MSc in Photonics

Universitat Politècnica de Catalunya (UPC)
Universitat Autònoma de Barcelona (UAB)
Universitat de Barcelona (UB)
Institut de Ciències Fotòniques (ICFO)

http://www.photonicsbcn.eu

PHOTONICS**BCN**

Master in Photonics

MASTER THESIS WORK

BOUNDING HILBERT SPACE DIMENSION FROM TEMPORAL CORRELATIONS

Rodrigo Gallego López

Supervised by Dr. Antonio Acín, (ICFO)

Presented on date 9th September 2009

Registered at

Escola Tècnica Superior d'Enginyeria de Telecomunicació de Barcelona

Bounding Hilbert Space dimension from Temporal Correlations

Rodrigo Gallego López

ICFO-Institut de Ciencies Fotoniques, Mediterranean Technology Park, 08860 Castelldefels (Barcelona), Spain

E-mail: `rodrigo.gallego@icfo.es`

Abstract. In this work, we tackle the problem of assessing the Hilbert space dimension from the set of correlations obtained when measuring in a nonlocal black box scheme. The concept of a dimension witness and its recent applications are explored. We also extend these new ideas to the case of a single local box with measurements at different times, and provide some examples of dimension criteria for this case.

1. Introduction

Physicists usually try to explain experimental results using models that assume a specific size for the dimension of the Hilbert Space. Thus, the model itself determines the dimension of the system. In this work the inverse approach is considered: Can we assess the dimension without a prior model?

The fundamental motivation for this question is that when physical systems are described, the dimension of the Hilbert space used in the description usually depends on approximations, whose validity is confronted with experimental results. In the approach considered here, correlations obtained can be used for deriving bounds on the dimension, which will directly evaluate convenience of some approximations and help us to find an effective model. Besides, in Quantum Information Science the dimension of the accessible Hilbert space appears to have the character of a resource, and also is fundamental in some proofs of security of quantum key distribution [1], so it seems necessary to derive some model independent conditions for the Hilbert space dimension needed.

In the same spirit, Bell inequalities address the problem of whether some correlations can be attained by a classical description, and establish conditions which are model independent. Indeed, Bell inequalities offer us an example of how to gather information about the system without considering specifically any of the elements of which it comprises. We can reduce the description of all the devices present in a certain non-local experiment to two measuring apparatuses, in which one can choose among a finite number of measurements and obtain a finite number of outcomes for each measurement. We repeat the experiment many times in order to obtain objects

$P(ab|xy)$ representing the probabilities of obtaining outcomes a,b for measurements x, y. Generalized Bell inequalities [2] can be used to build a frontier between classical and quantum behavior for this scenario, and also limits for the quantum and non-quantum behavior can be considered studying maximal quantum violation of these inequalities [3], however, in these cases the Hilbert Space dimension is not considered, or in other words, they establish the maximal violation without restrictions on the dimension.

The raw idea which supports most of the results in this work is that with higher dimension in the Hilbert space, greater violations of Bell inequalities can be attained, so we can set a device-independent frontier among behavior of systems with different dimensions.

This work is organized into two main sections. In Section 2 we present previous work done in relation with the characterization of Hilbert space dimension. In particular, we present the concept of a dimension witness and introduce its most characteristic example using violation of Bell inequalities. In Section 3 we explore the possibility of applying this concept to the case of a single local box, performing measurements at different times. Temporal Bell inequalities are presented and some examples of dimension witnesses are introduced.

2. Dimension witnesses

2.1. Scenario description

Let us describe the situation by saying that the two parties, Alice and Bob, have access to a "black box". When Alice inputs a number x into the black box, she obtains as output a measurement outcome a; similarly, when Bob inputs a number y, he receives an output b. The behavior of the box is characterized by the joint detection probabilities $P(ab|xy)$. If there is a quantum representation of dimension d for underlying system, Alice and Bob share a quantum system in a joint state ρ in $\mathbb{C}^d \otimes \mathbb{C}^d$ and inputs correspond with m possible measurements for each system $x, y \in \{0, 1, ..., m-1\}$ and v possible outcomes $a, b \in \{0, 1, ..., v-1\}$ for each measurement. In that case, the set of probabilities can be written as

$$P(ab|xy) = \mathrm{tr}(\rho M_a^x \otimes M_b^y) \tag{1}$$

with M_a^x representing the measurement operator acting on $\mathbb{C}^d$ corresponding to outcome a and measurement x. We can consider also linear combinations of elements from the set of probabilities, usually called correlations:

$$c_{xy} = P(a = b|xy) - P(a \neq b|xy) \tag{2}$$

2.2. Definition

We are interested in a criterion able to establish some bounds on the dimension of the Hilbert space used to explain a given set of probabilities in the scenario described above. With this spirit the concept of d-dimensional witness is introduced in [4].

