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Author: Xavier Ramos Olivé

Advisor: Narciso Román-Roy

Department: Matemàtica Aplicada IV

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Bachelor Thesis - Treball de Fi de Grau

An Introduction to General Relativity

Xavier Ramos Olivé

Advisor: Narciso Román-Roy

Matemàtica Aplicada IV

Nature likes theories that are simple when stated in coordinate-free, geometric language.

Nilsk Oeijord

Preface

This is a bachelor's degree thesis of the Bachelor on Mathematics ("Grau de Matemàtiques"), a degree included in the EHEA (European Higher Education Area), offered by the Technical University of Catalonia ("Universitat Politècnica de Catalunya"). As I belong to the first generation of EHEA students, it is the first year that this kind of thesis is done in this university, so I would like to thank the effort that all the professors, employees and students of the university, and specifically of the Faculty of Mathematics and Statistics ("Facultat de Matemàtiques i Estadística"), have made to help us during the elaboration of these thesis.

I would like to specially acknowledge and thank the effort of my tutor Professor Narciso Román-Roy, for his time and patience, and to Professor Xavier Gràcia, for allowing me to assist to his master lectures on Differentiable Manifolds that helped me a lot in this thesis. I would like to extend these acknowledgements to all the people that have helped me to write this paper in one or another way. I hope you will enjoy it!

Abstract

Key words: Pseudoriemannian Geometry, Differential Geometry, General Relativity

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This bachelor's degree thesis is an introduction to the Theory of General Relativity (GR), a relativistic theory of gravity, from the point of view of a recently graduated mathematitian. The principles of GR are stated and some motivation on the formulation of the theory is provided. It is shown that freely-falling particles move along geodesics of spacetime and Einstein's equations are derived as a generalization of Newton's gravity. The uniqueness of Einstein's equations and the presence of the cosmological constant are discussed.

No knowledge is assumed neither in Special or General Relativity, nor in Pseudoriemannian Geometry, but knowledge of basic Differential Geometry on Manifolds is highly recommended, although there is a brief introduction on this topic at the beginning.

The thesis concludes finding Schwarzschild solution by assuming that there exists a spherically symmetric metric that is a solution to Einstein's equations in vacuum and seeing what properties should this metric have. The boundary conditions imposed are the existence of a punctual uncharged mass at the origin and flatness of the metric at infinity. The result is a particular solution that can be applied in many contexts, such as in the Solar System.

Contents

Introduction

This text is a bachelor thesis about the Theory of General Relativity. Many articles have been written on this topic and there are plenty of related fields that can be studied, but the goal of this paper is not so ambicious; its objective is to give an introduction to the topic comprehensible from the point of view of a recently graduated mathematician. More precisely, we will study the geometric background needed to deal with the theory of General Relativity, we will introduce some results of Special Relativity, we will deduce Einstein's Equations and we will find a special solution to them: Schwarzschild's solution. The only requirement to understand it is to have some knowledge of differential geometry on manifolds (although there is a brief review on it in the thesis), and curiosity about the natural phenomena that makes us keep our feet on the ground: gravity.

Einstein's papers from 1905 to 1915 were actually a revolution in the scientific community; his new theory wasn't only important because of solving the inconsistencies between classical mechanics and electromagnetism, but also because his new theory brought to physicists a new point of view of the mathematics involved in physical fenomena: the noneuclidean geometries studied during the 19th century by Gauss and Riemann, among others, were not just a mathematical game. Space and time were described in Einstein's Theory of Special Relativity (SR) as a single 4-dimensional manifold (called spacetime) provided with a non-positive-defined constant and uniform metric.

However, Einstein realized that SR, although being successful in many experiments, was not able to describe gravity; this brought him to develop the Theory of General Relativity (GR), that is essentially a theory of gravity in the context of SR (GR reduces to SR locally). This theory is, probably, one of the most beautiful theories from a mathematical point of view. Contrasting with SR, where there existed a special class of frames, called inertial frames (bodies moving at a constant velocity from us), in GR any reference frame is valid for describing physics. Moreover, in general, the variables involved in GR (the coordinate functions of spacetime) have no longer a clear physical meaning, they are, as physicists would say, generalized variables: there is no distinction between time and space variables. This is beautiful from a mathematician's point of view: GR is a geometrical theory.

The outgrowth of SR and GR not only revolutionised physics, but also mathematics. The geometry of the new theories needed the study of pseudoriemannian manifolds: manifolds with non-positive-defined metrics, that were not so popular as they are today before Einstein's articles, maybe because of their unnatural properties (such as that they have some vectors with negative norm, or the existence of non-zero vectors with null norm). The first chapter on this text is devoted entirely to the study of pseudoriemannian geometries.

The second chapter formulates SR and introduces some relevant results for GR. It also contains the principles of GR, some simple ideas that guided Einstein towards his theory. One of them is specially remarkable from the point of view of geometry: the laws of physics must be invariant under changes of coordinates. This points out that they should be formulated in a coordinate-free environment: for example, using equalities between tensor fields.

GR considers gravity as a result of spacetime curvature. As it was known since Newton's times, mass is the source of gravity, so as gravity is now explained in terms of the metric tensor, GR has to provide the relation between the metric of spacetime and its mass distribution. Not only mass, but also, as Einstein saw in SR that there exists an equivalence between mass and energy (with the famous formula $E = mc^2$), any sort of energy influences the metric; all these sources of gravity are expressed in a compact tensor called the Energy-Momentum Tensor. In the third chapter of this thesis, I deduce Einstein's Equation, that gives the relation between this tensor and the metric tensor, and that constitutes the heart of GR. I also show why freely falling particles move along geodesics of spacetime (another beautiful result of GR).

To conclude with, we should notice that Einstein's equations are not easy to solve in general: they are a system of 10 second-order nonlinear PDEs (although just 6 of them are independent). Nevertheless, assuming spherical symmetry, we can find easily Schwarzschild's solution. This solution is really important, it was found by the astronomer Karl Schwarzschild in 1916, and it describes the metric of spacetime in vacuum, when there is just a spherical (or punctual) mass at the origin. It was the first nontrivial solution found and it describes the behavour of gravity near spherical bodies, such as stars, planets or satellites. In the last chapter I will discuss how to find this solution.

Notation remarks: In the following, we will use Einstein summation convention, i.e., if equal indices appear on different levels, summation over all the indices will be assumed. For example: $a_i dx^i = a_1 dx^1 + a_2 dx^2 + \ldots + a_n dx^n$. On the other hand, all the objects that we will use (manifolds, maps, fields, etc.) are supposed to be as regular as necessary; if it is not specified, we can assume that they are always smooth objects.

Chapter 1
Pseudoriemannian Geometry

In this section we are going to see the main geometrical tools needed to properly formulate the Theory of General Relativity (GR); it will be a rather short introduction to Pseudoriemannian (or Semiriemannian) Geometry, so the readers interested in a deeper insight into this field should see the references [**15**], [**13**] or [**10**].

1. General notions on Manifolds

This is a brief review from the basic notions learned in the courses of Differential Geometry and Differentiable Manifolds; you can use the notes on [**7**] as a trustable source. Let's remember some concepts:

A **differentiable manifold** is a topological space M together with an m-dimensional differentiable structure, i.e., a maximal collection of pairs $\{(U_\alpha, \phi_\alpha)\}$ where U_α is an open set of M, $M = \bigcup_\alpha U_\alpha$, and $\phi_\alpha : M \to \mathbb{R}^m$ are homeomorphisms such that, in the corresponding domain, $\phi_\alpha \phi_{\alpha'}^{-1}$ and $\phi_{\alpha'} \phi_\alpha^{-1}$ are differentiable functions (they are diffeomorphisms).

On every point $p \in M$ we can define its **tangent space** T_pM as the classes of equivalence of curves $\gamma : I \subset \mathbb{R} \to M$ that have the same tangent vector, i.e., given a chart (U, ϕ) with $p \in U$, $\gamma \sim \hat{\gamma}$ if $D(\phi \circ \gamma)(0) = D(\phi \circ \hat{\gamma})(0)$, assuming $\gamma(0) = \hat{\gamma}(0) = p$. In fact, there exists an isomorphism between T_pM and the space of **punctual derivations** of $C^\infty(M)$ in p, i.e., the $\mathbb{R}$-linear maps $\delta : C^\infty(M) \to R$ such that $\delta(fg) = \delta(f)g + f\delta(g)$ (the isomorphism is the one that brings a tangent vector u to the directional derivative along u, that we will denote $\mathcal{L}_u$). From this new point of view, fixed a chart (U, ϕ), we can define a basis of T_pM as the directional derivatives along the coordinate vectors $\{\frac{\partial}{\partial x^i}(p)\}$. The corresponding dual space, T_p^*M, is called **cotangent space** and has an associate dual basis $\{dx^i(p)\}$.

Then we can define the **tangent bundle** as the set of all possible tangent vectors on M, $TM = \bigsqcup_{p\in M} T_pM$, and analogously the **cotangent bundle**, $T^*M = \bigsqcup_{p\in M} T_p^*M$. These sets are, indeed, manifolds of dimension $2m$. Then we can define the projection $\pi_M : TM \to M$, and the set of **vector fields** $\mathfrak{X}(M)$ as the set of all sections of the tangent bundle; i.e., a vector field is a map $X : M \to TM$ such that $\pi_m \circ X = Id_M$. Similarly, we define the projection $\tau_M : T^*M \to M$ and the set of **one-forms** $\Omega^1(M)$ is the set of all sections of the cotangent bundle, so a one-form is a map $\omega : M \to T^*M$ such that $\tau_M \circ \omega = Id_M$.

