently large degrees d, which proves (b). In the case (c) when $D = (d+2)/4$ we combine the fourth part of Theorem 2 with (1), which yields

$$AC(d,2) \geq 6q(q-1) > 6(D - D^{0.525} - 1)^2 > (3/8)d^2 - 3 \cdot 4^{-0.525} d^{1.525} ,$$

that is, $AC(d,2) > (3/8)d^2 - 1.45d^{1.525}$ for all sufficiently large d. □

4 Remarks

The construction of the extended graphs of Brown together with Proposition 1 leads to the following interesting question. Given a graph G of maximum degree d, what is the smallest number $\delta = \delta_G$ such that there exists a vertex-transitive graph H of degree $d+\delta$ that contains G as a spanning subgraph? The problem is equivalent to finding a vertex-transitive spanning subgraph of largest degree in the complement of G. In this terminology, Proposition 1 says that $\delta_G > 0$ if $G = B^*(q)$, and it would be interesting to determine the value of δ_G in this case.

Our second remark concerns the number-theoretic approximation bound of [2]. Unfortunately, there appears to be no result on the existence of a prime *from a given congruence class* in the interval $[x - x^\theta, x]$ for some $\theta < 1$ and all sufficiently large x. If there was such a result, the proof of Corollary 3 would apply also to the cases (2) and (5) of Theorem 2 and we would obtain bounds on $vt(d,2)$ and $CC(d,2)$ of the form $(8/9)d^2 - O(d^{1+\theta})$ and $(9/25)d^2 - O(d^{1+\theta})$ for all sufficiently large odd degree d and general degree d, respectively.

In the vertex-transitive case we have stated part (2) of Theorem 2 just for $q \equiv 5 \bmod 8$ and odd degrees d. Recalling that n has been defined as the largest integer such that 2^n divides $q-1$, with $q - 1 = 2^n(2\ell+1)$, our statement of (2) corresponds to the instance when $n = 2$. We can extended (2) to all $n \geq 2$ as follows:

(2') *If q is the largest prime power such that $q \equiv 2^n + 1 \bmod 2^{n+1}$ and $(d+1)/2 < q < (2d+1)/3$, then $vt(d,2) \geq 2q^2$ for all d of the form $d = (3q-1)/2 + 2^{n-1}m$ for $m \leq 2\ell$.*

Indeed, to establish (2'), the only change required in the proof of (2) is to take the set Y to be a union of m orbits of the permutation of F^* given by $y \mapsto \lambda^2 y$ consisting of non-squares. Such a more general version, however, does not appear as appealing as the simplest form for $n = 2$ that we have used in the presentation of part (2) of Theorem 2.

Finally, we remark that an analysis of the orbit structure of the group G of automorphisms of the McKay-Miller-Širáň graphs $M(q)$, introduced in the last paragraph of Section 2, shows that it is not possible to add an extra G-invariant perfect matching to $M(q)$ and hence extend the statement (2) of Theorem 2 to even degrees.

Acknowledgement

The authors acknowledge the support of this research by the VEGA Research Grants 1/0489/08 and 1/0280/10, the APVV Research Grants 0040-06 and 0104-07, and the APVV LPP Research Grants 0145-06 and 0203-06.

References

[1] M. Abreu, C. Balbuena and D. Labbate. Adjacency matrices of polarity graphs and of other C_4-free graphs of large size. *Des. Codes Cryptogr.*, 55:221-233, 2010.

[2] R. C. Baker, G. Harman and J. Pintz. The diference between consecutive primes. II. *Proc. London Math. Soc.*, 83:532–562, 2001.

[3] W. G. Brown. On graphs that do not contain a Thompsen graph. *Canad. Math. Bull.*, 9:281–285, 1996.

[4] P. J. Cameron. *Permutation Groups.* Cambridge Univ. Press, 1999.

[5] C. Delorme. Examples of products giving large graphs with given degree and diameter. *Discrete Applied Math.*, 37-38:157–167, 1992.

[6] G. Dirac. Some theorems on abstract graphs. *Proc. Lond. Math. Soc.*, 2:69–81, 1952.

[7] P. Erdös, S. Fajtlowicz and A.J. Hoffman. Maximum degree in graphs of diameter 2. *Networks*, 10:87–90, 1980.

[8] P. Erdös, A. Rényi, and V. T. Sós. On a problem of graph theory. *Stud. Sci. Math. Hung.*, 1:215-235, 1966.

[9] P. Hafner. Geometric realization of the graphs of McKay-Milller-Širáň. *J. Combin. Theory Ser. B*, 90:223–232, 2004.

[10] A. J. Hoffman and R. R. Singleton. On Moore graphs with diameters 2 and 3. *IBM J. Res. Dev.*, 4:497–504, 1960.

[11] G. Hoheisel. Primzahlprobleme in der Analysis. *S. Preuss. Ak. Wiss.*, 2:1–13, 1930.

[12] E. Loz and and J. Širáň. New record graphs in the degree-diameter problem. *Australas. J. Combin.*, 41:63–80, 2008.

[13] M. Mačaj and J. Širáň. Search for properties of the missing Moore graph. *Linear Algebra Appl.*, 432:2381-2398, 2010.

[14] H. Macbeth, J. Šiagiová and J. Širáň. Cayley graphs of given degree and diameter for cyclic, Abelian, and metacyclic groups. *Submitted*, 2009.

[15] B.D. McKay, M. Miller and J. Širáň. A note on large graphs of diameter two and given maximum degree. *J. Combin. Theory Ser. B*, 74:110–118, 1998.

[16] M. Miller and J. Širáň. Moore graphs and beyond: A survey of the degree-diameter problem. *Electronic J. Combinat.*, Dynamic survey No. D14 (61pp), 2005.

[17] J. Šiagiová. A note on the McKay-Miller-Širáň graphs. *J. Combin. Theory Ser. B*, 81:205–208, 2001.

[18] J. Šiagiová and J. Širáň. A note on large Cayley graphs of diameter two and given degree. *Discrete Math.*, 305:379–382, 2005.

[19] On-line tables of the current largest graphs for the degree-diameter problem at `www.eyal.com.au/wiki/The_Degree/Diameter_Problem`.

Fiedler's Clustering on m-dimensional Lattice Graphs

Stojan Trajanovski and Piet Van Mieghem
Delft University of Technology
Delft

Abstract

We consider the partitioning of m-dimensional lattice graphs using Fiedler's approach [1], that requires the determination of the eigenvector belonging to the second smallest eigenvalue of the Laplacian. We examine the general m-dimensional lattice and, in particular, the special cases: the 1-dimensional path graph P_N and the 2-dimensional lattice graph. We determine the size of the clusters and the number of links, which are cut by this partitioning as a function of Fiedler's threshold α.

