

# MSc in Applied Mathematics

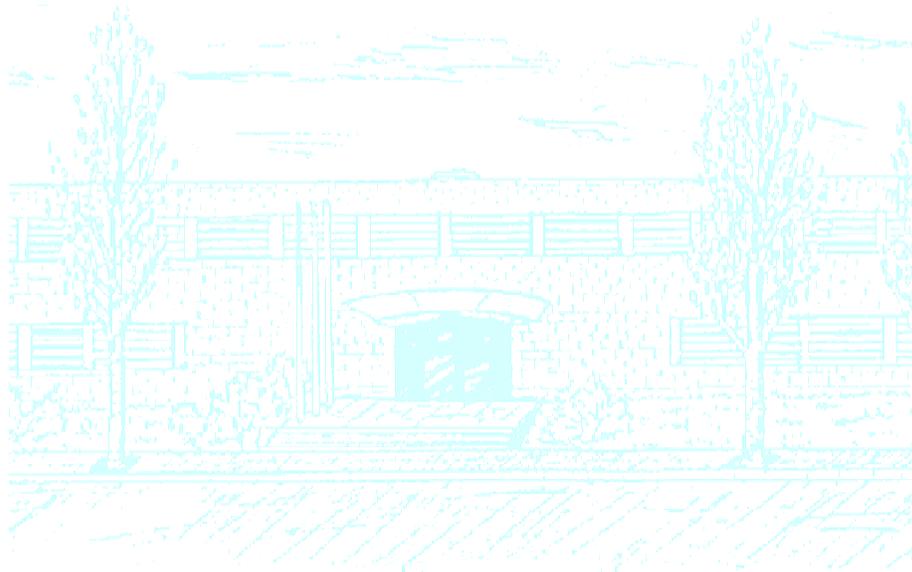
**Title:** Ordinary CM forms and local Galois representations

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# Ordinary CM forms and local Galois representations

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## Abstract

The  $p$ -adic Galois representation  $\rho_f$  attached to a  $p$ -ordinary newform  $f$  of weight  $k \geq 2$  is known to be reducible when restricted to a decomposition group  $D_p$  at  $p$ . Ralph Greenberg asked for a characterization of those  $f$  for which  $\rho_f$  actually splits when restricted to  $D_p$ . In 2004, Eknath Ghate suggested this happens exactly when  $f$  has CM, and in collaboration with Vinayak Vatsal, he has obtained strong, though not definitive, results towards his guess. Here, we describe these results and introduce a line of thought about Greenberg's question. This is in accordance with the  $R = \mathbb{T}$  philosophy, and would confirm new instances of the solution proposed by Ghate.

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# Introduction

## A question of Ralph Greenberg

Let  $f = \sum_{n=1}^{\infty} a_n(f)q^n \in S_k(\Gamma_0(N), \chi)$  be a normalized newform of weight  $k \geq 2$ , level  $N \geq 1$  and Nebentypus character  $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ . We take the algebraic closure  $\overline{\mathbb{Q}}$  of  $\mathbb{Q}$  in  $\mathbb{C}$ . Fix an odd prime  $p$ , a choice of algebraic closure  $\overline{\mathbb{Q}}_p$  of  $\mathbb{Q}_p$ , and an embedding  $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ , which we use to single out a decomposition group  $D_p$  at  $p$  inside the absolute Galois group  $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  of  $\mathbb{Q}$ . Let  $K_f \subset \overline{\mathbb{Q}}$  be the number field generated by the set of Hecke eigenvalues  $a_n(f)$  of  $f$ . Denote by  $\mathfrak{p}$  the prime above  $p$  in  $K_f$  induced by  $\iota_p$ , and let  $K_{\mathfrak{p}} \subset \widehat{\overline{\mathbb{Q}}_p}$  be the completion of  $K_f$  at  $\mathfrak{p}$ , where  $\widehat{\overline{\mathbb{Q}}_p}$  denotes the completion of  $\overline{\mathbb{Q}}_p$  at the prime  $\mathfrak{P} \subset \overline{\mathbb{Q}}_p$  above  $p$ . Denote by  $c$  the complex conjugation in  $\text{Aut}(\mathbb{C})$ .

Following the work of Shimura and Deligne, one can attach to  $f$  a two-dimensional continuous (irreducible) Galois representation

$$\rho = \rho_{f, \mathfrak{p}} : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(K_{\mathfrak{p}})$$

which is unramified away from  $pN$  and such that for every prime  $\ell$  not dividing  $pN$ ,

$$\det(\text{Id} - \rho(\text{Frob}_{\ell})X) = 1 - a_{\ell}(f)X + \chi(\ell)\ell^{k-1}X^2, \quad (1)$$

where  $\text{Frob}_{\ell}$  denotes an *arithmetic* Frobenius element at  $\ell$ , i.e. any element  $\sigma_{\mathfrak{L}} \in G_{\mathbb{Q}}$ , depending on the choice of a prime  $\mathfrak{L} \subset \overline{\mathbb{Q}}$  above  $\ell$ , determined modulo the inertia group  $I_{\mathfrak{L}} \subset G_{\mathbb{Q}}$  corresponding to this choice, by the requirement that  $\sigma_{\mathfrak{L}}x \equiv x^{\ell} \pmod{\mathfrak{L}}$  for all  $x$  in the ring of integers of the completion of  $\overline{\mathbb{Q}}$  at  $\mathfrak{L}$ .

We say that  $f$  is *ordinary* (at  $\mathfrak{p}$ ) if the Hecke eigenvalue  $a_p(f) \in K_f \subset K_{\mathfrak{p}}$  is a  $\mathfrak{p}$ -adic unit. Then it follows from the work of Wiles [Wil88] that the restriction of  $\rho$  to the decomposition group  $D_p \subset G_{\mathbb{Q}}$  at  $p$  is reducible, and that its module of coinvariants by the inertia group  $I_p \subset D_p$  is free of rank one. Thus after a suitable choice of basis of the representation space for  $\rho$ ,

$$\rho|_{D_p} = \begin{pmatrix} \eta' & a \\ & \eta \end{pmatrix}, \quad (2)$$

for certain continuous characters  $\eta', \eta : D_p \rightarrow K_{\mathfrak{p}}^\times$  and a continuous function  $a : D_p \rightarrow K_{\mathfrak{p}}$ ; and the character  $\eta$  is unramified, mapping  $\text{Frob}_p$  to the unique  $p$ -adic unit root of the equation

$$X^2 - a_p(f)X + \chi(p)p^{k-1} = 0.$$

Here we convey  $\chi(p) = 0$  if  $p|N$ , so that  $\eta(\text{Frob}_p) = a_p(f)$  in that case. The representation  $\rho$  is said to be (*locally*) *split* at  $p$  if  $\rho|_{D_p}$  is semi-simple (or equivalently, since it is reducible, if it is in fact decomposable).

Recall that the newform  $f$  is said to have *complex multiplication* (CM) by a Dirichlet character  $\varphi$  if

$$\varphi(\ell)a_\ell(f) = a_\ell(f) \quad (3)$$

for all but finitely many primes  $\ell$ . It is easy to see that there exists at most one non-trivial character  $\varphi$  satisfying (3), and that  $\varphi$  is then necessarily odd and quadratic. We call  $\varphi$  the CM character of  $f$ , and the quadratic imaginary field cut out by  $\varphi$  (thought of as a finite order Galois character), the CM field of  $f$ .

Ralph Greenberg asked for a characterization of those ordinary newforms  $f$  of weight  $k \geq 2$  for which the associated representation  $\rho_{f,\mathfrak{p}}$  is split at  $p$ . Eknath Ghate suggests in [Gha04] that a natural “guess” seems to be the following:

$$\rho_{f,\mathfrak{p}} \text{ splits at } p \stackrel{?}{\iff} f \text{ has complex multiplication.} \quad (4)$$

It is easy to see that  $\rho = \rho_{f,\mathfrak{p}}$  splits at  $p$  if  $f$  is ordinary and of CM type. Indeed, let  $\varphi$  be the CM character of  $f$ ,  $M$  the CM field of  $f$ , and set  $G_M = \text{Gal}(\overline{\mathbb{Q}}/M)$ . Then

$$\rho = \text{Ind}_M^{\mathbb{Q}} \psi \quad (5)$$

for a certain  $p$ -adic Hecke character  $\psi : G_M \rightarrow K_{\mathfrak{p}}^\times$  of  $M$ , and it follows that  $\rho$  is semisimple when restricted to  $G_M$ ; concretely,

$$\rho|_{G_M} = \psi \oplus \psi^c,$$

for the complex conjugate Hecke character  $\psi^c$  of  $\psi$  given by  $g \mapsto \psi(cgc)$ . Considering the determinant of  $\rho$ , it follows from (5) that, as Galois characters  $G_M \rightarrow K_{\mathfrak{p}}^\times$ ,

$$\chi \varepsilon_{\text{cyc}}^{k-1}|_{G_M} = \psi \psi^c, \quad (6)$$

where  $\varepsilon_{\text{cyc}}$  denotes the  $p$ -adic cyclotomic character  $G_{\mathbb{Q}} \rightarrow \mathbb{Z}_p^\times \subset K_{\mathfrak{p}}^\times$ , giving the Galois action on the group of  $p$ -power roots of unity in  $\overline{\mathbb{Q}}$ . Denote by  $D_{\mathfrak{p}} \subset D_p$  the decomposition group in  $G_M$  attached to  $\mathfrak{p}$  by our fixed embedding  $\iota_p$ . Due to the unramifiedness of the character  $\eta$  appearing in (2) we see that the characters  $\psi, \psi^c$  cannot both be ramified at  $\mathfrak{p}$ . Since  $\varepsilon_{\text{cyc}}$  is totally ramified at  $p$ , by (6) we may assume that  $\psi$  actually ramifies at  $\mathfrak{p}$ , and that  $\psi^c$  does not; but then  $p$  can only possibly split in  $M$ , and therefore  $D_p = D_{\mathfrak{p}} \subset G_M$  and the claim follows.

The notion of the representation  $\rho$  being split at  $p$  can be rephrased in cohomological terms. With the previous notations, let  $\alpha : D_p \rightarrow K_{\mathfrak{p}}^\times$  be the continuous character on  $D_p$  given by  $\eta'/\eta$ , and let  $K_{\mathfrak{p}}(\alpha)$  be the one-dimensional  $D_p$ -representation space  $K_{\mathfrak{p}}$  with  $D_p$ -action via the character  $\alpha$ . Denote by  $V_f$  a representation space for  $\rho$ . After twisting by  $\eta^{-1}$  it follows from (2) that one has the following exact sequence of  $K_{\mathfrak{p}}[D_p]$ -modules:

$$0 \rightarrow K_{\mathfrak{p}}(\alpha) \rightarrow V_f(\eta^{-1}) \rightarrow K_{\mathfrak{p}} \rightarrow 0. \quad (7)$$

This sequence splits (i.e.  $V_f(\eta^{-1})$  is isomorphic to the trivial extension  $K_{\mathfrak{p}}(\alpha) \oplus K_{\mathfrak{p}}$  of  $K_{\mathfrak{p}}$  by  $K_{\mathfrak{p}}(\alpha)$ ) if and only if the sequence (7) remains exact after taking  $D_p$ -invariants, or equivalently, if the coboundary map

$$\delta : K_{\mathfrak{p}}^{D_p} = K_{\mathfrak{p}} \rightarrow H^1(D_p, K_{\mathfrak{p}}(\alpha))$$

induced by (7) is identically zero. If the extension (7) is non-trivial, then  $\delta$  is injective and the line  $\delta(K_{\mathfrak{p}}) \subset H^1(D_p, K_{\mathfrak{p}}(\alpha))$  depends only on the isomorphism class of  $V_f(\eta^{-1})$  as an extension of  $K_{\mathfrak{p}}$  by  $K_{\mathfrak{p}}(\alpha)$ . We refer to  $\delta(1)$  as the *extension class* of  $V_f$ , and one can explicitly produce a generator of the line  $K_{\mathfrak{p}} \cdot \delta(1)$  using the matrix representation (2). Indeed, the map  $\xi_f : D_p \rightarrow K_{\mathfrak{p}}$  defined by  $\xi_f(g) = a(g)\eta(g)^{-1}$  is a 1-cocycle on  $D_p$  with values in  $K_{\mathfrak{p}}(\alpha)$  (as one can check using the multiplicativity of the matrix representation); a different choice of matrix representation affects the resulting  $\xi_f$  by a coboundary, and its cohomology class  $[\xi_f] \in H^1(D_p, K_{\mathfrak{p}}(\alpha))$  is non-zero if and only if  $\rho_{f,p}$  is non-split at  $p$ . One can thus restate the guess in (4) as

$$[\xi_f] = 0 \in H^1(D_p, K_{\mathfrak{p}}(\alpha)) \stackrel{?}{\iff} f \text{ has complex multiplication,}$$

and we see from the above discussion that the difficult problem is in the forward direction. Note also that  $[\xi_f]$  is non-zero in  $H^1(D_p, K_{\mathfrak{p}}(\alpha))$  if and only if its restriction to the inertia subgroup  $I_p \subset D_p$  is non-zero. Indeed, in the exact sequence

$$0 \rightarrow H^1(D_p/I_p, K_{\mathfrak{p}}(\alpha)^{I_p}) \rightarrow H^1(D_p, K_{\mathfrak{p}}(\alpha)) \rightarrow H^1(I_p, K_{\mathfrak{p}}(\alpha))$$

coming from the inflation-restriction exact sequence, the first non-trivial term vanishes, since  $I_p$  acts non-trivially on the one-dimensional space  $K_{\mathfrak{p}}(\alpha)$ .

Until the work of Ghate and Vatsal, almost all known evidence in support of (4) came from the weight two case, and when the newform  $f$  has rational Fourier coefficients, thus being attached to an elliptic curve over  $\mathbb{Q}$ . Recall that if  $E/\mathbb{Q}$  is a rational elliptic curve with CM then  $E$  was already known to be modular (cf. [Shi71]), attached to a CM newform of weight 2. In fact, the more recent proof of the Shimura-Taniyama conjecture tells us that nothing more than the solution to Greenberg's question for that class of newforms (and for a prime  $p$  not dividing the level) is contained in the following result (cf. [Ser89] IV. A.2.4):

**Theorem 1** (Serre). *Let  $E/\mathbb{Q}$  be an elliptic curve of conductor  $N$ , with good ordinary reduction at the prime  $p$ . Denote by  $\rho_{E,p}$  be  $p$ -adic Galois representation of the local Galois group  $D_p = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  given by the Tate module  $\text{Ta}_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  of  $E$ . Then  $\rho_{E,p}$  is semi-simple if and only if  $E$  has CM.*

## A $\Lambda$ -adic analogue

Let  $\Gamma = 1 + p\mathbb{Z}_p$  be the group units in  $\mathbb{Z}_p^\times$  congruent to 1 modulo  $p$ , and denote by  $\Lambda$  the completed group ring  $\mathbb{Z}_p[[\Gamma]]$ , identified with the power series ring  $\mathbb{Z}_p[[X]]$  by setting

$1 + X = 1 + p$ . It follows from the work of Hida recalled in Section 1.3, that there exists a unique *ordinary  $\Lambda$ -adic form*

$$\mathcal{F}_{\mathbb{I}} = \sum_{n=1}^{\infty} a_{\mathbb{I}}(n; \mathcal{F}) q^n \in \mathbb{I}[[q]],$$

for  $\mathbb{I}$  a finite flat extension of  $\Lambda$ , that *specializes* to  $f$  at weight  $k$ ; under the simplifying assumption that  $\mathbb{I}$  equals  $\Lambda$  itself, this means that the formal  $q$ -expansion obtained by applying the algebra homomorphism

$$\begin{aligned} \varphi_k : \mathbb{I} &\rightarrow \mathbb{Z}_p \\ X &\mapsto (1 + p)^k - 1 \end{aligned}$$

to the coefficients of  $\mathcal{F}_{\mathbb{I}}$  coincides with the  $q$ -expansion of  $f$ . Note that if we endow  $\mathbb{Z}_p$  with the  $\Gamma$ -module structure induced by letting  $1 + p$  act via multiplication by  $(1 + p)^k - 1$ , then  $\varphi_k$  becomes a  $\Lambda$ -algebra homomorphism. When  $f$  has complex multiplication, and therefore is the theta series

$$\theta(\psi) = \sum_{(\mathfrak{a}, \mathfrak{c})=1} \psi(\mathfrak{a}) q^{\text{Norm}(\mathfrak{a})}$$

attached to a Hecke character  $\psi$  of an imaginary quadratic field  $M$ , where  $\mathfrak{c}$  denotes the integral ideal of  $M$  that is the conductor of  $\psi$ , then the  $\Lambda$ -adic form  $\mathcal{F}_{\mathbb{I}}$  specializing to  $f$  admits a more explicit description which is recalled in Section 1.5. We then say that  $\mathcal{F}_{\mathbb{I}}$  itself has complex multiplication, and one has the easy but crucial fact that *all* its arithmetic specializations are of CM type.

Hida also attaches to any  $\Lambda$ -adic form  $\mathcal{F}_{\mathbb{I}}$  a continuous Galois representation

$$\rho_{\mathbb{I}} : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(\mathbb{I}),$$

which is unramified outside  $Np$ , and such that  $\text{tr} \rho_{\mathbb{I}}(\text{Frob}_{\ell}) = a_{\mathbb{I}}(\ell; \mathcal{F})$  for every prime  $\ell \nmid Np$ . Here the work of Mazur and Wiles [MW86] shows that the restriction of  $\rho_{\mathbb{I}}$  to the decomposition group  $D_p$  is also reducible with free rank one  $\mathbb{I}$ -module of  $I_p$ -coinvariants. When  $\mathcal{F}_{\mathbb{I}}$  is of CM type, the corresponding representation  $\rho_{\mathbb{I}}$  can be shown to be locally split at  $p$  using the above arguments. Therefore, a  $\Lambda$ -adic analogue of Greenberg's question naturally arises, and it makes sense to wonder whether Ghate's guess (4) has some chance to be true in this setting. In fact, by studying the weight one specializations of ordinary  $\Lambda$ -adic forms, Ghate and Vatsal were lead to the following result:

**Theorem 2** (Ghate-Vatsal). *Let  $p$  be an odd prime and let  $N$  be a positive integer with  $(N, p) = 1$ . Let  $\mathcal{F}_{\mathbb{I}}$  be an ordinary  $\Lambda$ -adic form attached to a local component  $\mathbb{I}$ . Denote by  $\mathfrak{m}$  the maximal ideal of  $\mathbb{I}$ , and by  $\rho_{\mathfrak{m}}$  the reduction modulo  $\mathfrak{m}$  of the Galois representation  $\rho_{\mathbb{I}}$  associated to  $\mathcal{F}_{\mathbb{I}}$ , and assume that  $\rho_{\mathfrak{m}}$  satisfies the following properties:*

- $\rho_{\mathfrak{m}}|_{D_p}$  has non-scalar semi-simplification;
- $\rho_{\mathfrak{m}}$  is absolutely irreducible when restricted to  $G_{\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})}$ .

Then

$$\rho_{\mathbb{I}} \text{ splits at } p \iff \mathcal{F}_{\mathbb{I}} \text{ has complex multiplication.}$$

This theorem is the main result in [GV04], and by a descend argument it allows Ghate and Vatsal to show that under the corresponding technical hypothesis, Ghate’s guess (4) is “generically true”:

**Corollary 3.** *Fix an odd prime  $p$  and a positive integer  $N$  with  $(N, p) = 1$ . Let  $\mathcal{S}_{\text{bad}} = \mathcal{S}_{\text{bad}}(p^\infty)$  be the set of ordinary non-CM newforms of weight  $k$  and level  $Np^r$ , for any integers  $k \geq 2$  and  $r \geq 1$ , satisfying the following properties:*

- $\rho_f|_{D_p}$  splits;
- $\bar{\rho}_f|_{D_p}$  has non-scalar semi-simplification;
- $\bar{\rho}_f$  is absolutely irreducible when restricted to  $G_{\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})}$ .

Then  $\mathcal{S}_{\text{bad}}$  is a finite set.

Note that this solution to Greenberg’s question, though almost complete (when coupled with the above analysis for ordinary CM newforms), has the obvious drawback of giving no indication of the truth or falsity of Ghate’s guess (4) for a given newform  $f$ , since one has almost no control at all over the set of possible exceptional non-CM newforms that appear in the corollary. These results are discussed in Section 1.6.

## Deformation theoretic approaches

Given an ordinary newform  $f$ , it is a hard problem to decide (i.e. to prove) whether the local representation  $\rho_f|_{D_p}$  is indecomposable. Ghate and Vatsal were able in [GV08] to show the first nontrivial examples of non-CM ordinary newforms of weight strictly  $> 2$  whose associated Galois representation is non-split at  $p$ , actually proving that (4) holds for each of the classical specializations of certain Hida families of tame level 1. This was achieved by means of a study of what they termed the *universal locally split (ordinary) deformation ring*.

Let  $f$  be an ordinary newform as above,  $R_{\mathfrak{p}}$  the ring of integers of  $K_{\mathfrak{p}}$  with maximal ideal  $\mathfrak{m}_{\mathfrak{p}}$ , and denote by  $k$  the residue field  $R_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$ . In general, we say that a Galois representation  $\rho : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(R)$  over a topological ring  $R$  is *ordinary* if is unramified outside a finite set of primes and is realized over a free  $R$ -module of rank 2 with  $R$ -module of  $I_p$ -coinvariants

free of rank one. Choose a  $G_{\mathbb{Q}}$ -stable lattice  $T_f \subset V_f$  and consider the ordinary residual representation

$$\bar{\rho}_f : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(k)$$

obtained by reducing modulo  $\mathfrak{m}_{\mathfrak{p}}$  the resulting ordinary representation on  $T_f$ . Fix  $S$  a finite set of primes containing  $\infty$ ,  $p$  and the prime divisors of  $N$ , and denote by  $G_{\mathbb{Q},S}$  the Galois group of the maximal extension of  $\mathbb{Q}$  unramified outside  $S$ .

**Assumption 1.**  $\bar{\rho}_f$  is absolutely irreducible.

Then it follows from the work of Mazur [Maz89] the existence of a complete Noetherian local ring  $(R^{\circ} = R^{\circ}(\bar{\rho}_f), \mathfrak{m}_{R^{\circ}})$  equipped with an ordinary Galois representation

$$\rho^{\circ} : G_{\mathbb{Q},S} \rightarrow \mathbf{GL}_2(R^{\circ}),$$

reducing modulo  $\mathfrak{m}_{R^{\circ}}$  to  $\bar{\rho}_f$ , which parametrizes all isomorphism classes of ordinary Galois representations  $\rho_A$  with values in an Artinian local ring  $(A, \mathfrak{m}_A)$  with residue field  $k$  reducing modulo  $\mathfrak{m}_A$  to  $\bar{\rho}_f$ , in the sense that each such  $\rho_A$  is obtained from  $\rho^{\circ}$  by composition with a unique specialization  $\varphi_A \in \text{Spec}(R^{\circ})(A)$ .

Denote also by  $\rho_f$  the Galois representation  $G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(R_{\mathfrak{p}})$  given by  $T_f$ ; the absolute irreducibility hypothesis ensures that the isomorphism class of the resulting representation does not depend on the choice of the the  $G_{\mathbb{Q}}$ -stable lattice  $T_f$ .

**Assumption 2.**  $\rho_f$  is  $p$ -distinguished, meaning that (after taking a suitable basis for a convenient  $T_f$ ) the characters  $\eta', \eta$  appearing in (2) have different reductions modulo  $\mathfrak{m}_{\mathfrak{p}}$ .

Under the two assumptions above, a result due to Ghate (cf. [Gha05], Prop. 6) assures that if  $\rho_f$  splits at  $p$ , then  $\bar{\rho}_f$  also does, and therefore we can write

$$\bar{\rho}_f|_{D_p} = \begin{pmatrix} \bar{\eta}' & \\ & \bar{\eta} \end{pmatrix},$$

where  $\bar{\eta}', \bar{\eta}$  denote the reductions modulo  $\mathfrak{m}_{\mathfrak{p}}$  of  $\eta', \eta$ , respectively. But it is certainly possible for a non-split representation  $\rho_f|_{D_p}$  to become semi-simple when reduced modulo  $\mathfrak{m}_{\mathfrak{p}}$ . Thus for the purposes of this paragraph we make the weaker

**Assumption 3.**  $\bar{\rho}_f$  is locally split at  $p$ .

One can then consider the functor  $\mathcal{D}_{\bar{\rho}_f}^{\text{split}}$  from the category  $\mathcal{A}(k)$  of Artinian local rings with residue field  $k$  into sets that attaches to every  $A \in \mathcal{A}(k)$  the set of isomorphism classes of ordinary representations

$$\rho_A : G_{\mathbb{Q},S} \rightarrow \mathbf{GL}_2(A)$$

satisfying the following two properties:

- $\rho_A \bmod \mathfrak{m}_A = \bar{\rho}_f$ ;
- $\rho_A|_{D_p} = \begin{pmatrix} \eta'_A & \\ & \eta_A \end{pmatrix}$ , with  $\eta'_A, \eta_A : G_{\mathbb{Q},S} \rightarrow A^\times$  reducing modulo  $\mathfrak{m}_A$  to  $\bar{\eta}', \bar{\eta}$ , respectively.

The original arguments of Mazur can be readily adapted to this *refined deformation problem* to show the pro-representability of  $\mathcal{D}_{\bar{\rho}_f}^{\text{split}}$ , thus yielding a complete Noetherian local ring  $R_{\bar{\rho}_f}^{\text{split}}$  with residue field  $k$ , together with an ordinary Galois representation

$$\rho_{\bar{\rho}_f}^{\text{split}} : G_{\mathbb{Q},S} \rightarrow \mathbf{GL}_2(R_{\bar{\rho}_f}^{\text{split}})$$

satisfying the above two properties, and such that  $\mathcal{D}_{\bar{\rho}_f}^{\text{split}}(A) = \text{Spec}(R_{\bar{\rho}_f}^{\text{split}}(A))$  for every  $A \in \mathcal{A}(k)$  via  $[\rho_A] \mapsto \varphi_A$ , where  $[\rho_A]$  denotes the *strict* isomorphism class of  $\rho_A$  and  $\varphi_A$  is the unique local algebra homomorphism  $R_{\bar{\rho}_f}^{\text{split}} \rightarrow A$  such that  $[\rho_A] = [\varphi_A \circ \rho_{\bar{\rho}_f}^{\text{split}}]$ .