In a more general context, we can define at each point the tensor product space $Tens^k_l(T_pM) = \bigotimes^l T^*_pM \bigotimes^k T_pM$, with $k, l \in \mathbb{N}$, and the associate **tensor bundle** $Tens^k_l(TM) = \bigsqcup_{p\in M} \bigotimes^l T^*_pM \bigotimes^k T_pM$, that brings us to define a **k-contravariant l-covariant tensor field** as a section of $Tens^k_l(TM)$, and we will denote the set of this kind of tensor fields as $\mathfrak{T}^k_l(M)$, and in general, the set of tensor fields of any order as $\mathfrak{T}(M)$. Notice that, $\forall p \in U \subset M$, we can write in local coordinates a tensor field $R \in Tens^k_l(TM)$ as:

$$R(p) = R^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_l}(p) \frac{\partial}{\partial x^{j_1}}\Big|_p \otimes \frac{\partial}{\partial x^{j_2}}\Big|_p \otimes \ldots \otimes \frac{\partial}{\partial x^{j_l}}\Big|_p \otimes d_px^{i_1} \otimes d_px^{i_2} \otimes \ldots \otimes d_px^{i_k}$$

Remembering that $\mathcal{L}_u$ is the directional derivative along u, we can understand vector fields as derivations:

Definition 1.1. *Given $X \in \mathfrak{X}(M)$, if $W \subset M$ is an open set and $f : W \to \mathbb{R}$ is a function, the* **Lie derivative** *along X is the function $\mathcal{L}_X f : W \to \mathbb{R}$ defined as $(\mathcal{L}_X f)(p) = \mathcal{L}_{X_p} f(p)$.*

With this new tool, we can define a useful operator that measures up to what point two vector fields commute:

Definition 1.2. *If $X, Y \in \mathfrak{X}(M)$ the* **Lie bracket** *of X and Y is the derivation $[X, Y] := \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X$.*

In coordinates, if (U, φ) is a chart, $X\Big|_U = f^i \frac{\partial}{\partial x^i}$, $Y\Big|_U = g^i \frac{\partial}{\partial x^i}$, then:

$$[X, Y]\Big|_U = \left(f^i \frac{\partial g^j}{\partial x^i} - g^i \frac{\partial f^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$$

In particular, $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$

2. Covariant Derivative

In this thesis, we will be interested in writing down differential equations that are invariant under changes of coordinates to describe physical phenomena in the same way from any reference frame. However, the natural concept of partial derivative doesn't fit this requirement. The concept of a covariant derivative is one of the main geometrical tools that we are going to use and it has to be regarded as a generalisation of the concept of partial derivative.

Definition 1.3. *A* **covariant derivative** *or a* **connection** *defined on M is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $\nabla(X, Y) = \nabla_X Y$, satisfying:*

a) $\nabla_X Y$ is $\mathbb{R}$-linear on Y

b) $\nabla_X Y$ *is* $C^\infty(M)$*-linear on* X
c) $\nabla_X(fY) = (L_X f)Y + f\nabla_X Y$, $\forall f \in C^\infty(M)$

Definition 1.4. *If* (U,ϕ) *is a chart of* M *and* $\frac{\partial}{\partial x^i}$ *are the corresponding coordinate vector fields, that form a basis of* $\mathfrak{X}(U)$*, we define the* **Christoffel symbols of** ∇ *with respect to this basis,* $\Gamma^k_{ij} \in C^\infty(M)$*, as:*

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}$$

This definition can be extended to an arbitrary basis (not necessarily a coordinate basis) of $\mathfrak{X}(U)$ $\{E_i\}$.

Proposition 1.5. *There exists a bijection between the set of connections on* M *and the set of Christoffel symbols (sets of* n^3 *functions* $\Gamma^k_{ij} \in C^\infty(M)$*). If* $X = f^i E_i$ *and* $Y = g^j E_j$, $\nabla_X Y = (L_X g^k + \Gamma^k_{ij} f^i g^j)E_k$.

The notion of covariant derivative of vector fields can be extended to the **covariant derivative of a tensor field**.

Theorem 1.6. *There exists a unique mapping* $\nabla_X : \mathfrak{T}(M) \to \mathfrak{T}(M)$ *such that:*

- $\forall f \in C^\infty(M)$, $\nabla_X f = L_X f$
- $\forall Y \in \mathfrak{X}(M)$, $\nabla_X Y$ *is the connection defined in 1.1*
- ∇_X *is* $\mathbb{R}$*-linear*
- ∇_X *applies* $\mathfrak{T}^k_l(M)$ *to* $\mathfrak{T}^k_l(M)$
- $\nabla_X(R \otimes S) = (\nabla_X R) \otimes S + R \otimes (\nabla_X S)$
- ∇_X *commutes with inner contractions*

In particular, $\nabla_{\frac{\partial}{\partial x^i}} dx^j = -\Gamma^j_{ik} dx^k$.

It can be seen that the covariant derivative $\nabla_X Y(p)$ depends only on the value of X_p and the value of Y in a neighbourhood of p. Moreover, it only depends on the value of Y along a path through $\gamma(t)$ such that $\gamma(0) = p$, when $t \in (-\epsilon, \epsilon)$. We will denote the set of vector fields along a path γ as $\mathfrak{X}(\gamma)$.

Proposition 1.7. *If* $\gamma : I \to M$ *is a path, there exists a unique map* $\nabla_t : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$ *such that:*

(1) *It is* $\mathbb{R}$*-linear*
(2) $\forall f \in C^\infty(I),\ \forall \omega \in \mathfrak{X}(\gamma),\ \nabla_t(f\omega) = \frac{df}{dt}\omega + f\nabla_t \omega$
(3) *If* $Y \in \mathfrak{X}(M), \nabla_t(Y \circ \gamma)(t) = \nabla_{\gamma'(t)} Y$

The operator ∇_t *is called the* **covariant derivative along** γ.

This operator can also be generalized to a **covariant derivation of a tensor field along** γ: the additional axioms we need to define it are the same as in the definition of the covariant derivation of a tensor field.

3. Parallel Transport and Geodesics

When studying GR, geodesics (in a certain way, the straightest possible curves in the manifold) are an imprescindible topic to cover, because in fact, as we will see later, bodies influenced by a gravitational field travel along a geodesic of spacetime (this is the simplest way they can behave, it is very similar to the idea of Newton's first law of motion). To define a geodesic, first we have to introduce the concept of parallel transport: if we have the notion of parallel transporting a vector field along a path, then a path whose tangent vector is always parallel (the straightest possible curve) will be defined as a geodesic.

Definition 1.8. *Given a path $\gamma : I \to M$, a vector field $X \in \mathfrak{X}(\gamma)$ is said to be* **parallel transported** *along $\gamma(t)$ if it satisfies:*

$$\nabla_t X = 0, \ \forall t \in I$$

In a chart of coordinates $(U, \phi = (x^i))$, where $X = X^k \frac{\partial}{\partial x^k}$ (assuming γ is entirely defined in U), this condition can be expressed as a 1st order linear system of differential equations:

$$\nabla_t X = \left(\frac{dX^k}{dt} + (\Gamma^k_{ij} \circ \gamma) \frac{d(x^i \circ \gamma)}{dt} X^j \right) \frac{\partial}{\partial x^k} \circ \gamma = 0$$

As an immediate result of writing down the definition of parallel transport in coordinates we obtain the next result, using Picard's theorem:

Proposition 1.9. *Given $t_0 \in I$ and $X_0 \in T_{\gamma(t_0)}M$, there exists a unique $X \in \mathfrak{X}(\gamma)$ such that it is parallel transported along γ and $X(t_0) = X_0$. We say that X is the* **parallel transport of X_0 with respect to the connection**.

Then we can define the concept of geodesic. The definition comes as a generalisation of the concept of a straight line in euclidean space; one of the properties of straight lines is that their tangent vector is always parallel to themselves.

Definition 1.10. *A path $\gamma : I \to M$ is a* **geodesic** *of (M, ∇) if γ' is parallel along γ, i.e., $\nabla_t \gamma' = 0$. This can also be written in coordinates, and we obtain the 2nd order non linear autonomous system:*

$$\nabla_t \gamma' = \left(\frac{d^2}{dt^2}(x^k \circ \gamma) + (\Gamma^k_{ij} \circ \gamma) \frac{d(x^i \circ \gamma)}{dt} \frac{d(x^j \circ \gamma)}{dt} \right) \frac{\partial}{\partial x^k} \circ \gamma = 0$$

4. Curvature and Torsion

So far, we have introduced the necessary tools to construct some objects that describe the local geometry of M: the curvature and the torsion.

Definition 1.11. *Given a manifold M with a connection ∇, the* **Riemann curvature tensor** *or* **Riemann tensor** *of M is defined as the map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ (notice that $R \in \mathfrak{T}^1_3(M)$). Sometimes it is denoted as $R(X,Y)Z$.*

We can compute its expression in coordinates. If E_i is a basis of $\mathfrak{X}(U)$, E^i being the corresponding dual basis of $\mathfrak{X}^*(U)$, and writing $\nabla_{E_i} E_j = \Gamma^\sigma_{ij} E_\sigma$, $[E_i, E_j] = c^\sigma_{ij} E_\sigma$ and $R = R^l{}_{ijk} E_l \otimes E^i \otimes E^j \otimes E^k$, we have:

$$\begin{aligned} R^l{}_{ijk} &=< E^l, R(E_i,E_j)E_k > \\ &=< E^l, \nabla_{E_i}\nabla_{E_j}E_k - \nabla_{E_j}\nabla_{E_i}E_k - \nabla_{[E_i,E_j]}E_k > \\ &=< E^l, \nabla_{E_i}(\Gamma^\sigma_{jk}E_\sigma) - \nabla_{E_j}(\Gamma^\sigma_{ik}E_\sigma) - \nabla_{c^\sigma_{ij}E_\sigma}E_k > \\ &=< E^l, L_{E_i}(\Gamma^\sigma_{jk})E_\sigma + \Gamma^\sigma_{jk}\Gamma^\rho_{i\sigma}E_\rho - L_{E_j}(\Gamma^\sigma_{ik})E_\sigma - \Gamma^\sigma_{ik}\Gamma^\rho_{j\sigma}E_\rho - c^\sigma_{ij}\Gamma^\rho_{\sigma k}E_\rho > \\ &= L_{E_i}(\Gamma^l_{jk}) + \Gamma^\sigma_{jk}\Gamma^l_{i\sigma} - L_{E_j}(\Gamma^l_{ik}) - \Gamma^\sigma_{ik}\Gamma^l_{j\sigma} - c^\sigma_{ij}\Gamma^l_{\sigma k} \end{aligned}$$

In a coordinate basis, $c^k_{ij} = 0$, so the coefficients $R^l{}_{ijk}$ are antisymmetric with respect to the first two indices, i.e., $R^l{}_{ijk} = -R^l{}_{jik}$. Notice that the position of the indices is important, and that the notation that I have decided to use may differ from other notations used in the sources that appear in the bibliography.