1 Introduction

There are many methods and approaches for graph partitioning. Here, we shall focus only on Fiedler's approach to clustering, which theoretically determines the relation between the size of the obtained clusters and the number of links that are cut by this partitioning as a function of a threshold α and of graph properties such as the number of nodes and links. When applying Fiedler's beautiful results [1] to the Laplacian matrix Q of a graph, the eigenvector belonging to the second smallest eigenvalue, known as the algebraic connectivity, needs to be computed. We apply Fiedler's approach to the m-dimensional lattice graph and determine the cluster size as a function of the threshold α. Following the notation in [2], a graph G consists of a set $\mathcal{N}$ of $N = |\mathcal{N}|$ nodes and a set $\mathcal{L}$ of $L = |\mathcal{L}|$ links. We denote by $x_i = \begin{bmatrix} (x_i)_1 & (x_i)_2 & \dots & (x_i)_N \end{bmatrix}^T$ the eigenvector of the $N \times N$ symmetric Laplacian Q belonging to the eigenvalue μ_i. Since eigenvectors of a symmetric matrix are orthogonal, we normalize the eigenvectors of Q by requiring that

$$\|x_i\|_2^2 = x_i^T x_i = 1 \text{ for each } i = 1, ..., N \tag{1}$$

The last condition ensures that the eigenvector is unique. The eigenvalues of the Laplacian are nonnegative with at least one eigenvalue equal to zero [1] and they can be ordered as $0 = \mu_N \leq \mu_{N-1} \leq ... \leq \mu_1$. If the graph is connected, then $\mu_{N-1} > 0$ and the components of the correspondent eigenvector x_{N-1} determine the Fiedler partitioning with respect to the threshold α: the set of nodes $\mathcal{M} = \left\{ j \in \mathcal{N} : (x_{N-1})_j \geq \alpha \right\}$ defines the first (connected) cluster and the set $\mathcal{N} \backslash \mathcal{M}$ determines the second (connected) cluster. Our interest concerns the size (or the number of nodes) of the obtained clusters and the number of links that will be cut by Fiedler's partitioning. The end points of those links are nodes in two separate clusters. We denote by $c(G)$ the number of links in G that will be cut by this partitioning. Furthermore, we define the "ratio of cut links"

$$r(G) = \frac{c(G)}{L} \tag{2}$$

where $L = |\mathcal{L}|$ is the total number of links in the graph.
Clearly, $0 \leq r(G) \leq 1$.

2 The path and lattice graphs

In this Section, we examine the effect of Fiedler's clustering on the lattice graph. We will start with the 1-dimensional path graph P_N on N nodes and containing $(N-1)$ links or hops, which we subsequently will generalize to m dimensions. Finally, we will apply the results to a 2-dimensional lattice.

2.1 A path P_N of $(N-1)$ hops

In [3], the Laplacian eigenvalues (as well as the eigenvectors) of the path P_N are derived as $\mu_{N-m}(P_N) = 2\left(1 - \cos\left(\frac{\pi m}{N}\right)\right)$ for $m = 0, 1, 2, ..., N-1$. The second smallest eigenvalue of the Laplacian is

$$\mu_{N-1}(P_N) = 2\left(1 - \cos\left(\frac{\pi}{N}\right)\right) = 4 \sin^2\left(\frac{\pi}{2N}\right)$$

and the corresponding Laplacian eigenvector components are [3]

$$(x_{N-1})_j = \sqrt{\frac{2}{N}} \cos \frac{\pi}{2N} (2j-1)$$

where $1 \le j \le N$. The corresponding Fiedler partitioning rule for the components of the eigenvector x_{N-1} with respect to the threshold α is

$$(x_{N-1})_j \ge \alpha$$

Clustering into two separate, non-empty sets of nodes will exist if and only if $|\alpha| \le \sqrt{\frac{2}{N}}$. Because $\cos \frac{\pi}{2N}(2j-1)$ decreases with j, the nodes labeled by j will belong to the first cluster provided

$$j \le \left[\frac{1}{2} + \frac{N}{\pi} \arccos \left(\alpha \sqrt{\frac{N}{2}} \right) \right] \tag{3}$$

Relation (3) shows for $\alpha = 0$ that one half of the nodes will belong to both clusters. In all cases only one link will be cut, thus $c(P_N) = 1$.

2.2 The general m dimensional lattice

We consider the m-dimensional lattice $\mathcal{C}_m = \mathrm{La}_{(z_1+1)\times(z_2+1)\times\ldots\times(z_m+1)}$ with lengths $z_1, z_2, \ldots, z_m$ in each dimension, respectively, and where at each lattice point with integer coordinates a node is placed that is connected to its nearest neighbors whose coordinates only differ by 1 in only 1 components. The total number of nodes in $\mathcal{C}_m$ is $N = (z_1+1) \times (z_2+1) \times \ldots \times (z_m+1)$. The lattice graph is a Cartesian product [7] of m path graphs, denoted by $\mathcal{C}_m = P_{(z_1+1)} \square P_{(z_2+1)} \square \ldots \square P_{(z_m+1)}$. According to [3, 4, 5, 6], the eigenvalues of C_m can be written as a sum of one combination of eigenvalues of path graphs and the corresponding eigenvector is the Kronecker product of the corresponding eigenvectors of the same path graphs,

$$\begin{aligned} &\mu_{i_1 i_2 \ldots i_N}(\mathcal{C}_m) = \textstyle\sum_{j=1}^{m} \mu_{i_j}\left(P_{(z_j+1)}\right) \\ &x_{i_1 i_2 \ldots i_m}(\mathcal{C}_m) = x_{i_1}\left(P_{(z_j+1)}\right) \otimes x_{i_2}\left(P_{(z_2+1)}\right) \otimes \ldots \otimes x_{i_m}\left(P_{(z_m+1)}\right) \end{aligned} \tag{4}$$

where $i_j \in \{1, 2, \ldots, z_j+1\}$ for each $j \in \{1, 2, \ldots, m\}$. Without loss of generality we can assume that $z_1 \le z_2 \le \ldots \le z_m$. In Section 2.1, we obtained the Laplacian eigenvalues of the path on N nodes and for the second smallest eigenvalues μ_{N-1} of $P_{(z_1+1)}, P_{(z_2+1)}, \ldots, P_{(z_m+1)}$, we have that

$$\mu_{z_1}\left(P_{(z_j+1)}\right) \geq \mu_{z_2}\left(P_{(z_j+1)}\right) \geq \ldots \geq \mu_{z_m}\left(P_{(z_j+1)}\right)$$