Each  $p$ -stabilized newform  $g$  obtained as a classical specialization of the  $\Lambda$ -adic form  $\mathcal{F}_{\mathbb{I}}$  containing  $f$  has an attached Galois representation  $\rho_g$  which is ordinary and reduces to  $\bar{\rho}_g = \bar{\rho}_f$ ; therefore, if  $\rho_g$  is locally split at  $p$ , it lies in the *locally split locus*  $\text{Spec}(R_{\bar{\rho}_f}^{\text{split}})$ .

**Assumption 4.** *f has no CM.*

We note that there do exist many examples of newforms  $f$  which are ordinary for a certain prime number  $p$  such that Assumptions 1–4 are satisfied for  $f$  and  $p$ . Indeed, in [GV08] Ghate and Vatsal dealt with an explicit list of six level one eigenforms for which these assumptions are satisfied for certain prime numbers in a bounded range. Note further that then *every*  $p$ -stabilized newform  $g$  of weight  $k \geq 2$  specialization of the  $\Lambda$ -adic form containing  $f$  also satisfies each of these four assumptions. The expectation (4) is that none of the corresponding specialized representations  $\rho_g$  gives rise to points on  $\text{Spec}(R_{\bar{\rho}_f}^{\text{split}})$ , and Ghate and Vatsal were able to obtain, for some of their explicit examples, a sufficient control on the size of  $R_{\bar{\rho}_f}^{\text{split}}$  to conclude that none of its characteristic zero points corresponds to any of these  $\rho_g$ , obtaining in particular the following result:

**Theorem 4** (Ghate-Vatsal). *Let  $f = \Delta = \sum_{n=1}^{\infty} \tau(n)q^n$  be the unique newform of weight 12 and level 1 (thus of non-CM type). Then every prime  $p$  with  $7 < p < 1000$  is an ordinary prime for  $\Delta$ , and the Galois representation attached to each classical member in the  $\mathbb{Z}_p[[X]]$ -adic form containing  $\Delta$  is non-split at  $p$ .*

This is obtained by the following analysis. For every ordinary prime of  $\Delta$  in the above range, the local residual representation  $\bar{\rho}_\Delta|_{D_p}$  is non-split, except for the primes  $p = 23$  and  $p = 691$ . In the first exceptional case,  $\bar{\rho}_\Delta$  is induced from a finite order character  $\psi_o$  of the imaginary quadratic field  $\mathbb{Q}(\sqrt{-23})$ , and the fact that  $\text{Spec}(R_{\bar{\rho}_\Delta}^{\text{split}})$  consists of a unique point corresponding to the induction from  $G_M$  to  $G_{\mathbb{Q}}$  of a lifting of  $\psi_o$  to characteristic zero (cf. [GV08] Thm. 4.1.7), allows to conclude that  $\rho_\Delta$  does not lie in the split locus of  $\bar{\rho}_\Delta$ . In the second one,  $\bar{\rho}_\Delta$  is globally reducible, and by a straight argument using Ribet's converse to Herbrand's Theorem one also deduces the indecomposability of  $\rho_\Delta|_{D_p}$ .

## Approach to the problem

We first remark some features of the context in which Ghate and Vatsal worked in [GV08]. Suppose that Assumption 1 is satisfied (which is not supposed by them at the outset). Then in their examples of level one forms, Assumptions 2 and 4 are automatically satisfied, and the cases that need to be studied are those when Assumption 3 is also satisfied (the “full” and “dihedral” cases in their terminology). Under these assumptions, the expectation (4) is that any point in  $\text{Spec}(R_{\bar{\rho}_f}^{\text{split}})$ , if coming from an ordinary newform  $f$  of weight *greater* than 1, would be explained by the CM-ness of  $f$ ; but since the local component attached to an imaginary quadratic field is *primitive*, such  $f$  would be necessarily of level 1, which is impossible. Thus one expects that  $R_{\bar{\rho}_f}^{\text{split}}$  has no points at all corresponding to ordinary (either CM or not) newforms of weight  $> 1$ , and the result of Ghate and Vatsal is in accordance with this expectation.

In a quite orthogonal direction, one may try to put oneself in a situation where one knows a priori the existence of many locally split classical points, and try to show that all of them are of the type one expects. More precisely, suppose now that  $f$  is an ordinary newform of weight  $k \geq 2$  satisfying Assumptions 1–3 above and the following

**Assumption 5.**  $\rho_f$  is locally split at  $p$ .

**Assumption 6.**  $\bar{\rho}_f = \text{Ind}_M^{\mathbb{Q}} \psi_o$ , for a character  $\psi_o : G_M \rightarrow k^\times$  of infinite order of an imaginary quadratic field  $M$ .

Now Assumption 3 becomes redundant due to the result of Ghate alluded to above, so we actually just need to suppose that  $f$  satisfies Assumptions 1, 2, 5 and 6. Note also that by the preceding discussion, ordinary CM forms of weight  $k \geq 2$  fulfill all these requirements, and that some non-CM forms would certainly also fulfill them, if we did not ask for the character  $\psi_o$  to be of infinite order, as evidenced by the “dihedral” primes for the non-CM instances studied in [GV08]. In fact, under the additional requirement on  $\psi_o$ , Ghate’s guess (4) predicts that an ordinary newform of weight  $k \geq 2$  satisfy all the four assumptions above *only* if it is of CM type.

Let  $\psi$  be the Teichmüller lift of  $\psi_o$  to the ring of Witt vectors  $W(k)$  of  $k$ . In Section 1.5 it is recalled how to attach to the pair  $M, \psi$ , a unique primitive local component  $\mathbb{I}$  equipped with an ordinary Galois representation

$$\rho_{\mathbb{I}} : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(\mathbb{I}).$$

By the properties of the universal ordinary Hecke algebra discussed in Section 2.1, the representation  $\rho_{\mathbb{I}}$  is such that for any  $p$ -stabilized newform  $g$  belonging to the CM type  $\Lambda$ -adic form containing  $f = \theta(\psi)$ , there exists a unique specialization  $\varphi_g$  such that

$$\rho_g = \varphi_g \circ \rho_{\mathbb{I}},$$

where  $\rho_g$  denotes the  $p$ -adic representation associated to the CM form  $g$ . Furthermore, the representation  $\rho_{\mathbb{I}}$  itself is the induction from  $G_M$  to  $G_{\mathbb{Q}}$  of a  $\Lambda$ -adic character  $\Psi : G_M \rightarrow \mathbb{I}^{\times}$  with  $(\Psi \bmod \mathfrak{m}) = \psi_o$ . Thus  $\rho_{\mathbb{I}}$  is locally split at  $p$ , and by the universal property of  $R_{\bar{\rho}_f}^{\text{split}}$ , there exists a unique local ring homomorphism

$$\varphi_{\Psi} : R_{\bar{\rho}_f}^{\text{split}} \rightarrow \mathbb{I}$$

such that  $\rho_{\mathbb{I}} = \varphi_{\Psi} \circ \rho_{\bar{\rho}_f}^{\text{split}}$ . This homomorphism can be easily seen to be surjective, and Ghate's guess leads one to believe the following

**Conjecture 5.**  *$\varphi_{\Psi}$  is an isomorphism.*

We remark the similarity with the conjecture raised by Mazur and Tilouine in [MT90]. There the rings involved are the ordinary deformation ring  $R_{\bar{\rho}_f}^{\text{ord}}$  attached to a modular ordinary residual representation  $\bar{\rho}_f$ , and a corresponding (non-necessarily irreducible) local component  $R$  of the universal ordinary Hecke algebra of Hida. The local homomorphism

$$\varphi : R_{\bar{\rho}_f}^{\text{ord}} \rightarrow R$$

is similarly deduced from the universality of  $R_{\bar{\rho}_f}^{\text{ord}}$  and the properties of the representation attached to the local component  $R$ . Of course, after the pioneering work of Wiles, this older conjecture is known to be true, and thus  $R_{\bar{\rho}_f}^{\text{ord}} = R$ . We may then view Conjecture 5 as asserting that  $R_{\bar{\rho}_f}^{\text{split}}$  sits as an irreducible component in the ordinary deformation space of  $\bar{\rho}_f$ .

For the clarity of exposition, we end this introduction stating the following conjecture with all its running hypothesis. These hypothesis are the same as those in Conjecture 5, and its conclusion, which would be a direct consequence of it, is predicted by Ghate's guess.

**Conjecture 6.** *Let  $p > 2$  be an odd prime, and  $f \in S_k(\Gamma_0(N), \chi)$  a  $p$ -ordinary newform of level  $N > 0$  and weight  $k \geq 2$ . Let  $K/\mathbb{Q}_p$  be a finite extension of  $\mathbb{Q}_p$  containing the Hecke eigenvalues of  $f$  (via a fixed choice of embedding  $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ ), denote by  $\mathcal{O}$  the ring of integers of  $K$ , and by  $k$  its residue field  $\mathcal{O}/\mathfrak{m}_{\mathcal{O}}$ . Let  $\rho_f : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(K)$  be the Galois representation attached to  $f$ , and denote by  $\bar{\rho}_f$  a choice of residual representation  $(\rho_f \bmod \mathfrak{m}_{\mathcal{O}}) : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(k)$ . Suppose that the following hypothesis are satisfied:*

- $\bar{\rho}_f$  is absolutely irreducible;
- $\rho_f$  is  $p$ -distinguished;
- $\bar{\rho}_f = \text{Ind}_M^{\mathbb{Q}} \psi_o$ , for a certain character  $\psi_o : G_M \rightarrow k^{\times}$  of infinite order of an imaginary quadratic field  $M$ .

*Then if  $\rho_f$  is locally split at  $p$ ,  $f$  is of CM type.*

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## 1 Hida deformation theory

### 1.1 The space of $p$ -adic modular cusp forms

Let  $p$  be an odd prime, and  $N \geq 1$  an integer relatively prime to  $p$ . For each pair of integers  $k \geq 2$  and  $r > 0$ , denote by  $S_k(Np^r)$  the space

$$S_k(\Gamma_1(Np^r); \overline{\mathbb{Q}}_p) := S_k(\Gamma_1(Np^r); \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}_p,$$

where the tensor product is taken along the fixed embedding  $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ . The  $q$ -expansion of cusp forms furnishes an inclusion  $S_k(Np^r) \subset \overline{\mathbb{Q}}_p[[q]]$ . For a  $\mathbb{Z}_p$ -algebra  $R \subset \overline{\mathbb{Q}}_p$  we let  $S_k(Np^r; R) \subset S_k(Np^r)$  be the submodule consisting of cusp forms whose  $q$ -expansion lies in  $R[[q]]$ . Let  $R$  be the ring of integers in a finite extension  $K/\mathbb{Q}_p$  inside  $\overline{\mathbb{Q}}_p$ . We first describe an object which collects all  $R$ -rational cusp forms with fixed prime to  $p$  tame level  $N$  and arbitrary  $p$ -power in the level. We let the group  $(\mathbb{Z}/Np^r\mathbb{Z})^\times$  act on  $S_k(Np^r)$  via the *weight zero* action  $\langle \cdot \rangle_0$  (cf.[Til88]), i.e. the product of the usual Nebentypus action  $\langle \cdot \rangle$  and the character  $\ell \mapsto \ell_p^k$ , where  $\ell = (\ell_N, \ell_p) \in (\mathbb{Z}/Np^r\mathbb{Z})^\times = (\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/p^r\mathbb{Z})^\times$ , so that

$$a_n(f|\langle \ell \rangle_0) = \ell_p^k a_n(f|\langle \ell \rangle).$$

Let  $T_\ell = T_{\ell, r}^{(k)}$  and  $U_\ell = U_{\ell, r}^{(k)}$ , for  $\ell \nmid Np$  and  $\ell | Np$  respectively, be the Hecke operators on  $S_k(Np^r; R)$  whose action on the Fourier coefficients of  $f = \sum_{n=1}^{\infty} a_n(f)q^n$  is given by the formulas

$$\begin{aligned} a_n(f|T_\ell) &= a_{n\ell}(f) + \ell^{-1} a_{n/\ell}(f|\langle \ell \rangle_0); \\ a_n(f|U_\ell) &= a_{n\ell}(f), \end{aligned}$$

with the convention that  $a_{n/\ell}(f|\langle \ell \rangle_0) = 0$  if  $\ell \nmid n$ . These operators preserve the subspace  $S_k(Np^r; R)$ , and we denote by  $\mathbb{T}_{Np^r}^{(k)}$  the subalgebra of  $\text{End}_R(S_k(Np^r; R))$  generated over  $R$  by the resulting Hecke operators  $T_\ell$  and  $U_\ell$ , together with the operators  $\langle \ell \rangle$  for  $(\ell, N) = 1$ , acting on  $S_k(Np^r; R)$ . For every pair of integers  $r' \geq r > 0$ , there is a commutative diagram

$$\begin{array}{ccc} S_k(Np^r) & \hookrightarrow & S_k(Np^{r'}) \\ \downarrow & & \downarrow \\ S_k(Np^r) & \hookrightarrow & S_k(Np^{r'}), \end{array}$$

where the horizontal arrows are the natural inclusion maps, and the vertical arrows are given by the same (with a slight abuse of terminology) Hecke operator. This diagram remains commutative when restricted  $R$ -rational cusp forms, thus showing that the space

$$S_k(Np^\infty; R) := \varinjlim_r S_k(Np^r; R)$$

is naturally endowed with an action of the Hecke algebra

$$\mathbb{T}_{Np^\infty}^{(k)} := \varprojlim_r \mathbb{T}_{Np^r}^{(k)},$$

where the projective limit is taken along the restriction homomorphisms.

We can complete  $S_k(Np^\infty; R)$  with respect to the natural norm  $\| \cdot \|$  on  $R[[q]]$  defined, for  $f = \sum_{n_1}^\infty a_n(f)q^n$ , by the formula

$$\| f \| := \sup_n |a_n(f)|_p \quad (8)$$

where  $| \cdot |_p$  is the valuation on  $R$  normalized with  $|p|_p = 1/p$ . We denote the resulting space with the script notation  $\mathcal{S}_k(Np^\infty; R)$ . Note that it follows readily from the above description of the action of the Hecke operators on cusp forms, that  $\mathbb{T}_{Np^\infty}^{(k)}$  acts uniformly continuously on  $S_k(Np^\infty; R)$ , and therefore the Hecke action extends to a continuous action on  $\mathcal{S}_k(Np^\infty; R)$ .

The weight zero action of  $(\mathbb{Z}/Np^r\mathbb{Z})^\times$  on  $S_k(Np^r; R)$  is also compatible with the natural inclusion map, and thus  $\mathcal{S}_k(Np^\infty; R)$  is also equipped with an action of the profinite group

$$\mathbb{Z}_{p,N}^\times := \varprojlim_r (\mathbb{Z}/Np^r\mathbb{Z})^\times.$$

Note that since  $(p, N) = 1$ , we have the decomposition

$$\mathbb{Z}_{p,N}^\times = (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times = (\mathbb{Z}/Np\mathbb{Z})^\times \times \Gamma,$$

where  $\Gamma = 1 + p\mathbb{Z}_p$  is the group on  $p$ -adic 1-units in  $\mathbb{Z}_p^\times$ .

For every integer  $r > 0$ , denote by  $\mathcal{S}(Np^r; R)$  the subspace of  $R[[q]]$  given by the completion under the norm  $\| \cdot \|$  defined by (8) of

$$S_\infty(Np^r; R) := \varinjlim_j \left( \bigoplus_{k=1}^j S_k(Np^r; R) \right).$$

Note that  $\bigcup_{k \geq 1} S_k(Np^r; R) \subset R[[q]]$  is naturally contained in  $\mathcal{S}(Np^r; R)$ . One easily checks that  $S_\infty(Np^r; R)$  is a continuous module over the Hecke algebra

$$\mathbb{T}_{Np^r}^{(\infty)} = \varprojlim_j \mathbb{T}_{Np^r}^{(j)},$$

where  $\mathbb{T}_{Np^r}^{(j)}$  is the  $R$ -subalgebra of  $\text{End}_R(\bigoplus_{k=1}^j S_k(Np^r; R))$  generated by the Hecke operators via the diagonal action, and the projective limit is taken along the natural restriction maps induced for every pair of integers  $j' \geq j \geq 1$  by the obvious commutative diagram. As before, the Hecke action extends to a continuous action on  $\mathcal{S}(Np^r; R)$ .

It is a result due to Katz that for every pair of integers  $r' \geq r > 0$ , the natural inclusion

$$S_\infty(Np^r; R) \subset S_\infty(Np^{r'}; R),$$

when extended to the completion under the topology defined by  $\|\cdot\|$ , allows us to identify

$$\mathcal{S}(Np^r; R) = \mathcal{S}(Np^{r'}; R).$$

We denote by  $\mathcal{S}(Np^\infty; R)$  this common space  $\mathcal{S}(Np^r; R)$  for every  $r > 0$ , and by  $\mathbb{T}_N$  the corresponding common algebra  $\mathbb{T}_{Np^r}^{(\infty)} \subset \text{End}_R(\mathcal{S}(Np^r; R))$  for every  $r > 0$ . We refer to  $\mathcal{S}(Np^\infty; R)$  as the space of *p-adic modular cusp forms of tame level N* (and arbitrary weight). Note that for every integer  $k \geq 1$ , we have

$$S_k(Np^r; R) \subset \mathcal{S}(Np^\infty; R)$$

naturally for every  $r > 0$ , and therefore also  $\mathcal{S}_k(Np^\infty; R) \subset \mathcal{S}(Np^\infty; R)$ .

## 1.2 The universal ordinary Hecke algebra

Let  $\mathcal{S}(Np^\infty; R)$  be the space of  $p$ -adic cusp forms  $\mathbb{T}_N$  its corresponding Hecke algebra as defined in Section 1.1. The pairing

$$\begin{aligned} \mathcal{S}(Np^\infty; R) \times \mathbb{T}_N &\rightarrow R \\ (f, T) &\mapsto a_1(f|T) \end{aligned}$$

is a perfect  $R$ -duality, thus inducing a linear isomorphism

$$\mathbb{T}_N \cong \text{Hom}_R(\mathcal{S}(Np^\infty; R), R).$$

We use the superscript “ord” to denote the maximal quotient of a Hecke algebra in which the Hecke operator  $U_p$  becomes a unit. One then has an algebra direct sum decomposition (and similarly for  $\mathbb{T}_{Np^\infty}^{(k)}$ ):

$$\mathbb{T}_N = \mathbb{T}_N^{\text{ord}} \oplus \mathbb{T}_N^{\text{ss}},$$

with  $U_p$  having topologically nilpotent image in the second direct summand. Denote by  $e_{\text{ord}}$  the idempotent attached to the *ordinary part*. The image

$$\mathcal{S}^{\text{o}}(Np^\infty; R) := e_{\text{ord}}\mathcal{S}(Np^\infty; R)$$

is called the space of *p-adic ordinary cusp forms of tame level N* (and arbitrary weight), and one has that an eigenform  $f \in S_k(Np^r; R)$ , with  $r \geq 0$ , is ordinary (i.e.  $f$  is contained in  $\mathcal{S}^o(Np^\infty; R)$ ) if and only if its  $p$ -th Fourier coefficient satisfies  $|a_p(f)|_p = 1$ . The restriction of operators via  $\mathcal{S}(Np^\infty; R) \supset \mathcal{S}_k(Np^\infty; R)$  furnishes the commutative diagram

$$\begin{array}{ccc} \mathbb{T}_N & \longrightarrow & \mathbb{T}_{Np^\infty}^{(k)} \\ \cup \Big\downarrow & & \Big\downarrow \cup \\ \mathbb{T}_N^{\text{ord}} & \longrightarrow & \mathbb{T}_{Np^\infty}^{(k), \text{ord}}, \end{array}$$

where the horizontal arrows put in correspondence the respective Hecke operators, and are therefore surjective, since these operators topologically generate the Hecke algebras.

**Proposition 1.1** ([H86b] Thm. 1.1). *The above algebra homomorphism  $\mathbb{T}_N^{\text{ord}} \rightarrow \mathbb{T}_{Np^\infty}^{(k), \text{ord}}$  is an isomorphism, independently of the integer  $k \geq 2$ .*

We denote by  $\mathbb{T}_N^o$  the *universal ordinary Hecke algebra*  $\mathbb{T}_N^{\text{ord}} \subset \text{End}_R(\mathcal{S}^o(Np^\infty; R))$ , and we see that by the above proposition it is naturally isomorphic, for every integer  $k \geq 1$ , to the Hecke subalgebra of  $\text{End}_R(\mathcal{S}_k(Np^\infty; R))$ .

**Remark 1.** For each integer  $r > 0$ , let  $X_1(Np^r)/\mathbb{Q}$  be the usual modular curve, compactification of the affine curve  $Y_1(Np^r)$ , and consider its jacobian variety  $J_1(Np^r) = \text{Jac}(X_1(Np^r))$ . Denote by  $\mathbb{T}_{Np^r} = \mathbb{T}_{Np^r}^{(2)}(\mathbb{Z})$  the ring generated over  $\mathbb{Z}$  by the Hecke operators acting as rational endomorphisms on  $J_1(Np^r)$ . For every pair of integers  $r' \geq r > 0$  the covering maps  $X_1(Np^{r'}) \rightarrow X_1(Np^r)$  induce by Picard (contravariant) functoriality Hecke-equivariant endomorphisms  $J_1(Np^r) \rightarrow J_1(Np^{r'})$ . Thus we see that

$$J_\infty := \varinjlim_r J_1(Np^r)$$

is naturally endowed with an action of the Hecke algebra  $\mathbb{T}_{Np^\infty} = \varprojlim_r \mathbb{T}_{Np^r}$ . Then we denote by  $J_\infty^o$  the maximal submodule of  $J_\infty$  on which the ordinary part  $\mathbb{T}_{Np^\infty}^{\text{ord}}$  of  $\mathbb{T}_{Np^\infty}$  acts faithfully. Then

$$\mathbb{T}_N^o = \mathbb{T}_{Np^\infty}^{\text{ord}} \otimes_{\mathbb{Z}} R.$$

### 1.3 Ordinary $\Lambda$ -adic forms: $p$ -adic families

Let  $f = \sum a_n(f)q^n \in S_k(Np^r; R)$ , for certain integers  $k > 1$  and  $r > 0$ , be a normalized Hecke eigenform; or equivalently, an  $R$ -algebra homomorphism

$$\begin{aligned} \lambda_f : \mathbb{T}_{Np^r}^{(k)} &\rightarrow R \\ T &\mapsto \lambda_f(T) = a_1(f|T). \end{aligned}$$

Suppose that  $f$  is ordinary, so that  $a_p(f) = \lambda_f(U_p)$  is a  $p$ -adic unit. We say that  $f$  is an *ordinary  $p$ -stabilized newform* if either it is a newform of level  $Np^r$ , or it is obtained from an ordinary newform  $g$  of level  $N$  by the following process of  *$p$ -stabilization*: let  $\alpha_p$  and  $\beta_p$  be the reciprocal of the equation

$$\det(\text{Id} - \rho_g(\text{Frob}_p)X) = 0,$$

where  $\rho_g$  denotes the  $p$ -adic representation attached to  $g$ ; the ordinarity of  $g$  and the assumption that  $k > 1$  implies that exactly one of these roots is a  $p$ -adic unit; we suppose that this is  $\alpha_p$ , and then we have

$$f(z) = g(z) - \beta_p g(pz).$$

Note that in the latter case,  $f$  is an old form (despite the terminology) of level  $Np$ ,  $f$  and  $g$  have the same eigenvalues  $a_\ell(f) = a_\ell(g)$  for all primes  $\ell \neq p$ ,  $g$  is uniquely determined by this property, and  $f|U_p = \alpha_p f$ .

The weight zero action of  $(\mathbb{Z}/Np^r\mathbb{Z})^\times$  on  $S_k(Np^r; R)$  can be expressed in terms of Hecke operators, giving rise to an homomorphism

$$(\ell \mapsto \langle \ell \rangle_0) : (\mathbb{Z}/Np^r\mathbb{Z})^\times \rightarrow \mathbb{T}_{Np^r}^{(k)}.$$

Thus the universal ordinary Hecke algebra  $\mathbb{T}_N^o$  adopts the structure of a module over the completed group ring  $R[[\mathbb{Z}_{p,N}^\times]]$ ; so, in particular  $\mathbb{T}_N^o$  is naturally a module over the Iwasawa algebra

$$\Lambda_R := R[[\Gamma]] = R[[1 + p\mathbb{Z}_p]].$$

**Theorem 1.2** ([H86a] Thm. 3.1).  $\mathbb{T}_N^o$  is finite free over  $\Lambda_R$ .

Fix for definiteness the topological generator  $\gamma_o = 1 + p$  of  $\Gamma$ ; then  $\Lambda_R$  is identified with the power series ring  $R[[X]]$  via  $\gamma_o = 1 + X$ . Recall there is a natural identification

$$\text{Hom}_{\text{gp}}(\Gamma, \overline{\mathbb{Q}}_p^\times) = \text{Hom}_{R\text{-alg}}(\Lambda_R, \overline{\mathbb{Q}}_p);$$

specifically, if  $\varphi : \Gamma \rightarrow \overline{\mathbb{Q}}_p^\times$  is a continuous group homomorphism, then  $\varphi(\gamma_o)$  is a principal unit if a finite extension inside  $\overline{\mathbb{Q}}_p^\times$ , and one defines the corresponding  $R$ -algebra homomorphism  $\varphi \in \text{Hom}(\Lambda_R, \overline{\mathbb{Q}}_p)$  by setting

$$\varphi(\phi(T)) := \phi(\varphi(\gamma_o) - 1)$$

for every  $\phi(T) \in \Lambda_R = R[[X]]$ . The elements in  $\text{Hom}(\Lambda_R, \overline{\mathbb{Q}}_p)$  are also called *specializations* (of  $\Lambda_R$ ), and we will occasionally use the notation  $\text{Spec}(\Lambda_R)(\overline{\mathbb{Q}}_p)$ . If  $\varphi$  is a specialization, its kernel  $P_\varphi$  is a height one prime in  $\Lambda_R$ , and the reduction maps on  $\Lambda_R$  modulo a height one

prime exhaust all specializations. We say that a specialization  $\varphi$ , or that its corresponding height one prime  $P_\varphi$ , is *arithmetic* if  $\varphi \in \text{Hom}(\Gamma, \overline{\mathbb{Q}}_p^\times)$  is of the form

$$\gamma \mapsto \varepsilon(\gamma)\gamma^k,$$

for a certain finite order character  $\varepsilon : \Gamma \rightarrow \overline{\mathbb{Q}}_p^\times$  and a certain integer  $k \geq 1$ ; if further  $k \geq 2$  we may refer to  $\varphi$  as a *classical* specialization. We note that  $\varepsilon$  is necessarily of  $p$ -power order, and sometimes we will write  $\varepsilon_r$  to indicate that it has exact order  $p^r$ . We say that  $\varepsilon$  is the *character* of  $P_\varphi$  and that  $k$  is the *weight* of  $P_\varphi$ ; we then also denote  $\varphi$  by  $\varphi_{k,\varepsilon}$ , and  $P_\varphi$  by  $P_{k,\varepsilon}$ . More generally, if  $T$  is a finite  $\Lambda_R$ -algebra, and  $P$  is a height 1 prime of  $T$ , we say that  $P$  is an arithmetic specialization if  $P \cap \Lambda_R = P_{k,\varepsilon}$  for some  $k$  and  $\varepsilon$ .

