Dominating Sets in Planar Graphs: Branch-Width and Exponential Speed-up*

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Abstract

Graph minors theory, developed by Robertson & Seymour, provides a list of powerful theoretical results and tools. However, the wide spread opinion in Graph Algorithms community about this theory is that it is mainly of theoretical importance. The main purpose of this paper is to show how very deep min-max and duality theorems from Graph Minors can be used to obtain essential speed-up to many known practical algorithms on different domination problems.

Keywords: Branch-width, Tree-width, Dominating Set, Planar Graph, Fixed Parameter Algorithm.

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1 Introduction

The main tool of this paper is the branch-width of a graph. Branch-width was introduced by Robertson & Seymour in their Graph Minors series papers several years after tree-width. These parameters are rather close but surprisingly many Graph Minors theorems are much more easy to prove by using branch-width instead of tree-width. Wonderful examples of using branch-width in proof techniques can be found in [21] and [22]. Another powerful property of branch-width is that it can be naturally generalized for hypergraphs and matroids. A good example of generalization of Robertson & Seymour theory for matroids by using branch-width is the recent paper by Geelen et al. [13]. Algorithms for problems expressible in MSOL on matroids of bounded branch-width are discussed by Hlineny [16]. Alekhnovich & Razborov [5] uses branch-width of hypergraphs to design algorithms for SAT.

From a practical point of view, branch-width is also promising. For some problems branch-width is more suitable for actual implementations. Cook & Seymour [9] used branch decompositions to solve ring routing problem arising in the design of reliable cost effective SONET networks. (See also [7].) In theory, there is not a big difference between tree-width and branch-width based algorithms. However in practice, branch-width is sometimes more easy to use. The question due to Hans Bodlaender (private communication) is: Are there examples where the constant factors for branch-width algorithms are significantly smaller than those when using tree-width? Also one of the challenges risen during the workshop "Optimization Problems, Graph Classes and Width Parameters" (Centre de Recerca Matemàtica, Bellaterra, Spain, November 15–17, 2001) was the question whether using the concept of branch-width instead of tree-width might lead to more efficient solutions for Planar Dominating Set and other parameterized problems on planar graphs. This paper is partially motivated by these questions.

Previous results. A *k-dominating* set D of a graph G is a set of k vertices such that every vertex outside D is adjacent to a vertex of D. The Planar Dominating Set problem is the task to compute, given a planar graph G and a positive integer k, a k-dominating set or to report that no such a set exists. It is well known that the Planar Dominating Set (as well as several variants of it) is NP-hard, and hence cannot be solved in polynomial time unless P=NP. Dominating Set is one of the NP-complete core problems. The book of Haynes et al. [14] is a nice source for further references on the dominating problem.

The last five years were the evidence of dramatic improvements of fixed parameter algorithms for the Planar Dominating Set problem. Downey and Fellows [11] suggested an algorithm with running time $O(11^k n)$. Later the running time was reduced to $O(8^k n)$ [2]. The first algorithm with a *sublinear* exponent for the problem with running time $O(4^{6\sqrt{34k}} n)$ (which is approximately $O(2^{70\sqrt{k}} n)$) was given by Alber et al. [1]. Recently, Kanj & Perkovuć [17] announced a faster algorithm of running time $O(2^{27\sqrt{k}}n)$.

The main idea to handle PLANAR DOMINATING SET which was used in several papers is that every planar graph with a domination set of size k has tree-width at most $c\sqrt{k}$, where c is a constant. With some work (sometimes very technical) a tree decomposition of width $c\sqrt{k}$ is constructed and standard dynamic programming techniques on graphs of bounded tree-width are implemented. The running time of the dynamic programming algorithm for dominating set on graphs of tree-width t is $O(2^{2t}n)$. (See Alber et al. [1].) The main disadvantage of this approach is that the constant c is too large for practical applications. The best known constant $c = 6\sqrt{34} = 34.98$ due to Alber et al. [1] was very recently beaten by Kanj & Perkovuć [17] who announced the proof that the tree-width of a planar graph G with dominating set of size k is $\leq 15.6\sqrt{k} + 50$.

Our results. In this paper we introduce a new approach for solving the PLANAR DOMINATING SET problem. As the result of this approach we obtain an algorithm of running time $O(k^4 + 2^{15.13\sqrt{k}}k + n^3)$, which is a significant step towards a practical algorithm. Instead of constructing a tree decomposition and proving that the width of the obtained decomposition is bounded by $c\sqrt{k}$, we prove a combinatorial result relating the branch-width with the domination number of a planar graph.

Our proof is not constructive in the sense that it can not be turned into a polynomial algorithm able to *construct* the corresponding branch decomposition. Fortunately, there is a well known algorithm due to Seymour & Thomas computing such a branch-decomposition of a planar graph in $O(n^4)$ time. This algorithm has not the so-called "enormous hidden constants" and is really practical. (We refer to the work of Hicks [15] on implementations of Seymour & Thomas algorithm.)