Substituted into (4), the second smallest Laplacian eigenvalue of $\mathcal{C}_m$ is obtained for $i_j = z_j + 1$, $j \in \{1, 2, \ldots, m-1\}$ and $i_m = z_m$. Since $\mu_N = 0$ or, equivalently, $\mu_{z_1+1}\left(P_{(z_j+1)}\right) = \mu_{z_2+1}\left(P_{(z_j+1)}\right) = \ldots = \mu_{z_{m-1}+1}\left(P_{(z_j+1)}\right) = 0$, we find that

$$\mu_{(z_1+1)(z_2+1)\ldots z_m}\left(\mathcal{C}_m\right) = \mu_{z_m}\left(P_{(z_j+1)}\right) = 2\left(1 - \cos\left(\frac{\pi}{z_m+1}\right)\right)$$

From (4), the corresponding eigenvector is

$$x_{(z_1+1)(z_2+1)\ldots z_m}\left(\mathcal{C}_m\right) = x_{z_1+1}\left(P_{(z_1+1)}\right) \otimes x_{z_2+1}\left(P_{(z_2+1)}\right) \otimes \ldots \otimes x_{z_m}\left(P_{(z_m+1)}\right)$$

To shorten the notation, we define $s = (z_1+1)(z_2+1)\ldots(z_{m-1}+1)$ and

$$t = \left[\frac{1}{2} + \frac{z_m+1}{\pi}\arccos\left(\alpha\sqrt{\frac{s(z_m+1)}{2}}\right)\right]$$

All components of $x_{z_i+1}\left(P_{(z_i+1)}\right) = \frac{1}{\sqrt{z_i+1}}$ for $i \in \{1, 2, \ldots, m-1\}$ are equal, so their final result is Kronecker product of a vector with all equal components and $y = x_{z_m}\left(P_{(z_m+1)}\right)$. Hence, we have

$$x_{(z_1+1)(z_2+1)\ldots z_m}\left(\mathcal{C}_m\right) = K\left[\underbrace{y \quad y \quad \ldots \quad y}_{s \text{ times}}\right]^T \tag{5}$$

After proper normalization using (1), we obtain $K = \sqrt{\frac{2}{s(z_m+1)}}$ (see Appendix A.1). According to (5), $x_{z_m}\left(P_{(z_m+1)}\right)$ occurs $(z_1+1)\ldots(z_{m-1}+1)$ times in $x_{(z_1+1)(z_2+1)\ldots z_m}\left(\mathcal{C}_m\right)$. This last result illustrates that every component of $x_{(z_1+1)(z_2+1)\ldots z_m}\left(\mathcal{C}_m\right)$ repeats periodically after (z_m+1) next components, such that the Fiedler partitioning condition reads

$$\sqrt{\frac{2}{s(z_m+1)}}\cos\frac{(2j-1)\pi}{2(z_m+1)} \geq \alpha \tag{6}$$

only for $j = 1, 2, ..., (z_m + 1)$. Thus, clustering of the m-dimensional lattice $\mathcal{C}_m$ into two non-empty subsets exists if and only if $|\alpha| \leq \sqrt{\frac{2}{s(z_m+1)}}$ in which case $j \leq t$. Because every component periodically repeats after $(z_m + 1)$ components, the final condition for the node labeled by j to belong to the first cluster is $j \bmod (z_m + 1) \in \{1, 2, \ldots, t\}$. Hence, those nodes are

$$\begin{aligned}
&j \in \{1, 2, \ldots, t, \\
&z_m + 2, \ldots, z_m + 1 + t, \\
&\vdots \\
&(s-1)(z_m + 1) + 1, \ldots, (s-1)(z_m + 1) + t\}
\end{aligned}$$

It could be written in a shorter form

$$j \in \{w + v | \forall w = 0, \ldots (s-1)(z_m + 1) \text{ and } \forall v = 1, \ldots t\} \tag{7}$$

This means that the number of nodes in the first cluster equals $s \cdot t$ and that in the second clusters equals $s \cdot (z_m + 1 - t)$. The $(m-1)$-dimensional hyperplane divides the m-dimensional lattice $\mathcal{C}_m$ into two clusters. Let us consider the links that will be cut by this partitioning. Those links are

$$\begin{aligned}
&t \leftrightarrow (t+1), \\
&(z_m + 1) + t \leftrightarrow (z_m + 1) + t + 1, \\
&\vdots \\
&(s-1)(z_m + 1) + t \leftrightarrow (s-1)(z_m + 1) + t + 1
\end{aligned}$$

Shortly those links are

$$w + t \leftrightarrow w + (t+1), \text{ for } w = 0, z_m + 1, \ldots, (s-1)(z_m + 1)$$

Hence, the number of cut links is

$$c(\mathcal{C}_m) = s = \prod_{i=1}^{m-1} (z_i + 1) \tag{8}$$

Finally, the total number of links in the general lattice $\mathcal{C}_m$ is specified in

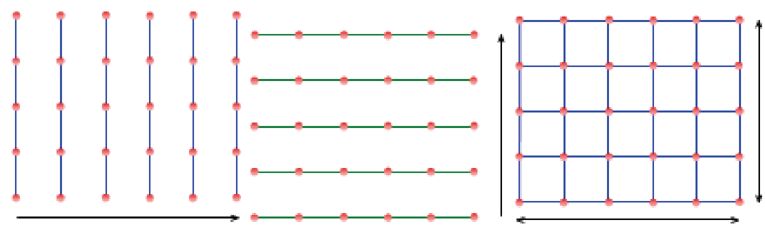
Lemma 1 *The number of links in the* $\mathcal{C}_m = La_{(z_1+1)\times(z_2+1)\times\ldots\times(z_m+1)}$ *is*

$$L = \left[\prod_{i=1}^{m} (z_i + 1)\right] \sum_{i=1}^{m} \frac{z_i}{z_i + 1}$$

Proof: We will prove the lemma by induction. Let the number of links in the k-dimensional lattice $\mathrm{La}_{(z_1+1)\times(z_2+1)\times\ldots\times(z_k+1)}$ be $l(z_1, z_2, \ldots, z_k)$.

1) For $k = 1$, we have a path graph P_{z_1+1} and its number of links is $L = l(z_1) = z_1 = (z_1+1)\frac{z_1}{z_1+1}$.