Note that if  $\varepsilon$  has exact order  $p^r$ ,  $r \geq 0$ , so that it (optimally) factors through  $\Gamma/\Gamma^{p^r} \cong \mathbb{Z}/p^r\mathbb{Z}$ , where  $\Gamma^{p^r} = 1+p^{r+1}\mathbb{Z}_p$ , then  $\varepsilon$  is completely determined by the data of the primitive  $p^r$ -th root of unity  $\zeta_r = \varepsilon(\gamma_o)$  and  $\varphi_{k,\varepsilon}$  obviously restricts to  $\gamma \mapsto \gamma^k$  on  $\Gamma^{p^r}$ . Conversely, note that for every positive integer  $r \geq 1$  one has a canonical decomposition

$$(\mathbb{Z}/p^{r+1}\mathbb{Z})^\times = (\mathbb{Z}/p\mathbb{Z})^\times \times \mathbb{Z}/p^r\mathbb{Z}, \quad (9)$$

resulting from writing  $m = \omega(m) \cdot \frac{m}{\omega(m)}$ , where  $\omega(m) = m \bmod p$ . The residue class of  $1+p$  modulo  $p^r$  gives a generator of the factor  $\mathbb{Z}/p^r\mathbb{Z}$ , and for each primitive  $p^r$ -th root of unity  $\zeta_r \in \mu_{p^r}(\overline{\mathbb{Q}}_p)$  one can define a primitive character  $\varepsilon_{\zeta_r}$  modulo  $p^{r+1}$  of exact order  $p^r$  by

$$\varepsilon_{\zeta_r}(\lambda, \mu) = \zeta_r^{s(\mu)},$$

where  $(\lambda, \mu)$  corresponds to the decomposition (9) and  $s(\mu)$  is an integer, well defined modulo  $p^r$ , such that  $\mu = s(\mu)(1+p)$  (or equivalently, such that  $\mu = (\gamma_o \bmod p^{r+1})^{s(\mu)}$ ). Then for any finite index subgroup  $\Gamma'$  of  $\Gamma$ , which will be of the form  $\Gamma' = \Gamma^{p^r}$  for some  $r \geq 0$ , one can construct a unique specialization of weight  $k$  and primitive character  $\varepsilon_{\zeta_r}$  for each choice of a primitive  $p^r$ -th root of unity  $\zeta_r$ , by just setting  $\gamma \mapsto \varepsilon_{\zeta_r}(\gamma)\gamma^k$ , where one uses the decomposition (9) and the isomorphism  $\Gamma/\Gamma^{p^r} \cong \mathbb{Z}/p^r\mathbb{Z}$  to define  $\varepsilon_{\zeta_r}(\gamma)$  for  $\gamma \in \Gamma$ .

Let  $P_{k,\varepsilon}$  be a classical specialization of  $\Lambda_R$  for some integer  $k \geq 2$  and a finite order character  $\varepsilon$  on  $\Gamma/\Gamma^{p^{r-1}}$ , seen as a Dirichlet character modulo  $Np^r$ . Consider the congruence subgroup  $\Delta_r = \Gamma_1(Np) \cap \Gamma_0(p^r)$ . Note that

$$\Gamma_1(Np^r) \subset \Delta_r \subset \Gamma_0(Np^r),$$

where the latter inclusion holds because  $N$  and  $p$  are assumed to be relatively primes, and that  $\varepsilon$  may be naturally regarded as a character on  $\Delta_r$  via the chain

$$\begin{aligned} \Delta_r &\twoheadrightarrow \Delta_r/\Gamma_1(Np^r) \subset \Gamma_0(Np^r)/\Gamma_1(Np^r) \cong (\mathbb{Z}/Np^r\mathbb{Z})^\times \twoheadrightarrow (\mathbb{Z}/p^r\mathbb{Z})^\times \twoheadrightarrow \\ &\twoheadrightarrow \mathbb{Z}/p^{r-1}\mathbb{Z} \cong \Gamma/\Gamma^{p^{r-1}} \xrightarrow{\varepsilon} \overline{\mathbb{Q}}_p^\times. \end{aligned}$$

Consider the subspace  $S_k(\Delta_r, \varepsilon) \subset S_k(Np^r)$ , where

$$S_k(\Delta_r, \varepsilon) := \{f \in S_k(Np^r) \mid f|(\ell)_0 = \varepsilon(\ell)f \text{ for all } \ell \in \Delta_r\},$$

and denote by  $S_k(\Delta_r, \varepsilon; R) \subset S_k(Np^r; R)$  the corresponding submodule of  $R$ -rational cusp forms. The restriction of Hecke operators via  $\mathcal{S}^o(Np^\infty; R) \supset S_k^{\text{ord}}(\Delta_r, \varepsilon; R)$  gives rise to a surjective algebra homomorphism

$$\varphi_{k,\varepsilon} : \mathbb{T}_N^o / P_{k,\varepsilon} \mathbb{T}_N^o \rightarrow \mathbb{T}_{Np^r}^{(k),\text{ord}}(\varepsilon),$$

where  $\mathbb{T}_{Np^r}^{(k),\text{ord}}(\varepsilon)$  denotes the ordinary part of the quotient of  $\mathbb{T}_{Np^r}^{(k)}$  that acts faithfully on  $S_k(\Delta_r, \varepsilon; R)$ . Indeed, from the definition of the action of  $\mathbb{Z}_{p,N}^\times$  we readily see that  $\mathbb{T}_{Np^r}^{(k),\text{ord}}(\varepsilon)$  is the maximal  $R$ -submodule of  $\mathbb{T}_{Np^r}^{\text{ord}}$  where  $\Gamma$  acts via  $\gamma \mapsto \varepsilon(\gamma)\gamma^k$ .

**Theorem 1.3** (Hida's Control Theorem, [H86b] Thm. 1.2). *For every integer  $k \geq 2$  and every character  $\varepsilon : \Gamma \rightarrow \overline{\mathbb{Q}}_p^\times$  of exact order  $p^r$ , the  $R$ -algebra homomorphism  $\varphi_{k,\varepsilon}$  is an isomorphism.*

Let  $\mathcal{L}_R$  be the fraction field of  $\Lambda_R$ . By Theorem 1.2,  $\mathbb{T}_N^o \otimes_{\Lambda_R} \mathcal{L}_R$  is a finite dimensional Artinian  $\mathcal{L}_R$ -algebra, and therefore decomposes as an algebra direct sum of the finitely many local components obtained as the localization of  $\mathbb{T}_N^o \otimes_{\Lambda_R} \mathcal{L}_R$  at each one of its maximal ideals. Therefore, singling out one such local component  $\mathbb{I}_{\mathcal{L}}$ , we can write

$$\mathbb{T}_N^o \otimes_{\Lambda_R} \mathcal{L}_R = \mathbb{I}_{\mathcal{L}} \oplus \mathbb{J}_{\mathcal{L}} \tag{10}$$

as  $\mathcal{L}_R$ -algebras. We denote by  $\mathbb{T}_N^{\text{o,new}}$  the quotient of the Hecke algebra  $\mathbb{T}_N^o$  acting faithfully on the space of newforms; it is a reduced finite torsion-free  $\Lambda_R$ -algebra. We say that the local component  $\mathbb{I}_{\mathcal{L}}$  as above is *primitive* if it corresponds by pullback via  $\mathbb{T}_N^o \twoheadrightarrow \mathbb{T}_N^{\text{o,new}}$  to a local component in  $\mathbb{T}_N^{\text{o,new}} \otimes_{\Lambda_R} \mathcal{L}_R$ . If the local component  $\mathbb{I}_{\mathcal{L}}$  is primitive, then it is a finite dimensional  $\mathcal{L}_R$ -vector space. Denote by  $\mathbb{I}$  the projected image of  $\mathbb{T}_N^o$  on  $\mathbb{I}_{\mathcal{L}}$ , i.e. the image of the composite map

$$\mathbb{T}_N^o \hookrightarrow \mathbb{T}_N^o \otimes_{\Lambda_R} \mathcal{L}_R \twoheadrightarrow \mathbb{I}_{\mathcal{L}},$$

and denote by  $\tilde{\mathbb{I}}$  the normalization of  $\mathbb{I}$  in  $\mathbb{I} \otimes_{\Lambda_R} \mathcal{L}_R = \mathbb{I}_{\mathcal{L}}$ , so that we have the chain of inclusions

$$\mathbb{I} \subset \tilde{\mathbb{I}} \subset \mathbb{I}_{\mathcal{L}},$$

with  $\tilde{\mathbb{I}}$  being finite flat as a  $\Lambda_R$ -algebra with Krull dimension 2. We remark that when  $\mathbb{I}_{\mathcal{L}} = \mathcal{L}_R$ , then  $\mathbb{I} = \Lambda_R$ , and therefore we also have  $\tilde{\mathbb{I}} = \Lambda_R$ .

Recall that there is a bijection between ordinary cuspidal eigenforms  $f \in S_k^{\text{ord}}(\Delta_r, \varepsilon)$  and  $R$ -algebra homomorphisms  $\lambda_f : \mathbb{T}_{Np^r}^{(k),\text{ord}}(\varepsilon) \rightarrow \overline{\mathbb{Q}}_p$ , the correspondence being determined by the requirement that

$$f|T = \lambda_f(T)f$$

holds for every  $T \in \mathbb{T}_{Np^r}^{(k),\text{ord}}(\varepsilon)$ .

If  $P_{k,\varepsilon}$  is a classical specialization of  $\Lambda_R$ , then there is an  $R$ -algebra homomorphism  $\mathbb{T}_N^o/P_{k,\varepsilon}\mathbb{T}_N^o \rightarrow \tilde{\mathbb{I}}/P_{k,\varepsilon}\tilde{\mathbb{I}}$  induced by the natural map  $\mathbb{T}_N^o \rightarrow \mathbb{I} \subset \tilde{\mathbb{I}}$ . Thus, by Theorem 1.3 we have an  $R$ -algebra homomorphism

$$\mathbb{T}_{Np^r}^{(k),\text{ord}}(\varepsilon) \rightarrow \tilde{\mathbb{I}}/P_{k,\varepsilon}\tilde{\mathbb{I}}.$$

**Theorem 1.4** (Hida, [H86b] Cor. 1.3). *Let  $k$ ,  $\varepsilon$  and  $\varphi_{k,\varepsilon}$  be as in Theorem 1.3.*

- *Let  $\mathbb{I}_{\mathcal{L}}$  be a primitive local component of  $\mathbb{T}_N^o \otimes_{\Lambda_R} \mathcal{L}_R$  and let  $d = [\mathbb{I}_{\mathcal{L}} : \mathcal{L}_R]$ . Then there are exactly normalized ordinary eigenforms  $f \in S_k^{\text{ord}}(\Delta_r, \varepsilon)$  belonging to  $\mathbb{I}_{\mathcal{L}}$ , in the sense that they correspond to the  $R$ -algebra homomorphisms  $\lambda_f$  and  $\tilde{\lambda}_f$  fitting into the commutative diagram*

$$\begin{array}{ccc} & & \tilde{\mathbb{I}}/P_{k,\varepsilon}\tilde{\mathbb{I}} \\ & \nearrow & \downarrow \tilde{\lambda}_f \\ \lambda_f : \mathbb{T}_{Np^r}^{(k),\text{ord}}(\varepsilon) & \longrightarrow & \overline{\mathbb{Q}}_p \end{array}$$

and each such  $f$  is the ordinary  $p$ -stabilized newform attached to a unique ordinary eigenform (depending on  $f$ ) of conductor dividing  $N$ .

- *Conversely, for every ordinary  $p$ -stabilized newform  $f \in S_k^{\text{ord}}(\Delta_r, \varepsilon)$  there exists a unique local algebra  $\mathbb{I}_{\mathcal{L}}$  direct summand of  $\mathbb{T}_N^o \otimes_{\Lambda_R} \mathcal{L}_R$  which is primitive and corresponds to  $f$  by the above process.*

We next discuss how this theorem due to Hida allows one to construct  $p$ -adic analytic families of ordinary  $p$ -stabilized newforms. Let the notations be as in Theorem 1.4, and assume for simplicity that  $[\mathbb{I}_{\mathcal{L}} : \mathcal{L}_R] = 1$ , so that  $\tilde{\mathbb{I}} = \Lambda_R = R[[\Gamma]]$ , and therefore  $\tilde{\mathbb{I}} = R[[X]]$  after the identification  $1 + p = 1 + X$ . Denote by  $\lambda_{\mathbb{I}}$  the composite map

$$\mathbb{T}_N^o \hookrightarrow \mathbb{T}_N^o \otimes_{\Lambda_R} \mathcal{L}_R \twoheadrightarrow \mathbb{I} = \tilde{\mathbb{I}} = \Lambda[[X]],$$

and form the formal  $q$ -expansion

$$\mathcal{F}_{\mathbb{I}} = \sum_{n=1}^{\infty} a_{\mathbb{I}}(n; \mathcal{F})q^n := \sum_{n=1}^{\infty} \lambda_{\mathbb{I}}(T_n)q^n \in R[[X]][[q]].$$

For each integer  $k \geq 2$  and finite order character  $\varepsilon$  on  $\Gamma/\Gamma^{p^{r-1}}$ , the specialization map  $\varphi_{k,\varepsilon} : \gamma \mapsto \varepsilon(\gamma)\gamma^k$  corresponds to the evaluation homomorphism

$$\begin{aligned} R[[X]] &\rightarrow \overline{\mathbb{Q}}_p \\ X &\mapsto \varepsilon(\gamma_o)\gamma_o^k - 1. \end{aligned}$$

Thus we see that in virtue of Theorem 1.3, for every  $k \geq 2$  and  $\varepsilon$  as above, the formal  $q$ -expansion

$$P_{k,\varepsilon}(\mathcal{F}_{\mathbb{I}}) := \sum_{n=1}^{\infty} \varphi_{k,\varepsilon}(a_{\mathbb{I}}(n; \mathcal{F}))q^n \in \overline{\mathbb{Q}}_p[[q]]$$

resulting from specializing the coefficients  $a_{\mathbb{I}}(n; \mathcal{F})$  to  $P_{k,\varepsilon} = \text{Ker}(\varphi_{k,\varepsilon})$  is in fact the  $q$ -expansion of an ordinary  $p$ -stabilized newform  $f_{k,\varepsilon} \in S_k^{\text{ord}}(\Delta_r, \varepsilon; R)$ , and we will usually identify the formal  $q$ -expansion  $P_{k,\varepsilon}(\mathcal{F}_{\mathbb{I}})$  with  $f_{k,\varepsilon}$  itself. We call  $\mathcal{F}_{\mathbb{I}}$  an ordinary primitive  $\Lambda$ -adic form (and often simply a  $\Lambda$ -adic form) and refer to  $f_{k,\varepsilon}$  as a *classical specialization* of  $\mathcal{F}_{\mathbb{I}}$  of weight  $k$ . Note that it follows in particular from the second assertion in Theorem 1.4 that for any ordinary newform of weight  $k \geq 2$  and level  $Np^r$  there exists a unique  $\Lambda$ -adic form which specializes to it.

Recall that  $\mathbb{T}_N^{\circ}$  has a module structure over the completed group ring  $R[[\mathbb{Z}_{p,N}^{\times}]]$ , and hence one may also regard  $\mathbb{T}_N^{\circ} \otimes_{\Lambda_R} \mathcal{L}_R$  as a  $R[[\mathbb{Z}_{p,N}^{\times}]]$ -module considering the action on the left factor. Since  $\mathbb{Z}_{p,N}^{\times} = \Gamma \times (\mathbb{Z}/Np\mathbb{Z})^{\times}$ , one has that

$$R[[\mathbb{Z}_{p,N}^{\times}]] = \Lambda_R \otimes_R R[(\mathbb{Z}/Np\mathbb{Z})^{\times}],$$

and therefore (assuming all characters of  $(\mathbb{Z}/Np\mathbb{Z})^{\times}$  take values contained in  $R$ ) we see that, as  $\mathcal{L}_R$ -algebras,

$$\mathbb{T}_N^{\circ} \otimes_{\Lambda_R} \mathcal{L}_R = \bigoplus_{\chi \in \widehat{(\mathbb{Z}/Np\mathbb{Z})^{\times}}} (\mathbb{T}_N^{\circ} \otimes_{\Lambda_R} \mathcal{L}_R)^{(\chi)},$$

where  $(\mathbb{T}_N^{\circ} \otimes_{\Lambda_R} \mathcal{L}_R)^{(\chi)}$  denotes the maximal  $\mathcal{L}_R$ -algebra direct summand of  $\mathbb{T}_N^{\circ} \otimes_{\Lambda_R} \mathcal{L}_R$  on which  $(\mathbb{Z}/Np\mathbb{Z})^{\times}$  acts via the character  $\chi$ . We say that the local component  $\mathbb{I}_{\mathcal{L}}$  of  $\mathbb{T}_N^{\circ} \otimes_{\Lambda_R} \mathcal{L}_R$  has *character*  $\chi$  if  $\mathbb{I}_{\mathcal{L}}$  is a direct summand of  $(\mathbb{T}_N^{\circ} \otimes_{\Lambda_R} \mathcal{L}_R)^{(\chi)}$ . Then, if an ordinary  $p$ -stabilized newform  $f \in S_k^{\text{ord}}(\Delta_r, \varepsilon; R)$  belongs to  $\mathbb{I}_{\mathcal{L}}$ , we see that

$$f \in S_k^{\text{ord}}(\Gamma_0(Np^r), \chi\varepsilon\omega^{-k}),$$

where  $\omega : (\mathbb{Z}/p\mathbb{Z})^{\times} \xrightarrow{\sim} \mu_{p-1} \subset \mathbb{Z}_p^{\times}$  denotes the Teichmüller character, since for

$$\ell = (\omega(\ell), \ell_N) \in (\mathbb{Z}/Np\mathbb{Z})^{\times} \cong \mu_{p-1} \times (\mathbb{Z}/N\mathbb{Z})^{\times},$$

we have  $f|\langle \ell \rangle_0 = \chi(\ell)f$  for the weight zero action, and therefore  $f|\langle \ell \rangle = \omega^{-k}(\ell)\chi(\ell)f$  for the usual Nebentypus action, and it follows that  $f|\langle \ell \rangle = \omega^{-k}(\ell)\chi(\ell)\varepsilon(\ell)f$  for  $\ell \in (\mathbb{Z}/Np^r\mathbb{Z})^{\times}$ .

## 1.4 $p$ -adic families of Galois representations

Recall that one can attach to every newform  $f = \sum_{n=1}^{\infty} a_n(f)q^n \in S_k(Np^r, \psi)$  an irreducible Galois representation  $\rho_f : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(\overline{\mathbb{Q}}_p)$  unramified outside  $Np$  and such that the relation

$$\det(\text{Id} - \rho_f(\text{Frob}_{\ell})X) = 1 - a_{\ell}(f)X + \psi(\ell)\ell^{k-1}X^2$$

holds for every prime  $\ell \nmid Np$ .

Let  $\mathbb{I}_{\mathcal{L}}$  be a primitive local algebra direct summand of  $\mathbb{T}_N^{\circ} \otimes_{\Lambda_R} \mathcal{L}_R$ , and let  $\tilde{\mathbb{I}}$  be the integral closure of  $\Lambda_R$  in  $\mathbb{I}_{\mathcal{L}}$ . We know that  $\mathbb{I}_{\mathcal{L}}$  is a field extension of  $\mathcal{L}_R$  of finite degree, and that  $\mathbb{I}_{\mathcal{L}}$  is an integrally closed complete Noetherian local  $R$ -algebra of Krull dimension 2. We denote by  $\mathfrak{m}$  the maximal ideal of  $\mathbb{I}$ , and say that a Galois representation  $\rho : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(\mathbb{I}_{\mathcal{L}})$  is *continuous* if it is realizable over a  $\mathbb{I}_{\mathcal{L}}$ -vector space  $V$  of dimension 2 having a  $G_{\mathbb{Q}}$ -stable  $\tilde{\mathbb{I}}$ -lattice of rank 2 such that the resulting representation  $G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(\tilde{\mathbb{I}})$  is continuous for the  $\mathfrak{m}$ -adic topology of  $\tilde{\mathbb{I}}$ . As before, denote by  $\lambda_{\mathbb{I}}$  the composite map

$$\mathbb{T}_N^{\circ} \hookrightarrow \mathbb{T}_N^{\circ} \otimes_{\Lambda_R} \mathcal{L}_R \rightarrow \mathbb{I} \subset \tilde{\mathbb{I}}.$$

**Theorem 1.5** ([H86b] Thm. 2.1). *There exists a continuous irreducible Galois representation*

$$\rho_{\mathbb{I}} : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(\tilde{\mathbb{I}})$$

*uniquely determined by the following properties:*

- $\rho_{\mathbb{I}}$  is unramified outside  $Np$ , and for every prime  $\ell \nmid Np$ ,

$$\det(\mathrm{Id} - \rho_{\mathbb{I}}(\mathrm{Frob}_{\ell})X) = 1 - \lambda_{\mathbb{I}}(T_{\ell})X + \lambda_{\mathbb{I}}(\ell^{-1}\langle \ell \rangle)X^2;$$

- For every ordinary  $p$ -stabilized newform  $f \in S_k^{\mathrm{ord}}(\Delta_r, \varepsilon)$  belonging to  $\mathbb{I}_{\mathcal{L}}$ , and for the unique  $R$ -algebra homomorphism  $\tilde{\lambda}_f : \tilde{\mathbb{I}}/P_{k,\varepsilon}\tilde{\mathbb{I}} \rightarrow \overline{\mathbb{Q}}_p$  attached to it,

$$\det(\mathrm{Id} - (\tilde{\lambda}_f \circ \rho_{\mathbb{I}})(\mathrm{Frob}_{\ell})X) = \det(\mathrm{Id} - \rho_f(\mathrm{Frob}_{\ell})X), \quad (11)$$

for every prime  $\ell \nmid Np$ ; or equivalently, if  $P_f$  denotes the height one prime in  $\tilde{\mathbb{I}}$  above  $P_{k,\varepsilon} \in \mathrm{Spec}(\Lambda_R)(\overline{\mathbb{Q}}_p)$  given by  $\mathrm{Ker}(\tilde{\lambda}_f)$ , then

$$(\rho_{\mathbb{I}} \bmod P_f) \cong \rho_f,$$

as Galois representations into  $\mathbf{GL}_2(\overline{\mathbb{Q}}_p)$ .

We note that in (11) above, we denote by  $\tilde{\lambda}_f$  the homomorphism  $\mathbf{GL}_2(\tilde{\mathbb{I}}) \rightarrow \mathbf{GL}_2(\overline{\mathbb{Q}}_p)$  given by applying the  $R$ -algebra homomorphism  $\tilde{\lambda}_f : \tilde{\mathbb{I}} \rightarrow R$  to each matrix entry.

**Corollary 1.6.** *If the local component  $\mathbb{I}$  has character  $\chi$  modulo  $Np$ , then*

$$\det(\rho_{\mathbb{I}}) = \chi \kappa \varepsilon_{\mathrm{cyc}}^{-1}, \quad (12)$$

where  $\chi$  denotes the finite Galois character induced by  $\chi$ , and  $\kappa$  denotes the Galois character  $G_{\mathbb{Q}} \rightarrow \mathbb{I}^{\times}$ , factoring through  $\mathrm{Gal}(\mathbb{Q}_{\infty}/\mathbb{Q})$ , given by  $\sigma \mapsto (1 + X)^{s(\sigma)}$ ,  $s(\sigma) \in \mathbb{Z}_p$  being uniquely determined by the condition

$$\langle \varepsilon_{\mathrm{cyc}}(\sigma) \rangle = \gamma_o^{s(\sigma)}.$$

Note that in fact  $s(\sigma) = \frac{\log(\langle \varepsilon_{\text{cyc}}(\sigma) \rangle)}{\log(\gamma_o)}$ .