Definition 2.2.1 *A d-dimensional witness is a linear combination of the elements $P(ab|xy)$ described by a tensor M, such that:*

$$M \cdot P = \sum_{a,b,x,y} M_{a,b,x,y} P(ab|xy) \leq W_d \tag{3}$$

for all probabilities of the form (1) and ρ in $\mathbb{C}^d \otimes \mathbb{C}^d$, and such that there exist probabilities of the form (1) and ρ in $\mathbb{C}^{d+l} \otimes \mathbb{C}^{d+l}$ for which $M \cdot P \geq W_d$

If for a set of inputs and outputs we find a d-dimensional witness, we can perform experiments, obtain the set of probabilities and calculate the value of the quantity $M \cdot P$. If we obtain a violation of the inequality, we can deduce that there is no d-dimensional model able to explain our system. It must be remarked that on this definition the value l does not need to be determined for establishing a dimension witness. We will present a situation in which the inequality is violated for dimension higher than d, but still undetermined. In both cases, wether determined or not l, we can only conclude that the dimension needed is larger than d and we do not obtain an upper bound for the effective dimension used.

2.3. Previous Tools

Numerous examples of dimension witnesses are presented in references [4, 5, 6, 7], most of them obtaining a higher violation of two-outcomes m-measurements Bell-type inequalities for qudits than for qubits. In this work we are presenting in detail the existence of a dimension witness for the case of m possible measurements and two outcomes for each measurement. The motivation is that some tools used in the proof will be interesting for tackling the problem of single local box with measurements at different times, and also because its simplicity in comparison with complex schemes will allow us to understand the underlying idea in order to apply it to a new case. Let us state some lemmas and definitions used in this approach

Definition 2.3.1 *The real Grothendieck constant of order n, is the smallest real number $K_G(n)$ such that: for all positive integers and all real $r \times r$ matrices M, the inequality*

$$\max_{a_1,...,a_r,b_1,...,b_r} \sum_{i,j} M_{ij} a_i \cdot b_j \leq K_G(n) \max_{\alpha_1,...,\alpha_r,\beta_1,...,\beta_r} \sum_{i,j} M_{ij} \alpha_i \beta_j \tag{4}$$

holds, where the maximum on the left-hand side is taken over all sequences $a_1, ..., a_r, b_1, ..., b_r$ of n-dimensional real unit vectors, and the maximum on the right-hand side is taken over all sequences $\alpha_1, ..., \alpha_r, \beta_1, ..., \beta_r$ of real numbers in the set $\{-1, +1\}$. The real Grothendieck constant, denoted K_G is defined as $\lim_{n\to\infty} K_G(n)$.

It was Tsirelson [8] who first found a relation between maximal violation of Bell inequalities and Grothendieck's inequality. As we will justify in following lemmas, Hilbert space dimension is related with the order of Grothendieck constant.

Lemma 2.3.2 *[8]. Let us consider correlations of the form (2) such that $c_{xy} = \vec{x} \cdot \vec{y}$ with $\vec{x}, \vec{y} \in \mathbb{R}^n$. These correlations can be implemented with a maximally entangled state $|\psi\rangle$ on $\mathbb{C}^d \otimes \mathbb{C}^d$ and observables X, Y acting on $\mathbb{C}^d$ if $d \geq 2^{\lfloor n/2 \rfloor}$, such that:*

$$c_{xy} = \langle\psi|X \otimes Y|\psi\rangle \tag{5}$$

Lemma 2.3.3 *[9].Let us consider $|\psi\rangle$ on $\mathbb{C}^d \otimes \mathbb{C}^d$, observables X, Y acting on $\mathbb{C}^d$ and quantum correlations $c_{xy} = \langle\psi|X \otimes Y|\psi\rangle$. These correlations can be implemented with dot product of vectors, $c_{xy} = \vec{x} \cdot \vec{y}$ with $\vec{x}, \vec{y} \in \mathbb{R}^n$, if $n \geq 2d^2$. Moreover, if $|\psi\rangle$ is a maximally entangled state, dot product representation is possible if $n \geq d^2 - 1$.*