2) Let us assume that the lemma holds for k-dimensional lattices. We consider the $(k+1)$-dimensional lattice $\mathrm{La}_{(z_1+1)\times(z_2+1)\times\ldots\times(z_{k+1}+1)}$, that is constructed from k different k-dimensional lattices ($\mathrm{La}_{(z_{i_1}+1)\times(z_{i_2}+1)\times\ldots\times(z_{i_k}+1)}$, where $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, (k+1)\}$) in the following way. We position a total of $\left(z_{i_{k+1}}+1\right)$ such k-dimensional lattices $\mathrm{La}_{(z_{i_1}+1)\times(z_{i_2}+1)\times\ldots\times(z_{i_k}+1)}$ in next to each other in the direction of i_{k+1} dimension. In this way, every link is counted k-times in all of the dimensions. Intuitively, this construction is easier to imagine in two or three dimensions. The 2-dimensional

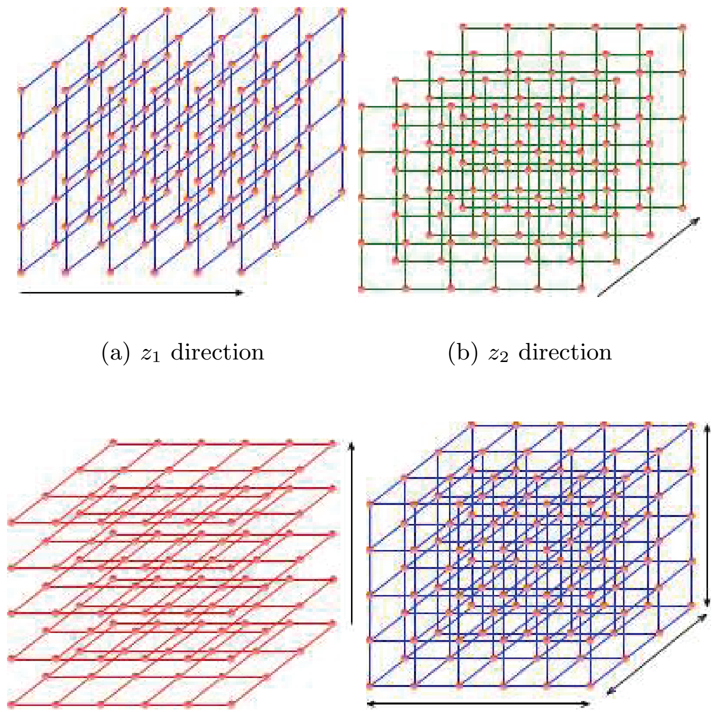


(a) z_1 direction (b) z_2 direction (c) $\mathrm{La}_{(z_1+1)\times(z_2+1)}$

Figure 1: Construction of 2-dimensional lattice

lattice $\mathrm{La}_{(z_1+1)\times(z_2+1)}$(Figure 1(c)) is constructed by positioning (z_1+1) consecutive path graphs P_{z_2+1} vertically(on the Figure 1(a)) and (z_2+1) consecutive path graphs P_{z_1+1} horizontally(on the Figure 1(b)). The 3-dimensional lattice $\mathrm{La}_{(z_1+1)\times(z_2+1)\times(z_3+1)}$(Figure 2) is constructed by (z_3+1) consecutive 2-dimensional $\mathrm{La}_{(z_1+1)\times(z_2+1)}$ planes that are positioned next to each other in the direction of the third dimension(on the Figure 2(a)), (z_2+1) consecutive 2-dimensional $\mathrm{La}_{(z_1+1)\times(z_3+1)}$ planes that are positioned next to each other in the direction of the second dimension(on the Figure 2(b))and, finally, (z_1+1) consecutive 2-dimensional La $_{(z_2+1)\times(z_3+1)}$ planes that are positioned next to each other in the direction of the first dimension(on the Figure 2(c)). In the process of constructing of $\mathrm{La}_{(z_1+1)\times(z_2+1)\times(z_3+1)}$

(on Figure 2(d)) all links in are counted twice. Returning to the k-dimensional



(a) z_1 direction (b) z_2 direction

(c) z_3 direction (d) $\mathrm{La}_{(z_1+1)\times(z_2+1)\times(z_3+1)}$

Figure 2: Construction of 3-dimensional lattice

case, we deduce that

$$l(z_1, z_2, \ldots, z_{k+1}) = \frac{1}{k}\sum_{i=1}^{k+1}(z_i+1)l(z_{j_1}, z_{j_2}, \ldots, z_{j_k})$$

where $j_w \neq i$ for each $i = 1, 2, \ldots, (k+1)$ and $w = 1, 2, \ldots, k$. Introducing the induction hypothesis for k-dimension lattices, we obtain

$$
\begin{aligned}
l(z_1, z_2, \ldots, z_{k+1}) &= \frac{1}{k}\sum_{i=1}^{k+1}(z_i+1)\prod_{j=1, j\neq i}^{k+1}(z_j+1)\sum_{j=1, j\neq i}^{k+1}\frac{z_j}{z_j+1} \\
&= \frac{1}{k}\prod_{j=1}^{k+1}(z_j+1)\sum_{i=1}^{k+1}\sum_{j=1, j\neq i}^{k+1}\frac{z_j}{z_j+1} \\
&= \frac{1}{k}\prod_{j=1}^{k+1}(z_j+1)k\sum_{i=1} k+1\frac{z_j}{z_j+1}
\end{aligned}
$$

which illustrates that the induction hypothesis is true for $(k+1)$, and consequently it is true for each dimension $m \geq 1$. $\square$

Using (4), the ordering $z_1 \leq z_2 \leq \ldots \leq z_m$ and Lemma 1, the "ratio of cut links" is

$$
r(\mathcal{C}_m) = \frac{1}{(z_m+1)\sum_{i=1}^{m}\frac{z_i}{z_i+1}}
$$

For the most common case of $\alpha = 0$ in (7), both clusters have almost the same number of nodes. For a 3-dimensional lattice $\mathrm{La}_{(z_1+1)\times(z_2+1)\times(z_3+1)}$, a plane divides $\mathrm{La}_{(z_1+1)\times(z_2+1)\times(z_3+1)}$ into two clusters with the same number of links. Figure 3 is an example for $m = 2$, in which $z_1 = 6$ and $z_2 = 7$

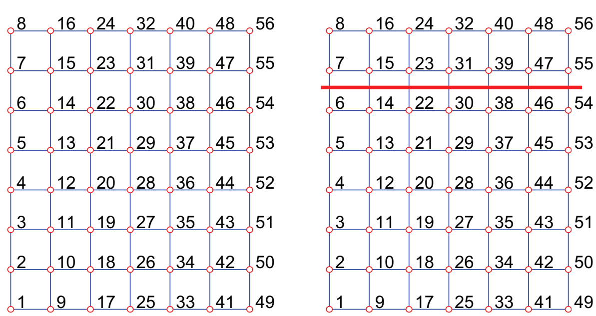
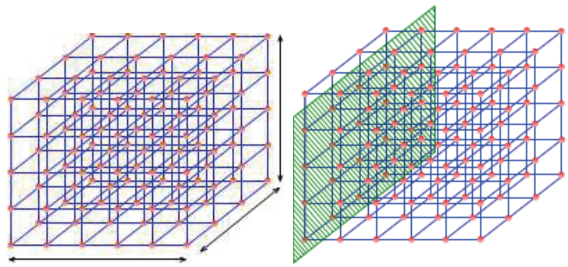