*Proof.* By Theorem 1.5, we just need to check that the left hand side in (12) interpolates  $\det(P_{k,\varepsilon}(\rho_{\mathbb{I}})) = \det(\rho_{P_{k,\varepsilon}}(\mathcal{F}_{\mathbb{I}})) = \chi \varepsilon \omega^{-k} \varepsilon_{\text{cyc}}^{k-1}$  as  $k$  varies, which is a straightforward calculation. Indeed,

$$(1 + X)^{s(\sigma)}|_{X=\varepsilon(\gamma_o)\gamma_o^k-1} = (\varepsilon(\gamma_o)\gamma_o^k)^{s(\sigma)} = \varepsilon(\langle \varepsilon_{\text{cyc}}(\sigma) \rangle) \langle \varepsilon_{\text{cyc}}(\sigma) \rangle^k,$$

and the claim follows when we note that the composition  $G_{\mathbb{Q}} \twoheadrightarrow \Gamma \xrightarrow{\varepsilon} \Lambda^{\times}$  is given by  $\varepsilon(\sigma) = \varepsilon(\langle \varepsilon_{\text{cyc}}(\sigma) \rangle)$ .  $\square$

Mazur and Wiles studied the geometry of the big Galois representations constructed by Hida, obtaining in particular a control on the action of the inertia group at  $p$  in these representations.

**Theorem 1.7** ([MW86] §8, Prop. 2). *Let  $\mathcal{F}_{\mathbb{I}} = \sum a_{\mathbb{I}}(n; \mathcal{F})q^n \in \mathbb{I}[[q]]$  be a  $\Lambda$ -adic form attached to a primitive local component  $\mathbb{I}_{\mathcal{L}}$  of  $\mathbb{T}_N^{\circ} \otimes_{\Lambda_R} \mathcal{L}_R$  of character  $\chi$ . Let*

$$\rho_{\mathbb{I}} : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(\mathbb{I}_{\mathcal{L}})$$

*be the  $\Lambda$ -adic representation attached to  $\mathcal{F}_{\mathbb{I}}$  by Theorem 1.5, and denote by  $V$  a representation space for  $\rho_{\mathbb{I}}$ . Then  $V$  has a  $D_p$ -stable line with unramified quotient. More precisely, after a suitable choice of basis for  $V$ ,*

$$\rho_{\mathbb{I}}|_{D_p} = \begin{pmatrix} \chi \kappa \varepsilon_{\text{cyc}}^{-1} \nu^{-1} & * \\ & \nu \end{pmatrix}, \quad (13)$$

where

$$\nu : D_p \rightarrow \mathbb{I}^{\times}$$

*is the unramified character given by  $\nu(\text{Frob}_p) = a_{\mathbb{I}}(p; \mathcal{F})$ , and  $\kappa : G_{\mathbb{Q}} \rightarrow \mathbb{I}^{\times}$  is as in Corollary 1.6.*

## 1.5 Local components of CM type

Fix a prime  $p$ , and maintain the assumption that  $R$  is the ring of integers of a finite extension  $K/\mathbb{Q}_p$ . Let  $M$  be an imaginary quadratic field of discriminant  $-D$ ,  $\mathcal{O}_M$  its ring of integers, and suppose that  $p$  splits in  $M$  as  $p\mathcal{O}_M = \mathfrak{p}\bar{\mathfrak{p}}$ , with  $\mathfrak{p} \neq \bar{\mathfrak{p}}$ . Let  $\mathfrak{c}$  be an integral ideal of  $M$  relatively prime to  $\mathfrak{p}$ , and set

$$N = \text{Norm}(\mathfrak{c}) \cdot D$$

throughout this section. For each prime  $\mathfrak{l}$  in  $\mathcal{O}_M$  denote by  $\mathcal{O}_{\mathfrak{l}}$  its  $\mathfrak{l}$ -adic completion  $\varprojlim \mathcal{O}_M/\mathfrak{l}^n \mathcal{O}_M$ . Let  $\mathbb{A}_M^{\times}$  be the idèle group of  $M$  and consider the compact quotient

$$W(\mathfrak{c}) := \mathbb{A}_M^{\times} / M^{\times} \mathbb{C}^{\times} U(\mathfrak{c}),$$

where  $U(\mathfrak{c})$  is defined to be the product  $\prod_{\mathfrak{l} \neq \mathfrak{p}} U_{\mathfrak{l}}(\mathfrak{c})$ , with  $U_{\mathfrak{l}}(\mathfrak{c}) = \{x \in \mathcal{O}_{\mathfrak{l}}^{\times} \mid x \equiv 1 \pmod{\mathfrak{c}}\}$ . Note that since  $p$  splits in  $M$ , there is an isomorphism  $\Gamma = 1 + p\mathbb{Z}_p \cong 1 + \mathfrak{p}\mathcal{O}_{\mathfrak{p}}$  induced by the inclusion  $\mathbb{Z} \subset R$ , and thus a natural injection  $\Gamma \hookrightarrow W(\mathfrak{c})$ . Denote by  $W$  a maximal  $p$ -profinite torsion free subgroup of  $W(\mathfrak{c})$  containing  $\Gamma$ ; such a subgroup is independent of  $\mathfrak{c}$  in a clear sense, and the index  $(W : \Gamma)$  is finite. The quotient  $\Delta_{\mathfrak{c}} = W(\mathfrak{c})/W$  is a finite group of order prime to  $p$ , and we have a non-canonical decomposition

$$W(\mathfrak{c}) = W \times \Delta_{\mathfrak{c}}.$$

Denote by  $I(\mathfrak{p}\mathfrak{c})$  the set of integral ideals of  $M$  relatively prime to  $\mathfrak{p}\mathfrak{c}$  and note that we have a natural injective map  $I(\mathfrak{p}\mathfrak{c}) \hookrightarrow W(\mathfrak{c})$  with dense image. Let  $\mathcal{C}(W(\mathfrak{c}); K)$  be the Banach space of continuous functions from  $W(\mathfrak{c})$  into  $K$ , and put

$$\mathcal{S}(Np^{\infty}; K) := \mathcal{S}(Np^{\infty}; R) \otimes_R K.$$

Consider the *base change map* from  $\mathbf{GL}_{1/M}$  to  $\mathbf{GL}_{2/\mathbb{Q}}$  given by the linear form

$$\begin{aligned} \theta : \mathcal{C}(W(\mathfrak{c}); K) &\rightarrow \mathcal{S}(Np^{\infty}; K) \\ \phi &\mapsto \theta(\phi) = \sum_{\mathfrak{a} \in I(\mathfrak{p}\mathfrak{c})} \phi(\mathfrak{a}) q^{\text{Norm}(\mathfrak{a})}. \end{aligned}$$

That the image of  $\theta$  actually lies in  $\mathcal{S}(Np^{\infty}; K)$  follows from the density of  $I(\mathfrak{p}\mathfrak{c})$  in  $W(\mathfrak{c})$ , a well-known theorem due to Hecke and Shimura, and the density in  $\mathcal{C}(I(\mathfrak{p}\mathfrak{c}); K)$  of the  $K$ -subspace spanned by the Hecke characters  $\phi : I(\mathfrak{p}\mathfrak{c}) \rightarrow K^{\times}$  (with the topology on  $I(\mathfrak{p}\mathfrak{c})$  induced from that of  $W(\mathfrak{c})$ ).

The Hecke action on  $\theta(\phi)$  is expressed in terms of the natural translation action of  $W(\mathfrak{c})$  on  $\mathcal{C}(W(\mathfrak{c}); K)$  by the following formulas:

$$\theta(\phi)|T_{\ell} = \begin{cases} \theta(\phi|\mathfrak{l}) + \theta(\phi|\bar{\mathfrak{l}}) & \text{if } \ell\mathcal{O}_M = \mathfrak{l}\bar{\mathfrak{l}}, \mathfrak{l} \neq \bar{\mathfrak{l}} \\ 0 & \text{if } \text{Norm}(\mathfrak{l}) = \ell^2 \\ \theta(\phi|\mathfrak{l}) & \text{if } \ell\mathcal{O}_M = \mathfrak{l}^2 \text{ or } \ell\mathcal{O}_M = \mathfrak{l}\bar{\mathfrak{l}} \text{ with } \bar{\mathfrak{l}} \notin I(\mathfrak{p}\mathfrak{c}). \end{cases}$$

And further we have that  $\theta(\phi)|\langle z \rangle_0 = z\theta(\phi|z)$  for  $z \in \Gamma \subset W(\mathfrak{c})$ . Thus by the duality between modular forms and the corresponding Hecke algebra on the one side, and Mahler's theorem on the other, giving a duality between  $\mathcal{C}(W(\mathfrak{c}); R)$  and  $R[[W(\mathfrak{c})]]$ , we see that  $\theta$  induces an  $R$ -algebra homomorphism

$$\varphi = \theta^* : \mathbb{T}_N \rightarrow R[[W(\mathfrak{c})]]. \quad (14)$$

Note that it follows from the third of the above formulas that the following relations holds for every  $\phi \in \mathcal{C}(W(\mathfrak{c}); R)$ :

$$\phi|_{\varphi(U_p)} = \theta(\phi)|_{U_p} = \theta(\phi|\bar{\mathfrak{p}}),$$

and thus  $\varphi(U_p)$  is the image of  $\bar{\mathfrak{p}} \in I(\mathfrak{pc})$  in  $R[[W(\mathfrak{c})]]$ , which is invertible, and therefore  $\varphi$  factors through the ordinary part  $\mathbb{T}_N^o$ .

Recall that there is a bijection between complex-valued ideal Hecke characters

$$\psi : I(\mathfrak{pc}) \rightarrow \overline{\mathbb{Q}}^\times$$

such that  $\psi(a\mathcal{O}_M) = a^{\kappa-1}$  for all  $a \in M^\times$  with  $a \equiv 1 \pmod{\times \mathfrak{pc}}$ , and continuous  $p$ -adic Hecke characters

$$\psi_{\mathfrak{P}} : W(\mathfrak{c}) \rightarrow \overline{\mathbb{Q}}_p^\times$$

such that  $\psi_{\mathfrak{P}}(w) = w^{\kappa-1}$  for all  $w \in 1 + \mathfrak{p}\mathcal{O}_{\mathfrak{p}} \cong \Gamma$ . (Here  $\mathfrak{P}$  denotes the prime in  $\overline{\mathbb{Q}}$  above  $p$  induced by  $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ .) The relation  $\psi_{\mathfrak{P}}|_{I(\mathfrak{pc})} = \psi$  holds.

Note that the decomposition  $W(\mathfrak{c}) = W \times \Delta_{\mathfrak{c}}$  induces an isomorphism

$$R[[W(\mathfrak{c})]] = R[[W]] \otimes_R R[\Delta_{\mathfrak{c}}],$$

and that, since the cardinality of  $\Delta_{\mathfrak{c}}$  is prime to  $p$ , every primitive  $R$ -valued character of  $\Delta_{\mathfrak{c}}$  determines a unique irreducible component of  $R[[W(\mathfrak{c})]]$ , isomorphic to  $R[[W]]$ , in which  $\Delta_{\mathfrak{c}}$  acts via  $\chi$ . We denote it by  $R[[W]]^{(\chi)}$ .

**Theorem 1.8** ([H86b] Prop. 2.3). *Fix a  $p$ -adic Hecke character  $\lambda : I(\mathfrak{p}) \rightarrow \overline{\mathbb{Q}}_p^\times$  such that  $\lambda(a\mathcal{O}_M) = a$  for all  $a \in M^\times$  with  $a \equiv 1 \pmod{\times \mathfrak{p}}$ , and assume that  $R$  contains all the values of  $\lambda$ . Then for each primitive character  $\chi : \Delta_{\mathfrak{c}} \rightarrow R^\times$ , there is a unique primitive local ring  $\mathbb{I}_{\mathcal{L}}$  of  $\mathbb{T}_N^o \otimes_{\Lambda_R} \mathcal{L}_R$  characterized by the following properties:*

- The algebra homomorphism (14) induces an isomorphism:

$$\mathbb{I}_{\mathcal{L}} \cong R[[W]]^{(\chi)} \otimes_{\Lambda_R} \mathcal{L}_R;$$

- For every finite character  $\varepsilon_r : W \rightarrow \overline{\mathbb{Q}}_p^\times$  of exact order  $p^r$ , and every integer  $k \geq 1$ , the theta series

$$\theta(\chi\lambda^{k-1}\varepsilon_r) = \sum_{\mathfrak{a} \in I(\mathfrak{pc})} \chi(\mathfrak{a})\lambda(\mathfrak{a})^{k-1}\varepsilon_r(\mathfrak{a})q^{\text{Norm}(\mathfrak{a})},$$

is a cusp form in  $S_k^{\text{ord}}(Np^r)$  which belongs to  $\mathbb{I}_{\mathcal{L}}$ .

One can give an explicit description of the Galois representation attached to local components  $\mathbb{I}_{\mathcal{L}}$  of the Hecke algebra coming from imaginary quadratic fields. Let the notations and assumptions be as in Theorem 1.8. Consider the character

$$\begin{aligned} \Psi : W(\mathfrak{c}) &\rightarrow R[[W]]^\times \\ w = (w_o, w_t) &\mapsto \chi(w_t)\lambda(w_o)^{-1} \cdot [w_o], \end{aligned}$$

where  $[w_o]$  is the image of  $w_o$  in  $R[[W]]$  under the tautological character  $W \rightarrow R[[W]]^\times$ . For each integer  $k \geq 1$ , denote by  $\varphi_{k,\varepsilon_r}$  the algebra homomorphism  $R[[W]] \rightarrow \overline{\mathbb{Q}}_p^\times$  induced by  $w_o \mapsto \varepsilon_r(w_o)\lambda(w_o)^k$ , and let  $P_{k,\varepsilon_r}$  be the height one prime ideal of  $R[[W]]$  given by  $\text{Ker}(\varphi_{k,\varepsilon_r})$ . We put  $\psi_{k,\varepsilon_r}$  to denote the Hecke character  $\chi\lambda^{k-1}\varepsilon_r$ , although it is clear that it actually depends on  $\chi$ , as  $\Psi$  does. Note that by construction we have a commutative diagram

$$\begin{array}{ccc} W(\mathfrak{c}) & \xrightarrow{\Psi} & R[[W]]^\times \\ & \searrow \psi_{k,\varepsilon_r} & \downarrow \varphi_{k,\varepsilon_r} \\ & & \overline{\mathbb{Q}}_p^\times \end{array}$$

The height one prime  $P_{k,\varepsilon_r}$  of  $R[[W]]$  lies above the arithmetic prime  $P_{k,\varepsilon'_r}$  of  $\Lambda_R$ , where  $\varepsilon'_r = \varepsilon_r|_\Gamma$ , and we see that the  $\Lambda$ -adic character  $\Psi$  specializes to every  $\psi_{k,\varepsilon_r}$ , and that these arithmetic specializations in turn uniquely determine  $\Psi$ .

Recall that by Class Field Theory,  $W(\mathfrak{c})$  is naturally identified with a quotient of  $G_M^{\text{ab}}$ , and therefore  $\Psi$  may be regarded as a Galois character on  $G_M$ .

**Proposition 1.9.** *Let  $\mathbb{I}_{\mathcal{L}}$  be a local component of  $\mathbb{T}_N^\circ \otimes_{\Lambda_R} \mathcal{L}_R$  attached to the imaginary quadratic field  $M$ . The Galois representation  $\rho_{\mathbb{I}} : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(\mathbb{I}_{\mathcal{L}})$  attached to  $\mathbb{I}$  by Theorem 1.5 is given by*

$$\rho_{\mathbb{I}} = \text{Ind}_M^{\mathbb{Q}} \Psi.$$

Since the argument in the introduction explaining the splitting of the local Galois representation attached to an ordinary newform applies verbatim to the present situation, we obtain the following result:

**Corollary 1.10.** *If  $\mathcal{F}_{\mathbb{I}}$  is a primitive ordinary  $\Lambda$ -adic form of CM type, then its associated Galois representation  $\rho_{\mathbb{I}}$  splits at  $p$ .*

## 1.6 Greenberg's question for $\Lambda$ -adic forms

Let  $\mathcal{F} = \sum_{n=1}^{\infty} a(n; \mathcal{F})$  be a  $\Lambda$ -adic form attached to a primitive local component  $\mathbb{I}_{\mathcal{L}}$  of the field of fractions  $\mathbb{T}_N^\circ \otimes_{\Lambda_R} \mathcal{L}_R$  of the universal ordinary Hecke algebra  $\mathbb{T}_N^\circ$ . After Theorem 1.5 we know that there is attached to  $\mathcal{F}$  a continuous  $\Lambda$ -adic representation

$$\rho_{\mathcal{F}} : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(\mathbb{I}_{\mathcal{L}}),$$

and by Theorem 1.7, this representation becomes reducible when restricted to a decomposition group  $D_p$  at  $p$ :

$$\rho_{\mathcal{F}}|_{D_p} = \begin{pmatrix} \varepsilon_{\mathcal{F}} & a_{\mathcal{F}} \\ & \delta_{\mathcal{F}} \end{pmatrix}. \quad (15)$$

On the other hand, if  $\mathcal{F}$  is of CM type, Corollary 1.10 asserts that  $\rho_{\mathcal{F}}$  is actually semi-simple when restricted to  $D_p$ . The following result of Ghate and Vatsal answers in many cases the natural extension of Greenberg's question to the  $\Lambda$ -adic setting:

**Theorem 1.11** (Ghate-Vatsal). *Let  $p$  an odd prime and  $N > 0$  an integer with  $(N, p) = 1$ . Let  $\mathcal{F} = \mathcal{F}_{\mathbb{1}}$  be a  $\Lambda$ -adic form of tame level  $N$  attached to a local component  $\mathbb{1}$  with maximal ideal  $\mathfrak{m}$ , and denote by  $\rho_{\mathcal{F}}$  its associated Galois representation into  $\mathbf{GL}_2(\mathbb{1})$ . Define  $\bar{\rho}_{\mathcal{F}} := (\rho_{\mathcal{F}} \bmod \mathfrak{m})$  and suppose that the following properties are satisfied:*

- $\bar{\rho}_{\mathcal{F}}|_{D_p}$  has non-scalar semi-simplification;
- $\bar{\rho}_{\mathcal{F}}$  is absolutely irreducible when restricted to  $G_{\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})}$ .

Then the following are equivalent:

- (i)  $\rho_{\mathcal{F}}|_{D_p}$  splits;
- (ii)  $\mathcal{F}$  has infinitely many weight one classical specializations;
- (iii)  $\mathcal{F}$  has infinitely many weight one classical specializations of CM type;
- (iv)  $\mathcal{F}$  is of CM type.

**Remark 2.** There are explicit examples of  $\Lambda$ -adic forms  $\mathcal{F}$  which indicate that the second assumption is not necessary for the the properties (i) and (iv) being both satisfied by  $\mathcal{F}$ .

Recall that by Hida's Theorem 1.4 the specialization of the  $\Lambda$ -adic form  $\mathcal{F}$  at any arithmetic point of  $\mathbb{1}$  of weight *at least 2* is a classical cusp form, but that no information is given about the specialization at an arithmetic point of weight one. In fact, Mazur and Wiles first showed in [MW86] explicit examples of  $\Lambda$ -adic forms with weight one arithmetic specializations that do not give rise to the  $q$ -expansion of any classical cusp form.

We now give a sketch of the proof of Theorem 1.11. Suppose that  $\mathcal{F}$  has character  $\chi$  modulo  $Np$ . By (the proof of) Corollary 1.6,  $\det(\rho_{\mathcal{F}})$  specializes at  $P_{k, \varepsilon_r}$  to  $\chi_{\varepsilon_r} \omega^{-k} \varepsilon_{\text{cyc}}^{k-1}$ . Thus if  $\rho_{\mathcal{F}}$  splits at  $p$ , by Theorem 1.7 we see that

$$P_{1, \varepsilon_r}(\rho_{\mathcal{F}})|_{I_p} \cong \begin{pmatrix} \psi_{\varepsilon_r} \omega^{-1} & \\ & 1 \end{pmatrix} \quad (16)$$

The finite order Galois character  $\chi_{\varepsilon_r} \omega^{-1}$  has conductor divisible by  $p^{r+1}$  for each integer  $r \geq 1$ ; if for varying  $r$  infinitely many of the representations  $P_{1, \varepsilon_r}(\rho_{\mathcal{F}})$  are isomorphic to the  $p$ -adic Galois representation attached by Deligne-Serre to a weight one cusp form, it follows from this and the fact that

$$P_{1, \varepsilon_r}(\rho_{\mathcal{F}}) \cong \rho_{P_{1, \varepsilon_r}(\mathcal{F})}$$

that  $\mathcal{F}$  has infinitely many weight one classical specializations. More precisely, this is indeed the case, since the two assumptions made on  $\bar{\rho}_{\mathcal{F}}$  allows us to directly invoke the following result of Buzzard [Bu03]:

**Theorem 1.12** (Buzzard). *Let  $\rho : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(\mathcal{O})$  be a continuous representation unramified outside a finite set of primes, with  $\mathcal{O}$  being the ring of integers of a finite extension of  $\mathbb{Q}_p$  with maximal ideal  $\mathfrak{m}_{\mathcal{O}}$ . Suppose that*

$$\rho|_{D_p} \cong \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$$

for two continuous characters  $\alpha, \beta : D_p \rightarrow \mathcal{O}^{\times}$  with finite image by the inertia group  $I_p \subset D_p$  at  $p$ , and assume that the following conditions are satisfied:

- $(\alpha \bmod \mathfrak{m}_{\mathcal{O}}) \neq (\beta \bmod \mathfrak{m}_{\mathcal{O}})$ ;
- $(\rho \bmod \mathfrak{m}_{\mathcal{O}})$  is modular;
- $(\rho \bmod \mathfrak{m}_{\mathcal{O}})$  is absolutely irreducible when restricted to  $G_{\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})}$ .

Then there exists a newform  $f$  of weight one and an embedding  $\iota : \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$  such that  $\rho_f \cong \iota \circ \rho$ , where  $\rho_f : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(\mathbb{C})$  is the complex representation attached to  $f$  by Deligne and Serre.

Thus we know that (ii) follows from (i). Now suppose that (ii) holds, and fix an embedding  $\iota : \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$ . Then  $\mathcal{F}$  has infinitely many classical weight one specializations whose Deligne-Serre representation has image a finite subgroup of  $\mathbf{PGL}_2(\mathbb{C})$  obtained from the composition of the reduction of  $\rho_{\mathcal{F}}$  modulo  $P_{1,\varepsilon_r}$  with the composite map

$$\pi \circ \iota : \mathbf{GL}_2(\overline{\mathbb{Q}}_p) \rightarrow \mathbf{GL}_2(\mathbb{C}) \rightarrow \mathbf{PGL}_2(\mathbb{C}).$$

Any finite subgroup of  $\mathbf{PGL}_2(\mathbb{C})$  is well-known to be either cyclic, dihedral or one of  $A_4$ ,  $A_5$  or  $S_4$ . One immediately rules out the first possibility due to irreducibility, and a previous observation about the conductor of the specialization  $P_{1,\varepsilon_r}(\rho_{\mathcal{F}})$  allows us to exclude the alternating and symmetric possibilities for  $r$  sufficiently large, noting that at least the image of  $I_p$  by any of the specialized representations has trivial intersection with the scalar matrices, and therefore injects into  $\mathbf{PGL}_2(\mathbb{C})$ . Thus for  $r$  large enough, all possibilities for the finite group  $\mathrm{Im}((\pi \circ \iota)(P_{1,\varepsilon_r}(\rho_{\mathcal{F}}))) \subset \mathbf{PGL}_2(\mathbb{C})$  but the dihedral one, which can actually hold, are precluded. Therefore the projectivization  $\pi \circ \iota(\rho_{f_{\varepsilon_r}}) : G_{\mathbb{Q}} \rightarrow \mathbf{PGL}_2(\mathbb{C})$  is dihedral for infinitely many weight one specializations  $f_{\varepsilon_r} = P_{1,\varepsilon_r}(\mathcal{F})$  of  $\mathcal{F}$ . It follows that

$$\rho_{f_{\varepsilon_r}} = \mathrm{Ind}_K^{\mathbb{Q}} \phi_{\varepsilon_r}$$

for a Hecke character  $\phi_{\varepsilon_r}$  of finite order of the quadratic field  $K$  cut out by  $(\pi \circ \iota(\rho_{f_{\varepsilon_r}}))^{-1}(\langle c \rangle)$  for any element  $c$  of order two in the image of  $\pi \circ \iota(\rho_{f_{\varepsilon_r}})$ . We may (and do) fix a quadratic field  $K$  which appears for infinitely many weight one specializations, since by the conductor formula  $\mathrm{disc}(K)$  must divide  $Np^r$ , and therefore for a fixed  $N$  and varying  $r > 0$  there are only finitely many possibilities for  $K$ . The ordinarity of  $\mathcal{F}$  forces  $p$  to split in  $K$ , say

$p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$ , where  $\bar{\mathfrak{p}} = \sigma\mathfrak{p}$  for the non-trivial automorphism  $\sigma \in G_{\mathbb{Q}} \setminus G_K$ , and  $\phi$  to ramify at exactly one of  $\mathfrak{p}$  or  $\bar{\mathfrak{p}}$ . We thus assume that infinitely many weight one specializations of  $\rho_{\mathcal{F}}$  are of the form

$$P_{1,\varepsilon_r}(\rho_{\mathcal{F}}) = \text{Ind}_K^{\mathbb{Q}} \phi_{\varepsilon_r},$$

for a our fixed quadratic field  $K$ , and certain (varying) Hecke character  $\phi_{\varepsilon_r}$  of  $K$  ramified at  $\mathfrak{p}$  and unramified at  $\bar{\mathfrak{p}}$ . Then  $K$  can not be *real* quadratic, since for a fixed  $N$  and varying  $r > 0$  a real quadratic field only admits *finitely* many Hecke characters of conductor dividing  $N\mathfrak{p}^r \infty_1 \infty_2$ , as it follows from the following:

**Lemma 1.13.** *Let  $F$  be a real quadratic field with infinite places  $\infty_1$  and  $\infty_2$ . Let  $\mathfrak{p}$  be a split prime of  $F$ , and  $\mathfrak{n}$  and integral ideal of  $F$  relatively prime to  $\mathfrak{p}$ . Then the ray class field of  $F$  modulo  $\mathfrak{n}\mathfrak{p}^r \infty_1 \infty_2$  has bounded order as  $r$  tends to infinity.*

Indeed, the exact sequence

$$1 \rightarrow (\mathcal{O}_F/\mathfrak{p}^r)^{\times}/U_F \rightarrow \text{Cl}_F(\mathfrak{p}^r) \rightarrow \text{Cl}_F \rightarrow 1,$$

where  $U_F$  denote the image of to unit group  $\mathcal{O}_F^{\times}$  under the natural projection  $\mathcal{O}_F^{\times} \rightarrow (\mathcal{O}_F/\mathfrak{p}^r)^{\times}$ , shows that

$$\#\text{Cl}_F(\mathfrak{p}^r) = [(\mathcal{O}_F/\mathfrak{p}^r)^{\times} : U_F] \cdot h_F,$$

for the class number  $h_F = \#\text{Cl}_F$  of  $F$ . Thus it suffices to bound the index appearing in the above formula, since we can easily reduce to prove the claim corresponding to a conductor of the form  $\mathfrak{p}^r$ . Let  $U_{\mathfrak{p},1} \subset U_{\mathfrak{p}}$  be the subgroup of local principal one-units in  $U_{\mathfrak{p}}$ , and denote by  $\bar{U}_{F,1}$  the closure of the image  $U_{F,1} \subset U_{\mathfrak{p},1}$  of the global units in  $\mathcal{O}_F$  mapping to  $U_{\mathfrak{p},1}$  under the natural inclusion map  $U_F \hookrightarrow U_{\mathfrak{p}}$ . It follows from Leopoldt's conjecture (which is trivial for real quadratic  $F$ ) that  $[U_{\mathfrak{p},1} : \bar{U}_{F,1}]$  is finite, and thus  $[U_{\mathfrak{p}} : \bar{U}_F]$  is finite as well, say bounded by  $M$ . Since  $U_{\mathfrak{p}} = \varprojlim_r (\mathcal{O}_F/\mathfrak{p}^r)^{\times}$ , we see that  $[(\mathcal{O}_F/\mathfrak{p}^r)^{\times} : U_F]$  is bounded by  $M$  independently of  $r$ , as needed.