Definition 1.12. *The* **torsion tensor** *of ∇ is the map $T : \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$, $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ (notice that $T \in \mathfrak{T}^1_2(M)$).*

In coordinates, similarly as what we have done before, if $T = T^k_{ij} E_k \otimes E^i \otimes E^j$, we obtain:

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji} - c^k_{ij}$$

Again, using a coordinate basis ($c^k_{ij} = 0$) we obtain the symmetry $T^k_{ij} = -T^k_{ji}$, and also that $T^k_{ij} = 0 \iff \Gamma^k_{ij} = \Gamma^k_{ji}$, i.e., the Christoffel symbols are symmetric in their lower indices. This is the reason to call ∇ **symmetric** or **torsionless** when $T = 0$.

When dealing with GR, we need to define some objects that derive from the curvature; one of those objects is the Ricci tensor.

Definition 1.13. *We define the* **Ricci tensor** *as $Ric(Y,Z) :=< E^i, R(E_i,Y)Z >$. Then, in a coordinate basis, we will usually use the notation for the components $R_{jk} := R^i{}_{ijk}$.*

5. Pseudoriemannian Manifolds

As we mentioned above, in Einstein's theory of relativity space and time are hold together in a 4-dimensional manifold that has associated a metric; this metric can be though as a kind of inner product in the tangent bundle, but has some special characteristics that doesn't allow us to say that it is an inner product in the usual sense that we studied in Linear Algebra. A manifold together with this kind of metric is called a Pseudoriemannian Manifold.

Definition 1.14. *Given a manifold M, a* **pseudoriemannian** *or* **semiriemannian metric** *in M is a tensor field $g \in \mathfrak{T}_2^0(M)$ symmetric and nondegenerate, i.e., it satisfies:*

(1) $g_p(U,V) = g_p(V,U),\ \forall U,V \in T_pM,\ \forall p \in M$
(2) $g_p(U,V) = 0,\ \forall U \in T_pM \implies V = 0,\ \forall p \in M$

The couple (M,g) is called a **pseudoriemannian manifold***. If g is also positive defined ($g_p(U,U) \geq 0,\ g_p(U,U) = 0 \iff U = 0,\ \forall U \in TpM,\ \forall p \in M$), then the metric and the manifold are called* **riemannian***.*

In coordinates $g = g_{\mu\nu} E^\mu E^\nu$. Notice that the metric allows us to stablish an isomorphism $\hat{g} : TM \to T^*M$: $< \hat{g_p}(u_p), v_p >= g_p(u_p, v_p)$; this is what is known colloquially as the operations of **rising and lowering indices** (because $g_{\mu\nu}E^\mu = E_\nu$ and using the inverse of the metric $g^{\mu\nu}E_\mu = E^\nu$, just by deffinition of the metric coefficients). Using this new operation, the metric allows us to compute a new parameter related to the curvature of the manifold, the scalar curvature:

Definition 1.15. *The* **scalar curvature** *is defined as $R = R^\nu{}_\nu = g^{\mu\nu} R_{\mu\nu} \in \mathfrak{T}_0^0(M) = C^\infty(M)$.*

We see that the metric is a tool that allows us to define the inner product of two vectors. Now we may be interested in studying the covariant derivatives ∇ on M that make the inner product of X and Y remain constant when they are parallel transported.

Definition 1.16. *A connection ∇ is said to be* **metric compatible** *with (M,g) or a* **metric connection** *if it satisfies $\nabla g = 0$, or equivalently $\nabla_X g = 0,\ \forall X \in \mathfrak{X}(M)$.*

Lemma 1.17. $\nabla g = 0 \iff L_Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y),\ \forall X,Y,Z \in \mathfrak{X}(M)$

Proof. Using that $g(X,Y) =< \hat{g}(X), Y >$ and the commutativity of ∇ with respect to inner contractions:

$$\begin{aligned}
L_Z(g(X,Y)) &= \nabla_Z < \hat{g}(X), Y > \\
&= \nabla_Z << g, X >, Y > \\
&=<< \nabla_Z g, X >, Y > + << g, \nabla_Z X >, Y > + << g, X >, \nabla_Z Y > \\
&= (\nabla_Z g)(X,Y) + g(\nabla_Z X, Y) + g(X, \nabla_Z Y)
\end{aligned}$$

And then both implications are trivial. ∎

In general, if $N \subset M$ is a submanifold, N is not a semiriemannian manifold with the semiriemannian metric induced by the metric on M (this is always true in a riemannian manifold), so allowing the metric to be negative defined has deep consequences. Nevertheless, there are some interesting properties that hold for any pseudoriemannian manifold, such as the **fundamental theorem of pseudoriemannian geometry**:

Theorem 1.18. *On a pseudoriemannian manifold (M, g) there exists a unique torsionless connection wich is compatible with the metric g. This connection is called the* **Levi-Civita connection**.

Proof. If ∇ exists, then by the condition of being metric compatible and using the previous lemma, we can write the following expressions:

$$L_X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$
$$L_Y(g(Z,X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X)$$
$$L_Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

Adding up the first two equations and subtracting the third one, we obtain:

$$L_X(g(Y,Z)) + L_Y(g(Z,X)) - L_Z(g(X,Y)) =$$
$$= g(X, \nabla_Y Z - \nabla_Z Y) + g(Y, \nabla_X Z - \nabla_Z X) + g(Z, \nabla_X Y + \nabla_Z X)$$

Using now that ∇ is torsionless: $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$, we obtain:

$$L_X(g(Y,Z)) + L_Y(g(Z,X)) - L_Z(g(X,Y)) =$$
$$= g(X, [Y,Z]) + g(Y, [X,Z]) + g(Z, 2\nabla_X Y - [X,Y])$$

Thus, now we can compute the Christoffel symbols choosing $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$ and $Z = \frac{\partial}{\partial x^k}$, where de Lie brackets are 0 because of Schwarz's Theorem:

$$2g(\Gamma^l_{ij}\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}) = \frac{\partial}{\partial x^i} g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) + \frac{\partial}{\partial x^j} g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}) - \frac{\partial}{\partial x^k} g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$$

So, as $g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = g_{ij}$, we obtain:

$$2\Gamma^l_{ij} g_{lk} = \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}$$

And thus we find the corresponding Christoffel symbols (so, by virtue of the bijection between Christoffel symbols and connections, we have effectively defined a unique connection, and it satisfies de properties, as it can be checked):

$$\Gamma^l_{ij} = \frac{1}{2} g^{lk} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right)$$

□

The curvature tensor has n^4 components, but it contains many symmetries. Some of them appear just when considering the Levi-Civitta connection. For instance, we have that with the Levi-Civitta connection:

Proposition 1.19. *The Riemann curvature tensor has the properties:*

(1) ${R^\rho}_{\mu\nu\sigma} = -{R^\sigma}_{\mu\nu\rho}$
(2) ${R^\rho}_{\mu\nu\sigma} = {R^\mu}_{\rho\sigma\nu}$
(3) ${R^\rho}_{\mu\nu\sigma} + {R^\rho}_{\nu\sigma\mu} + {R^\rho}_{\sigma\mu\nu} = 0$
(4) The **Bianchi identity**: $\nabla_\lambda {R^\rho}_{\mu\nu\sigma} + \nabla_\rho {R^\sigma}_{\mu\nu\lambda} + \nabla_\sigma {R^\lambda}_{\mu\nu\rho} = 0$

We will not prove these identities here, but notice that to prove them we just need to compute them in an adequate frame: since they are tensorial equations, they will hold in any other frame. These symmetries induce symmetries on the Ricci tensor too:

Lemma 1.20. *The Ricci tensor is symmetric.*

Proof. It is deduced from the Riemann tensor's symmetries: in particular, from the second property, that says that ${R^\rho}_{\mu\nu\sigma} = {R^\mu}_{\rho\sigma\nu}$, so: $R_{\nu\sigma} = {R^\rho}_{\rho\nu\sigma} = {R^\rho}_{\rho\sigma\nu} = R_{\sigma\nu}$. Notice that we are assuming that ∇ is the Levi-Civitta connection. □

6. Example

An easy but useful example of a pseudoriemannian manifold is the **Minkowski spacetime**; it is the spacetime of Special Relativity (SR), a 4-dimensional manifold with the metric $\eta_{\mu\nu}$ defined as $\eta_{0\nu} = \delta^0_\nu$ and $\eta_{ij} = -\delta^i_j$, i.e.:

$$\eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Sometimes it is alternatively defined as $-\eta$, but we won't care about that here. Notice that this metric allows a vector to have positive, null or negative norm; we will call it **timelike**, **lightlike or null** or **spacelike** respectively.

Using the formula of the section above, we can find the Christoffel symbols corresponding to its associated Levi-Civita connection; since the metric is constant, it is clear that they are the constant functions:

$$\Gamma^l_{ij} = 0$$

Now, using the coordinate expression of Riemann's curvature tensor:

$$ {R^\mu}_{\nu\sigma\rho} = 0$$

And, thus, $R_{\mu\nu} = 0$ and $R = 0$. This explains why the Minkowski spacetime is said to be a **flat spacetime**.

Chapter 2
Foundations of General Relativity

Now that we have defined the main tools we need to formulate the Theory of General Relativity, we are going to see what ideas and postulates are used to build it up. There are many sources covering this topic; I have used mainly [**3**], [**14**], [**2**] and [**4**], but also, specially for a rigorous discussion, [**8**] and [**12**]. To start with, we are going to see a brief review of the Theory of Special Relativity (SR), that mainly can be found in [**9**].

1. Special Relativity

SR is the solution to the incongruencies between classical mechanics and electromagnetism that appeared in the 19^{th} century. It was a problem that had been studied by many physicists and mathematicians, such as Lorentz, Poincaré or Hilbert. But it was Albert Einstein who in his famous articles from 1905 proposed the solution (Minkowski gave it a geometrical interpretation later on).

One of the problems that 19^{th} century physicists dealt with is the noninvariance of Maxwell's equations under galilean transformations. SR tries to reformulate newtonian mechanics accepting as true the Maxwell equations, and looking for a new set of coordinate transformations (the **Lorentz transformations**) that leave all the physical laws invariant in any inertial frame transformation.

Its startpoint is the fact that the velocity of light is a constant of the Universe (something intrinsic in Maxwell equations and confirmed by several experiments), together with the natural idea that the laws of physics have to be the same in any inertial frame (a specific equivalence class of frames). This discards the idea of an absolute time and space, it has to be replaced by a 4-dimensional space, called **spacetime**.