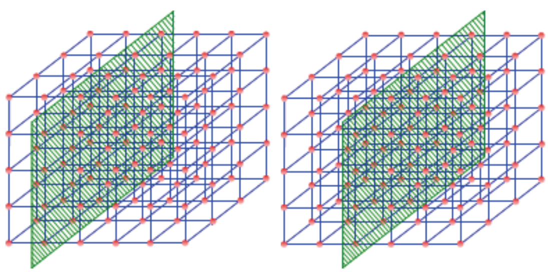
Figure 3: Partitioning of two-dimensional lattice $\mathrm{La}_{7\times 8}$ for $\alpha = \frac{1}{20}$.

and Fiedler's partitioning for $\alpha = 0.05$. In this case $c\left(\mathrm{La}_{7\times 8}\right) = 7$ and

$L = z_1(z_2+1) + z_2(z_1+1) = 97$. Hence $r(\mathrm{La}_{7\times 8}) = \frac{c(\mathrm{La}_{7\times 8})}{L} = \frac{7}{97} \approx 7.22\%$ of all links will be cut by Fiedler's partitioning. On the Figure 4 are given partitions of $\mathrm{La}_{6\times 4\times 5}$(Figure 4(a)) for different values of $\alpha = 0.1$(Figure 4(b)), 0.05(Figure 4(c)) and 0(Figure 4(d)).



(a) $\mathrm{La}_{6\times 4\times 5}$ (b) $\alpha = 0.1$



(c) $\alpha = 0.05$ (d) $\alpha = 0$

Figure 4: Partitioning of three-dimensional lattice $\mathrm{La}_{6\times 4\times 5}$

3 Conclusion

We have applied Fiedler's partitioning algorithm to an m-dimensional lattice $\mathrm{La}_{(z_1+1)\times(z_2+1)\times\ldots\times(z_m+1)}$ and have calculated the size of the two clusters, the number of links that are cut by this partitioning and the percentage of cut links as a function of the Fiedler threshold α and the characteristic dimensions of the lattice. In the most common case of $\alpha = 0$, both clusters have equal sizes. The number of cut links does not depend on α.

References

[1] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal.*, Vol.23(98):298-305, 1973.

[2] P. Van Mieghem. *Performance Analysis of Communications Networks and Systems.* Addison-Wesley, Cambridge University Press, 2006.

[3] P. Van Mieghem. *Graph Spectra of Complex Networks.* Addison-Wesley, Cambridge University Press, 2010 to appear.

[4] D. Cvetković, P. Rowlinson, S. Simić. *An introduction to the Theory of Graph Spectra.* Addison-Wesley, Cambridge University Press, 2010.

[5] A. Kaveh, H. Rahami. A Unified Method of Eigendecomposition of Graph Product. *Asian journal of Civil Engineering.*, Vol.7, No.6, 2006.

[6] B. Mohar, The Laplacian Spectra of Graphs. *Graph Theory, Combinatorics, and Applications*, Wiley, pp. 871, 1991.

[7] F. Aurenhammer, J. Hagauer, W. Imrich. Cartesian graph factorization at logarithmic cost per edge. *Comput. Complexity*, 2:331–349, 1992.

Appendix: The normalization coefficient of $\mathcal{C}_m$

According to (1), we normalize the eigenvector of $\mathcal{C}_m$ as

$$\sum_{j=1}^{(z_1+1)\times(z_2+1)\times\ldots\times(z_m+1)} \left(x_{(z_1+1)\times(z_2+1)\times\ldots\times z_m}\left(\mathcal{C}_m\right)\right)_j^2 = 1$$

which is equivalent to determining K such that

$$\underbrace{\sum_{j=0}^{z_m} \left(x_{(z_1+1)(z_2+1)\ldots z_m}\left(\mathcal{C}_m\right)\right)_j^2}_{(z_1+1)(z_2+1)\ldots(z_{m-1}+1)\text{ times}} = 1$$

$$\underbrace{\sum_{j=0}^{z_m} K^2 \cos^2 \frac{(2j+1)\pi}{2\left(Z_m+1\right)}}_{(z_1+1)(z_2+1)\ldots(z_{m-1}+1)\text{ times}} = 1$$

and, explicitly,

$$K^2 (z_1+1)(z_2+1)\ldots(z_{m-1}+1)\sum_{j=0}^{z_m} \cos^2 \frac{(2j+1)\pi}{2(z_m+1)} = 1$$

Now, since

$$\begin{aligned}
\sum_{j=0}^{z_m} \cos^2 \frac{(2j+1)\pi}{2(z_m+1)} &= \sum_{j=0}^{z_m} \frac{1+\cos\left(\frac{(2j+1)\pi}{z_m+1}\right)}{2} \\
&= \frac{z_m+1}{2} + \mathrm{Re}\left\{\sum_{j=0}^{z_m}\left(e^{\frac{(2j+1)\pi}{z_m+1}}\right)^j\right\} \\
&= \frac{z_m+1}{2} + \mathrm{Re}\left\{e^{\frac{\pi}{Z_m+1}i}\sum_{j=0}^{z_m}\left(e^{\frac{2\pi}{Z_m+1}i}\right)^j\right\} \\
&= \frac{z_m+1}{2} + \mathrm{Re}\left\{e^{\frac{\pi}{Z_m+1}i}\frac{e^{\frac{2\pi\cdot i}{Z_m+1}(Z_m+1)}-1}{e^{\frac{2\pi\cdot i}{Z_m+1}}-1}\right\} \\
&= \frac{z_m+1}{2} + \mathrm{Re}\left\{e^{\frac{\pi}{Z_m+1}i}\frac{1-1}{e^{\frac{2\pi\cdot i}{Z_m+1}}-1}\right\} = \frac{z_m+1}{2}
\end{aligned}$$

We find that

$$K = \sqrt{\frac{2}{(z_1+1)(z_2+1)\ldots(z_{m-1}+1)(z_m+1)}}$$

Large digraphs of given diameter and degree from coverings

Mária Ždímalová and Ľubica Staneková
Slovak University of Technology
Bratislava

Abstract

We show that a construction of Comellas and Fiol for large vertex-transitive digraphs of given degree and diameter from small digraphs preserves the properties of being a Cayley digraph and being a regular covering.

1 Introduction

There age numerous ways to construct 'large' digraphs of a given in- and out-degree and a given diameter; we refer to [5] for a fairly recent survey. The two most prominent contribution for digraphs that, in addition, are *vertex-transitive*, are the constructions by Comellas and Fiol [1], Gómez [3], and Faber, Moore and Chen [2].