We finally assume (iii) and show that (iv) follows. Thus we let  $\mathcal{F}$  be a primitive  $\Lambda$ -adic form of level  $N$  with infinitely many weight one specializations of CM type,

$$f_{\varepsilon_r} = P_{1,\varepsilon_r}(\mathcal{F}) = \sum \phi_{\varepsilon_r}(\mathfrak{a})q^{\text{Norm}(\mathfrak{a})},$$

coming from finite order Hecke characters  $\phi_{\varepsilon_r}$ , of a fixed imaginary quadratic field  $K$ , that are assumed to be ramified at  $\mathfrak{p}$  and unramified at  $\bar{\mathfrak{p}}$ , where  $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$ . We will construct a  $\Lambda$ -adic form  $\mathcal{F}'$  of CM type which interpolates infinitely many of these  $f_{\varepsilon_r}$ , and we will conclude by the following lemma:

**Lemma 1.14.** *Let  $\mathcal{F}$  and  $\mathcal{F}'$  be two  $\Lambda$ -adic forms of level  $N$  such that*

$$P_{k,\varepsilon_r}(\mathcal{F}) = P_{k,\varepsilon_r}(\mathcal{F}') \quad \text{for infinitely many } P_{k,\varepsilon_r} \text{ with } k, r \geq 1.$$

*Then  $\mathcal{F}$  and  $\mathcal{F}'$  are Galois conjugates.*

*Proof.* Denote by  $\mathbb{T}_N^o$  the universal ordinary Hecke algebra of tame level  $N$  constructed by Hida. The  $\Lambda$ -adic form  $\mathcal{F}$  corresponds to an algebra homomorphism  $\lambda_{\mathcal{F}} : \mathbb{T}_N^o \rightarrow \tilde{\mathbb{I}}$ , with  $\tilde{\mathbb{I}}$  being the normalization of a local component  $\mathbb{I}$  of  $\mathbb{T}_N^o$  in its fraction field. We know that  $\tilde{\mathbb{I}}$  is of Krull dimension 2 and that it corresponds to the localization of  $\mathbb{T}_N^o$  at a minimal prime  $\mathfrak{p}$ . Absolutely analogous assertions also hold for  $\mathcal{F}'$  replacing  $\mathcal{F}$ . By hypothesis, there exist infinitely many height one primes  $P = \text{Ker}(\varphi : \tilde{\mathbb{I}} \rightarrow \overline{\mathbb{Q}}_p)$ ,  $P' = \text{Ker}(\varphi' : \tilde{\mathbb{I}}' \rightarrow \overline{\mathbb{Q}}_p)$  such that

$$\lambda_{\mathcal{F}} \circ \varphi = \lambda_{\mathcal{F}'} \circ \varphi' : \mathbb{T}_N^o \rightarrow \overline{\mathbb{Q}}_p,$$

thus corresponding to infinitely many height one primes of  $\mathbb{T}_N^o$  containing both  $\mathfrak{p}$  and  $\mathfrak{p}'$ . If  $\mathfrak{p} \neq \mathfrak{p}'$ , such primes of  $\mathbb{T}_N^o$  correspond bijectively minimal primes of  $\mathbb{T}_N^o/(\mathfrak{p} + \mathfrak{p}')$ , but these are finite in number. Thus  $\mathfrak{p} = \mathfrak{p}'$ , and it follows that  $\mathcal{F}$  and  $\mathcal{F}'$  are Galois conjugates.  $\square$

This lemma allows us to conclude that (iv) holds one  $\mathcal{F}'$  is constructed, since it is clear that if some conjugate of  $\mathcal{F}$  is of CM type, then  $\mathcal{F}$  itself is of CM type.

We proceed now to the construction of the above mentioned  $\mathcal{F}'$ . Write  $\phi$  for some  $\phi_{\varepsilon_r}$  as before. Since there are only a finite number of ideals in  $K$  with norm dividing  $N$ , we may assume that all such  $\phi$  have conductor a fixed ideal  $\mathfrak{c}$ , and therefore

$$\phi = (\phi_w, \phi_t) : W(\mathfrak{c}) = W \times \Delta_{\mathfrak{c}} \rightarrow \overline{\mathbb{Q}}_p^{\times}.$$

The component  $\phi_t$  has order prime to  $p$ , and since the reduction of all  $\theta(\phi)$  are the same, it does not depend on  $\phi$ . On the other hand, since  $(p, N) = 1$ , for each integral ideal  $\mathfrak{q}$  of  $K$  dividing  $N$ ,  $\phi_w|_{I_{\mathfrak{q}}}$  cuts out a finite and tamely ramified  $p$ -extension of  $K$ , and there are only finitely many possibilities for it; we therefore assume that  $\phi_w|_{I_{\mathfrak{q}}}$  does not depend on  $\phi$  neither. Fixing one such  $\phi = \phi_o$ , and letting  $\phi' = \phi\phi_o^{-1}$  for each  $\phi$ , we restrict first our attention to characters

$$\phi' = (\phi'_w, \phi'_t) : W(1) = W \times \Delta_1 \rightarrow \overline{\mathbb{Q}}_p^{\times}$$

of  $p$ -power order. By a previous reasoning, we assume now that  $\phi'_t$  does not depend on  $\phi'$ .

Let  $\Lambda$  be the completed group ring  $\mathbb{Z}_p[[W]]$ , identified with the power series ring  $\mathbb{Z}_p[[Y]]$  via  $w_o = 1 + Y$  for a fixed choice of topological generator  $w_o \in W$ , and consider the character  $G_K \rightarrow \Lambda^{\times}$  obtained by composition of the Artin map  $[\cdot, K_{\mathfrak{p}^{\infty}}/K] : G_K \rightarrow W(1)$  with the tautological character  $W(1) \rightarrow W \rightarrow \Lambda^{\times}$ . Then  $\Phi$  factors through  $\text{Gal}(K\mathbb{Q}_{\infty}/K)$  for the cyclotomic  $\mathbb{Z}_p$ -extension  $K\mathbb{Q}_{\infty}$  of  $K$ , and the image by  $\Phi$  of the inertia group  $I_{\mathfrak{p}}$  is a finite index closed subgroup  $\Gamma_{\mathbb{Q}} \subset \Gamma$  which is identified with  $\text{Gal}(\mathbb{Q}_{\infty}/\mathbb{Q})$ . Write  $p^{\gamma} = [\Gamma : \Gamma_{\mathbb{Q}}]$ , and put  $\Lambda_{\mathbb{Q}} = \mathbb{Z}_p[[\Gamma_{\mathbb{Q}}]] \cong \mathbb{Z}_p[[X]]$ , where  $u_o = 1 + X$  for a fixed choice of topological generator  $u_o \in \Gamma_{\mathbb{Q}}$ . Then  $\Lambda$  can be regarded as a  $\Lambda_{\mathbb{Q}}$ -algebra. Since  $\phi' = \phi'_{\varepsilon_r}$  is uniquely determined by the image  $\zeta_{\varepsilon_r} = \phi'_{\varepsilon_r}(w) \in \mu_{p^{\infty}}(\overline{\mathbb{Q}}_p)$ , we see that the character  $\Phi_o$  simultaneously interpolates all  $\phi'_{\varepsilon_r}$  under the specialization  $Y = \zeta_{\varepsilon_r} - 1$ .

We now turn to the general case  $\phi_{\varepsilon_r}$ . Let  $f_o$  be a CM form, weight one specialization of  $\mathcal{F}$ , associated to the finite order Hecke character  $\phi_o$  of  $K$  fixed previously, and write the character of  $f_o$  as  $\psi\omega^{-1}\chi_{\nu_o}$ , for  $\nu_o = \varepsilon_1(u_o) \in \mu_{p^\infty}(\overline{\mathbb{Q}_p})$ . Let  $\mathcal{O}$  be the ring of integers of a finite extension of  $\mathbb{Q}_p$  containing the values of  $\phi_o$ ,  $\beta_T$ ,  $\zeta_o := \nu_o^{1/p^\gamma}$  and  $t_o := u_o^{1/p^\gamma}$ , and let  $L$  be the power series ring  $\mathcal{O}[[Y]]$ . Define  $\Phi := \phi_o\beta_T\tau(\Phi_o)$ , where  $\tau$  is the automorphism of  $L$  determined by the change of variables  $1 + Y \mapsto \zeta_o^{-1}t_o^{-1}(1 + Y)$ . Now we see that  $\Phi|_{Y=\zeta_{\varepsilon_r}t_o-1}$  interpolates the characters  $\phi_{\varepsilon_r}$  attached to the weight one CM forms  $f_{\varepsilon_r} = P_{1,\varepsilon_r}(\mathcal{F})$  of character  $\psi\omega^{-1}\chi_{\nu_{\varepsilon_r}}$ , where  $\zeta_{\varepsilon_r}^{p^\gamma} = \nu_{\varepsilon_r}$ . Since  $(Y - (\zeta_{\varepsilon_r}t_o - 1)) \subset L$  clearly lies over  $(X - (\nu_o u_o - 1)) \subset \Lambda$ , we deduce that the formal  $q$ -expansion

$$\mathcal{F}' := \sum \Phi(\mathfrak{a})q^{\text{Norm}(\mathfrak{a})}$$

is a  $\Lambda$ -adic CM form interpolating infinitely weight one CM specializations of  $\mathcal{F}$ , thus concluding that (iv) holds after Lemma 1.14. This finishes the proof of Theorem 1.11.

One can use Theorem 1.11 to study Greenberg's question for  $p$ -stabilized newforms, seen as classical specializations of  $\Lambda$ -adic forms. Recall that the expectation (4) is that a  $p$ -ordinary newform  $f$  has locally split at  $p$  Galois representation  $\rho_f$  exactly when  $f$  has CM. A descend argument from the  $\Lambda$ -adic setting allows to conclude that this is indeed the case for all the  $p$ -stabilized newforms obtained as classical specializations of a  $\Lambda$ -adic form, except possibly for finitely many *unspecified* classical points. In fact, Ghate and Vatsal deduced from their Theorem 1.11 a slightly stronger result:

**Theorem 1.15.** *Let  $p$  be an odd prime number and  $N > 0$  an integer with  $(N, p) = 1$ . Let  $\mathcal{S}(N)$  denote the set of all ordinary newforms  $f$  of weight  $k \geq 2$  and level  $Np^r$  for some  $r \geq 0$  satisfying the following two properties:*

- $f$  is  $p$ -distinguished; and
- $\bar{\rho}_f$  is absolutely irreducible when restricted to  $G_{\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})}$ .

*Then, for all but finitely many  $f \in \mathcal{S}(N)$ , if  $\rho_f|_{D_p}$  splits, then  $f$  has CM.*

*Proof.* Each  $f \in \mathcal{S}(N)$  is an arithmetic specialization of a unique  $\Lambda$ -adic form  $\mathcal{F} \in \mathbb{I}[[q]]$  of level  $N$ , with  $\mathbb{I}$  being a finite flat extension of  $\Lambda$  with fraction field  $\mathbb{I}_{\mathcal{L}}$ . Therefore, if we denote by  $\mathcal{S}_{\mathcal{F}}$  the set of those ordinary newforms  $f$  that are an arithmetic specialization of the  $\Lambda$ -adic forms  $\mathcal{F}$ , then it suffices to restrict our attention to  $f \in \mathcal{S}_{\mathcal{F}}$  for a fixed primitive  $\mathcal{F}$  of level  $N$ .

Fix such a  $\Lambda$ -adic form  $\mathcal{F}$ , and suppose that  $f \in \mathcal{S}_{\mathcal{F}}$  has no CM. Then  $\mathcal{F}$  itself has no CM. Furthermore, by the hypothesis in Theorem 1.15,  $\mathcal{F}$  verifies the analogous assumptions in Theorem 1.11, and therefore  $\rho_{\mathcal{F}}|_{D_p}$  is non-split. Then  $\rho_f$  can possibly split at  $p$  only for finitely many  $f \in \mathcal{S}_{\mathcal{F}}$ . Indeed, the decomposition

$$\tilde{\mathbb{I}}^\times = \boldsymbol{\mu}_{q-1} \times 1 + \mathfrak{m},$$

for the maximal ideal  $\mathfrak{m}$  of  $\tilde{\mathbb{I}}$ , and  $\mu_{q-1} \cong \mathbb{F}^\times$  for the residue field  $\mathbb{F}$  of  $\tilde{\mathbb{I}}$ , allows one to deduce that the characters  $\delta_{\mathcal{F}}$  and  $\varepsilon_{\mathcal{F}}$  appearing in

$$\rho_{\mathcal{F}}|_{D_p} = \begin{pmatrix} \delta_{\mathcal{F}} & a_{\mathcal{F}} \\ & \varepsilon_{\mathcal{F}} \end{pmatrix}$$

factor through  $\text{Gal}(E/\mathbb{Q}_p)$ , where  $E$  is the composite of  $E_t$ , the union of the finitely many tamely ramified abelian extensions of  $\mathbb{Q}_p$  of order dividing  $q-1$ , and  $E_w$ , the maximal abelian pro- $p$ -extension of  $\mathbb{Q}_p$ . Thus if  $H$  denotes the group  $\text{Gal}(\overline{\mathbb{Q}_p}/E) \subset D_p$ , then

$$\rho_{\mathcal{F}}|_H \cong \begin{pmatrix} 1 & \lambda \\ & 1 \end{pmatrix},$$

for a certain continuous homomorphism  $\lambda : H \rightarrow \tilde{\mathbb{I}}^\times$ . Since  $\rho_{\mathcal{F}}|_{D_p}$  is non-split, the cohomology class  $[\xi_{\mathcal{F}}] \in H^1(D_p, \mathbb{I}_{\mathcal{L}}(\delta_{\mathcal{F}}\varepsilon_{\mathcal{F}}^{-1}))$  is nonzero. Write  $\Delta = \text{Gal}(E_t/\mathbb{Q}_p)$  and  $\Gamma = \text{Gal}(E_w/\mathbb{Q}_p) \cong \mathbb{Z}_p^2$ , so that  $\text{Gal}(E/\mathbb{Q}_p) \cong \Delta \times \Gamma$ . Consider the inflation-restriction exact sequence

$$0 \rightarrow H^1(\text{Gal}(E/\mathbb{Q}_p), \mathbb{I}_{\mathcal{L}}(\delta_{\mathcal{F}}\varepsilon_{\mathcal{F}}^{-1})^H) \rightarrow H^1(D_p, \mathbb{I}_{\mathcal{L}}(\delta_{\mathcal{F}}\varepsilon_{\mathcal{F}}^{-1})) \rightarrow H^1(H, \mathbb{I}_{\mathcal{L}}(\delta_{\mathcal{F}}\varepsilon_{\mathcal{F}}^{-1})). \quad (17)$$

The decomposition  $\text{Gal}(E/\mathbb{Q}_p) = \Delta \times \Gamma$  gives rise to the decomposition

$$H^1(\text{Gal}(E/\mathbb{Q}_p), \mathbb{I}_{\mathcal{L}}(\delta_{\mathcal{F}}\varepsilon_{\mathcal{F}}^{-1})^H) = H^1(\Delta, \mathbb{I}_{\mathcal{L}}(\delta_{\mathcal{F}}\varepsilon_{\mathcal{F}}^{-1})^\Delta) \times H^1(\Gamma, \mathbb{I}_{\mathcal{L}}(\delta_{\mathcal{F}}\varepsilon_{\mathcal{F}}^{-1})^\Gamma). \quad (18)$$

The first group in the right of (18) vanishes, and the second also does, unless  $\Delta$  acts trivially on  $\mathbb{I}_{\mathcal{L}}(\delta_{\mathcal{F}}\varepsilon_{\mathcal{F}}^{-1})$ ; we therefore assume that  $\mathbb{I}_{\mathcal{L}}(\delta_{\mathcal{F}}\varepsilon_{\mathcal{F}}^{-1})^\Delta = \mathbb{I}_{\mathcal{L}}(\delta_{\mathcal{F}}\varepsilon_{\mathcal{F}}^{-1})$  and use the decomposition  $\Gamma = \Gamma_1 \times \Gamma_2$  with  $\Gamma_i \cong \mathbb{Z}_p$  to decompose further

$$H^1(\Gamma, \mathbb{I}_{\mathcal{L}}(\delta_{\mathcal{F}}\varepsilon_{\mathcal{F}}^{-1})) = H^1(\Gamma_1, \mathbb{I}_{\mathcal{L}}(\delta_{\mathcal{F}}\varepsilon_{\mathcal{F}}^{-1})^{\Gamma_2}) \times H^1(\Gamma_2, \mathbb{I}_{\mathcal{L}}(\delta_{\mathcal{F}}\varepsilon_{\mathcal{F}}^{-1})^{\Gamma_1}). \quad (19)$$

Since  $p$  is odd, the tame part of the characters  $\delta_{\mathcal{F}}$  and  $\varepsilon_{\mathcal{F}}$  must reduce modulo  $\mathfrak{m}_{\mathcal{O}}$  to the reduction of the tame part of the characters  $\delta_f$  and  $\varepsilon_f$  corresponding to any of the specializations  $f$  of  $\mathcal{F}$ . Using the  $p$ -distinguishedness hypothesis we see that  $\delta_t \neq \varepsilon_t \pmod{\mathfrak{m}}$ , and we may thus assume that  $\mathbb{I}_{\mathcal{L}}(\delta_{\mathcal{F}}\varepsilon_{\mathcal{F}}^{-1})^{\Gamma_2} = 0$ , so that the first cohomology group in the right of (19) vanishes. If  $\Gamma_1$  also acts nontrivially on  $\mathbb{I}_{\mathcal{L}}(\delta_{\mathcal{F}}\varepsilon_{\mathcal{F}}^{-1})$  we conclude by (18) and (19) that

$$H^1(\text{Gal}(E/\mathbb{Q}_p), \mathbb{I}_{\mathcal{L}}(\delta_{\mathcal{F}}\varepsilon_{\mathcal{F}}^{-1})^H) = 0; \quad (20)$$

otherwise we have  $H^1(\Gamma_2, \mathbb{I}_{\mathcal{L}}(\delta_{\mathcal{F}}\varepsilon_{\mathcal{F}}^{-1})) \cong \mathbb{I}_{\mathcal{L}}/(\delta_{\mathcal{F}}\varepsilon_{\mathcal{F}}^{-1}(\gamma) - 1)\mathbb{I}_{\mathcal{L}} = 0$ , where the isomorphism is induced by the evaluation of cocycles at the topological generator  $\gamma \in \Gamma_2$ , and we conclude that (20) holds anyway, and therefore that the restriction map in (17) is injective. It follows that  $\rho_{\mathcal{F}}|_H$  is non-split, and therefore  $\lambda$  cannot be identically zero. Finally, if  $I$  denotes the nonzero ideal of  $\tilde{\mathbb{I}}$  generated by the image of  $\lambda$ , then  $I$  is contained in only finitely many height one primes  $P$  of  $\tilde{\mathbb{I}}$ , and it is clear that if  $f = P(f_{\mathcal{F}}) \in \mathcal{S}_{\mathcal{F}}$ , then  $\rho_f|_H$  splits if and only if  $I \subseteq P$ . Therefore,  $\rho_f|_H$  is non-split for all but finitely many  $f \in \mathcal{S}_{\mathcal{F}}$ , and in particular the same assertion holds for  $\rho_f|_{D_p}$ .  $\square$

## 2 The $R = \mathbb{T}$ philosophy

### 2.1 Modular Galois representations

It follows from Theorem 1.2 that the universal ordinary Hecke algebra  $\mathbb{T}_N^o$  can be decomposed

$$\mathbb{T}_N^o = \bigoplus_{\mathfrak{m}} \mathbb{T}_{\mathfrak{m}}$$

as the direct sum of local rings finite free over  $\Lambda$ , with  $\mathfrak{m}$  running over the finite set of maximal ideals in  $\mathbb{T}_N^o$ . For each  $\mathbb{T}_{\mathfrak{m}}$ , the residue ring  $\mathbb{T}_{\mathfrak{m}}/P_{k,\varepsilon}\mathbb{T}_{\mathfrak{m}}$  is a local ring direct summand of the Hecke algebra  $\mathbb{T}_{Np^r}^{(k),\text{ord}}(\varepsilon)$ , and if  $f \in S_k^{\text{ord}}(\Delta_r, \varepsilon)$  is a normalized cuspidal eigenform attached to  $\mathbb{T}_{\mathfrak{m}}^{(k)} := \mathbb{T}_{\mathfrak{m}}/P_{k,\varepsilon}\mathbb{T}_{\mathfrak{m}}$ , i.e. an algebra homomorphism

$$\lambda_f : \mathbb{T}_{Np^r}^{(k),\text{ord}}(\varepsilon) \rightarrow R \subset \overline{\mathbb{Q}}_p,$$

with denoting the ring of integers of a finite extension  $K/\mathbb{Q}_p$  containing the Hecke eigenvalues of  $f$ , factoring through  $\mathbb{T}_{\mathfrak{m}}^{(k)}$ , the composition with the reduction map  $R \rightarrow \kappa = R/\pi R$ , where  $\pi$  denotes a uniformizer of  $R$ , induces a field embedding

$$\begin{aligned} \mathbb{T}_{\mathfrak{m}}^{(k)}/\mathfrak{m}\mathbb{T}_{\mathfrak{m}}^{(k)} &\hookrightarrow \kappa \\ T \bmod \mathfrak{m} &\mapsto \lambda_f(T) \bmod \pi, \end{aligned}$$

with  $\mathfrak{m}$  denoting here the maximal ideal in  $\mathbb{T}_{\mathfrak{m}}^{(k)}$ . Therefore, once fixed an algebraic closure  $\overline{\mathbb{F}}_p$  of the prime field with  $p$  elements and an embedding  $\kappa \hookrightarrow \overline{\mathbb{F}}_p$ , we see that for each pair of such forms  $f, f'$  attached to  $\mathbb{T}_{\mathfrak{m}}^{(k)}$  either one of the following two conditions holds:

- $a_n(f') \equiv \sigma a_n(f) \pmod{\pi}$  for all  $n > 0$ , for a certain  $\sigma \in \text{Aut}(\overline{\mathbb{Q}}_p/K)$ ;
- $a_n(f') \equiv a_n(f) \pmod{\pi}$  for all  $n > 0$ ,

and further, that one can use each such  $f$  to attach a *residual representation*

$$\rho_{\mathfrak{m}} : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(\kappa)$$

unramified outside  $Np$  with semi-simplification uniquely determined by the requirement that

$$\text{tr} \rho_{\mathfrak{m}}(\text{Frob}_{\ell}) = T_{\ell} \bmod \mathfrak{m} \tag{21}$$

for every prime  $\ell \nmid Np$ . If  $\rho_{\mathfrak{m}}$  turns out to irreducible, then  $\rho_{\mathfrak{m}}$  itself is uniquely determined by (21). If we assume  $\text{char}(\kappa) \neq 2$ , then  $\rho_{\mathfrak{m}}$  is automatically *absolutely* irreducible, if irreducible, since it is odd. This is highly desirable due to the following result.

**Theorem 2.1.** *Let  $R_{\mathfrak{m}}$  be a local component of  $\mathbb{T}_N^o$  with residue field  $\kappa$ , and denote by  $\rho_{\mathfrak{m}} : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(\kappa)$  a residual representation attached to  $\mathbb{T}_{\mathfrak{m}}$ . If  $\rho_{\mathfrak{m}}$  is absolutely irreducible, then:*

- The  $\Lambda$ -module  $\mathrm{Hom}_\Lambda(\mathbb{T}_m, \Lambda)$  is free over  $\mathbb{T}_m$ , and therefore  $\mathbb{T}_m$  is a Gorenstein ring;
- The Pontryagin dual  $J_\infty(\mathbb{T}_m)^\wedge = \mathrm{Hom}(J_\infty(\mathbb{T}_m), \mu_{p^\infty})$  is free of rank two over  $\mathbb{T}_m$ , and the  $G_\mathbb{Q}$ -action on  $J_\infty(\mathbb{T}_m)^\wedge$  gives rise to a Galois representation:

$$\rho^{\mathrm{mod}} : G_\mathbb{Q} \rightarrow \mathbf{GL}_2(\mathbb{T}_m)$$

which is unramified outside  $Np$ , and such that for every prime  $\ell \nmid Np$ ,

$$\det(\mathrm{Id} - \rho^{\mathrm{mod}}(\mathrm{Frob}_\ell)X) = 1 - T_\ell X + \ell^{-1} \langle \ell \rangle X^2.$$

Furthermore, the module of coinvariants of  $J_\infty(\mathbb{T}_m)^\wedge$  by any inertia group at  $p$  is free of rank one over  $R_m$ .