2.4. Example of a dimension witness

Let us restrict ourselves to the simplest scenario. Two possible incomes associated with two measurement operators for Alice, and the same situation for Bob. They share a pure quantum state (as long as we are dealing with maximal violation of Bell inequalities there is no lost of generality) thus the set of probabilities can be written as

$$P(ab|xy) = \langle\psi|M_a^x \otimes M_b^y|\psi\rangle \tag{6}$$

The correlator (2) can be written in terms of observables X and Y with eigenvalues ± 1 such that $c_{xy} = \langle\psi|X \otimes Y|\psi\rangle$. Let us consider the following linear function of such correlators:

$$I = \sum_{i,j=1} M_{ij} c_{x_i y_j} \tag{7}$$

with M verifying the normalization condition $\sum_{i,j} M_{ij}\alpha_i\beta_j = 1$ with $\alpha_i, \beta_j \in \{-1, 1\}$.

Now we have the necessary elements for combination with the previous lemmas. If Alice and Bob share a quantum state of dimension two, using lemma 2.3.3 and definition 2.3.1 we have,

$$\max_{q_2} I \leq \max_{a_1,\ldots,a_r,b_1,\ldots,b_r} \sum_{i,j} M_{ij} a_i \cdot b_j = K_G(3) \tag{8}$$

with q_d representing all the possible quantum strategies, i.e. choice of operators, state and M, which use a state of dimension d. On the other hand, we can consider n-dimensional unit vectors with n arbitrarily large. If we consider a quantum representation with $d \geq 2^{\lfloor n/2 \rfloor}$ applying lemma 2.3.2 we obtain,

$$\max_{q_d} I \geq \max_{a_1,\ldots,a_r,b_1,\ldots,b_r} \sum_{i,j} M_{ij} a_i \cdot b_j = K_G \tag{9}$$

Although exact values of the Grothendieck constants are still unknown, it is proven that $K_G(3) \leq K_G$ [10], so equations (8) and (9) constitute a dimension witness.

This type of 2-dimensional witness is extended using a generalized definition of Grothendieck constant in [11], deriving the following theorem.

Theorem 2.4.1 *For any d, there are two-outcome correlations that are finitely quantum-realizable, but which are not d-quantum-realizable.*

This theorem show us that we can find d-dimension witnesses using only the simplest strategy that we can construct in a non-local scheme, i.e performing two different measurements with two possible outcomes for each measurement.

3. Single box dimension

In previous section we have presented the concept of dimension witness and some of its direct examples. As has been highlighted, dimension witnesses arise from the existence of Bell-type inequalities, and for their construction a nonlocal box scenario seems to be necessary. However, as presented in [12] it is possible to construct some Bell inequality analogues for the single box scenario, with measurement at different times, which defines the concept of "Quantum Entanglement in Time". New work in this thesis is supported by the idea of extending methods for construction of dimension witness to the case of single box, using as in previous section, maximal violation of Bell-type inequalities. In this direction, is worth mentioning the work of Wolf and Perez-García [13]. They show how to determine the dimensionality of a quantum system from its dynamics in a model-independent way. A certain number of conserved quantities are assumed to be known, and from these data one can derive the dimension of the system. Note that our approach is very different, in the sense that we only deal with measurement correlations -always possible to collect- therefore the evolution of the box is not unitary and there is no place for considering conserved quantities.

Let us introduce first a detailed overview of Quantum Time Entanglement and the procedure to obtain Bell inequalities in this scheme.

3.1. Temporal correlations scenario

Let us consider a particle in a certain quantum state $|\psi\rangle$. At time t_1, we choose among m possible measurements to be performed on the particle, and we obtain an outcome $a^i_{t_1}$ for i-th measurement choice. The state of the particle is in general modified to the state $|\psi'\rangle$ after this measurement. At a latter time t_2, we choose among m' possible measurements to be performed on the same particle and we obtain another outcome $b^j_{t_2}$. Scenario can be regarded as two black boxes. We press a button for the choice of measurement and it produces a certain outcome given an pair of input choices $(i, j) \in (\{1, ..., m\}, \{1, ..., m'\})$, see Fig. (1).