Our interest in these constructions has been motivated by the remark in [5] to the extent that *graph coverings* appear to be an appropriate language for presenting a number of constructions in the degree-diameter problem. Two constructions of Comellas and Fiol have already been studied from this point of view in [6]. For completeness we note that digraphs and graphs of Faber, Moore and Chen have been studied in depth in [4].

The underlying idea of all constructions of [1] is to take a 'small' digraph as input and by a kind of 'composition', as the operation is called in [1], produce a 'large' output digraph that depends on various parameters. Here we examine a construction of the paper [1] that has not been considered in [6], with the aim to show that the construction preserves coverings and the property of being Cayley in the sense that if the input digraph has the property, then so does the output digraph.

2 Constructions

Throughout, we assume that the reader is familiar with the concepts of a Cayley digraph as well as of regular coverings of digraphs and their description in the language of voltage assignments; we recommend [5] for details especially in connection with the directed version of the degree-diameter problem.

We begin with a description of the construction of Comellas and Fiol [1] which will be considered from now on. Let G be a digraph with vertex set V and dart set D, the *input digraph* for the construction. Let m and $n \geq 2$ be positive integers such that m is divisible by n. Further, let ℓ be a fixed element of the cyclic group Z_m such that $\ell \neq 0, 1$. The *output digraph* G' will have vertex set V' and dart set D' defined as follows. The vertex set V' consists of all ordered $(n+1)$-tuples $(p_0p_1 \dots p_{n-1}|j)$ such that $j \in Z_m$ and $p_i \in V$ for all i, $0 \leq i \leq n-1$. The dart set D' consists precisely of the darts of the form

$$(p_0p_1 \dots p_{j-1}\, u\, p_{j+1} \dots p_{n-1}|j) \rightarrow \left\{ \begin{array}{l} (p_0p_1 \dots p_{j-1}\ v\ p_{j+1} \dots p_{n-1}|(j+1)) \\ (p_0p_1 \dots p_{j-1}\ u\ p_{j+1} \dots p_{n-1}|(j+\ell)), \end{array} \right.$$

whenever v is adjacent from u in G.

The role of this construction in the directed version of the degree-diameter problem is clear from the results of [1] which imply that if the input digraph G of order, say, c, is vertex-transitive, regular of degree d, and k-reachable, meaning that any ordered pair of vertices is connected by a directed walk of length precisely k, then the output digraph G' is vertex transitive, regular of degree $d+1$, of order mc^n and diameter at most $kn+b$ where b is the diameter of the Cayley digraph $Cay(Z_m, \{1, \ell\})$.

For our purposes it will be more convenient to work with an isomorphic image of G', defined by means of the bijection $\varphi : V' \rightarrow V'$ given by

$$(p_0p_1 \dots p_{j-1}\ u\ p_{j+1} \dots p_{n-1}|j) \mapsto (u\ p_{j+1} \dots p_{n-1}p_0p_1 \dots p_{j-1}|j)\ .$$

The isomorphic copy $G^* = \varphi(G')$ of G' thus has the same vertex set $V^* = V'$ but all the darts in its dart set D^* have the form

$$(p_0p_1 \dots p_{n-1}|j) \rightarrow \left\{ \begin{array}{l} (p_1 \dots p_{n-1}q_0|(j+1)) \\ (p_\ell p_{\ell+1}p_{\ell+2} \dots p_{\ell-1}|(j+\ell)) \end{array} \right.$$

where q_0 is adjacent from p_0 in G and, as before, ℓ is a fixed element of Z_m such that $\ell \neq 0, 1$.

In order to have a more explicit notation, we denote the digraph G^* by writing $G^* = CF(G, n, m, \ell)$, where CF stands for Comellas-Fiol and G, n, m and ℓ are the parameters upon which the construction of G^* depends.

We note that a similar amendment of the description of this construction of Comellas and Fiol was suggested by Gómez [3] but the isomorphism proposed in [3] appears to be inconsistent with the actual graph description given therein.

3 Preservation of Cayley digraphs

We are now in position to show that the construction described in the previous section preserves the property of being Cayley in the sense outlined in the Introduction. We keep to all the notation introduced earlier.

Theorem 1 *If G is a Cayley digraph, then $G^* = CF(G, n, m, \ell)$ is a Cayley digraphs as well.*

Proof: Let $G = Cay(H, X)$ be a Cayley digraph for a group H and a generating set X. Let H^* be the semidirect product of $H^n = H \times H \times \ldots \times H$ (n times) by Z_m, with elements of the form $(a; j)$ where $a = (a_0, \ldots, a_{n-1}) \in H^n$ and $j \in Z_m$ and with the action j of Z_m on H^n defined by

$$j(a) = (a_j, a_{j+1}, \ldots, a_{n-1}, a_0, \ldots, a_{j-1})$$

for any $a = (a_0, \ldots, a_{n-1}) \in H^n$; at this point we recall that m is a multiple of n and hence the action is well defined. Multiplication in the semidirect product $H^* = H^n \rtimes Z_m$ is given by

$$(a; j)(a'; j') = (j'(a).a'; j + j') \quad . \tag{1}$$

Now, for any $x \in X$ we let $x^* = (e, e, \ldots, e, x) \in H^n$ where e is the unit element of A; also, let $e^* = (e, \ldots, e) \in H^n$. Finally, define $X^* = \{(x^*; 1), x \in X\} \cup \{(e^*, \ell)\}$ where $\ell \in Z_m$ is the fixed element different from 0 and 1. Note that the action of $j = 1 \in Z_m$ on H^n is given by $j(a) = 1(j_0, \ldots, j_{n-1}) = (j_1, \ldots, j_{n-1}, j_0)$.

We show that the Cayley digraph $Cay(H^*, X^*)$ is isomorphic to the digraph G^* for $G = Cay(H, X)$. The key is to observe that, for any $x \in X$,

$$\begin{aligned} 1(a).x^* &= 1(a_0, \ldots, a_{n-1})(e, \ldots, e, x) \\ &= (a_1, \ldots, a_{n-1}, a_0 x) \end{aligned}$$

On the other hand, right multiplication by $(x^*, 1) \in X^*$ gives:

$$\begin{aligned}(a; j)(x^*; 1) &= (1(a).x^*; j+1)\\ &= ((a_1, \ldots, a_{n-1}, a_0).(e, \ldots, e, x); j+1)\\ &= (a_1, \ldots, a_{n-1}, a_0x; j+1).\end{aligned}$$

It follows that for any $(x^*, 1) \in X^*$ the vertex $(a; j)$ is adjacent to the vertex $(a; j)(x^*; 1) = (1(a).x^*; j+1)$ in the Cayley digraph $Cay(H^*, X^*)$. This, however, is precisely the first adjacency rule in the definition of the output digraph G^*. Similarly, $(a; j)$ is in the Cayley digraph adjacent to the vertex $(a; j)(e^*; \ell) = (\ell(a).e^*; j+\ell)$, which gives the second adjacency rule for G^*. Consequently, G^* is isomorphic to the Cayley digraph $Cay(H^*, X^*)$ if the input digraph G is a Cayley digraph $Cay(H, X)$, as claimed. □

4 Preservation of coverings

We continue with the result regarding preservation of regular coverings.