Moreover, we can observe that, if the velocity of light has to be the same in any frame, there exists a quantity relating the points of spacetime that are connected by a light ray that is invariant under changes of frames: the **relativistic interval** $\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$. In fact, in SR this is always an invariant quantity, and can be thought as the pseudoriemannian metric of Minkowski spacetime that we saw in Chapter 1. It is a **Lorentzian metric**, a metric with signature $(1, n)$ (it has one positive and n negative eigenvalues) in a manifold of dimension $n+1$. This shows a hyperbolic symmetry on spacetime: the set of points of constant distance is given by the equation of the hyperbola $(\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2 = C$.

Summarizing, we can formulate the basic postulates of SR as:

(1) Time and space are not independent. They are held together in the **Minkowski spacetime** M, a 4-dimensional semiriemannian manifold with signature (1,3), who's metric is constant and is denoted as η ($\eta_{00} = 1$, $\eta_{\mu i} = -\delta_{\mu i}$).
(2) There exists an equivalence class of frames (the **inertial frames**) where the laws of physics (mechanics and electromagnetism) are valid. The velocity between them is always constant.
(3) The velocity of light is a constant of the Universe (it takes the same value in any inertial frame).

From here, the Lorentz transformations are deduced as the linear isometries of (M, η), as they have to keep constant Δs (the metric); the ones that do not reverse neither the time nor any spacial variable are of special interest in physics. If $v = (v_x, v_y, v_z)$ is the velocity between two inertial frames, and we denote $\beta_i = \frac{v_i}{c}$ (c is the velocity of light in vacuum), $\beta = \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2}$ and $\gamma = \frac{1}{1-\beta^2}$, then a general Lorentz transformation that does not reverse any variable would be:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+(\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & (\gamma-1)\frac{\beta_y\beta_x}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & (\gamma-1)\frac{\beta_z\beta_x}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^2}{\beta^2} \end{pmatrix} \cdot \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} + \begin{pmatrix} ct_0 \\ x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

1.1. Relativistic Kinematics

Definition 2.1. *The* **proper time interval** *between two events is the elapsed time measured by an observer for which the two events happen in the same point of space. The* **proper time** *for a worldline $\gamma : I \subset R \to M$ between two events $s_1 = \gamma(t_1)$ and $s_2 = \gamma(t_2)$ is:*

$$\tau = \frac{1}{c}\int_{t_1}^{t_2} ds$$

where $ds = \eta_{\mu\nu} dx^\mu dx^\nu$.

Definition 2.2. *The* **proper length** *between two events is the spacial distance between them measured in an inertial frame S where both events happen in points that are at rest.*

Some easy results that can be derived from the Lorentz transformations (when written in a friendlier way, choosing adequately the two inertial frames) are the **Proper Time Dilation** and the **Proper Length Contraction**:

Proposition 2.3. *If γ relates the inertial frame in which a time T is measured between two events and the frame where proper time T_0 is measured, then:*

$$T = T_0\gamma$$

Proposition 2.4. *If γ relates the inertial frame in which a length L is measured between two events and the frame where proper length L_0 is measured, then:*

$$L = \frac{L_0}{\gamma}$$

Notice that, since Galilean transformations are no longer valid, if a first inertial frame S sees a second one S' moving with velocity v, and the second one observes a particle moving with velocity v' on the same direction (it can be considered as a third inertial frame), it is not true that the first one sees the third one moving at velocity $v + v'$, as one would expect. The correct version of the **Law of addition of velocities** is:

Theorem 2.5. *Consider three inertial reference frames S_0, S_1 and S_2 such that S_1 moves with velocity v_1 with respect to S_0, S_2 moves with velocity v_2 with respect to S_1, their axes are parallel and they are moving along one of them. Then, the velocity between S_2 and S_0 is:*

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

1.2. Relativistic Dynamics

Consider a worldline $x^\mu(\tau)$ in M (a curve in M with timelike tangent vector) parameterised by the proper time; it can be though to be the trajectory of some massive particle.

Definition 2.6. *The* **four-velocity** *of a particle with worldline $x^\mu(\tau)$ is $U^\mu(\tau) = \frac{\partial x^\mu}{\partial \tau}$.*

Using the chain rule, and that $\frac{\partial t}{\partial \tau} = \gamma$ (because of time dilation), it is clear that in an inertial frame where a particle moves with velocity $v = (v^1, v^2, v^3)$ (not necessarily constant) its four-velocity is:

$$U = \gamma(c, v^1, v^2, v^3)$$

Definition 2.7. *The* **four-acceleration** *of a particle with worldline $x^\mu(\tau)$ is $A^\mu = \frac{\partial U^\mu}{\partial \tau} = \frac{\partial^2 x^\mu}{\partial \tau^2}$.*

Definition 2.8. *The* **rest mass** *or* **proper mass** *of a particle is the mass m_0 measured in a reference frame where the particle is at rest.*

Definition 2.9. *The* **four-momentum** *or* **four-vector enregy-momentum** *is $P^\mu = m_0 U^\mu$, so $P = m_0\gamma(c, v^1, v^2, v^3)$. The magnitude $m = m_0\gamma$ is called* **inertial relativistic mass**.

Notice that in many of the definitions we are giving is implicit the fact that c is a limit velocity for massive particles; for instance, the inertial relativistic mass represents the mass measured by an inertial frame in which the particle is moving at velocity v, where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. So we have that $m \to \infty$ when $v \to c$.

Definition 2.10. *The* **Total Relativistic Energy** *is* $E = P^0 c = mc^2$.

Then $P = (E/c, \mathbf{p})$, where $\mathbf{p}$ represents the linear momenum of the particle, and it can be seen that $E = \sqrt{(m_0c^2)^2 + c^2||\mathbf{p}||^2}$. As $E = m_0c^2$, $E_0 = m_0c^2$ is called the **energy at rest**.

Definition 2.11. *The* **four-force** *is defined as* $F^\mu = m_0 A^\mu$.

Theorem 2.12. *The relation between the four-momentum and the four-force is:*

$$\frac{\partial P}{\partial \tau} = F$$

This is the **general equation of relativistic dynamics**.

1.3. Gravity

Up to this point, we may ask ourselves: if we have constructed a dynamical theory of relativity, why doesn't SR describe gravity?

One of the main experimental facts that currently supports GR is that light is affected by gravity; a gravitational field can bend a light ray, so one of the postulates of SR is not true! This was first observed by Eddington in 1919, and has been observed more recently using the Hubble telescope. Of course, as this was not observed until 1919, it wasn't the reason for constructing GR.

Another reason is Bondi's thought experiment, whih provided a perpetual motion machine. He imagined a vertical circular belt submitted to gravity, with some atoms attached. The atoms in the right-hand side are excited, so by the formula $E = mc^2$ they are heavier than the ones in the left-hand side; this makes the belt move. When an excited atom reaches the lowest point in the belt, it emmits a photon that, using a mirror, is absorbed by the non-excited atom in the top, and this keeps the belt in a perpetual motion. The way out in GR is that photons lose energy as they climb through the gravitational field: they are red-shifted. Such a shift was measured directly by Pound and Rebka in 1959. But this red-shift is incompatible with SR, as it considers that gravity can not affect light's velocity.

But the definitive reason was the one concerning the equivalence principle, that we will explain in the next section.

2. Postulates of General Relativity

Classical mechanics and the special theory of relativity distinguish between two kinds of bodies: reference-bodies relative to which the recognised "laws of nature" can be said to hold, and reference-bodies relative to which these laws do not hold. We would like to generalize the theory of SR to accelerating frames. In particular, Einstein's principle of relativity has to be modified, it has to be extended to these frames.

Let me reproduce an example that appears in [**4**]. Consider a Galileian reference-body, for example a room with an observer inside in a large portion of empty space, far from stars and other appreciable masses. A rope attached in the middle of the ceiling of the room is pulled with a constant force by an "immaterial being". The room together with the observer, then, begin to move "upwards" with uniformly accelerated motion, from the point of view of an external observer.

But the observer inside the room notices the acceleration by the reaction on the floor. He is then standing in the room as anyone stands in a room on earth. If he releases a body which he previously had in his hand, the acceleration of the room will no longer be transmitted to this body, and will then approach the floor of the room with an accelerated relative motion. After several experiments, the observer will convince himself that the acceleration of the body towards the floor of the room is always of the same magnitude, whatever kind of body he may happen to use for the experiment. Then, because of the experimental fact that inertial and gravitational masses are equal (this is called the **Weak Equivalence Principle**), the man can not distinguish this situation from being at rest in a gravitational field, because two bodies in the same gravitational field behave in the same way independently of their inertial masses, just as the observer in the room is measuring. Notice that the observer can not measure the absence of tidal forces, as he is inside a room (i.e., mathematically, these measurements are local, not global).

This mental experiment justifies adopting the **General Postulate of Relativity** (also called the **Equivalence Principle**), the main idea of which is that "all bodies of reference are equivalent for the description of natural phenomena, whatever may be their state of motion." An accelerated observer with respect to us will just consider the accelerations of the bodies that are at rest from our point of view as the result of some gravitational field (not necessarily produced by Newton's law of universal gravitation).

This postulate has deep implications: some basic postulates of SR, such as the invariance of the velocity of light, are no longer valid. For instance, if an observer K sees a light ray at constant velocity c travelling in a straight line path, an observer K' in rotation with respect to K will see it travelling along a curved path, and he will explain this phenomena because of the presence of a gravitational field. So the idea of straight line has to be reconsidered.

Moreover, the way of measuring time and distances has to be changed too: consider an observer K in a "nearly Galileian" frame as before, and an observer K' located on a rotating disc (from the point of view of K). By the time dilation, from the point of view of K, a clock located in the center of the rotating disc and a clock located on the disc at any other point will not measure the same time (the latter goes at a rate permanently slower than the first one); as this would be oberved too by any other observer, in general, we can say that in every gravitational field, a clock will go more quickly or less quickly according to the position in which the clock (at rest) is situated. A similar result is obtained when measuring distances: the size of a rod to measure distances depends on the position where the rod is situated.