3.2. Temporal Bell inequalities

In quantum mechanics, time and space play a substantially different role. While spatial coordinates are regarded as quantum-mechanical observables, time acts as an external parameter in the evolution of the system. With respect to Bell inequalities, this difference does not play an essential role, due to the fact that spatial coordinate operators are not necessary for the theoretical implementation of such inequalities. However, spatially separated quantum systems are associated with the tensor product

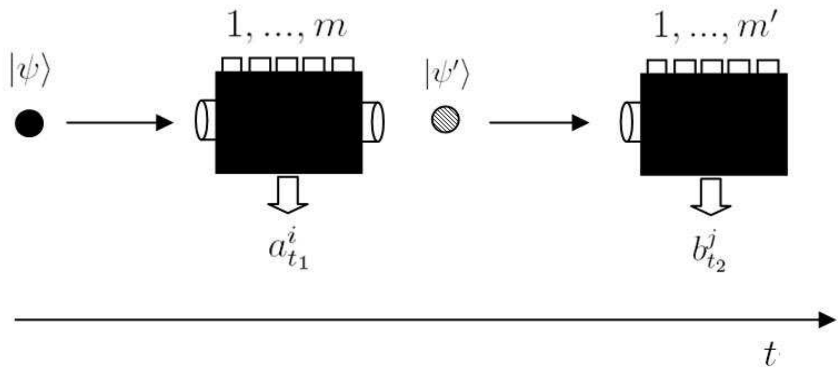

Figure 1. Schematic view of the process for obtaining temporal correlations.

structure which cannot always be separated into spatially separated components, thus it is from this that the mathematical concept of entanglement arises. Temporally separated quantum systems live in the same Hilbert space and an analogous tensor product structure would be meaningless, hence is not possible to consider entanglement in the same way as in spatially separated system. Despite this, it is worth asking oneself whether it is possible to construct temporal Bell-type inequalities which are satisfied by a hidden variable model but violated by Quantum mechanics.

Spatial Bell inequalities are built on a scenario with two spatially-separated observers, namely Alice and Bob, who perform measurements on their physical state. Two fundamental assumptions give rise to the existence of such inequalities,

(i) *Realism:* The measurement results are determined by hidden properties carried by the physical system and independent of observation.

(ii) *Locality in space:* Observer measurement results are independent of any measurement performed by the other spatially separated observer.

The second assumption is ensured for both classical and quantum models by Special Relativity, therefore violation of Bell inequalities highlight invalidity of *Realism* [14].

Temporal Bell inequalities can be constructed using the assumption of *Realism* and substituting *Locality in space* for its temporal analogue,

- *Locality in time:* The result of measurement performed at time t_2 is independent of any measurement performed at some earlier or later time t_1.

In this case, there is no such principle that ensures validity of this assumption so violation of inequalities derived from this cannot be attributed to violation of Realism. Indeed, we will confirm that quantum mechanics violates *Locality in time*. Temporal Bell inequalities implementation has been formally introduced in [15] and has led to strong criticism and an interesting debate [16] about the possibility of inferring violation of *Realism*.

We shall now review the derivation of the temporal analogue of the CHSH inequality given in ref. [17]. Consider an observer choosing at time t_1 between two observables $\vec{a}_1$ and $\vec{a}_2$ with outcomes in the set $\{-1,1\}$. Let us note predetermined values $A^1_{t_1}$ and $A^2_{t_1}$ for $\vec{a}_1$ and $\vec{a}_2$ respectively. At a later time t_2, the observer chooses between another two observables $\vec{b}_1$ and $\vec{b}_2$ obtaining similarly $B^1_{t_2}$ and $B^2_{t_2}$ in the set $\{-1,1\}$. Note that *Locality in time* is assumed in this classical model. Predetermined values of second measurement are not influenced by first measurement. It is easy to check that $A^1_{t_1}B^1_{t_2} + A^1_{t_1}B^2_{t_2} + A^2_{t_1}B^1_{t_2} - A^2_{t_1}B^2_{t_2} = \pm 2$. If we repeat the experiment many times with a physical system with the same predetermined values, one obtains the temporal CHSH inequality

$$B \equiv |\langle A^1_{t_1}B^1_{t_2}\rangle + \langle A^1_{t_1}B^2_{t_2}\rangle + \langle A^2_{t_1}B^1_{t_2}\rangle - \langle A^2_{t_1}B^2_{t_2}\rangle| \leq 2 \tag{10}$$