Since  $\mathbb{T}_m/\mathfrak{m}\mathbb{T}_m \cong \mathbb{T}_m^{(k)}/\mathfrak{m}\mathbb{T}_m^{(k)}$ , the above discussion allows us to note that the pair  $(\mathbb{T}_m, \rho^{\mathrm{mod}})$  consisting of the local Noetherian  $\Lambda$ -algebra  $\mathbb{T}_m$  and the degree 2 Galois representation  $\rho^{\mathrm{mod}}$  over  $\mathbb{T}_m$  satisfies the following universal property:

“for any  $p$ -stabilized ordinary cuspidal eigenform  $f \in S_k^{\mathrm{ord}}(\Delta_r, \varepsilon)$  whose associated Galois representation  $\rho_f : G_\mathbb{Q} \rightarrow \mathbf{GL}_2(\overline{\mathbb{Q}}_p)$  is such that

$$\rho_f \bmod \pi = \rho_m$$

as representation into  $\mathbf{GL}_2(\overline{\mathbb{F}}_p)$ , therefore corresponding to an algebra homomorphism  $\lambda_f : \mathbb{T}_m^{(k)} \rightarrow \overline{\mathbb{Q}}_p$ , there exists a unique algebra homomorphism

$$\varphi_f : \mathbb{T}_m \rightarrow \mathbb{T}_m^{(k)}$$

such that  $\rho_f = (\lambda_f \circ \varphi_f) \circ \rho^{\mathrm{mod}}$ .”

Considerations of this sort lead Mazur to develop his foundational work [Maz89] on the systematic study of deformation of Galois representations.

## 2.2 The theorem by Greenberg and Stevens

Let  $E/\mathbb{Q}$  be an elliptic curve of conductor  $Np$  with  $(N, p) = 1$ , and assume that  $p$  is a prime of split multiplicative reduction for  $E$ . Thus if  $f_E = \sum a_n(E)q^n \in \mathbb{Z}[[q]]$  is the newform attached to  $E$ , then  $a_p(E) = 1$  and therefore  $f_E$  is ordinary at  $p$ . Under these circumstances, Mazur and Swinnerton-Dyer attached to  $f_E$  a  $p$ -adic analytic  $L$ -function which we denote by  $L_p(f_E, s)$ . It is defined in terms of a (nontrivial) power series  $\mathcal{L}(f_E, T) \in \Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  by means of the formula

$$L_p(f_E, s) := \mathcal{L}(f_E, \kappa(\gamma)^{1-s} - 1),$$

for any choice of topological generator  $\gamma_o \in \Gamma = 1 + p\mathbb{Z}_p$ , and where  $\kappa$  denotes the canonical isomorphism  $\text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p)) \cong \Gamma$  induced by the cyclotomic character  $\varepsilon_{\text{cyc}} : \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \rightarrow \mathbb{Z}_p^\times$ . Here  $s$  denotes a  $p$ -adic variable in  $\mathbb{Z}_p$ . For a character  $\eta \in \text{Hom}(\Gamma, \mu_{p^\infty})$ , denote by  $L(f_E, \eta, s)$  the  $L$ -function of  $f_E$  twisted by the Dirichlet character of  $p$ -power conductor associated to  $\eta$ . Then it is known that for the positive Néron period  $\Omega_E$  of  $E$  the special values  $L(f_E, \eta, 1)/\Omega_E$  for  $\eta$  running in  $\text{Hom}(\Gamma, \mu_{p^\infty})$  are all algebraic, and the above power series  $\mathcal{L}(f_E, T)$  is then characterized by the interpolation property

$$\mathcal{L}(f_E, \eta(\gamma_o) - 1) = \tau(\eta^{-1}) \cdot \alpha_p^{-m} \cdot \frac{L(f_E, \eta, 1)}{\Omega_E} \quad \text{for all } \eta \in \text{Hom}(\Gamma, \mu_{p^\infty}),$$

where  $\tau(\eta^{-1})$  denotes the Gauss sum of  $\eta$ , and the integer  $m \geq 1$  is such that  $\eta(\gamma_o)$  is a root of unit of exact order  $p^{m-1}$ . In particular, one has the evaluation formula

$$L_p(f_E, 1) = (1 - a_p(E)^{-1}) \cdot \frac{L(f_E, 1)}{\Omega_E}. \quad (22)$$

Since  $E$  is assumed to be split multiplicative, we have  $a_p(E) = 1$ , and (22) becomes the identity  $0 = 0$  due to the extra factor  $(1 - a_p(E)^{-1})$ . Despite this, one still hopes to recover the algebraic  $L$ -value (assuming it to be nonzero) in terms of the  $p$ -adic  $L$ -function  $L_p(f_E, s)$ . More precisely, Mazur, Tate and Teitelbaum conjectured the following relation between the orders of vanishing:

$$\text{ord}_{s=1} L_p(f_E, s) = 1 + \text{ord}_{z=1} L(f_E, z), \quad (23)$$

and they further gave a precise recipe for the “error term” – the so-called  $\mathcal{L}$ -invariant of  $E$  – between the arithmetical leading coefficients of their Taylor expansions in the rank zero case, i.e. the scalar  $\mathcal{L}(E) \in \mathbb{Q}_p$  for which the following equality holds:

$$\left. \frac{d}{ds} L_p(f_E, s) \right|_{s=1} = \mathcal{L}(E) \cdot \frac{L(f_E, 1)}{\Omega_E}. \quad (24)$$

It follows from Tate’s uniformization theory that one can write

$$E(\overline{\mathbb{Q}}_p) = \overline{\mathbb{Q}}_p^\times / q_E^{\mathbb{Z}} \quad (25)$$

for a certain Tate period  $q_E \in p\mathbb{Z}_p$ , and it was conjectured in [MTT86] that the expression

$$\frac{\log_p(q_E)}{\text{ord}_p(q_E)}$$

is precisely the error term  $\mathcal{L}(E)$  in (24). This was proved by Greenberg and Stevens [GS93], and an important point in the argument was a purely cohomological description of the  $\mathcal{L}$ -invariant  $\mathcal{L}(E)$  in terms of the isomorphism class of the Tate module  $V(E) = \text{Ta}_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  as a (non-trivial) extension of  $\mathbb{Q}_p$  by  $\mathbb{Q}_p(1)$ .

More generally, Greenberg proposed in [Gr94a] a similar cohomological definition of an  $\mathcal{L}$ -invariant attached to a wide class of  $p$ -adic ordinary representations  $V$ , which is conjectured by him to account for the error between the algebraic  $L$ -value and the leading term of the archimedean and  $p$ -adic  $L$ -functions attached to  $V$ , in the presence of an exceptional zero phenomenon. Unfortunately, the existence and even the precise interpolation properties that  $p$ -adic  $L$ -functions attached to more general  $p$ -adic (motivic) representations should satisfy is still largely conjectural.

A crucial point in the proof by Greenberg and Stevens (another equally important one being the construction of a suitable 2-variable  $p$ -adic  $L$ -function) is the formula

$$\mathcal{L}(E) = -2 \cdot \left. \frac{da_p(\kappa)}{d\kappa} \right|_{\kappa=2}, \quad (26)$$

where  $a_p(\kappa)$  is the  $p$ -adic analytic function of the  $p$ -adic variable  $\kappa \in U$ , for a certain open disc  $U \subset \mathbb{Z}_p$ , given by

$$a_p(\kappa) := a_{\mathbb{I}}(p; \gamma_o^{2-\kappa} - 1),$$

where  $\mathcal{F}_{\mathbb{I}} = \sum a_{\mathbb{I}}(n; X)q^n \in \mathbb{I}[[q]]$ , for  $\mathbb{I}$  a finite flat extension of  $\mathbb{Z}_p[[X]]$ , is the  $\Lambda$ -adic form attached to  $f_E$ , which is the specialization of  $\mathcal{F}_{\mathbb{I}}$  at  $X = 0$ . The proof of (26) consists in using the Galois representation attached to  $\mathcal{F}_{\mathbb{I}}$  by Theorem 1.5 to explicitly compute an annihilator under the local Tate pairing of the extension class in  $H^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$  defined by the Tate module of  $E$ . We next discuss this point in further detail.

Let  $z \in E[p^n]$ , so that  $z^{p^n} = q_E^{d(z)}$  for certain integer  $d(z)$  due to the uniformization (25). Conversely, given  $d \in \mathbb{Z}/p^n\mathbb{Z}$ , one can solve the previous equation for some  $z \in \overline{\mathbb{Q}_p}^\times$  which will be well defined modulo  $\mu_{p^n} = \mu_{p^n}(\overline{\mathbb{Q}_p})$ . Thus the association  $z \mapsto d(z)$  yields an exact sequence of  $D_p$ -modules

$$0 \rightarrow \mu_{p^n} \rightarrow E[p^n] \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow 0,$$

with  $D_p$  acting trivially on  $\mathbb{Z}/p^n\mathbb{Z}$  and via the mod  $p^n$  cyclotomic character on  $\mu_{p^n}$ . Taking the projective limit along  $n$  and tensoring with  $\mathbb{Q}_p$ , the above sequence becomes

$$0 \rightarrow \mathrm{Ta}_p(\mu_{p^\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \rightarrow V_p(E) \rightarrow \mathbb{Q}_p \rightarrow 0, \quad (27)$$

where  $\mathrm{Ta}_p(\mu_{p^\infty}) \otimes \mathbb{Q}_p$  is identified with  $\mathbb{Q}_p(1)$  after a (non-canonical) choice of  $\mathbb{Z}_p$ -basis of  $\mathrm{Ta}_p(\mu_{p^\infty})$ . Thus we see that  $V_p(E)$  is a  $\mathbb{Q}_p[D_p]$ -module extension of  $\mathbb{Q}_p$  by  $\mathbb{Q}_p(1)$ , and it therefore yields an extension class

$$\kappa(V_p(E)) \in \mathrm{Ext}_{\mathbb{Q}_p[D_p]}^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) = H^1(D_p, \mathbb{Q}_p(1))$$

given by the image of 1 under the coboundary map  $d_0 : H^0(D_p, \mathbb{Q}_p) = \mathbb{Q}_p \rightarrow H^1(D_p, \mathbb{Q}_p(1))$ .

On the other hand, it follows from Kummer theory that one has a canonical isomorphism

$$H^1(D_p, \mathbb{Q}_p(1)) \cong \varprojlim_n (\mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^{p^n}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Therefore, each element  $q \in \mathbb{Q}_p^\times$  determines a *Kummer cohomology class*

$$\kappa_{\text{Kum}}(q) \in H^1(D_p, \mathbb{Q}_p(1)),$$

explicitly given by the natural image under  $H^1(D_p, \mathbb{Z}_p(1)) \rightarrow H^1(D_p, \mathbb{Q}_p(1))$  of the cocycle  $\sigma \mapsto \{(q^{1/p^n})^{\sigma-1}\}_n$  for the choice of a compatible sequence  $\{q^{1/p^n}\}_n$  of  $p$ -power roots of  $q$ . In particular, one can associate to the Tate period  $q_E$  its Kummer class  $\kappa_{\text{Kum}}(q_E)$ .

**Lemma 2.2.** *The sequence (27) is non-split, and its extension class  $\kappa(V_p(E))$  coincides with the Kummer class  $\kappa_{\text{Kum}}(q_E)$ .*

*Proof.* The equality  $\kappa(V_p(E)) = \kappa_{\text{Kum}}(q_E)$  follows immediately from the previous description of  $\kappa_{\text{Kum}}(q)$ , since  $\delta_{0,n}(1) \in H^1(D_p, \mathbb{Z}/p^n\mathbb{Z}(1))$  for the degree zero coboundary map  $\delta_{0,n}$  of the exact sequence  $0 \rightarrow \mu_{p^n} \rightarrow E[p^n] \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow 0$  is given by  $\xi_n = (\sigma \mapsto (q_E^{1/p^n})^{\sigma-1})$ . For the first assertion, note that we may identify  $\xi = \{\xi_n\} \in \varprojlim_n H^1(D_p, \mathbb{Z}/p^n\mathbb{Z}(1))$  with its (pre-)image  $\{q_E \bmod (\mathbb{Q}_p^\times)^n\}_n$  under the Kummer isomorphism. Then we note that the valuation homomorphism  $v_p : \mathbb{Q}_p^\times \rightarrow \mathbb{Z}$  yields a well-defined mapping  $\mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^{p^n} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$ , and thus also a map

$$v : \varprojlim_n \mathbb{Q}_p / (\mathbb{Q}_p^\times)^{p^n} \rightarrow \mathbb{Z}_p.$$

Under the previous identification,  $v(\xi) = v_p(q_E)$ , and therefore  $\xi$  must be of infinite order in  $\mathbb{Z}_p$ , since  $q_E \in p\mathbb{Z}_p$ . Suppose now that the sequence (27) splits. Then there exists a  $D_p$ -stable subspace  $X$  of  $V_p(E)$  which is mapped isomorphically onto  $\mathbb{Q}_p$  under the projection in (27); the image of  $\text{Ta}_p(E) \cap X$  in  $\mathbb{Z}_p$  under this mapping is of the form  $p^N\mathbb{Z}_p$  for some  $N \geq 0$ , and it follows that  $p^N\xi = 0$ , contradicting the fact that  $\xi$  is of infinite order.  $\square$

Let  $\rho_{\mathbb{I}} : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(\mathbb{I}_{\mathcal{L}})$  be the Galois representation attached by Theorem 1.5 to the  $\Lambda$ -adic form  $\mathcal{F}_{\mathbb{I}} = \sum a_{\mathbb{I}}(n; X)q^n \in \mathbb{I}[[q]]$ , and denote by  $V = \mathbb{I}_{\mathcal{L}}^2$  a representation space for  $\rho_{\mathbb{I}}$ . Recall that Theorem 1.7 asserts the existence of an exact sequence of  $D_p$ -modules

$$0 \rightarrow \mathbb{I}_{\mathcal{L}}(\kappa\varepsilon_{\text{cyc}}^{-1}\nu^{-1}) \rightarrow \mathbb{I}_{\mathcal{L}}^2 \rightarrow \mathbb{I}_{\mathcal{L}}(\nu) \rightarrow 0, \quad (28)$$

where  $\nu : D_p \rightarrow \mathbb{I}_{\mathcal{L}}^\times$  is the unramified character mapping an arithmetic Frobenius  $\text{Frob}_p$  to  $a_{\mathbb{I}}(p; X)$ , and  $\kappa : D_p \rightarrow \mathbb{I}_{\mathcal{L}}^\times$  is such that  $\kappa \bmod P_{k, \varepsilon_r} = \varepsilon_r \langle \varepsilon_{\text{cyc}} \rangle^k$ . The representation  $\rho_{\mathbb{I}}$  can be descended to the integral closure  $\tilde{\mathbb{I}}$  of  $\mathbb{I}$  in  $\mathbb{I}_{\mathcal{L}}$ , and if  $P$  is a height one prime in  $\tilde{\mathbb{I}}$ , we have a canonical isomorphism  $\tilde{\mathbb{I}}_P \cong \mathbb{I}_P$ . Thus if  $P$  denotes the arithmetic prime in  $\tilde{\mathbb{I}}$  above

$P_2$  corresponding to  $f_E$ , we see that the exact sequence (28) gives rise to the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{I}_P(\kappa\varepsilon_{\text{cyc}}^{-1}\nu^{-1}) & \longrightarrow & \mathbb{I}_P^2 & \longrightarrow & \mathbb{I}_P(\nu) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & V_p(E) & \longrightarrow & \mathbb{Q}_p \longrightarrow 0, \end{array} \quad (29)$$

where the vertical arrows are given by specialization at  $P$ . After twisting the first and second arrows in (29) by the  $\Lambda$ -adic character  $\nu\varepsilon_{\text{cyc}}^2\kappa^{-1}$  and its (trivial) specialization at  $P_2$  respectively, we obtain the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{I}_P(\varepsilon_{\text{cyc}}) & \longrightarrow & \mathbb{I}_P^2(\nu\varepsilon_{\text{cyc}}^2\kappa^{-1}) & \longrightarrow & \mathbb{I}_P(\nu^2\varepsilon_{\text{cyc}}^2\kappa^{-1}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & V_p(E) & \longrightarrow & \mathbb{Q}_p \longrightarrow 0. \end{array} \quad (30)$$

Finally, letting  $t := \kappa - 2 \in \Lambda$ , we see that  $\Lambda/(t^2) \cong \mathbb{Q}_p[\varepsilon]$ , where  $\varepsilon = t \bmod(t^2)$ , and the reduction mod  $t^2$  of (30) becomes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{\mathbb{Q}}_p(1) & \longrightarrow & \tilde{V} & \longrightarrow & \tilde{\mathbb{Q}}_p(\psi) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & V_p(E) & \longrightarrow & \mathbb{Q}_p \longrightarrow 0, \end{array} \quad (31)$$

where  $\psi$  denotes the character  $\nu_k^2\langle\varepsilon\rangle^{2-k} \bmod(t^2)$ . We have an exact sequence of  $\mathbb{Q}_p[D_p]$ -modules

$$0 \rightarrow \mathbb{Q}_p \xrightarrow{\varepsilon} \tilde{\mathbb{Q}}_p(\psi) \rightarrow \text{coker}(\varepsilon) = \mathbb{Q}_p \rightarrow 0 \quad (32)$$

induced by the mapping  $\varepsilon : q \mapsto \varepsilon \cdot q$ . The degree zero coboundary  $\mathbb{Q}_p$ -linear map in the long exact sequence associated to (32) is explicitly given by

$$\begin{aligned} \delta_\psi : \mathbb{Q}_p^{D_p} = \mathbb{Q}_p & \rightarrow \mathbb{H}^1(D_p, \mathbb{Q}_p) \\ 1 & \mapsto \left( \sigma \mapsto \frac{d\psi}{dt}(\sigma) \right), \end{aligned}$$

where  $\frac{d\psi}{dt}$  is the composite map

$$D_p \xrightarrow{\psi} 1 + \varepsilon \cdot \mathbb{Q}_p \subset \tilde{\mathbb{Q}}_p \xrightarrow{\frac{d}{dt}} \mathbb{Q}_p,$$

seen as an element in  $\text{Hom}(D_p, \mathbb{Q}_p) = \mathbb{H}^1(D_p, \mathbb{Q}_p)$ . Note that  $\psi(\sigma) = \psi_0(\sigma) + \varepsilon \cdot \frac{d\psi}{dt}(\sigma)$ , where  $\psi_0(\sigma) = \psi(\sigma)$  modulo  $\varepsilon$ , and that  $\frac{d\psi}{dt}$  is nonzero, since  $\psi$  is nontrivial. Write  $\kappa(\psi)$  for cocycle  $\delta_\psi(1)$ . The expression of  $\kappa(\psi) \in \mathbb{H}^1(D_p^{\text{ab}}, \mathbb{Q}_p)$  in the basis  $\kappa_{\text{nr}}, \kappa_{\text{cyc}}$  is obviously

$$\kappa(\psi) = \frac{d\psi}{dt}([p, \mathbb{Q}_p]) \cdot \kappa_{\text{nr}} + \frac{d\psi}{dt}([\gamma, \mathbb{Q}_p]) \cdot \kappa_{\text{cyc}}, \quad (33)$$

and we see that the ratio  $(\frac{d\psi}{dt}([p, \mathbb{Q}_p]) : \frac{d\psi}{dt}([\gamma, \mathbb{Q}_p]))$  is uniquely determined by the infinitesimal deformation  $\tilde{\mathbb{Q}}_p(\psi)$  of  $\mathbb{Q}_p$ .

The cohomology groups  $H^1(D_p, \mathbb{Q}_p)$  and  $H^1(D_p, \mathbb{Q}_p(1))$  are both 2-dimensional  $\mathbb{Q}_p$ -vector spaces, and the local Tate pairing

$$\langle , \rangle : H^1(D_p, \mathbb{Q}_p(1)) \times H^1(D_p, \mathbb{Q}_p) \rightarrow H^2(D_p, \mathbb{Q}_p(1)) = \mathbb{Q}_p$$

furnishes a perfect duality between them.

**Lemma 2.3.** *The cohomology class  $\kappa(\psi)$  coming from the connecting homomorphism associated to the exact sequence (32) is an annihilator under the local Tate pairing of the cohomology class  $\kappa_{\text{Kum}}(q_E)$ .*

*Proof.* Consider the commutative diagram in cohomology induced by the diagram (31):

$$\begin{array}{ccccccc} H^1(D_p, \tilde{V}) & \longrightarrow & H^1(D_p, \tilde{\mathbb{Q}}_p) & \longrightarrow & H^2(D_p, \tilde{\mathbb{Q}}_p(1)) & \longrightarrow & 0 \\ \uparrow & & \uparrow d_1 & & \uparrow d_2 & & \\ H^1(D_p, V(E)) & \longrightarrow & H^1(D_p, \mathbb{Q}_p) & \xrightarrow{\delta_1} & H^2(D_p, \mathbb{Q}_p(1)) = \mathbb{Q}_p & \longrightarrow & 0 \\ & & \uparrow \delta_\psi & & \uparrow d & & \\ & & H^0(D_p, \mathbb{Q}_p) = \mathbb{Q}_p & \xrightarrow{\delta_0} & H^1(D_p, \mathbb{Q}_p(1)) & & \end{array}$$

The connecting homomorphism  $d$  vanishes, and therefore the cohomology class  $\kappa(\psi)$  is in the kernel of  $\delta_1$ , since it is in the image of  $\delta_\psi$ . On the other hand,  $\kappa_{\text{Kum}}(q_E)$  is in the image of  $\delta_0$ . Thus it suffices to show that the kernel of  $\delta_1$  is orthogonal under the local Tate pairing to the image of  $\delta_0$ . Indeed, the connecting homomorphisms  $\delta_0 : \mathbb{Q}_p \rightarrow H^1(D_p, \mathbb{Q}_p(1))$  and  $\delta_1 : H^1(D_p, \mathbb{Q}_p) \rightarrow \mathbb{Q}_p$  are transposes of one another, since the exact sequence

$$0 \rightarrow \mathbb{Q}_p(1) \rightarrow V(E) \rightarrow \mathbb{Q}_p \rightarrow 0 \quad (34)$$

is self dual with respect to the multiplication pairing  $\mathbb{Q}_p(1) \times \mathbb{Q}_p \rightarrow \mathbb{Q}_p(1)$  and the Weil pairing  $V(E) \times V(E) \rightarrow \mathbb{Q}_p(1)$ . Since the extension class in  $H^1(D_p, \mathbb{Q}_p(1))$  corresponding to (34) lies in the line spanned by  $\kappa_{\text{Kum}}(q_E)$ , the conclusion follows.  $\square$

**Corollary 2.4.**

$$\mathcal{L}(E) = -2 \cdot \frac{da_p(\kappa)}{dx} \Big|_{\kappa=2}.$$

Indeed, the local Tate pairing of the cohomology classes  $\kappa_{\text{Kum}}(q_E)$  and  $\kappa(\psi) = \frac{d\psi}{dt}$  can be computed as

$$\langle \kappa_{\text{Kum}}(q_E), \kappa(\psi) \rangle = \frac{d\psi}{dt}([q_E, \mathbb{Q}_p]).$$

Writting  $q_E = p^{\text{ord}_p(q_E)}u$  for  $u \in \mathbb{Z}_p^\times$ , we have  $\psi([q_E, \mathbb{Q}_p]) = a_p(k)^{-2n}\langle u \rangle^{2-k}$ . Since  $\langle u \rangle^{2-k} = \exp((2-k)\log_p(u))$ , we see that

$$\begin{aligned} \left. \frac{d\langle u \rangle^{2-k}}{dk} \right|_{k=2} &= -\log_p(u) \cdot \exp((2-k)\log_p(u))|_{k=2} \\ &= -\log_p(q_E), \end{aligned}$$

since  $\log_p(u) = \log_p(q_E)$  by our choice of  $p$ -adic logarithm (i.e. such that  $\log_p(p) = 0$ ). Since  $\psi([q_E, \mathbb{Q}_p]) = a_p(k)^{-2n}\langle u \rangle^{2-k}$  for  $n = \text{ord}_p(q_E)$ , we have

$$\begin{aligned} 0 &= \left. \frac{d\psi([q_E, \mathbb{Q}_p])}{dk} \right|_{k=2} \\ &= -2 \cdot \text{ord}_p(q_E) \cdot a'_p(k)\langle u \rangle^{2-k} + a_p(k)^{-2\text{ord}_p(q_E)} \cdot \log_p(\langle u \rangle)\langle u \rangle^{2-k}|_{k=2} \\ &= -2 \cdot \text{ord}_p(q_E) \cdot a'_p(k)|_{k=2} + \log_p(q_E), \end{aligned}$$

and this concludes the proof of the corollary.

### 2.3 The adjoint square Selmer group

Given an ordinary modular Galois representation  $\rho_f$ , its adjoint square representation  $\text{Ad}(\rho_f)$  can be defined, and the ordinarity assumption implies that  $\text{Ad}(\rho_f)$  presents an exceptional zero phenomenon in the sense of Greenberg. In this section we review the definition of a panoply of so-called Selmer groups attached to  $\text{Ad}(\rho_f)$ , which are certain cohomology groups obtained by imposing local restrictions on cocycles in the Galois cohomology of  $\text{Ad}(\rho_f)$ . The usefulness of these groups becomes apparent when one interprets them in terms of more explicit objects. In particular, we show how the various Selmer groups attached to  $\text{Ad}(\rho_f)$  relate to various deformation problems defined by  $\bar{\rho}_f$ .