This fact introduces the idea of allowing non-euclidean geometries as possible geometries for spacetime, and here appears the necessity of considering spacetime as a manifold with a given metric that contains all the gravitational field information. This point of view implies that gravity is omnipresent: there is no such thing as a "gravitationally neutral object", thus it makes no sense to try to measure an object's acceleration due to gravity; instead, it is better to define an unaccelerated body as a freely falling object.

Not only that, but also, notice that it is not necessary that the coordinates of spacetime have any physical significance as they had in Minkowski spacetime, where x^i represented space coordinates and x^0 the time coordinate. Why is that so? Any event is a point in spacetime, and a "particle" (a being with some duration) can be represented by a curve in spacetime, called its **worldline**; the only actual evidence of a time-space nature with which we meet in physical statements appears when two worldlines intersect, and the intersection of worldlines is well described even if the coordinates have no physical interpretation.

Putting all this information together, we can formulate some postulates or main ideas over which the whole theory is constructed:

(1) Spacetime is a 4-dimensional semiriemannian manifold whose metric g contains all the information about the gravitational field and is locally lorentzian. Notice that any frame is, locally, a Lorentz frame, using Riemann Normal Coordinates (see for example [**1**] p.335-341 or [**5**] p.158-160 to see the properties and definition of Riemann Normal Coordinates).
(2) **Einstein's Equivalence Principle**: In any and every local Lorentz frame, anywhere and anytime in the universe, all the (nongravitational) laws of physics must take on their familiar special-relativistic forms.

The relation between the metric and the mass and energy distributions will be discussed in chapter 3, and constitutes the main piece of the theory: Einstein's field equation.

3. The matter fields

If we want to formulate a physical law in a manifold satisfying Einstein's Equivalence Principle, we should announce it using an equality between tensors, because a law formulated in terms of tensors is independent of the coordinates chosen in the manifold. As we want to relate the geometry of the manifold with the mass and energy distribution, we have to single out adequate tensors for both concepts. We have already studied some tensor candidates containing geometrical information in the preceding chapter, such as Riemann's curvature tensor and its inner contractions. Now we should study how to define a tensor containing information about the mass and energy distribution of the Universe: a **matter field**. This tensor we are looking for is not a new concept: it appears in the Special Theory of Relativity, and it can be regarded as a generalization of the four-vector energy-momentum P^μ.

The definition of this tensor field will depend on the problem we are studying. There will be various fields on M which describe the matter content of space-time: a part from the mass distribution, we have to take into account the electromagnetic field, the neutrino field, etc. The theory of gravity one obtains depends on what matter fields one incorporates in it. These matter fields should obey a couple of postulates:

a) **Local causality:** the equations governing the matter fields must be such that a signal can be send between two points if and only if these points can be joined by a non-spacelike curve (a curve whose tangent vector is not spacelike).
b) **Local conservation of energy and momentum:** The equations governing the matter fields are such that there exists a symmetric tensor field $T \in \mathfrak{T}^2_0(M)$, called the **Energy-Momentum Tensor** or the **Stress-Energy Tensor**, which depends on the fields, their covariant derivatives, and the metric, and which has the properties:
    - T vanishes on an open set U if and only if all the matter fields vanish on U
    - $T^{\mu\nu}$ obey the equation $\nabla_\mu T^{\mu\nu} = 0$ (this expresses the conservation of energy in the presence of a gravitational field; in fact, the operator ∇_μ is the generalization of the divergence operator $\frac{\partial}{\partial x^\mu}$ to a curved space, as it will be discussed in the next chapter).

From the second postulate, we get that in vacuum conditions $T^{\mu\nu} = 0$, as anyone would expect. On the other hand, if we are considering a fluid with particle density $k \in C^\infty(M)$ at rest:

Definition 2.13. *The* **Particle Density 4-vector** N *is defined as* $P = kU$*, where* U *is the* 4*-velocity and* $k \in C^\infty(M)$ *is the density of particles of the medium at rest.*

In coordinates we can write it $N^\mu = k \cdot U^\mu$.

Definition 2.14. *The* **Energy-Momentum Tensor** *or* **Stress-Energy Tensor***, is the symmetric tensor field* $T \in \mathfrak{T}^2_0$ *defined as* $T = P \otimes N$*, where* P *and* N *are the* 4*-momentum and the particle density* 4*-vector, respectively.*

In coordinates, $T^{\mu\nu} = P^\mu \cdot N^\nu$.

Chapter 3
Einstein's Equations

Up to this point, it only remains to characterize the way the space-time is curved because of the matter fields' presence; we have to state Einstein's equation. Good discussions on this topic can be found on the same references that I used in chapter 2; particularly, I used [**3**] for the informal approach.

1. An informal approach

Let's first consider a freely falling particle. Freely falling particles in flat space (SR) move in straight lines γ, thus the second derivative of the parameterized path $x^\mu(\tau)$ vanishes:

$$\frac{d^2x^\mu}{d\tau^2} = 0$$

According to Einstein's Equivalence Principle, this equation should hold in curved space if we are using Riemann Normal Coordinates. But this equation is not an equation between tensors, so it doesn't hold in an arbitrary coordinate system. However, there is a unique tensorial equation which reduces to this one when the Christoffel symbols vanish; it is:

$$\nabla_\tau \gamma' = 0 \iff \frac{d^2x^k}{d\tau^2} + \Gamma^k_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0$$

This is, in fact, the usual way of generalizing the operator $\frac{\partial}{\partial\lambda}$ to a covariant expression. We have obtained the geodesic equation; therefore, in GR free particles move along geodesics. Here it is clear why sometimes the covariant derivative is thought as a possible way of defining the concept of acceleration on a manifold. So far, we have seen that curvature is necessary to describe the motion of freely falling particles. Let's see that it is sufficient: we will deduce the usual results of Newtonian gravity from it.

Let's define the Newtonian limit by the requirements:

(1) The particles are moving slowly with respect to the speed of light. This means that $\frac{dx^i}{d\tau} << \frac{dt}{d\tau}$, so the terms $\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}$ can be neglected, and then the geodesic equation becomes:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0$$

(2) The field is static, it doesn't depend on time. Then, the Christoffel symbols of the Levi-Civitta connection can be simplified:

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{0\nu}}{\partial t} + \frac{\partial g_{0\nu}}{\partial t} - \frac{\partial g_{00}}{\partial x^\nu}\right) = -\frac{1}{2}g^{\mu\nu}\frac{\partial g_{00}}{\partial x^\nu}$$

(3) The gravitational field is weak, it can be considered as a perturbation of flat space. Then, we can write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| << 1$, and its inverse becomes $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, where $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$. Substituting this expression into the previous one, and neglecting the terms where $h_{\mu\nu}$ appears multiplying (because of its small norm):

$$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\nu}\frac{\partial h_{00}}{\partial x^\nu}$$

The geodesic equation is therefore:

$$\frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\nu}\frac{\partial h_{00}}{\partial x^\nu}\left(\frac{dt}{d\tau}\right)^2$$

Using again that the field is static, $\frac{\partial h_{00}}{\partial t} = 0$, and that $\eta^{0i} = 0$, the equation above, when $\mu = 0$, becomes:

$$\frac{d^2t}{d\tau^2} = 0$$

So $\frac{dt}{d\tau}$ is constant. Then, the spacelike components of the geodesic equation ($\mu = i$) are:

$$\frac{d^2x^i}{d\tau^2} = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^i}\left(\frac{dt}{d\tau}\right)^2 \iff \frac{d^2x^i}{dt^2} = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^i}$$

And defining $\Phi = \frac{1}{2}h_{00}$ we recover Newton's field equation:

$$\frac{\partial^2 x}{\partial t^2} = -\nabla\Phi$$

Now, we know that the curvature of the manifold is sufficient to describe the gravity in the Newtonian limit. It remains, of course, to find field equations for the metric that allow it to take the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and to see that in the case of a single gravitating punctual body of mass M, $h_{00} = \frac{2GM}{r}$; but this will come soon.

We know how to generalize the laws of physics from a flat spacetime to a curved one: the only thing we have to do is to change partial derivatives by covariant derivatives. This is a good idea that comes from Einstein's Equivalence Principle, but it is not a well stablished rule: obviously, there are many ways the same physical law can be generalized to curved space. For instance, consider a law of the form:

$$Y^{\mu}\frac{\partial X^{\nu}}{\partial x^{\nu}\partial x^{\mu}} = 0$$

The partial derivatives are commutative operators, but the covariant ones are not! So there are two possible ways of generalizing this result, and they differ in:

$$Y^{\mu}\nabla_{\mu}\nabla_{\nu}X^{\nu} - Y^{\mu}\nabla_{\nu}\nabla_{\mu}X^{\nu} = -R_{\mu\nu}Y^{\mu}X^{\nu}$$

We will not care about that now; in order to decide which of them is the correct one we should consider their behaviour and their compatibility with other physical laws. Now, let's try to generalize, in an informal way, Newton's field equations of the gravitational potential to curved spacetime. Consider Poisson's equation for the gravitational potential:

$$\Delta\Phi = 4\pi G\rho$$

Where ρ represents the mass density. The equations we want to find has to reduce to this one in the Newtonian limit: on the left-hand side we have a second-order differential operator acting on the gravitational potential, whereas the right-hand side contains a measure of the mass distribution. As we have already discussed, a relativistic generalization should take the form of an equation between tensors. We know how to generalize the mass distribution, using the energy-momentum tensor $T^{\mu\nu}$, or analogously, lowering its indices, $T_{\mu\nu}$. On the other hand, the gravitational potential should get replaced by the metric tensor. Now, we have to choose a proper second-order differential operator to replace the laplacian.

2. Einstein's Tensor

This operator should be such that the resulting tensor, that we will call $G_{\mu\nu}$, is symmetric (as $T_{\mu\nu}$ is), and is compatible with energy-momentum conservation in curved spacetime, i.e., $\nabla^{\mu}G_{\mu\nu} = 0$. It should be derived using only geometric information: the metric tensor $g_{\mu\nu}$ and Riemann's curvature tensor $R^{\mu}{}_{\alpha\beta\gamma}$ (that in fact is derived from g); notice that I am not considering torsion here because we use the Levi-Civitta connection, that is torsionless. As it has to be a generalization of the laplacian, we will consider just 2nd order tensors, and as it has to equal $T_{\mu\nu}$, we are just going to consider second-rank tensors.