where $\langle\cdot\rangle$ denotes an average over many runs of the experiment. Let us calculate the maximum value attained for B when measuring on a qubit. The observer performs at time t_1 the measurement of the observable $\hat{A}_{t_1}(\vec{a}) = \vec{a}\cdot\vec{\sigma}$ and at a later time t_2 performs on the resultant state a measurement of the observable $\hat{B}_{t_2}(\vec{b}) = \vec{b}\cdot\vec{\sigma}$. An arbitrary mixed state of a qubit can be written $\rho = \frac{1}{2}(\mathbb{I} + \vec{r}\cdot\vec{\sigma})$ with $\vec{r}\in\mathbb{R}^3$ being the Bloch vector and $\vec{\sigma}$ the Pauli vector. Projectors associated with outcome k can be also written $P^k_{\vec{a}} = \frac{1}{2}(\mathbb{I} + k\vec{r}\cdot\vec{\sigma})$. On the other hand quantum-averaged value can be written as,

$$\langle\hat{A}_{t_1}(\vec{a})\hat{B}_{t_2}(\vec{b})\rangle = \sum_{k,l=\pm 1} kl\cdot \mathrm{Tr}(\rho P^k_{\vec{a}})\cdot \mathrm{Tr}(P^k_{\vec{a}}P^l_{\vec{b}}) \tag{11}$$

where we use the fact that after the first measurement the state projects onto the new state $P^k_{\vec{a}}$. We can calculate correlations, obtaining:

$$\langle\hat{A}_{t_1}(\vec{a})\hat{B}_{t_2}(\vec{b})\rangle = \vec{a}\cdot\vec{b} \tag{12}$$

It is an immediate conclusion that correlations are similar as those obtained in a usual scheme of two spatially-separated observers sharing a singlet state, independently of the initial state ρ used. Thus we obtain a violation of the temporal Bell inequality similar to the violation of Spatial Bell inequality with a maximally entangled state. We shall calculate it explicitly. Substituting (12) in expression (10) we obtain

$$B_{QM} = |\vec{a_1}\cdot(\vec{b_1}+\vec{b_2}) + \vec{a_2}\cdot(\vec{b_1}-\vec{b_2})| \tag{13}$$

The maximal violation of such an inequality can be attained for the choice of measurement settings: $\vec{a}_1 = \frac{1}{\sqrt{2}}(\vec{b}_1+\vec{b}_2)$ and $\vec{a}_2 = \frac{1}{\sqrt{2}}(\vec{b}_1-\vec{b}_2)$ and is equal to $2\sqrt{2}$.

Note also that this scheme does not obey non-signalling stated as $\sum_k P(kl|xy) = \sum_k P(kl|x'y)$. In fact,

$$\sum_k P(kl|\hat{A}_{t_1}(\vec{a})\hat{B}_{t_2}(\vec{b})) = \frac{1}{2}(1 + l(\vec{r}\cdot\vec{a})(\vec{a}\cdot\vec{b})) \tag{14}$$

which clearly depends on the first measurement.

3.3. Optimal signalling strategies

We have seen in previous section how the scheme of temporal Bell correlations assume non-signalling for a hidden variable model. In order to obtain a device-independent dimension witness, we need to relax this assumption and consider signalling strategies. We will show the existence of a dimension witness for a more general case, without imposing *Locality in time*. The underlying idea is that the information from the choice of measurement of the first observer has to be recorded in the particle, and it carries the information to the second observer using some physical dimension for the implementation of this memory. If we increase the number of possible measurements performed at t_1, a larger dimension is needed for efficient storage of the information, thus is possible to find strategies that only systems of larger dimension can attain.

Let us consider in a hidden variable model what would be needed for attaining the maximum algebraic violation 4. Assume that predetermined values in the second measurement depend on the measurement apparatus selection, such that $B^j_{t_2}(i)$ is the value for the j-th measurement chosen at t_2 when measurement i has been chosen previously at time t_1. With predetermined values:

$$A^1_{t_1} = 1, A^2_{t_1} = 1, B^1_{t_2}(1) = 1, B^2_{t_2}(1) = 1, B^1_{t_2}(2) = 1, B^2_{t_2}(2) = -1 \tag{15}$$

we can violate Bell inequality, such that

$$B_S \equiv |\langle A^1_{t_1} B^1_{t_2}(1)\rangle + \langle A^1_{t_1} B^2_{t_2}(1)\rangle + \langle A^2_{t_1} B^1_{t_2}(2)\rangle - \langle A^2_{t_1} B^2_{t_2}(2)\rangle| = 4 \tag{16}$$