Theorem 2 *If G regularly covers a digraph of a smaller order, then so does $G^* = CF(G, n, m, \ell)$.*

Proof: Let G be a regular covering space of a digraph of order smaller than the order of G. Equivalently, we assume that G is a lift of a base digraph $J = (V_J, D_J)$ by a voltage assignment α in some non-trivial group A.

We first introduce a new base digraph $L = (V_L, D_L)$. Its vertex set V_L will be the set $\{(r_0, \ldots, r_{n-1}); r_i \in V_J)\}$, where $i \in \{0, \ldots, n-1\}$. If there exists an arc b from a vertex $r' \in V_J$ to a vertex $s' \in V_J$ in the digraph J, then incidence in the digraph L is defined by the following rule. For every arc $b \in D_J$ from r' to s' in J and for every $(n-1)$-tuple $(r_1, \ldots, r_{n-1})$ of vertices in J there will be a dart labeled $\tilde{b} \in D_L$ from the vertex $r = (r', r_1, \ldots, r_{n-1}) \in V_L$ to the vertex $s = (r_1, \ldots, r_{n-1}, s') \in V_L$. Furthermore, for each vertex $r = (r_0, \ldots, r_{n-1}) \in V_L$ we include a dart $\tilde{c}_r$ from r to the vertex $r^* = (r_\ell, r_{\ell+1}, \ldots, r_{\ell-1})$.

Let us now introduce a voltage assignment β on L in the group $A^* = A^n \rtimes Z_m$, with the cyclic group Z_m acting on the n-fold direct product $A^n = A \times A \times \ldots \times A$ in the same way as it acted on H in the previous section.

Consider a dart $\tilde{b}: r \longrightarrow s$ of L that has originally come from a dart $b: r' \longrightarrow s'$ in J carrying voltage $\alpha(b)$. The new voltage of $\tilde{b}$ will be $\beta(\tilde{b}) = (e, \ldots, e, \alpha(b); 1)$; moreover, the darts $\tilde{c}_r$ will receive the voltage $\beta(\tilde{c}_r) = (e, \ldots, e|\ell)$.

Let us denote the lift of L with respect to this voltage assignment by L^β. By definition of a lift, vertices and darts of L^β have the form (r, a) and $(\tilde{b}, a)$, $(\tilde{c}_r, a)$ where $r \in V_L$, $\tilde{b} \in D_L$ and $a \in A^*$. A dart $(\tilde{b}, a)$ emanates from (r, a) and terminates at $(s, a.\beta(\tilde{b}))$ while a dart $(\tilde{c}_r, a)$ emanates from (r, a) and terminates at $(r^*, a.\beta(\tilde{c}))$. Note that the last two products evaluate as

$$\begin{aligned} a.\beta(\tilde{b}) &= (a_0, a_1, \ldots, a_{n-1}|j).(e, \ldots, e, \alpha(b)|1) \\ &= ((a_1, \ldots, a_{n-1}, a_0), (e, \ldots, e, \alpha(b))|j+1)) \\ &= (a_1, \ldots, a_{n-1}, a_0.\alpha(b)|j+1). \end{aligned}$$

and

$$\begin{aligned} a.\beta(\tilde{c}) &= (a_0, a_1, \ldots, a_{n-1})\beta(\tilde{c}) \\ &= (a_0, a_1, \ldots, a_{n-1})(e, \ldots, e|\ell) \\ &= (a_\ell, a_{\ell+1}, \ldots, a_{\ell-1}). \end{aligned}$$

We prove that the digraphs G^* and L^β are isomorphic. Having assumed that G is a lift of J by the voltage assignment α in the group A, it follows that the first type of incidence in the definition of the Comellas-Fiol digraph G^* can be described in the form

$$((r_0, a_0) \ldots (r_{n-1}, a_{n-1})|j) \longrightarrow ((r_1, a_1) \ldots (r_{n-1}, a_{n-1})(s_0, a_0.\alpha(b))|j+1)$$

for any $a_i \in A$, $0 \le i \le n-1$, whenever b is a dart from r_0 to s_0 in J. It is now a matter of routine to check that the mapping

$$((r_0, a_0)(r_1, a_1) \ldots (r_{n-1}, a_{n-1})|j) \mapsto ((r_0, \ldots, r_{n-1})(a_0, \ldots, a_{n-1})|j)$$

is an isomorphism from G^* onto L^β. $\square$

In conclusion, let us note that although it is true that every Cayley digraph is a regular covering of a one-vertex digraph, our Theorem 2 does *not* imply Theorem 1 because, in the proof of Theorem 2, the digraph L^* need not be a one-vertex digraph even when J is.

Acknowledgement

The authors acknowledge support by the APVV Research Grants No. 0040-06 and the LPP Research Grant No. 0145-06, as well as partial support from the VEGA Research Grant No. 1/0489/08.

The support by the "Program na podporu mladých vedeckých výskumníkov", FCE, Slovak University of Technology in Bratislava, 7601 Podpora mladých výskumníkov-Veľké vrcholovo-tranzitívne a cayleyovské grafy daného stupňa a priemeru-VVTCG, 781000 is acknowledged by the first author.

References

[1] F. Commellas and M.A. Fiol. Vertex-symmetric digraphs with small diameter. *Discrete Applied Mathematics*, 58:1–11, 1995.

[2] V. Faber, J.W. Moore and W.Y.C. Chen. Cycle prefix digraphs for symmetric interconnection networks. *Networks*, 23:641–649, 1993.

[3] J. Gómez. Large vertex-symmetric digraphs. *Networks*, 50:421–250, 2007.

[4] H. Macbeth, J. Šiagiová, J. Širáň and T. Vetrík. Large Cayley graphs and vertex-transitive non-Cayley graphs of given degree and diameter. *J. Graph Theory*, Volume 64, Issue 2, 87–98, 2010.

[5] M. Miller and J. Širáň. Moore graphs and beyond: A survey. *Electron. J. Combin. (Dynamic Survey DS)*, 14:61 pp., 2005.