Proposition 3.1. *The most general second-rank symmetric tensor constructable from $g_{\mu\nu}$ and $R^{\mu}{}_{\alpha\beta\gamma}$ and linear in $R^{\mu}{}_{\alpha\beta\gamma}$ is:*

$$G_{\mu\nu} = aR_{\mu\nu} + bRg_{\mu\nu} + \Lambda g_{\mu\nu}$$

And $\nabla_\mu G^\mu{}_\nu = 0 \iff b = -\frac{1}{2}a$. *Moreover, it vanishes in flat spacetime if and only if* $\Lambda = 0$.

Proof. Let's consider all the possible elements that we can construct from $g_{\mu\nu}$ and $R^\mu{}_{\alpha\beta\gamma}$ that have rank 2:

- First of all, we have $g_{\mu\nu}$, a second-order tensor that is symmetric and does not depend on the curvature. $G_{\mu\nu}$ will contain a multiple of it: $\Lambda g_{\mu\nu}$.
- The curvature tensor hasn't got second order. In order to obtain a second-order tensor, we have to contract two of its indices: this can be done in several ways.
  1. The Ricci tensor $R_{\mu\nu} = R^i{}_{i\mu\nu}$ is one of these contractions. It is symmetric, as it was seen in the first chapter, and depends linearly on the Riemann curvature tensor, so it can contribute to $G_{\mu\nu}$ in the form $aR_{\mu\nu}$.
  2. Using the same idea, we can construct two other second-order tensors: $R^i{}_{\mu i\nu}$ and $R^i{}_{\mu\nu i}$. On the other hand, we can consider other contractions using the metric to rising and lowering indices. But Ricci tensor is essentially the unique nontrivial way of contracting Riemann curvature tensor. I will not discuss this here.
- The scalar curvature $R = g^{\mu\nu}R_{\mu\nu}$ is another contraction that we can consider: as it is a scalar field it is symmetric, but it hasn't got second order. Nevertheless, we can multiply a second-order tensor by R and we will obtain another possible element of $G_{\mu\nu}$. Since we want the tensor G to be linear with $R^\mu{}_{\alpha\beta\gamma}$, R can only be multiplied by $g_{\mu\nu}$, and will contribute to $G_{\mu\nu}$ in the form $bRg_{\mu\nu}$.

So we have seen that the most general second-rank symmetric tensor constructable from $g_{\mu\nu}$ and $R^\mu{}_{\alpha\beta\gamma}$ and linear in $R^\mu{}_{\alpha\beta\gamma}$ is $G_{\mu\nu} = aR_{\mu\nu} + bRg_{\mu\nu} + \Lambda g_{\mu\nu}$. Now we have to see that $\nabla_\mu G^\mu_\nu = 0 \iff b = -\frac{1}{2}a$.

First of all, lets find a useful relation. Consider the Biancchi identity rising its indices:

$$0 = g^{\nu\sigma}g^{\mu\lambda}\left(\nabla_\lambda R_{\rho\mu\nu\sigma} + \nabla_\rho R_{\sigma\mu\nu\lambda} + \nabla_\sigma R_{\lambda\mu\nu\rho}\right)$$

Using that $R_{abcd} = R_{cdab}$ and that $R_{\nu\rho} = R_{\rho\nu}$:

$$0 = \nabla^\mu R_{\rho\mu} + \nabla_\rho R^{\nu\lambda}{}_{\nu\lambda} + \nabla^\nu R_{\rho\nu} = 2\nabla^\mu R_{\rho\mu} + \nabla_\rho R^{\nu\lambda}{}_{\nu\lambda}$$

Notice that:

$$R = g^{\nu\mu}R^\lambda{}_{\lambda\nu\mu} = g^{\nu\mu}g^{\sigma\lambda}R_{\sigma\lambda\nu\mu}$$

And using that $R_{abcd} = -R_{dbca}$:

$$R = -g^{\nu\mu}g^{\sigma\lambda}R_{\mu\lambda\nu\sigma} = -g^{\nu\mu}R_\mu{}^\sigma{}_{\nu\sigma} = -R^{\nu\sigma}{}_{\nu\sigma}$$

So introducing this in the previous form of the Biancchi identity (using that $\nabla^\mu R_{\rho\mu} = \nabla_\mu R^\mu{}_\rho$, that can be checked easily) we obtain that:

$$\nabla_\mu R^\mu{}_\rho = \frac{1}{2}\nabla_\rho R$$

Now we can compute $\nabla \cdot G = \nabla_\mu G^\mu{}_\nu$:

$$\begin{aligned} G_{\mu\nu} &= aR_{\mu\nu} + bRg_{\mu\nu} + \Lambda g_{\mu\nu} \\ G^\mu{}_\nu &= g^{\mu\rho}G_{\rho\nu} = aR^\mu{}_\nu + bR\delta^\mu_\nu + \Lambda\delta^\mu_\nu \\ \nabla_\mu G^\mu{}_\nu &= a\nabla_\mu R^\mu{}_\nu + b\nabla_\nu R = \left(\frac{1}{2}a + b\right)\nabla_\nu R \end{aligned}$$

So, at last, we see that $\nabla \cdot G = 0 \iff b = -\frac{1}{2}a$.

Finally, we should proof that in flat space $G = 0 \iff \Lambda = 0$. But this is trivial, as we have seen in the first chapter that in flat space $R_{\mu\nu} = 0$ and $R = 0$, so $G_{\mu\nu} = 0 \iff \Lambda\eta_{\mu\nu} = 0 \iff \Lambda = 0$. □

It can be proved that there exists no tensor with components constructable from the ten metric coefficients and their first derivatives. In fact, the only tensors constructable from the ten $g_{\alpha\beta}$, the fourty $\partial_\mu g_{\alpha\beta}$ and the one hundred $\partial_\mu\partial_\nu g_{\alpha\beta}$ that are linear in the latter are:

(1) $R^\mu_{\alpha\beta\gamma}$
(2) $g_{\mu\nu}$
(3) Tensors constructed from the Riemann curvature tensor that are linear in $R^\mu_{\alpha\beta\gamma}$ (such as the Ricci tensor).

The tensor $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta}$ is the only second-rank, symmetric tensor such that:

a) Has components constructable solely from $g_{\alpha\beta}$, $\partial_\mu g_{\alpha\beta}$ and $\partial_\mu\partial_\nu g_{\alpha\beta}$.
b) Has components linear in $\partial_\mu\partial_\nu g_{\alpha\beta}$.
c) $\partial_\alpha G^\alpha_\beta = 0$

If we add the condition that G vanishes in flat space, then $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}$.

The uniqueness of this tensor, that is called **Einstein's tensor**, was proven, according to [**6**], by Poincaré. From these propositions, we deduce the unicity (under the restrictions above) of Einstein's tensor $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}$, and thus we are able to formulate **Einstein's field equation**:

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

Where κ is some real constant.

These equations can be derived from a variational principle. In fact, Hilbert published the correct equations deriving them from a variational principle (using **Hilbert's action**) even before Einstein published the correct ones (althought Hilbert knew the work of Einstein, and always deffended Einstein's authorship of the equations). You can find a modern proof of the unicity of these equations in [**11**], where Lovelock shows using the variational formulation, that in a 4-dimensional space the Einstein field equations (with the term $\Lambda g_{\mu\nu}$) are the only permissible second order Euler-Lagrange equations.

Lemma 3.2. *Einstein's field equation is equivalent to the equation:*

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)$$

PROOF. From Einstein's field equation: $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}$. Rising its indices with the inverse of the metric:

$$g^{\mu\rho}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = g^{\mu\rho}(\kappa T_{\mu\nu})$$
$$R^{\rho}_{\nu} - \frac{1}{2}\delta^{\rho}_{\nu}R = \kappa T^{\rho}_{\nu}$$

Now, contracting the indices ν and ρ:

$$R^{\nu}_{\nu} - \frac{1}{2}\delta^{\nu}_{\nu}R = \kappa T^{\nu}_{\nu}$$
$$R - \frac{1}{2}4R = \kappa T$$

Then, we deduce that $R = -\kappa T$, where T is called **Laue's scalar**. So we can rewrite Einstein field equation, just substituting R by this expression:

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)$$

□

This alternative way of writing Einstein's equations is very useful, specially when we are looking for the solutions in vacuum space, because then Einstein's equation reads:

$$R_{\mu\nu} = 0$$

Although apparently simple, this equation is still very difficult to solve unless additional conditions are supposed. For instance, one of the first solutions that was found, Schwarschild's metric, has the additional assumption of spherical symmetry; this solution is very useful, as it describes spacetime near a celestial body such as a star, or a planet. We will study it later; now, let's find the constant κ.

To find it, we will impose that Einstein's equation predicts Newtonian gravity in the weak-field, time-independent, slowly-moving-particles limit. Notice that in a mass distribution ρ with slowly-moving particles, the terms $T_{i\mu}$ can be neglected in front of T_{00}, so we are going to study the case where $\mu = \nu = 0$ of Einstein's equation. As we are in the weak-field limit, $g_{00} = 1 + h_{00}$ and $g^{00} = 1 - h^{00}$. Then, up to first order, $T = g^{00}T_{00} \simeq T_{00}$. So:

$$R_{00} = \frac{1}{2}\kappa T_{00}$$

To obtain the explicit relation with the metric we need to compute $R_{00} = R^{\lambda}{}_{\lambda 00}$; as $R^{0}{}_{000} = 0$, we just need to compute $R^{i}{}_{j00}$:

$$R^{i}{}_{j00} = \frac{\partial}{\partial x^j}\Gamma^{i}_{00} - \frac{\partial}{\partial x^0}\Gamma^{i}_{j0} + \Gamma^{i}_{j\lambda}\Gamma^{\lambda}_{00} - \Gamma^{i}_{0\lambda}\Gamma^{\lambda}_{j0}$$

Using that we are in the static approximation, up to first order in Γ:

$$R^{i}{}_{j00} \simeq \frac{\partial}{\partial x^j}\Gamma^{i}_{00} = \frac{\partial}{\partial x^j}\left(-\frac{1}{2}g^{ik}\frac{\partial g_{00}}{\partial x^k}\right)$$

Now contracting its indices, substitutin the expression of g_{00} above and neglecting the terms with order $\simeq h$:

$$R_{00} = R^{i}{}_{i00} \simeq -\frac{1}{2}\eta^{ik}\frac{\partial^2}{\partial x^i \partial x^k}h_{00} = -\frac{1}{2}\Delta h_{00}$$

So comparing this with the expression $R_{00} = \frac{1}{2}\kappa T_{00}$, we find that:

$$\Delta h_{00} = -\kappa T_{00}$$

And this is precisely Newton's equation if we take $\kappa = 8\pi G$, because Newton's equation tells us $\Delta\Phi = 4\pi\rho = 4\pi T_{00}$ and we already knew that $h_{00} = 2\Phi$. Thus, we have computed the adequate constant κ that makes Einstein's equation predict Newtonian gravity in the specified limit conditions.