This signalling hidden variable model is inconceivable in the spatially separated boxes setup, due to the fact that *Locality in time* is ensured by relativity. In this temporal scenario, we can attain this violation with a classical bit. Let us consider a particle carrying a classical bit $\lambda \in \{-1, +1\}$. In this hidden variable model, operators are substituted by deterministic functions of the classical bit. We denote $f_1(\lambda) = 1$ and $f_2(\lambda) = \lambda$. We can also consider that a flip on the bit can be performed in any of the boxes which constitute the measuring device. With these elements we perform the following strategy:

The classical bit is initialized to $\lambda = 1$. Let us associate the outcome values $A^1_{t_1}(\lambda) = f_2(\lambda) = \lambda$ and $A^2_{t_1}(\lambda) = f_2(\lambda) = \lambda$. The bit is flipped when second measurement is performed, thus the bit prepared for a second measurement is either $\lambda^1 = 1$ or $\lambda^2 = -1$ depending on choice ($A^1_{t_1}$ and $A^2_{t_1}$ respectively) of the first measurement. At latter time t_2, we measure on the bit associating $B^1_{t_2}(i)(\lambda) = f_1(\lambda^i)$ and $B^2_{t_2}(i)(\lambda) = f_2(\lambda^i)$. One can easily check that we obtain the same values as in (15), thus the attained value for the Temporal Bell inequality is 4.

One can expect that analogous strategy can be performed using qubits and quantum operators, establishing a contradiction with (13). In fact, this strategy can be performed using another elements that have not been considered in derivation of (13) but cannot be dismissed in a completely device-independent study. These elements are the quantum analogue of the bit flipping and the function $f_1(\lambda) = 1$, respectively unitary transformations and degenerate measurements [18].

3.4. Unitary transformations and degenerate measurements

In the case of temporal inequalities we need to consider carefully the effect of unitary transformations between measurements in different ways and for different reasons.

- Firstly free temporal evolution driven by the Hamiltonian will act on the resulting states from the first measurement. The evolution operator is **independent of the measurement performed** at t_1 and would only depend on possible interactions of the particle with the media between measurements. It is easy to check that this effect is equivalent to choosing another measurement direction at t_2,

$$\langle \hat{A}_{t_1}(\vec{a})\hat{B}_{t_2}(\vec{b})\rangle = \sum_{k,l=\pm 1} kl \cdot \mathrm{Tr}(\rho P^k_{\vec{a}}) \cdot \mathrm{Tr}(U P^k_{\vec{a}} U^\dagger P^l_{\vec{b}}) = \\ = \sum_{k,l=\pm 1} kl \cdot \mathrm{Tr}(\rho P^k_{\vec{a}}) \cdot \mathrm{Tr}(P^k_{\vec{a}} U^\dagger P^l_{\vec{b}} U) = \sum_{k,l=\pm 1} kl \cdot \mathrm{Tr}(\rho P^k_{\vec{a}}) \cdot \mathrm{Tr}(P^k_{\vec{a}} P^l_{\vec{b}'}) \tag{17}$$

  thus the maximal violation would be attained for another pair of operators at t_2 but its value would be, as before, $2\sqrt{2}$.

- The other case in which unitary evolution has to be considered is when the operation is **dependent on the measuring device chosen**. We can always understand this as a rotation performed by the measurement apparatus. If we are dealing with device-independent strategies we should consider the case in which black boxes can interact with the state before and after the first measurement.
  In the usual non-local boxes scheme these rotations do not need to be considered in the same way. Local transformations of the state can be understood as another choice of operators and do not affect measurement results of a distant observer, thus maximal violation is not effected. Local transformations of the resultant particle after measurement are not relevant because the particle is discarded.
  In the temporal Bell inequalities scheme, transformations of the state before the first measurement are not relevant due the state-independent form of the correlations (12) for $d = 2$. Unitary transformations between measurements dependent on the measurement chosen at t_1 need to be considered in detail. As we can see from equation (17) the effect is equivalent to choosing another observable $\hat{B}_{t_2}(\vec{b}|\vec{a}) = U^\dagger(\vec{a})(\vec{b}\cdot\vec{\sigma})U(\vec{a})$. Unitary transformations on operators can be written as orthogonal transformations on Bloch vectors as shown in [19], thus equation (13) transforms to