[6] M. Ždímalová. Revisiting the Comellas-Fiol-Gómez Construction of Large Digraphs of Given Degree and Diameter. *Discrete Mathematics*, Volume 310, Issue 9, 1439–1444, 2010.

Participants

Abiad, Aida
Univ. Politècnica de Catalunya, Spain aida.abiad@ma4.upc.edu
Aguiló, Francesc
Univ. Politècnica de Catalunya, Spain matfag@ma4.upc.edu
Archdeacon, Dan
Univ. of Vermont, USA dan.archdeacon@uvm.edu
Balbuena, Camino
Univ. Politècnica de Catalunya, Spain m.camino.balbuena@upc.edu
Ball, Simeon
Univ. Politècnica de Catalunya, Spain simeon@ma4.upc.edu
Barguñó, Lluis
Univ. Politècnica de Catalunya, Spain luisbargu@gmail.com
Barrière, Lali
Univ. Politècnica de Catalunya, Spain lali@ma4.upc.edu
Baskoro, Edy Tri
Institut Teknologi Bandung, Indonesia ebaskoro@gmail.com
Boulet, Romain
Univ. de Toulouse, France boulet@lmtg.obs-mip.fr
Bras, Maria
Univ. Rovira i Virgili, Spain maria.bras@urv. cat
Brunat, Josep M.
Univ. Politècnica de Catalunya, Spain josep.m.brunat@upc.edu
Buset, Dominique
Univ. Libre Bruxelles, Belgium dbuset@ulb.ac.be
Cámara, Marc
Univ. Politècnica de Catalunya, Spain mcamara@ma4.upc.edu
Comellas, Francesc
Univ. Politècnica de Catalunya, Spain comellas@ma4.upc.edu
Dalfó, Cristina
Univ. Politècnica de Catalunya, Spain cdalfo@ma4.upc.edu
Delorme, Charles
Univ. Paris-Sud, France Charles.Delorme@lri.fr
Domínguez, David
Univ. Politècnica de Catalunya, Spain ddomings@ac.upc.edu

Fàbrega, Josep
Univ. Politècnica de Catalunya, Spain jfabrega@ma4.upc.edu
Fiol, Maria Lluisa
Univ. Autònoma de Barcelona, Spain marialluisa.fiol@uab.cat
Fiol, Miquel Àngel
Univ. Politècnica de Catalunya, Spain fiol@ma4.upc.edu
García-Vázquez, Pedro
Univ. de Sevilla, Spain pgvazquez@us.es
Garriga, Ernest
Univ. Politècnica de Catalunya, Spain egarriga@ma4.upc.edu
Gimbert, Joan
Univ. de Lleida, Spain joangim@matematica.udl.cat
Haemers, Willem
Tilburg Univ., Netherlands Haemers@uvt.nl
Hamidoune, Yahya
Univ. Pierre et Marie Curie, France hamidoune@math.jussieu.fr
Hansberg, Adriana
Univ. Politècnica de Catalunya, Spain adrana.hansberg@upc.edu
Jørgensen, Leif
Aalborg Univ., Denmark leif@math.aau.dk
Knor, Martin
Slovak Univ. of Technology, Slovakia knor@math.sk
Kouider, Mekkia
LRI-Orsay, France km@lri.fr
Lladó, Anna
Univ. Politècnica de Catalunya, Spain allado@ma4.upc.edu
López Lorenzo, Nacho
Univ. de Lleida, Spain nlopez@matematica.udl.es
López Masip, Susana
Univ. Politècnica de Catalunya, Spain susana@ma4.upc.edu
Marcote, Xavier
Univ. Politècnica de Catalunya, Spain francisco.javier.marcote@upc.edu
Martínez, Carmen
Univ. de Cantabria, Spain carmen.martinez@unican.es
Maureso, Montserrat
Univ. Politècnica de Catalunya, Spain montserrat.maureso@upc.edu
Mier, Anna de
Univ. Politècnica de Catalunya, Spain anna.de.mier@upc.edu

Miller, Mirka
Univ. of Newcastle, Australia Mirka.Miller@newcastle.edu.au
Miralles, Alícia
Univ. Politècnica de Catalunya, Spain almirall@ma4.upc.edu
Mitjana, Margarida
Univ. Politècnica de Catalunya, Spain margarida.mitjana@upc.edu
Mora Giné, Mercè
Univ. Politècnica de Catalunya, Spain merce.mora@upc.edu
Montejano, Luís Pedro
Univ. Politècnica de Catalunya, Spain luis.pedro.montejano@upc.edu
Moreno, Rocío
Univ. de Sevilla, Spain rociomc@us.es
Muntés, Victor
Univ. Politècnica de Catalunya, Spain vmuntes@ac.upc.edu
Muñoz, Xavier
Univ. Politècnica de Catalunya, Spain xml@ma4.upc.edu
Noy, Marc
Univ. Politècnica de Catalunya, Spain marc.noy@upc.edu
Ozón, Javier
Univ. Politècnica de Catalunya, Spain ozzy@entel.upc.edu
Perarnau, Guillem
Univ. Politècnica de Catalunya, Spain p.melliug@gmail.com
Pérez Mansilla, Sonia
Univ. Politècnica de Catalunya, Spain sonia@ma4.upc.edu
Raspaud, André
LABRI, Universit Bordeaux I, France raspaud@labri.fr
Rubio, Christian
Univ. Nacional Autónoma de México, México ok.rubio@gmail.com
Serra, Oriol
Univ. Politècnica de Catalunya, Spain oserra@ma4.upc.edu
Serratosa, Francesc
Univ. Rovira i Virgili, Spain francesc.serratosa@urv. cat
Šiagiová, Jana
Slovak Univ. of Technology, Slovakia siagiova@svf.stuba.sk
Širáň, Jozef
Slovak Univ. of Technology, Slovakia siagiova@svf.stuba.sk
Solé, Albert
Univ. Rovira i Virgili, Spain francesc.serratosa@urv.cat

Staneková, Lubica
Slovak Univ. of Technology, Slovakia ls@math.sk
Stokes, Clara
Univ. Rovira i Virgili, Spain klara.stokes@urv. cat
Trajanovski, Stojan
Delft Univ. of Technology, Netherlands S.Trajanovski@tudelft.nl
Valenzuela, Juan Carlos
Univ. de Cádiz, Spain jcarlos.valenzuela@uca.es
Vilaltella, Joan
Univ. Politècnica de Catalunya, Spain joanvilaltella@farinyes.net
Yebra, Jos Luís A.
Univ. Politècnica de Catalunya, Spain jose.luis.andres@upc.edu
Zaragozá, Marisa
Univ. Politècnica de Catalunya, Spain marisa@ma4.upc.edu
Ždímalová, Mária
Slovak Univ. of Technology, Slovakia zdimalova@svf.stuba.sk

Author Index