Let  $f$  be an ordinary modular form,  $K$  a finite extension of  $\mathbb{Q}_p$  with ring of integers  $\mathcal{O}$  containing all the Hecke eigenvalues of  $f$ . Let  $\rho_f$  be the Galois representation attached to  $f$  and  $U$  the representation  $K$ -space of  $\rho_f$ . The adjoint square representation  $\text{Ad}(\rho_f)$  is the Galois representation given by the adjoint action of  $G_{\mathbb{Q}}$  on  $V = \text{End}_K(U)$ , i.e. for  $\sigma \in G_{\mathbb{Q}}$  and  $\varphi \in V$ ,  $\sigma.\varphi$  is the  $K$ -endomorphism of  $U$  given by  $u \mapsto \sigma.\varphi(\sigma^{-1}.u)$ . Thus if  $\rho_f$  has a matrix representation

$$\rho_f : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(K) = \text{Aut}_K(U),$$

then  $\text{Ad}(\rho_f)$  has matrix representation

$$\text{Ad}(\rho_f) : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_4(K) = \text{Aut}_K(V),$$

where  $\text{Ad}(\rho_f)(\sigma) : M \mapsto \rho_f(\sigma).M.\rho_f(\sigma)^{-1}$  for  $M \in M_2(K) = V$ .

Denote by  $U^0$  the  $D_p$ -stable  $K$ -line in  $U$ , so that one has the two-step decreasing filtration of  $D_p$ -modules

$$U \supset U^0 \supset \{0\}. \quad (35)$$

By a suitable choice of basis, the restriction of  $\rho_f$  to the decomposition group at  $p$  takes the form

$$\rho_f|_{D_p} = \begin{pmatrix} \varepsilon_f & a_f \\ & \delta_f \end{pmatrix}, \quad (36)$$

with the character  $\varepsilon_f$  describing the action of  $D_p$  on  $U^0$ , and the unramified character  $\delta_f$  describing the corresponding action on the 1-dimensional unramified quotient  $U/U^0$ . Let  $W = \text{End}_K^0(U) \subset V$  be the representation space of the subrepresentation  $\text{Ad}^0(\rho_f) = (\text{Sym}^2 \otimes \det^{-1})(\rho_f)$  of  $\text{Ad}(\rho_f)$  given by trace-zero  $K$ -endomorphisms of  $U$ . The filtration (35) induces corresponding decreasing filtrations on  $V$  and  $W$ :

$$\begin{aligned} V &\supset V^0 \supset V^{00} \supset \{0\}; \\ W &\supset W^0 \supset W^{00} \supset \{0\}, \end{aligned}$$

where

$$\begin{aligned} V^0 &= \{\varphi \in V \mid \varphi(U^0) \subset U^0\}, \\ W^0 &= V^0 \cap W; \\ V^{00} &= \{\varphi \in V \mid \varphi(U^0) = \{0\}\}, \\ W^{00} &= V^{00} \cap W = V^{00}. \end{aligned}$$

With a choice of basis for  $U$  which gives a matrix form as in (36),  $V^0$  (resp.  $W^0$ ) is identified with the subspace of  $M_2(K)$  consisting of upper triangular matrices (resp. trace zero upper triangular matrices), and  $V^{00} = W^{00}$  with the nilpotent ones.

For our purposes, we also need to consider the following sets:

$$\begin{aligned} V^{\text{ord}} &= \{\varphi \in V \mid \varphi(U) \subset U^0\} = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\} \subset V^0; \\ V^{\text{split}} &= V^{\text{ord}}/V^{00} = \left\{ \begin{pmatrix} * & \\ & \end{pmatrix} \right\}. \end{aligned}$$

Note that  $V^{\text{ord}} \cap W^0 = W^{00}$ .

**Remark 3.** As can be checked for example with a straightforward calculation, the adjoint action of  $D_p$  on  $V^0/V^{00}$  is trivial, thus in particular the action of  $D_p$  on  $V^{\text{split}} \cong W^0/W^{00}$  is trivial. One then expects that the  $p$ -adic  $L$ -function  $L(\text{Ad}(\rho_f), s)$  has an exceptional zero at  $s = 1$ .

Fix a  $G_{\mathbb{Q}}$ -stable  $\mathcal{O}$ -lattice  $T \subset U$ , and define the representations

$$\begin{aligned} \mathrm{Ad}(\rho_f)_T : G_{\mathbb{Q}} &\rightarrow \mathbf{GL}_4(\mathcal{O}), \\ \mathrm{Ad}(\rho_f)_{U/T} : G_{\mathbb{Q}} &\rightarrow \mathbf{GL}_4(K/\mathcal{O}), \end{aligned} \quad (37)$$

given by the adjoint action of  $G_{\mathbb{Q}}$  on  $V_T = V \cap \mathrm{End}_{\mathcal{O}}(T)$  and on  $V_{U/T} = V/V_T = V_T \otimes_{\mathcal{O}} K/\mathcal{O}$  respectively. We put  $W_T = W \cap V_T$ , and intersecting with  $T$  the filtration (35) of  $U$  we define the two-step decreasing filtration of  $T$ :

$$T \supset T^0 \supset \{0\}. \quad (38)$$

Replacing  $V$  and  $W$  by  $V_T$  and  $W_T$  respectively in the above definitions, we define  $V_T^0$ ,  $V_T^{00}$ ,  $W_T^0$ ,  $V_T^{\mathrm{ord}}$  and  $V_T^{\mathrm{split}}$  and identifying  $V_T^?$  with its natural image in  $V^?$ , we define  $V_{U/T}^? = V^?/V_T^? = V_T^? \otimes_{\mathcal{O}} K/\mathcal{O}$  just as before, and  $W_{U/T}^? = W^?/W_T^? = W_T^? \otimes_{\mathcal{O}} K/\mathcal{O}$  for  $? = 0, 00, \mathrm{ord}$  or  $\mathrm{split}$ .

For every a rational prime  $q$  consider the inflation-restriction exact sequence

$$0 \rightarrow \mathrm{H}^1(\mathbb{Q}_q^{\mathrm{unr}}/\mathbb{Q}_p, V^{I_q}) \xrightarrow{\mathrm{inf}_q} \mathrm{H}^1(\mathbb{Q}_q, V) \xrightarrow{\mathrm{res}_q} \mathrm{H}^1(\mathbb{Q}_q^{\mathrm{unr}}, V).$$

Denote by  $L_q^{\mathrm{unr}}$  be the subspace of the local cohomology group  $\mathrm{H}^1(\mathbb{Q}_q, V)$  given by

$$L_q^{\mathrm{unr}} := \mathrm{Ker}(\mathrm{res}_q).$$

Let  $R$  be a finite set of primes different from  $p$ , and set  $S = R \cup \{p\}$ . A collection of *local conditions* is the data  $\mathcal{L}_S = \{L_q\}$  consisting in a subspace  $L_q$  of  $\mathrm{H}^1(\mathbb{Q}_q, V)$  for every prime  $q$  such that  $L_q = L_q^{\mathrm{unr}}$  for every  $q \notin S$ .

**Definition 2.5.** The *adjoint square Selmer group* of  $\rho_f$  with local condition  $\mathcal{L}_S$  is the subspace of  $\mathrm{H}^1(\mathbb{Q}^{(S)}, V)$  cut out by the local conditions  $\{L_q\}$ :

$$\begin{aligned} \mathrm{Sel}_R^{\mathcal{L}}(\mathbb{Q}, \mathrm{Ad}(\rho_f)) &:= \bigcap_q \mathrm{Ker} \left( \mathrm{H}^1(\mathbb{Q}, V) \xrightarrow{\mathrm{res}_q} \mathrm{H}^1(\mathbb{Q}_q, V) / L_q \right) \\ &= \{[\xi] \in \mathrm{H}^1(\mathbb{Q}^{(S)}, V) \mid \mathrm{res}_q[\xi] \in L_q \text{ for all } q\}. \end{aligned}$$

We will consider the following Selmer groups (with  $S = \{p\}$ ):

$$\mathrm{Sel}_{\emptyset}^s(\mathbb{Q}, \mathrm{Ad}(\rho_f)) \subset \mathrm{Sel}_{\emptyset}^{00}(\mathbb{Q}, \mathrm{Ad}(\rho_f)) \subset \mathrm{Sel}_{\emptyset}^0(\mathbb{Q}, \mathrm{Ad}(\rho_f)),$$

and

$$\mathrm{Sel}_{\emptyset}^{\mathrm{split}}(\mathbb{Q}, \mathrm{Ad}(\rho_f)) \subset \mathrm{Sel}_{\emptyset}^{\mathrm{ord}}(\mathbb{Q}, \mathrm{Ad}(\rho_f)),$$

We only need to specify the local condition at  $p$  in each one of the above cases. We set

$$\begin{aligned} L_p^s &:= \mathrm{Im} \left( \mathrm{H}^1(\mathbb{Q}_p, W^{00}) \rightarrow \mathrm{H}^1(\mathbb{Q}_p, V) \right) \\ L_p^{00} &:= \mathrm{Ker} \left( \mathrm{H}^1(\mathbb{Q}_p, V) \rightarrow \mathrm{H}^1(\mathbb{Q}_p^{\mathrm{unr}}, V/W^{00}) \right), \end{aligned} \quad (39)$$

and the definitions for  $L_p^0$ ,  $L_p^{\text{ord}}$  and  $L_p^{\text{split}}$  are made with  $W^{00}$  replaced in (39) by  $W^0$ ,  $V^{\text{ord}}$  and  $V^{\text{split}}$  respectively. These Selmer groups are subspaces of  $H^1(\mathbb{Q}^{(p)}, V)$ .

We will also need to consider certain  $\mathcal{O}$ -torsion Selmer groups. They are denoted by  $\text{Sel}_R^{\mathcal{L}}(\mathbb{Q}, \text{Ad}(\rho_f)_{U/T})$ , and are defined by taking cohomology with values in  $V_{U/T}$  and replacing  $V^?$  and  $W^?$  by  $V_{U/T}^?$  and  $W_{U/T}^?$  respectively in the above definitions.

Consider now the dual of the above constructions. Let  $V^*(1) = \text{Hom}_K(V, K)(1) \cong V(1)$  be the Kummer dual of  $V$  with Galois action induced by the adjoint action on  $V$  twisted by the cyclotomic character, and similarly for  $V_T^*(1) = \text{Hom}_{\mathcal{O}}(V_T, \mathcal{O})(1) \cong V_T(1)$ . Set  $V_{U/T}^*(1) = \text{Hom}(V_{U/T}, \mu_{p^\infty})$  for the Pontryagin dual of  $V_{U/T}$ . We define in the same way  $W^*(1)$ ,  $W_T^*(1)$  and  $W_{U/T}^*(1)$ . The filtrations on  $V$ ,  $W$  and  $T$  allows us to filtrations on their duals by taking orthogonal complement under the corresponding local Tate pairings, and we use these filters to define just as before the meaning of the dual Selmer groups  $\text{Sel}_{\mathfrak{h}}^?( \mathbb{Q}, \text{Ad}(\rho_f)^*(1) )$  for  $? = s, 00, 0, \text{ord}, \text{split}$ , and  $\cdot = , T$  or  $U/T$ .

## Deformation theoretic interpretation of classical Iwasawa theory

Since it is quite illuminating, we pause here to give an interpretation of the Selmer group of Greenberg in the cyclotomic setting of classical Iwasawa theory, following [Gr94b].

Let  $p$  be an odd prime. The cyclotomic character  $\chi : G_{\mathbb{Q}} \rightarrow \mathbb{Z}_p^\times$ , giving the Galois action on the group  $\mu_{p^\infty}$  of  $p$ -power roots of unity, factors through  $G_\infty = \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$ , inducing an isomorphism  $G_\infty \cong \mathbb{Z}_p^\times$ . The canonical decomposition

$$\mathbb{Z}_p^\times = \mu_{p-1} \times 1 + p\mathbb{Z}_p$$

is thus identified with the canonical decomposition  $G_\infty = \Delta \times \Gamma$  of  $G_\infty$  into its maximal torsion subgroup  $\Delta$  and its pro- $p$ -free quotient  $\Gamma = G_\infty/\Delta$ . The restriction homomorphism allows us to write  $\Delta = \text{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$  and  $\Gamma = \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ , where  $\mathbb{Q}_\infty = \bigcup_{r \geq 0} \mathbb{Q}_r$  is the cyclotomic  $\mathbb{Z}_p$  extension of  $\mathbb{Q}$ , its  $r$ th piece  $\mathbb{Q}_r$  being uniquely determined by the requirement that  $\text{Gal}(\mathbb{Q}_r/\mathbb{Q})$  is identified via restriction with the unique closed subgroup  $\text{Gal}(\mathbb{Q}(\mu_{p^{r+1}})/\mathbb{Q}(\mu_p))$  of  $\Gamma \cong \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p))$  of order  $p^r$ .

It is a consequence of Kummer congruences that for an *odd* integer  $i$ ,  $0 \leq i < p-1$ ,  $i \neq 1$ , there exists a unique continuous  $\mathbb{Z}_p$ -valued function  $L_{p,i}(s)$  of a  $p$ -adic variable  $s \in \mathbb{Z}_p$  characterized by the interpolation property

$$L_{p,i}(n) = (1 - p^{-n})\zeta(n), \tag{40}$$

for all odd integers  $n \leq -1$  with  $n \equiv i \pmod{p-1}$ . This is Kubota-Leopoldt's  $p$ -adic  $L$ -function, and Iwasawa showed that it is given by a so-called Iwasawa function, i.e.  $L_{p,i}(s) = \kappa^s(g_i)$ , for a certain  $g_i \in \Lambda$ .

Let  $L_\infty$  be the maximal abelian pro- $p$ -extension of  $K_\infty$  everywhere unramified, and put  $Y_\infty = \text{Gal}(L_\infty/K_\infty)$ . Since  $Y_\infty$  is abelian, the Galois group  $G_\infty = \Delta \times \Gamma$  naturally acts on  $Y_\infty$  via  $\gamma.y = \tilde{\gamma}y\tilde{\gamma}^{-1}$  for  $\gamma \in G_\infty$  and  $y \in Y_\infty$ , where  $\tilde{\gamma}$  denote any lifting of  $\gamma$  to  $\text{Gal}(L_\infty/\mathbb{Q})$ . In particular  $Y_\infty$  thus becomes a  $\Gamma$ -module, and since it is a pro- $p$ -group, a  $\Lambda = \mathbb{Z}_p[[\Gamma]]$ -modules. Since  $\#\Delta$  is prime to  $p$ , one has a natural decomposition of  $\Lambda$ -modules:

$$Y_\infty = \bigoplus_{i=0}^{p-2} Y_\infty^{(i)}, \quad (41)$$

where  $Y_\infty^{(i)}$  denotes the  $\omega^i$ -isotypic component of  $Y_\infty$ , and  $\omega$  is the Teichmüller character  $\omega : \Delta \rightarrow \mu_{p-1} \subset \mathbb{Z}_p^\times$  which generated the cyclic character group  $\tilde{\Delta}$ . It follows from Class Field Theory that  $Y_\infty$  is a finitely generated  $\Lambda$ -module, so the same is true for each  $Y_\infty^{(i)}$ ; since they are obviously torsion  $\Lambda$ -modules, the characteristic ideal  $\text{char}_\Lambda(Y_\infty^{(i)})$  of each  $Y_\infty^{(i)}$  is defined, which will be a principal ideal in  $\Lambda$ . Then the Main Conjecture of Iwasawa, as proved by Mazur and Wiles in [MW84], is the assertion that one may construct a generator of  $\text{char}_\Lambda(Y_\infty^{(i)})$  via interpolation of special values of  $L$ -functions: more precisely, that

$$\text{char}_\Lambda(Y_\infty^{(i)}) = g_i \Lambda.$$

Again by the congruences of Kummer, it follows that for *even* integers  $j$ ,  $0 \leq j < p-1$ ,  $j \neq 0$ , there exists a unique  $p$ -adic  $L$ -function  $L_{p,j}(s) = \kappa^s(\theta_j) : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$  characterized by the interpolation formulae

$$L_{p,j}(n) = (1 - p^{n-1}) \frac{\zeta(n)}{\Omega_n}, \quad (42)$$

for all even integers  $n \geq 2$  with  $n \equiv j \pmod{p-1}$ , where  $\Omega_n \in \mathbb{C}^\times$  is a nonzero constant such that the functional equation satisfied by  $\zeta(s)$  yields the identity

$$\frac{\zeta(n)}{\Omega_n} = \zeta(1-n). \quad (43)$$

It follows from (43) that one actually has  $L_{p,j} = \kappa^{1-s}(\theta_j)$  for  $i$  with  $i \equiv 1-j \pmod{p-1}$ .

Let  $M_\infty$  denote the maximal abelian pro- $p$ -extension of  $K_\infty$  everywhere unramified except the places in  $K_\infty$  above  $p$ , and put  $X_\infty = \text{Gal}(M_\infty/K_\infty)$ . In complete analogy with  $Y_\infty$ , the  $\mathbb{Z}_p$ -module  $X_\infty$  can be naturally endowed with a  $\Lambda$ -modules structure, and one has a decomposition  $X_\infty = \bigoplus_{j=0}^{p-2} X_\infty^{(j)}$  of finite torsion  $\Lambda$ -modules. Then a reformulation of the theorem of Mazur and Wiles is the assertion that the following identity holds:

$$\text{char}_\Lambda(X_\infty^{(j)}) = f_j \Lambda.$$

## 2.4 The locally split universal deformation ring

Suppose now that the modular representation  $\rho_f$  is *residually CM*, i.e. there exists a quadratic imaginary field  $M$  and a Hecke character  $\psi : G_M \rightarrow \mathcal{O}^\times$  such that

$$\bar{\rho}_f = \text{Ind}_M^{\mathbb{Q}} \bar{\psi}.$$

Then the  $p$ -adic representation attached to the character  $\psi$ :

$$\rho_\psi := \text{Ind}_M^{\mathbb{Q}} \psi : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(\mathcal{O})$$

is obviously a deformation of  $\bar{\rho}_f$ ; since it also split at  $p$ , there exists a unique local morphism of  $\mathcal{O}$ -algebras

$$\varphi_\psi : R^{\text{split}} \rightarrow \mathcal{O}$$

bringing the universal deformation  $\rho^{\text{split}}$  into  $\rho_\psi$ . Let  $I_\psi$  be the kernel of  $\varphi_\psi$ . The following proposition is a direct adaptation of the arguments in [Wil95] Prop.1.2:

**Proposition 2.6.** *There is a canonical isomorphism of  $\mathcal{O}$ -modules:*

$$\text{Hom}_{\mathcal{O}}(I_\psi/I_\psi^2, K/\mathcal{O}) \cong \text{Sel}_{\emptyset}^{\text{split}}(\mathbb{Q}, \text{Ad}(\rho_f)_{U/T}). \quad (44)$$

*Proof.* We first construct for each integer  $n > 0$  a morphism

$$\varphi : \text{Sel}_{\emptyset}^{\text{split}}(\mathbb{Q}, \text{Ad}(\rho_f)_{U/T}[\pi^n]) \rightarrow \text{Hom}_{\mathcal{O}}(I_\psi/I_\psi^2, \pi^{-n}\mathcal{O}/\mathcal{O}), \quad (45)$$

which will be compatible with the natural maps induced by the inclusions  $\text{Ker}(\pi^n) \hookrightarrow \text{Ker}(\pi^m)$  for  $m \geq n > 0$  thus yielding (44). Let  $\mathcal{O}_n[\varepsilon]$  be the ring of dual numbers  $\mathcal{O}/\pi^n[X]/(X^2) \cong \mathcal{O}/\pi^n \oplus \varepsilon \cdot \mathcal{O}/\pi^n$ , where  $\varepsilon = X \pmod{X^2}$ . Let  $[\xi]$  be a cocycle in  $\text{Sel}_{\emptyset}^{\text{split}}(\mathbb{Q}, \text{Ad}(\rho_f)_{U/T}[\pi^n])$ , and consider the lifting of  $\rho_\psi$  to  $\mathcal{O}_n[\varepsilon]$  given by the infinitesimal deformation

$$\begin{aligned} \rho_\xi : G_{\mathbb{Q}} &\rightarrow \mathbf{GL}_2(\mathcal{O}_n[\varepsilon]) \\ \sigma &\mapsto \rho_\psi(\sigma) + \xi(\sigma)\rho_\psi(\sigma).\varepsilon, \end{aligned}$$

where  $\rho_\psi$  is seen as taking values in  $\mathbf{GL}_2(\mathcal{O}_n[\varepsilon])$  by composition with the projection map  $\mathcal{O} \rightarrow \mathcal{O}/\pi^n \subset \mathcal{O}/\pi^n \oplus \varepsilon \cdot \mathcal{O}/\pi^n$ , the inclusion being along the first summand in  $\mathcal{O}_n[\varepsilon]$ , and one uses the natural inclusion of  $\mathcal{O}/\pi^n$ -algebras (since  $\varepsilon^2 = 0$ )

$$\text{M}_2(\mathcal{O}/\pi^n) \hookrightarrow 1_2 + \varepsilon \cdot \text{M}_2(\mathcal{O}/\pi^n) \subset \mathbf{GL}_2(\mathcal{O}_n[\varepsilon])$$

to see  $\xi(\sigma) \in \mathbf{GL}_2(\mathcal{O}_n[\varepsilon])$ . The correspondence is well-defined, since  $\rho_\xi$  is easily checked to be a continuous homomorphism, and affecting  $\xi$  by a coboundary does not change the equivalence class of the resulting lifting to  $\mathcal{O}_n[\varepsilon]$ . The local restrictions imposed on  $\xi$  imply

that  $\rho_\xi$  is a split lifting of  $\rho_0$ . Indeed, we only need to check the local condition at  $p$ . Consider the exact sequence

$$\begin{aligned} 0 &\rightarrow \mathrm{H}^1(D_p/I_p, (V_{U/T}[\pi^n]/W_{U/T}^{\mathrm{split}}[\pi^n])^{I_p}) \rightarrow \mathrm{H}^1(D_p, V_{U/T}[\pi^n]/W_{U/T}^{\mathrm{split}}[\pi^n]) \rightarrow \\ &\rightarrow \mathrm{H}^1(I_p, V_{U/T}[\pi^n]/W_{U/T}^{\mathrm{split}}[\pi^n])^{D_p/I_p} \end{aligned}$$

coming from the inflation-restriction exact sequence. By hypothesis, the cocycle  $\xi$  is in the image of the first term in the above sequence. Let  $X$  be the subspace of  $(V_{U/T}[\pi^n]/V_{U/T}^0[\pi^n])^{I_p}$  such that the natural inclusion map  $V_{U/T}^0 \subset V_{U/T}$  yields the exact sequence

$$0 \rightarrow (V_{U/T}^0[\pi^n]/W_{U/T}^{\mathrm{split}}[\pi^n])^{I_p} \rightarrow (V_{U/T}[\pi^n]/W_{U/T}^{\mathrm{split}}[\pi^n])^{I_p} \rightarrow X \rightarrow 0. \quad (46)$$

Since the action of  $D_p$  on  $V_{U/T}[\pi^n]/V_{U/T}^0[\pi^n]$  is given by  $\eta/\eta'$  and we have  $(\eta/\eta')|_{D_p} \neq 1 \pmod{\pi}$  due to the  $p$ -distinguishedness hypothesis, we see that  $X^{D_p/I_p} = X^{\mathrm{Frob}_p} = 0$  and  $\mathrm{H}^1(D_p/I_p, X) = X/(\mathrm{Frob}_p - 1)X = 0$  so that the long exact sequence of  $D_p/I_p$ -modules induced by (46) gives rise to an isomorphism

$$\mathrm{H}^1(D_p/I_p, (V_{U/T}^0[\pi^n]/W_{U/T}^{\mathrm{split}}[\pi^n])^{I_p}) \cong \mathrm{H}^1(D_p/I_p, (V_{U/T}[\pi^n]/W_{U/T}^{\mathrm{split}}[\pi^n])^{I_p}).$$

Thus  $\xi$  admits a representative mapping  $D_p$  into  $V_{U/T}^0[\pi^n]$  and  $I_p$  into  $W_{U/T}^{\mathrm{split}}[\pi^n]$ , and now we see that  $\rho_\xi$  is indeed a split lifting of  $\rho_0$  from the formula

$$\begin{pmatrix} * & \\ & \end{pmatrix} \begin{pmatrix} \eta & \\ & \eta' \end{pmatrix} = \begin{pmatrix} * \cdot \eta & \\ & \end{pmatrix}.$$

By the universal property of  $R^{\mathrm{split}}$ , the split lifting  $\rho_\xi$  induces a unique morphism of local  $\mathcal{O}$ -algebras

$$\varphi_\xi : R^{\mathrm{split}} \rightarrow \mathcal{O}_n[\varepsilon]$$

such that  $\rho_\xi = \varphi_\xi \circ \rho^{\mathrm{split}}$  for the universal split deformation  $\rho^{\mathrm{split}}$  of  $\rho_0$ . The image of  $\varphi_\xi$  is entirely contained in  $\varepsilon \cdot \mathcal{O}/\pi^n$  when restricted to  $I_\psi$ , since  $\rho_\xi \equiv \rho_\psi \pmod{\varepsilon}$ , and we have  $\varphi_\xi(I_\psi^2) = \{0\}$ . Thus the correspondence  $\xi \mapsto \rho_\xi$  induces a map  $\varphi : \xi \mapsto \varphi_\xi$  as in (45), which is easily checked to be an  $\mathcal{O}$ -linear map.

Suppose that  $\varphi_\xi(I_\psi) = 0$ . Then  $\varphi_\psi$  factor through  $R^{\mathrm{split}}/I_\psi \cong \mathcal{O}_\psi$ , and thus  $[\rho_\psi] = [\rho_\xi]$ , and it follows from Schur's lemma that  $\xi$  must be a coboundary. So  $\varphi$  is injective.