$$B_{QM} = |\vec{a_1} \cdot O_{\vec{a}_1}(\vec{b_1} + \vec{b_2}) + \vec{a_2} \cdot O_{\vec{a}_2}(\vec{b_1} - \vec{b_2})| \tag{18}$$

  with O being an orthonormal matrix. The maximal violation of such inequality can be attained for the choice of measurement settings: $\vec{a}_1 = \frac{1}{\sqrt{2}} O_{\vec{a}_1}(\vec{b}_1 + \vec{b}_2)$ and $\vec{a}_2 = \frac{1}{\sqrt{2}} O_{\vec{a}_2}(\vec{b}_1 - \vec{b}_2)$ and is equal to $2\sqrt{2}$.
  In conclusion, unitary transformations before or after measurements do not affect violation of Temporal Bell inequalities in the 2-dimensional case.

- Nevertheless, when we combine this rotations with **degenerate measurements**, we can attain maximal algebraic violation with qubits, with an strategy analogous to the previous one using a classical bit. Consider $\{|-1\rangle, |+1\rangle\}$ a basis of $\mathbb{C}^2$, a measurement operator such that $\hat{\sigma}_z|k\rangle = k|k\rangle$ and a unitary transformation such that $\hat{\sigma}_x|+1\rangle = |-1\rangle$. Let us measure the initial state $|1\rangle$ with the strategy $\hat{A}^1_{t_1} = \hat{A}^2_{t_1} = \hat{B}^2_{t_2} = \hat{\sigma}_z$ and $\hat{B}^2_{t_2} = \mathbb{I}$, and perform the rotation $\hat{\sigma}_x$ after measurement $\hat{A}^2_{t_1}$. In this case one obtains values as in (15), therefore temporal the CHSH inequality is violated up to 4.

3.5. *A new inequality for a dimension witness*

As we have seen, qubits (or classical bits) can maximally violate the temporal analogue of the CHSH inequality. In order to construct a dimension witness, we present a new inequality that can be maximally violated by qutrits (or classical trits) but no by qubits (or classical bits). Consider the expression:

$$\langle \hat{A}^1_{t_1}\hat{B}^1_{t_2}\rangle + \langle \hat{A}^1_{t_1}\hat{B}^2_{t_2}\rangle + \langle \hat{A}^2_{t_1}\hat{B}^1_{t_2}\rangle - \langle \hat{A}^2_{t_1}\hat{B}^2_{t_2}\rangle + \langle \hat{A}^1_{t_1}\rangle + \langle \hat{A}^2_{t_1}\rangle + \langle \hat{A}^3_{t_1}\rangle - \langle \hat{A}^3_{t_1}\hat{B}^1_{t_2}\rangle \quad (19)$$

In this case we choose among three different measurements at time t_1 and two different measurements at time t_2. Consider measurements on a qubit (or a classical bit); the four firsts terms make up temporal CHSH, which is only maximized if on employs degenerate measurements for $\hat{B}^1_{t_2}$ and rotations (or constant classical functions and flips in the classical bit case). To attain an algebraic maximization of the expression (19), one requires that the term $\langle \hat{A}^1_{t_1}\rangle + \langle \hat{A}^2_{t_1}\rangle + \langle \hat{A}^3_{t_1}\rangle$ give an outcome $+1$ for all possible choices of measurement at time t_1; furthermore, one requires that in the case that $\hat{A}^3_{t_1}$ has been chosen, the outcome of measurement $\hat{B}^1_{t_2}$ be -1, and if $\hat{A}^1_{t_1}$ or $\hat{A}^2_{t_1}$ have been performed, the outcome of $\hat{B}^1_{t_2}$ be $+1$. This is clearly not possible for a degenerate measurement (or constant function), thus we conclude that qubits (or classical bits) cannot achieve violation 8 in (19). We present a strategy which attains this maximal violation with a trit.