Summarizing, the equations that govern the curvature of spacetime are:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Here, as we have seen before, an additional term $\Lambda g_{\mu\nu}$ could be added; this term was considered at first by Einstein, but then he discarted it. Λ is the famous **cosmological constant** that Einstein considered one of the worst mistakes in his life. He introduced it to obtain a static (non-spanding) universe; but experimental data showed that the universe is actually expanding, so by introducing this term Einstein lost the opportunity of predicting the expansion of the universe. Nowadays, it is known that the universe is not only expanding, but its expansion is being accelerated (the source of this acceleration is not already understood, and is called the **dark energy**). Thus, some physicists have argued that the cosmological constant should be reintroduced to the equations to predict the accelerated expansion. Whether it has to be considered or not, I don't know, but it has been shown that, currently, if Λ is not zero, it has such a small value that if we are not working in a cosmological scale, the equation above is the only one need.

Chapter 4
The Schwarzschild Solution

In this chapter we will find a particular solution to Einstein's equation. This chapter is mainly based on Olesen's lecture notes [**14**], but similar discussions can be found in many other sources. Schwarzschild solution is a spherically symmetric solution that has many applications: for instance, it describes the spacetime metric near a massive spherically symmetric body such as a planet or a star, so it is useful to describe the Solar System. Our discussion will not be strictly rigorous, specially when dealing with the concept of symmetry, but it can be formalized using some additional differential geometry tools, such as Lie groups and Lie algbras.

1. The time-dependent spherically symmetric metric

A compact way of expressing the metric tensor is using the proper time interval: $d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu$. If we consider, first, the cartesian variables of space x^1, x^2 and x^3, together with the time t, a spherically symmetric metric intuitively is a metric for which its proper time only depends on t, r, dt, $\mathbf{x}d\mathbf{x} = rdr$ and $(d\mathbf{x})^2 = dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2)$, where $r = \sqrt{x^2+y^2+z^2}$. Thus:

$$d\tau^2 = A(r,t)dt^2 - B(r,t)dr^2 - C(r,t)drdt - D(r,t)r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

As we have said in chapter 3, the coordinates in GR are arbitrary, so we are free to make transformations $x^\mu \to x'^\mu$ to simpify the expression of $d\tau^2$. For example, we can do the transformation $r' = r\sqrt{D(r,t)}$ (here we assume that r' is not constant) and $d\tau$ becomes dependent on new functions A', B' and C', which are functions of t and r'. Dropping the primes for simplicity, we get:

$$d\tau^2 = A(r,t)dt^2 - B(r,t)dr^2 - C(r,t)drdt - r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

The term $drdt$ can also be removed by using a new time coordinate t' defined by:

$$dt'(r,t) = \mu(r,t)\left[A(r,t)dt - \frac{1}{2}C(r,t)dr\right]$$

Where $\mu(r,t)$ is an integrating factor that makes dt' be a perfect differential with $\mu A = \frac{\partial t'}{\partial t}$, $-\frac{1}{2}\mu C = \frac{\partial t'}{\partial r}$. Then, from here, we deduce that:

$$\frac{1}{A\mu^2}dt'^2 = Adt^2 - Cdtdr + \frac{C^2}{4A}dr^2$$

and the expression of the proper time becomes:

$$d\tau^2 = \frac{1}{\mu^2 A}dt'^2 - \left(B + \frac{C^2}{4A}\right)dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

or, renaming the corresponding functions:

$$d\tau^2 = E(r,t)dt^2 - F(r,t)dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

This is the **standard** form of the metric, derived for the first time by Weyl. Then we have:

$$\begin{array}{llll} g_{rr} = -F, & g_{\theta\theta} = -r^2, & g_{\varphi\varphi} = -r^2 sin^2\theta, & g_{tt} = E \\ g^{rr} = -\frac{1}{F}, & g^{\theta\theta} = -\frac{1}{r^2}, & g^{\varphi\varphi} = \frac{1}{r^2 sin^2}\theta, & g_{tt} = \frac{1}{E} \end{array}$$

1.1. The Christoffel Symbols

Now that we have a general expression for the metric, we can compute the Christoffel symbols. We will do this computation using the formula that we saw in chapter 1:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left[\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right]$$

There are some other methods to compute $\Gamma^\lambda_{\mu\nu}$ that may take less time, such as the variational method proposed in [**14**], but we will compute them in the classical way, using the good properties of this particular metric.

First of all, let's compute $\Gamma^r_{\mu\nu}$. There are 16 such functions, but as $\Gamma_{\mu\nu} = \Gamma_{\nu\mu}$ only 10 of them are independent. Notice that, since $g^{r\sigma} = -\delta^\sigma_r \frac{1}{F}$, the only terms in the sumation that do not vanish are those corresponding to $\sigma = r$. So the formula above becomes:

$$\Gamma^r_{\mu\nu} = \frac{1}{2}g^{rr}\left[\frac{\partial g_{\nu r}}{\partial x^\mu} + \frac{\partial g_{\mu r}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial r}\right]$$

Clearly, if μ, ν and r are all different, $\Gamma^r_{\mu\nu} = \Gamma^r_{\nu\mu} = 0$ because $g_{r\mu} = g_{r\nu} = g_{\nu\mu} = 0$, so we get that:

$$\Gamma^r_{t\theta} = \Gamma^r_{\theta t} = \Gamma^r_{t\varphi} = \Gamma^r_{\varphi t} = \Gamma^r_{\theta\varphi} = \Gamma^r_{\varphi\theta} = 0$$

If $\mu = \nu \neq r$, $g_{r\mu} = g_{r\nu} = 0$, thus $\Gamma^r_{\mu\mu} = -\frac{1}{2} g^{rr} \frac{\partial g_{\mu\mu}}{\partial r}$, and we obtain:

$$\Gamma^r_{tt} = \tfrac{1}{2F}\tfrac{\partial E}{\partial r}, \quad \Gamma^r_{\theta\theta} = -\tfrac{r}{F}, \quad \Gamma^r_{\varphi\varphi} = -\tfrac{r sin^2\theta}{F}$$

Finally, if $r = \mu$ and ν is arbitrary, as $\frac{\partial g_{r\nu}}{\partial r} = \frac{\partial g_{\nu r}}{\partial r}$, we have that $\Gamma^r_{r\nu} = \Gamma^r_{\nu r} = \frac{1}{2} g^{rr} \frac{\partial g_{rr}}{\partial x^\nu}$, and remembering that $F = F(r,t)$ we obtain:

$$\Gamma^r_{rr} = \tfrac{1}{2F}\tfrac{\partial F}{\partial r}, \quad \Gamma^r_{rt} = \Gamma^r_{tr} = \tfrac{1}{2F}\tfrac{\partial F}{\partial t}, \quad \Gamma^r_{r\theta} = \Gamma^r_{\theta r} = \Gamma^r_{r\varphi} = \Gamma^r_{\varphi r} = 0$$

Using a similar analysis, we can compute $\Gamma^t_{\mu\nu}$, $\Gamma^\theta_{\mu\nu}$ and $\Gamma^\varphi_{\mu\nu}$. The nonvanishing terms are:

$$\begin{array}{lll}
\Gamma^t_{tt} = \tfrac{1}{2E}\tfrac{\partial E}{\partial t} & \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \tfrac{1}{r} & \Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} = \tfrac{1}{r} \\
\Gamma^t_{tr} = \Gamma^t_{rt} = \tfrac{1}{2E}\tfrac{\partial E}{\partial r} & \Gamma^\theta_{\varphi\varphi} = -sin\theta cos\theta & \Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \tfrac{cos\theta}{sin\theta} \\
\Gamma^t_{rr} = \tfrac{1}{2E}\tfrac{\partial F}{\partial t} & &
\end{array}$$

1.2. The Ricci tensor

Once we have the connection coefficients, we need to compute the Riemann curvature tensor, and then contract its indices to get the Ricci tensor, that we need to study Einstein equation. This computation, even in the most simple cases, is really tedious. For example, let's compute R_{tr}; we only need the terms ${R^i}_{itr}$. Remember the formula:

$${R^l}_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x^i} + \Gamma^\sigma_{jk}\Gamma^l_{i\sigma} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} - \Gamma^\sigma_{ik}\Gamma^l_{j\sigma}$$

Then, we compute:

$$\begin{aligned}
{R^r}_{rtr} = & -\frac{1}{2F^2}\frac{\partial F}{\partial r}\frac{\partial F}{\partial t} + \frac{1}{2F}\frac{\partial F}{\partial t \partial r} + \left(\frac{1}{2F}\right)^2 \frac{\partial F}{\partial t}\frac{\partial F}{\partial r} + \frac{1}{2F}\frac{\partial F}{\partial t}\frac{1}{2E}\frac{\partial E}{\partial r} + \\
& - \left[-\frac{1}{2F^2}\frac{\partial F}{\partial t}\frac{\partial F}{\partial r} + \frac{1}{2F}\frac{\partial F}{\partial r \partial t}\right] - \left(\frac{1}{2F}\right)^2 \frac{\partial F}{\partial t}\frac{\partial F}{\partial r} - \frac{1}{2E}\frac{\partial F}{\partial t}\frac{1}{2F}\frac{\partial E}{\partial r} = 0
\end{aligned}$$

$${R^t}_{ttr} = \frac{\partial \Gamma^t_{tr}}{\partial t} + \Gamma^\sigma_{tr}\Gamma^t_{t\sigma} - \frac{\partial \Gamma^t_{tr}}{\partial t} - \Gamma^\sigma_{tr}\Gamma^t_{t\sigma} = 0$$

$${R^\theta}_{\theta tr} = \Gamma^\sigma_{tr}\Gamma^\theta_{\theta\sigma} - \frac{\partial \Gamma^\theta_{\theta r}}{\partial t} - \Gamma^\sigma_{\theta r} - \Gamma^\theta_{t\sigma} = \Gamma^r_{tr}\Gamma^\theta_{\theta r} = \frac{1}{2Fr}\frac{\partial F}{\partial t}$$

$$R^{\varphi}{}_{\varphi tr} = \Gamma^r_{tr}\Gamma^{\varphi}_{\varphi r} = \frac{1}{2Fr}\frac{\partial F}{\partial t}$$