We check the surjectivity of  $\varphi$ . Take  $\phi \in \mathrm{Hom}_{\mathcal{O}}(I_\psi/I_\psi^2, \mathcal{O}/\pi^n)$ , which obviously induces (by abuse of notation)  $\phi \in \mathrm{Hom}_{\mathcal{O}}(I_\psi/(I_\psi^2, \mathrm{Ker}\phi), \mathcal{O}/\pi^n)$ . Take a representative  $\rho$  of  $\rho^{\mathrm{split}} : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(R^{\mathrm{split}})$  such that

$$\rho_\psi = (\rho^{\mathrm{split}} \bmod I_\psi). \quad (47)$$

Then  $\phi$  induces a representation  $\rho_\phi : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(R^{\text{split}}/(I_\psi^2, \text{Ker}\phi))$  and we define

$$\xi_\phi : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(R^{\text{split}}/(I_\psi^2, \text{Ker}\phi)) : g \mapsto \xi_\phi(g) = \rho_\phi(g)\rho_\psi(g)^{-1}.$$

By (47),  $\xi_\phi$  actually take values in  $1_2 + M_2(I_\psi/(I_\psi^2, \text{Ker}\phi))$ , and we see that its image contained in  $V_{U/T}[\pi^n]$  via the composite  $\phi : I_\psi/(I_\psi^2, \text{Ker}\phi) \hookrightarrow \mathcal{O}/\pi^n \subset \mathcal{O}_n[\varepsilon]$ , where the inclusion is via the projection  $1 \mapsto \varepsilon$  to the second direct summand in  $\mathcal{O}_n[\varepsilon] = \mathcal{O}/\pi^n \oplus \varepsilon \cdot \mathcal{O}/\pi^n$ . It is easily checked that  $\xi_\phi$  is actually a cocycle. By construction, we have  $\varphi(\xi_\phi) = \rho_{\xi_\phi} = \phi$ , which proves the surjectivity of  $\varphi$ .  $\square$

## 2.5 Greenberg's adjoint square $\mathcal{L}$ -invariant

In this section we recall the cohomological definition of the  $\mathcal{L}$ -invariant of Greenberg particularized to the case of an adjoint square Galois representation attached to a modular form. When the modular representation  $\rho_f$  comes from the Tate module of a rational elliptic curve with split multiplicative reduction at  $p$ , this invariant coincides with the  $\mathcal{L}$ -invariant  $\mathcal{L}(E)$  first introduced by Mazur, Tate and Teitelbaum. Thus the Theorem due to Hida explained in the last part of this section provides a generalization of the original proof by Greenberg and Stevens of the exceptional zero conjecture of [MTT86].

**Lemma 2.7.** *The torsion group  $\text{Sel}_\emptyset^{00}(\mathbb{Q}, \text{Ad}(\rho_f)_{U/T})$  is finite.*

*Proof.* Let  $\bar{\rho} = \rho_f \pmod{\lambda}$ . Since the universal deformation ring  $R_{\bar{\rho}}$  is isomorphic to a universal ordinary Hecke algebra  $R^o$ , we see that  $R_k := R^o \otimes W/PW$ , for  $P = P_f \cap W[[\Gamma]]$ , is isomorphic to a suitable Hecke algebra of weight  $k$  and finite level. Thus  $R_k$  is a finite  $W$ -module, and its  $W$ -free quotient is reduced. It follows that  $\text{Sel}_{\mathcal{L}^{00}}(\mathbb{Q}, V_{U/T}^*(1))$  is isomorphic to the finite module  $\Omega_{R_k/W} \otimes_{R_k} W$ , where the tensor product is taken along the algebra homomorphism  $R_k \rightarrow W$  associated to  $\rho_f$ .  $\square$

**Corollary 2.8.** *The following  $K$ -vector spaces are all null*

$$\text{Sel}_\emptyset^s(\mathbb{Q}, \text{Ad}(\rho_f)) = \text{Sel}_\emptyset^s(\mathbb{Q}, \text{Ad}(\rho_f)^*(1)) = 0. \quad (48)$$

*Proof.* Indeed, it follows from Lemma 2.7 that the Selmer space  $\text{Sel}_\emptyset^{00}(\mathbb{Q}, \text{Ad}(\rho_f))$  vanishes, and Greenberg shows in [Gr94a] that  $\dim_K(\text{Sel}_\emptyset^s(\mathbb{Q}, \text{Ad}(\rho_f)) = \dim_K(\text{Sel}_\emptyset^s(\mathbb{Q}, \text{Ad}(\rho_f)^*(1)))$ . The assertion follows from the inclusion  $\text{Sel}_\emptyset^s(\mathbb{Q}, \text{Ad}(\rho_f)) \subset \text{Sel}_\emptyset^{00}(\mathbb{Q}, \text{Ad}(\rho_f))$ .  $\square$

Denote by  $S$  the set of ramified primes for  $\rho_f$ , including  $p$ . The exact sequence of Poitou and Tate yields the exact sequence

$$0 \rightarrow \text{Sel}_\emptyset^s(\mathbb{Q}, \text{Ad}(\rho_f)) \rightarrow H^1(\mathbb{Q}^{(S)}, V) \rightarrow \prod_{q \in S} H^1(\mathbb{Q}_q, V)/L_q^s \rightarrow \text{Sel}_\emptyset^s(\mathbb{Q}, \text{Ad}(\rho_f)^*(1))^*. \quad (49)$$

Thus by (48) the restriction map  $\bigoplus_{q \in S} \text{res}_q$  gives rise to the isomorphism

$$\mathrm{H}^1(\mathbb{Q}^{(S)}, V) \cong \prod_{q \in S} \mathrm{H}^1(\mathbb{Q}_q, V)/L_q^s. \quad (50)$$

Consider the subspace  $H$  of the right hand term of (50) determined by the image of the composite map  $\mathrm{H}^1(\mathbb{Q}_p, W^0) \rightarrow \mathrm{H}^1(\mathbb{Q}_p, V) \rightarrow \mathrm{H}^1(\mathbb{Q}_p, V)/L_p^s$  under the natural embedding  $\mathrm{H}^1(\mathbb{Q}_p, V)/L_p^s \hookrightarrow \prod_q \mathrm{H}^1(\mathbb{Q}_q, V)/L_q^s$ . This subspace is isomorphic to  $W^0/W^{00} \cong K$  (cf. [Gr94a]), and we denote by  $\mathrm{H}_p^1(\mathbb{Q}^{(S)}, V)^{\text{cyc}}$  the subspace of  $\mathrm{H}^1(\mathbb{Q}^{(S)}, V)$  given by the inverse image of  $H$  under the restriction map  $\text{res}_p$ . Thus  $\mathrm{H}_p^1(\mathbb{Q}^{(S)}, V)^{\text{cyc}}$  is mapped isomorphically via  $\text{res}_p$  onto a one-dimensional subspace of  $\mathrm{H}^1(\mathbb{Q}_p, W^0/W^{00}) = \text{Hom}(D_p^{\text{ab}}, K) \cong K^2$ , the last isomorphism being given by

$$\xi \mapsto \left( \frac{\xi([u, \mathbb{Q}_p])}{\log_p(u)}, \xi([p, \mathbb{Q}_p]) \right) = (x(\xi), y(\xi))$$

for the local Artin Symbol  $[\cdot, \mathbb{Q}_p] : \mathbb{Q}_p^\times \rightarrow D_p^{\text{ab}}$  and any  $u \in \mathbb{Z}_p^\times$  of infinite order. Consider the unramified cocycle  $\kappa_{\text{nr}} \in \text{Hom}(D_p/I_p, K)$  mapping  $\text{Frob}_p$  to 1, the ramified cocycle induced by restriction to  $D_p$  of  $\kappa_{\text{cyc}} = \log_p(\varepsilon_{\text{cyc}}) \in \text{Hom}(G_{\mathbb{Q}}, K) = \mathrm{H}^1(\mathbb{Q}, K)$ . The cocycles  $\kappa_{\text{cyc}}$  and  $\kappa_{\text{nr}}$  gives a basis of  $\mathrm{H}^1(\mathbb{Q}_p, K)$ , and the above ‘‘coordinates’’ attached to a cocycle  $\xi$  as above are determined by the expression

$$\text{res}_p(\xi) = x(\xi) \cdot \kappa_{\text{cyc}} + y(\xi) \cdot \kappa_{\text{nr}},$$

and the ratio  $x : y$  is independent of the choice of nonzero  $\xi$ . If a nonzero  $\xi \in \mathrm{H}_p^1(\mathbb{Q}, V)$  is unramified (i.e.  $x(\xi) = 0$ ), then it gives rise to a nonzero element of  $\text{Sel}_0^s(\mathbb{Q}, \text{Ad}(\rho_f))$ , contradicting Lemma 2.8. Thus  $\mathrm{H}_p^1(\mathbb{Q}^{(S)}, V)^{\text{cyc}}$  sits inside  $\text{Hom}(\mathbb{Q}_p, K) \cong K^2$  as the image of a  $K$ -linear map given by multiplication by  $\mathcal{L}(\text{Ad}(\rho_f)) \in K$ . This scalar is the so-called  $\mathcal{L}$ -invariant of Greenberg attached to the adjoint square representation of  $\rho_f$ , and we see that it can be explicitly given by the formula

$$\mathcal{L}(\text{Ad}(\rho_f)) = \xi([p, \mathbb{Q}_p]) \cdot \frac{\log_p(\gamma_o)}{\xi([\gamma_o, \mathbb{Q}_p])}$$

for  $\gamma_o = 1 + p \in 1 + p\mathbb{Z}_p$  and any nonzero  $\xi \in \mathrm{H}_p^1(\mathbb{Q}^{(S)}, V)$ .

To compute the  $\mathcal{L}$ -invariant of  $\text{Ad}(\rho_f)$  one must thus construct a global cohomology class  $[\xi] \in \mathrm{H}^1(\mathbb{Q}^{(S)}, V)$  which is unramified at each prime  $\ell \in S \setminus \{p\}$  and its restriction to  $D_p$  has image contained in upper triangular trace zero matrices, and such that one can explicitly calculate the above coordinates  $x(\xi)$ ,  $y(\xi)$ . The  $\Lambda$ -adic Galois representation  $\rho_{\mathcal{F}}$  attached to the Hida family  $\mathcal{F}$  containing  $f$ , and its relation to deformation problems defined by  $\bar{\rho}_f$  often allows one to explicitly produce a suitable cocycle whose coordinates one can compute and even relate to  $p$ -adic  $L$ -functions. This is the approach taken by Hida to obtain a formula for  $\text{Ad}(\rho_f)$  in many cases.

## 2.6 The exceptional zero conjecture: the proof by Hida

Hida obtained a generalization of the theorem by Greenberg and Stevens recalled in Section 2.2 that apply to the adjoint square representation attached to an ordinary modular form  $f$ . The original exceptional zero conjecture of Mazur, Tate and Teitelbaum is recovered when  $f$  correspond to a rational elliptic curve with split multiplicative reduction at  $p$ . A crucial point in the arguments of Hida is made by the universality of the ordinary Hecke algebra under certain conditions, which lightens a relation between the adjoint square Selmer groups and deformation problems attached to  $f$ .

**Proposition 2.9** (Hida [H04]). *Let  $f$  be an ordinary Hecke eigenform with trivial Nebentypus. Suppose that its associated Galois representation  $\rho_f$  is distinguished and absolutely irreducible when restricted to  $\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$ . Denote by  $\mathcal{F} = a_p(\mathcal{F})q^n \in \mathbb{I}[[q]]$  its associated  $\Lambda$ -adic form. Then*

$$\mathcal{L}(\mathrm{Ad}(\rho_f)) = -2\log_p(\gamma)a_p(f)^{-1} \frac{d}{dX} a_p(\mathcal{F})|_{X=0}.$$

In the above statement, recall that  $\mathbb{I}$  is a finite flat extension of  $\mathcal{O}[[x]]$ , where  $\mathcal{O}$  denotes the ring of integers of a finite extension  $K/\mathbb{Q}_p$  containing all the Hecke eigenvalues of  $f$ . Thus the derivative appearing in the right hand term makes sense.

**Lemma 2.10.** *As subspaces of  $H^1(\mathbb{Q}^{(S)}, V)$ , the following equality holds:*

$$H_p^1(\mathbb{Q}^{(S)}, V) = \mathrm{Sel}_\emptyset^0(\mathbb{Q}, \mathrm{Ad}(\rho_f)).$$

*Proof.* Denote by  $U_p^{s,0}(V)$  (respectively,  $U_p^{s,00}(V)$ ) be the natural image of  $H^1(\mathbb{Q}_p, W^0)$  (respectively,  $H^1(\mathbb{Q}_p, W^{00})$ ) in  $H^1(\mathbb{Q}_p, V)$ . Then we have the exact sequence

$$0 \rightarrow U_p^{s,0}(V)/U_p^{s,00}(V) \xrightarrow{\alpha} H^1(\mathbb{Q}_p, \mathrm{Ad}(\rho_f))/U_p^{00}(V) \rightarrow H^1(\mathbb{Q}_p, \mathrm{Ad}(\rho_f))/U_p^0(V) \rightarrow 0,$$

where  $\alpha$  is the natural map, and the last term is defined so that the sequence becomes exact. Let  $\xi$  be a cocycle in  $H_p^1(\mathbb{Q}^{(S)}, V)$ . By definition,  $\mathrm{res}_p[\xi]$  lies in  $U_p^{s,0}(V)/U_p^{s,00}(V)$ , and via the embedding  $\alpha$  we may regard it uniquely as an element in  $\mathrm{Sel}_\emptyset^0(\mathbb{Q}, \mathrm{Ad}(\rho_f))$ , since  $\mathrm{Im}(\alpha) = U_p^0(V)/U_p^{00}(V)$  and  $\mathrm{Sel}_\emptyset^{00}(\mathbb{Q}, \mathrm{Ad}(\rho_f)) = 0$  by Lemma 2.7. The lemma follows immediately from this.  $\square$

We can now proof Proposition 2.9:

*Proof.* Let  $U$  be the representation space of  $\rho_f$  and choose a  $D_p$ -stable  $\mathcal{O}$ -lattice  $T \subset U$  such that the resulting matrix representation, which with a slight abuse of notation we write  $\rho_f : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(\mathcal{O})$  has the local shape

$$\rho_f|_{D_p} = \begin{pmatrix} \varepsilon_f & \beta_f \\ & \delta_f \end{pmatrix}.$$

By the universality of the pair  $(R^{\text{univ}}, \rho^{\text{univ}})$ , there is a point  $P_T \in \text{Spf}(R^{\text{univ}})(\mathcal{O})$  corresponding to  $\rho_f$  such that  $\rho_f = \rho^{\text{univ}} \pmod{P_T}$ .

Since  $H^1(\mathbb{Q}^{(S)}, V) = \text{Sel}_0^1(\mathbb{Q}, \text{Ad}(\rho_f))$  by Lemma 2.10, it follows from Proposition 2.6 that the one-dimensional subspace  $H_p^1(G_S, V)^{\text{cyc}} \subset H^1(G_S, V)$  is isomorphic to the tangent space of  $R_{\rho_f}^{\text{univ}}$  at  $P_T$ :

$$H_p^1(\mathbb{Q}, V)^{\text{cyc}} \cong \text{Hom}_K(P_T/P_T^2, \text{Ad}(\rho_f)).$$

Recall that the isomorphism is explicitly given by the following correspondence. A cocycle  $\xi$  naturally defines an infinitesimal deformation  $\rho_\xi : G_{\mathbb{Q}} \rightarrow \mathbf{GL}_2(K[\varepsilon])$  of  $\rho_f$  by the formula

$$\rho_\xi(\sigma) = \rho_f(\sigma) + \xi(\sigma)\rho_f(\sigma) \cdot \varepsilon;$$

the cohomology relation for  $\xi$  corresponds to the strict equivalence relation for  $\rho_\xi$ , and the set of strict equivalence classes of such infinitesimal deformations naturally corresponds to the tangent space of  $\text{Spf}(R_{P_T})$  at  $P_T$ .

Let  $\kappa$  be a generator of  $H_p^1(\mathbb{Q}^{(S)}, V)^{\text{cyc}}$  and write

$$\kappa(\sigma) = \begin{pmatrix} -a(\sigma) & b(\sigma) \\ & a(\sigma) \end{pmatrix} \quad \text{for } \sigma \in D_p.$$

The  $\mathcal{L}$ -invariant of  $\text{Ad}(\rho_f)$  is then given by the formula

$$\mathcal{L}(\text{Ad}(\rho_f)) = a([p, \mathbb{Q}_p]) \cdot \frac{\log_p(\gamma)}{a([\gamma_o, \mathbb{Q}_p])} \quad (51)$$

for the topological generator  $\gamma_o \in \Gamma = 1 + p\mathbb{Z}_p$ . Note that  $a$  can not be identically on  $I_p$ , since if  $\kappa|_{D_p}$  were unramified modulo nilpotent matrices, then it would give rise to a nonzero element in  $\text{Sel}_0^s(\mathbb{Q}, \text{Ad}(\rho_f))$ , contradicting Corollary 2.8.

Since  $(R_{P_T}^{\text{univ}}, \rho^{\text{univ}}) \cong (\Lambda_{P_f}, \rho_{\mathcal{F}})$  by hypothesis, we see that  $\frac{d}{dX}\rho_{\mathcal{F}}$  gives rise to a generator of  $H_p^1(\mathbb{Q}^{(S)}, V)^{\text{cyc}}$ . Thus for all  $\sigma \in D_p$  the identity

$$\kappa(\sigma)\rho_f(\sigma) = C \cdot \frac{d}{dx}\rho_{\mathcal{F}}(\sigma) \quad (52)$$

holds for a certain constant  $C \in K^\times$ . Writing

$$\rho_{\mathcal{F}}(\sigma) = \begin{pmatrix} \varepsilon_{\mathcal{F}}(\sigma) & \beta_{\mathcal{F}}(\sigma) \\ & \delta_{\mathcal{F}}(\sigma) \end{pmatrix}$$

for  $\sigma \in D_p$ , a comparison of the lower right terms in both sides of (52) after expanding the product in the left hand side yields the formulae:

$$a([p, \mathbb{Q}_p]) \cdot \delta_f([p, \mathbb{Q}_p]) = C \cdot \frac{d}{dX}\delta_{\mathcal{F}}([p, \mathbb{Q}_p]) \Big|_{X=0}; \quad (53)$$

$$a([\gamma_o, \mathbb{Q}_p]) \cdot \delta_f([\gamma_o, \mathbb{Q}_p]) = C \cdot \frac{d}{dX} \delta_{\mathcal{F}}([\gamma_o, \mathbb{Q}_p]) \Big|_{X=0}. \quad (54)$$

We have  $\delta_{\mathcal{F}}^o([p, \mathbb{Q}_p]) = a_p(\mathcal{F})$ . Since the character  $\gamma^s \mapsto (1+X)^{s/2}$  interpolates the characters  $\varepsilon_{\text{cyc}}^{m(p-1)}$  for all integers  $m$ , and since  $\varepsilon_{\text{cyc}}([p, \mathbb{Q}_p]) = 1$ , we have  $\delta_{\mathcal{F}}([p, \mathbb{Q}_p]) = \delta_{\mathcal{F}}^o([p, \mathbb{Q}_p]) \cdot (1+x)^{s/2}([p, \mathbb{Q}_p]) = a_p(\mathcal{F})$ . Thus we see from (53) that

$$a([p, \mathbb{Q}_p]) = C \cdot a_p(f)^{-1} \cdot \frac{da_p(\mathcal{F})}{dX} \Big|_{X=0}, \quad (55)$$

given that  $\delta_f([p, \mathbb{Q}_p]) = \delta_{\mathcal{F}}^o([p, \mathbb{Q}_p])|_{X=0} = a_p(\mathcal{F})|_{X=0} = a_p(f)$ .

Since  $\det(\rho_{\mathcal{F}}^o)$  is the universal deformation of  $(\det(\rho_f) \bmod \mathfrak{m}) : G_{\mathbb{Q}, S} \rightarrow (\mathcal{O}/\mathfrak{m})^\times$ , we see that

$$\rho_{\mathcal{F}} = \rho_{\mathcal{F}}^o \otimes (1+X)^{-s/2} : G_{\mathbb{Q}, S} \rightarrow \mathbf{GL}_2(\Lambda_P),$$

where  $\gamma_o^s \mapsto (1+X)^s : G_{\mathbb{Q}(p)} \rightarrow \Lambda^\times$  is the universal deformation of the trivial character, sending  $\sigma$  to  $(1+X)^s$  if  $\varepsilon_{\text{cyc}}(\sigma) = \gamma_o^s$ . It follows that  $\delta_{\mathcal{F}}([\gamma_o^s, \mathbb{Q}_p]) = (1+X)^{-s/2}$  since  $\delta_{\mathcal{F}}^o$  is unramified at  $p$ . Therefore  $\frac{d}{dX} \delta_{\mathcal{F}}([\gamma_o^s, \mathbb{Q}_p]) = -s/2$  so that  $\frac{d}{dX} \delta_{\mathcal{F}}([\gamma_o, \mathbb{Q}_p]) = -1/2$ . Thus we see from (54) that

$$a([\gamma_o, \mathbb{Q}_p]) = C \cdot -\frac{1}{2}, \quad (56)$$

given that due to the unramifiedness of  $\delta_{\mathcal{F}}^o$  we have

$$\delta([\gamma_o, \mathbb{Q}_p]) = \delta_{\mathcal{F}}([\gamma_o^s, \mathbb{Q}_p])|_{X=0} = (\delta_{\mathcal{F}}^o \otimes (1+X)^{-s/2})([\gamma_o, \mathbb{Q}_p])|_{X=0} = 1.$$

Now the formula for  $\mathcal{L}(\text{Ad}(\rho_f))$  is obtained from (55) and (56) using the definition (51).  $\square$

## References

- [Bu03] Buzzard, K.: *Analytic continuation of overconvergent eigenforms*, Jour. Amer. Math. Soc., 16 no.1 (2003), 29–55.
- [DDP09] Darmon, H., Dasgupta, S., Pollack, R.: *Hilbert modular forms and the Gross-Stark conjecture*, preprint (2009).
- [Gha04] Ghate, R.: *On the local behaviour of ordinary modular Galois representations*. In *Modular curves and abelian varieties*, Progress in Mathematics vol. 224, pp.105–124. Birkhäuser 2004.
- [Gha05] Ghate, E.: *Ordinary forms and their local Galois representations*. In *Algebra and Number Theory*, pp.226–242. Hindustan Book Agency 2005.

- [GV04] Ghate, E., Vatsal, V.: *On the local behaviour of ordinary  $\Lambda$ -adic representations*, Ann. Inst. Fourier, 54 no.7 (2004), 2143–2162.
- [GV08] Ghate, E., Vatsal, V.: *Locally indecomposable Galois representations*, Canad. J. Math., to appear.
- [Gr94a] Greenberg, R.: *Trivial zeroes of  $p$ -Adic  $L$ -functions*. In  *$p$ -adic monodromy and the conjecture of Birch and Swinnerton-Dyer*, Contemporary Mathematics vol.165, pp.149–174. AMS 1994.
- [Gr94b] Greenberg, R.: *Iwasawa theory and  $p$ -adic deformation of motives*. In *Motives*, Proc. Symp. Pure Math. 55 vol.2, pp.193–223. AMS 1994.
- [GS93] Greenberg, G., Stevens, G.:  *$p$ -adic  $L$ -functions and  $p$ -adic periods of modular forms*, Invent. Math. 111 (1993), 407–447.
- [GS94] Greenberg, G., Stevens, G.: *On the conjecture of Mazur, Tate and Teitelbaum*. In  *$p$ -adic monodromy and the conjecture of Birch and Swinnerton-Dyer*, Contemporary Mathematics vol.165, pp.183–211. AMS 1994.
- [H86a] Hida, H.: *Iwasawa modules attached to congruences of cusp forms*, Ann. Scient. Éc. Norm. Sup., 4<sup>e</sup> série, t.19 (1986), 231–273.
- [H86b] Hida, H.: *Galois representations into  $\mathbf{GL}_2(\mathbb{Z}_p[[X]])$  attached to ordinary cusp forms*, Invent. Math. 85 (1986), 545–613.
- [H04] Hida, H.: *Greenberg’s  $\mathcal{L}$ -invariant of adjoint square Galois representations*, IMRN No. 89 (2004), 3177–3189.
- [HT] Hida, H., Tilouine, J.: *On the anticyclotomic main conjecture for CM fields*, Invent. Math. 117 (1994), 89–147.
- [Len93] Lenstra, H.W.: *Complete intersections and Gorenstein rings*. In *Proc. 1993 Conference on Elliptic Curves and Modular Forms*, Series in Number Theory vol.1, Academic Press 1995.
- [Maz89] Mazur, B.: *Deforming Galois representations*. In *Galois groups over  $\mathbb{Q}$* , MSRI Publications 16, pp. 385–437. Springer 1989.
- [MTT86] Mazur, B., Tate, J., Teitelbaum, J.: *On  $p$ -adic analogues of the conjecture of Birch and Swinnerton-Dyer*, Invent. Math. 84 (1986), 1–48.
- [MT90] Mazur, B., Tilouine, J.: *Représentations galoisiennes, différentielles de Kähler et “conjectures principales”*, Publ. Math. de l’I.H.É.S., tome 71 (1990), 65–103.

- [MW84] Mazur, B., Wiles, A.: *Class fields of abelian extensions of  $\mathbb{Q}$* , Invent. Math. 76 (1984), 179–330.
- [MW86] Mazur, B., Wiles, A.: *On  $p$ -adic analytic families of Galois representations*, Compositio Math. 59 (1986), 231–264.
- [Rib77] Ribet, K.: *Galois Representations attached to eigenforms with nebentypus*. In *Modular Functions of One Variable V*, LNM 601, pp.17–52. Springer 1977.
- [Ser89] Serre, J.-P.: *Abelian  $\ell$ -adic representations and elliptic curves*. Advanced Books Classics. Addison Wesley Publ. Co. 1989.
- [Shi71] Shimura, G.: *On elliptic curves with complex multiplication as factors of the Jacobians of modular function fields*, Nagoya Math. Jour. 43 (1973), 199–208.
- [Til88] Tilouine, J.: *Théorie d’Iwasawa classique et de l’algèbre de Hecke ordinaire*, Comp. Math., 65 (1988), 265–320.
- [Wil88] Wiles, A.: *On ordinary  $\lambda$ -adic representations associated to modular forms*, Invent. Math. 94 (1988), 529–573.
- [Wil95] Wiles, A.: *Modular elliptic curves and Fermat’s Last Theorem*, Ann. Math. (2) 141 (1995), 443–551.