Let us consider a particle carrying a classical trit $\lambda \in \{1, 2, 3\}$. In this hidden variable model, operators are substituted by deterministic functions of the classical trit. We consider two classical functions $f_1(\lambda)$ and $f_2(\lambda)$ such that $\{f_1(1) = 1, f_1(2) = 1, f_1(3) = -1\}$ and $\{f_2(1) = 1, f_2(2) = -1, f_2(3) = 1\}$. We can also consider that a flip on the trit can be performed in any of the boxes which constitute the measuring device. With these elements we perform the following strategy:

The classical bit is initialized to $\lambda = 2$. Outcome values are associated as $A^i_{t_1}(\lambda = 2) = f_1(\lambda = 2) = 1$ for $i \in \{1, 2, 3\}$. The bit is flipped to $\lambda = 1$ when $A^1_{t_1}$ is performed and to $\lambda = 3$ when $A^2_{t_1}$ is performed, thus the bit prepared for a second measurement is either $\lambda^1 = 1$ or $\lambda^2 = 3$ or $\lambda^3 = 2$ depending on choice ($A^1_{t_1}, A^2_{t_1}$ and $A^3_{t_1}$ respectively) of the first measurement. At a later time t_2, we measure on the bit associating $B^1_{t_2}(i)(\lambda) = f_2(\lambda^i)$ and $B^2_{t_2}(i)(\lambda) = f_1(\lambda^i)$. One can carefully check that we obtain outcomes which attain a value 8 (maximal algebraic violation) for expression (19).

4. Conclusions and further work

As presented in the last part of this work, it is possible to construct a way of assessing the dimension of a single box only from the set of probabilities of obtaining one specific outcome when measuring certain observables. We have also shown the existence of a dimension witness in the case of temporally separated measurements using temporal analogues of Bell inequalities. The effect of unitary transformations as part of the measurement process has also been remarked upon and used for a strategy which constitute a dimension witness.

Further work can be developed in this area and certain important questions remain unsolved. As the CHSH can be maximally violated with both qubits and classical bits, it does not play the role of a Bell inequality, in the sense that it does not distinguish between hidden variable models and quantum models for a given dimension. Thus, it is worth investigating if one can find a gap between such behaviors for a certain dimension.

One can also consider the possibility of inferring from temporal correlations, whether the particle measured at time t_1 is the same that we are measuring at t_2, and what kind of correlations can be attained if a new particle is created after first measurement.

References

[1] A. Acín, N. Gisin, L. Masanes. *Phys. Rev. Lett.* **97,** 120405 (2006)

[2] D. Collins et. al. *Phys. Rev. Lett.* **88,** 4 (2002).

[3] M. Navascues, S. Pironio and A. Acín N. J. Physics **10,** 073013 (2008).

[4] N. Brunner, S. Pironio, A. Acín, N. Gisin, A. Méthot, and V. Scarani. *Phys. Rev. Lett.* **100,** 210503 (2008).

[5] T. Vértesi and K. Pál, arXiv:quant-ph/0812.1572

[6] T. Vértesi and K. Pál *Phys. Rev. Lett.* **77,** 042105 (2008).

[7] T. Vértesi and K. Pál *Phys. Rev. Lett.* **77,** 042106 (2008).

[8] B. S. Tsirelson. *J. Soviet Math.* **36,** 557-570 (1987)

[9] A. Acín, N. Gisin and B. Toner *Phys. Rev.* A **73,** 062105 (2006)

[10] A. M. Davies (unpublished note, 1984) and J. A. Reeds (unpublished note, 1991) showed that $K_G \geq 1.6770$ while J. Krivine Adv. Math. **54,** 16 (1979) showed that $K_G(3) \leq 1.5163$.

[11] J. Briët, H. Bhurman and B. Toner arXiv:quant-ph/0901.2009

[12] C. Brukner, S. Taylor, S. Cheung and V. Vedral arXiv:quant-ph/0402127

[13] M. Wolf and D. Perez-García. arXiv:quant-ph/0901.2542

[14] J. S. Bell. "Speakable and Unspeakable in Quantum mechanics". *Cambridge University press* (1987)

[15] A. J. Leggett and A. Garg *Phys. Rev. Lett.* **54,** 857 (1985)

[16] A. Peres *Phys. Rev. Lett.* **61,** 2019 (1985)

[17] J. Clauser, M. Horne, A. Shimony and R. Holt, *Phys. Rev. Lett.* **23,** 880 (1969)

[18] A measurement is degenerate when it produces the same outcome for all the states in a certain Hilbert Space. In this case we are refering to measurements with outcome +1 (or -1) for all qubits. The matrix form of such operator is $\mathbb{I}$ (or $-\mathbb{I}$).

[19] J. Schlienz and G. Mahler *Phys. Rev.* A **52,** 4396 (1995)