So we finally obtain:

$$R_{tr} = R^{\mu}{}_{\mu tr} = \frac{1}{rF}\frac{\partial F}{\partial t}$$

In the lecture notes [**14**] Olesen derives a method for computing the Ricci tensor components in a more straighforward way. Be careful, because he uses a different sign convention for the metric coefficients and for the Ricci tensor components! If we compute the other components, the nonvanishing ones are:

$$\begin{aligned}
R_{rr} &= -\frac{1}{2E}\frac{\partial^2 E}{\partial r^2} + \frac{1}{4E^2}\left(\frac{\partial E}{\partial r}\right)^2 + \frac{1}{4EF}\frac{\partial E}{\partial r}\frac{\partial F}{\partial r} + \frac{1}{rF}\frac{\partial F}{\partial r} + \frac{1}{2E}\frac{\partial^2 F}{\partial t^2} \\
&\quad - \frac{1}{4E^2}\frac{\partial E}{\partial t}\frac{\partial F}{\partial t} - \frac{1}{4EF}\left(\frac{\partial F}{\partial t}\right)^2 \\
R_{tt} &= \frac{1}{2F}\frac{\partial^2 E}{\partial r^2} - \frac{1}{4F^2}\frac{\partial E}{\partial r}\frac{\partial F}{\partial r} + \frac{1}{rF}\frac{\partial E}{\partial r} - \frac{1}{4EF}\left(\frac{\partial E}{\partial r}\right)^2 - \frac{1}{2F}\frac{\partial^2 F}{\partial t^2} \\
&\quad + \frac{1}{4F^2}\left(\frac{\partial F}{\partial t}\right)^2 + \frac{1}{4EF}\frac{\partial E}{\partial t}\frac{\partial F}{\partial t} \\
R_{\theta\theta} &= 1 - \frac{1}{F} + \frac{r}{2F^2}\frac{\partial F}{\partial r} - \frac{r}{2EF}\frac{\partial E}{\partial r} \\
R_{\varphi\varphi} &= sin^2\theta R_{\theta\theta} \\
R_{tr} &= \frac{1}{rF}\frac{\partial F}{\partial t}
\end{aligned}$$

2. The Schwarzschild solution

Let's consider now the case where space is empty except for a mass M situated at $r = 0$. Thus, except for $r = 0$, we must satisfy the vacuum Einstein equations $R_{\mu\nu} = 0$. Using the expression of R_{tr} above, we see that $R_{tr} = 0 \implies \frac{\partial F}{\partial t} = 0$. Notice that then, in the expressions above, we can remove all the time derivatives of $R_{\mu\nu}$. As $R_{\varphi\varphi} = -sin^2\theta R_{\theta\theta}$, the equations will be satisfied if and only if $F = F(r)$ and $R_{rr} = R_{tt} = R_{\theta\theta} = 0$. Observing that R_{rr} and R_{tt} contain similar terms, we impose:

$$0 = \frac{R_{rr}}{F} + \frac{R_{tt}}{E} = \frac{1}{rF}\left[\frac{\partial F}{\partial r} + \frac{\partial E}{\partial r}\right]$$

This is equivalent to $\frac{\partial(\ln F)}{\partial r} = -\frac{\partial(\ln E)}{\partial r}$. Integrating this first order ODE we obtain:

$$E(r,t)F(r) = f(t)$$

Where $f(t)$ is a certain function. As we want the metric to be flat when $r \to \infty$, then, $E(r,t) = F(r) = 1$ at infinity, so $f(t) = 1$ everywhere (as $f(t)$ does not depend on r). So we have seen that $E(r,t) = E(r) = \frac{1}{F(r)}$.

Now, we just need to impose $R_{\theta\theta}$ and R_{rr} to vanish. Using the expression we have just derived, we get:

$$R_{\theta\theta} = 1 - E - r\frac{dE}{dr}$$
$$R_{rr} = -\frac{1}{2E(r)}\frac{d^2E}{dr^2} - \frac{1}{rE}\frac{dE}{dr} = \frac{1}{2rE}\frac{dR_{\theta\theta}}{dr}$$

Thus, if $R_{\theta\theta} = 0$, then automatically $R_{rr} = 0$. We can write $R_{\theta\theta} = 0$ as:

$$\frac{d}{dr}(rE(r)) = 1$$

So if C is a constant:

$$rE(r) = r + C$$

So we have found the functions $E(r)$ and $F(r)$ in terms of the constant C, which can be fixed by demanding compatibility with the Newton limit: $g_{00} \to 1 - 2GM/r$. So we obtain that $C = -2GM$, and then:

$$E(r) = 1 - \frac{2GM}{r};\ F(r) = \frac{1}{1 - \frac{2GM}{r}}$$

So we obtain the metric:

$$d\tau^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{2GM}{r}} - r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

This is the Schwarzschild solution. Notice that this expression contains two singularities: one of them, $r = 2MG$, is fictitious, it appears because of a bad choice of coordinates (why have we chosen spherical coordinates if flat spacetime has hyperbolic symmetry?), the other one $r = 0$ is, in fact, a physical singularity (it appears in any reference frame), and it can related to black holes' event horizon. Of course, as this solution was found from Einstein's Equations in vacuum, it makes no sense to consider what happens at $r = 0$, because there is matter there, so we can not derive any conclusion from it.

Conclusions

To conclude with, we have seen that GR is geometrical theory that describes gravity not as a force, but as a consequence of the curvature of the spacetime manifold. In this context, objects with no external forces acting on them, such as the Earth, are no longer considered as massive bodies submitted to a gravitational force made by the sun on them, but as freely falling bodies moving along the most similar thing to a straight line in spacetime: geodesics.

We have studied the geometrical tools needed to formulate GR, in particular we have seen what is a pseudoriemannian manifold, what is a connection, how can we construct a connection using the metric of the manifold, how this connection allows us to measure curvature using Riemann and Ricci tensors, etc. All these tools where used to formulate Einstein's equations: a tensorial (thus coordinate-free) equation that relates the geometry of the manifold with the mass and energy distribution of spacetime (the matter fields).

In the thesis it has been discussed the physical principles that brings us to formulate the theory in this way: it reduces to SR locally, it avoids the problem of distinguishing between inertial and non-inertial frames (how could we discover wheather we are in an inertial frame?), and the most important thing, it explains what SR can not explain. Moreover, we have studied the uniqueness, up to a certain point, of Einstein tensor and, thus, of Einstein's equation. Of course, many other equations can be proposed, as we have seen whean dealing with the cosmological constant. Moreover, why do we only consider secon-order tensors? Why don't we consider third-order tensors? Well, Einstein knew this possibility and it is still being studied nowadays, in fact [**11**] shows the unicity of the third-order tensor that can be constructed from the metric. I think that here is where Einstein would apply the criterion of choosing the most beautiful explanation available (thus, if the result is a good description of nature, it is always better to keep things simple and choose second-order tensors).

Finally, we have given a particular solution to the equations: Schwarzschild solution. This has been derived from the assumption of having a spherically symmetric solution in vacuum, with a massive particle in some point of space (that is just a boundary condition, the solution we get is not valid there). The importance of this solution should be clear, because as I said above, it describes, for example, the situation of any planet in the Solar System. Of course, this was already predicted by Newton's gravity, but in this new theory, we see that not only massive bodies are affected by the sun, but also the light we see from distant stars (and this is something that has been observed several times, and that supports GR).

Further work: once Einstein's equations are formulated, we can study many other topics that I don't even had time to comment. For example, one can study the three classical predictions that support GR: the perihelion precession of Mercury, the deflection of light by the Sun and the gravitational redshift of light. Not only this, but also many topics on cosmology can be studied such as the existence of the Big Bang, black holes or the cosmic microwave radiation background (although that, to do this, some other metrics should be studied, such as the Robertson-Walker metric). In addition, it could be studied a new system of coordinates that made nonsingular the Schwarzschild metric in $r = 2GM$, we could study the physical singularity that it has in $r = 0$, or we could look for new similar metrics for different boundary conditions, such as having a charged punctual mass. As you can see, many fields can be studied from this point! I hope to be able to study them in the future.

References

[1] William M. Boothby. *An Introduction to Differentiable Manifolds and Riemannian Geometry*. Academic Press, Inc., 2nd edition, 1986.

[2] James J. Callahan. *The Geometry of Spacetime - An Introduction to Special and General Relativity*. Springer, 2000.

[3] Sean M. Carroll. Lecture notes on general relativity, December 1997.

[4] Albert Einstein and Roger Penrose. *Relativity: The Special and General Theory*. Penguin Group US, 2006.

[5] Bernard F.Schutz. *A first course in general relativity*. Cambridge University Press, 7th edition, 1993.

[6] Joan Girbau. *Geometria diferencial i relativitat*. Universitat Autònoma de Barcelona, 1993.

[7] Xavier Gràcia. Geometria diferencial 2 - definicions i resultats, 2011.

[8] Stephen W. Hawking and G.F.R. Ellis. *The large scale structure of space-time*. Cambridge University Press, March 1975.

[9] M.C. Muñoz Lecanda and N. Román-Roy. Notas sobre teoría de la relatividad especial y general, 2013.

[10] John M. Lee. *Riemannian manifolds: an introduction to curvature*. Springer, 1997.

[11] David Lovelock. The uniqueness of the einstein field equations in a four-dimensional space. *Archive for Rational Mechanics and Analysis*, 33(1):54–70, 1969.

[12] Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler. *Gravitation*. W. H. Freeman, 1st edition, September 1973.

[13] Mikio Nakahara. *Geometry, Topology and Physics*. Taylor & Francis, 2nd edition, 2003.

[14] Poul Olesen. General relativity and cosmology, lecture notes, 2008.

[15] Barrett O'Neill. *Semi-Riemannian Geometry With Applications to Relativity*. Academic Press, 1st edition, July 1983.