Our main combinatorial result is that for every planar graph G with a dominating set of size $\leq k$, the branch-width of G is at most $3\sqrt{4.5}\sqrt{k}$. Combining our bound with the Seymour & Thomas algorithm and with recent results of Alber, Fellows & Niedermeier [3] on a liner problem kernel of PLANAR DOMINATING SET and with a dynamic programming approach on graphs of bounded branch-width, we obtain an algorithm of running time $O(k^4 + 2^{15.13\sqrt{k}}k + n^3)$.

Notice also that combining our bound for branch-width with the well known result of Robertson & Seymour [19] that for any graph G with at least 3 edges tree-width of G is always bounded by $\frac{3}{2}$ times its branch-width, we have that the tree-width of a planar graph with a dominating set of size k is at most $4.5^{\frac{3}{2}}\sqrt{k} = 9.546\sqrt{k}$. This is an improvement of all the previous known bounds for tree-width.

The paper and the proof of the main result are organized as follows. In Section 2 we give definitions, state some known theorems and observe how a theorem of Robertson, Seymour & Thomas can be used to prove that every planar graph with a dominating set of size $\leq k$ has branch-width at most $\leq 12\sqrt{k}+9$. This observation (combining with the results discussed in Section 4) implies an algorithm for the Planar Dominating Set problem with running time $O(2^{28.56\sqrt{k}}k + k^4 + n^3)$, where n is the number of vertices of G. This is already a strong improvement (for large k) of Alber et al. result and is close to the running time $O(2^{27\sqrt{k}}n)$ of Kanj & Perković's algorithm.

In Section 3 we prove the main combinatorial result of the paper. The proof of this result is complicated and we split it into several steps. In Section 3.1 we give technical results about branch decompositions. These results are based on the powerful theorem of Robertson & Seymour on the branch-width of dual graphs. We emphasize that these results are crucial for our proof. In Section 3.2 we describe the structures of graphs and hypergraphs used in the proof. We introduce the notion of nicely dominated graphs which is a suitable "normalization" of the structure of the dominated planar graphs. We describe how to reduce a nicely dominated graph G to a graph $\mathbf{red}(G)$ whose number of vertices depends only on the size of the domination set. Then we introduce two ways of turning planar graphs into hypergraphs and prove some relations between hypergraphs related to reduced graphs and hypergraphs of the "simplest possible" nicely dominated graphs called prime graphs. Finally, we prove how every nicely dominated graph can be constructed from prime graphs. All these results and constructions are merged in Section 3.7 to prove the main combinatorial result.

Section 4 contains discussions on algorithmic consequences of the combinatorial result. In this section we also give a dynamic programming algorithm solving dominating set problem on graphs of branch-width $\leq \ell$ and m edges in time $O(2^{3\log_4 3 \cdot \ell} m)$.

In Section 5 we provide some concluding remarks.

2 Definitions and preliminary results

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For every non-empty $W \subseteq V(G)$, the subgraph of G induced by W is denoted by $G[W]$. A vertex $v \in V(G)$ of a connected graph G is called a *cutvertex* if the graph $G - \{v\}$ is not connected. A connected graph on ≥ 3 vertices without a cutvertex is called *2-connected*.

Let Σ be a sphere. By Σ-*plane* graph G we mean a planar graph G with the vertex set $V(G)$ and the edge set $E(G)$ drawn in Σ. To simplify notations, we usually do not distinguish between a vertex of the graph and the point of Σ used in the drawing to represent the vertex or between an edge and the open line segment representing it. If $\Delta \subseteq \Sigma$, then $\overline{\Delta}$ denotes the *closure* of Δ, and the boundary of Δ is $\widehat{\Delta} = \overline{\Delta} \cap \overline{\Sigma - \Delta}$. We denote the set of the regions of the drawing by $R(G)$. (Every region is an open set.) An edge e (a vertex v) is incident with a region r if $e \subseteq \bar{r}$ ($v \subseteq \bar{r}$). We do not distinguish between a boundary of a region and the subgraph of the drawing induced by edges incident to the region. For a region r, by the length of the boundary $\widehat{r}$ we mean the number of edges incident to r. $\Delta \subseteq \Sigma$ is an open disc if it is homeomorphic to $\{(x, y) : x^2 + y^2 < 1\}$. Let C be a cycle in a Σ-plane graph G. By the Jordan curve theorem, C bounds exactly two discs. For a vertex $x \in V(G)$, we call a disc Δ bounded by C *x-avoiding* if $x \notin \overline{\Delta}$. We call a region $r \in R(G)$ *square region* if $\widehat{r}$ is a cycle of length 4.

A set $D \subseteq V(G)$ is a *dominating set* in a graph G if every vertex in $V(G) - D$ is adjacent to a vertex in D. Graph G is *D-dominated* if D is a dominating set in G.

For a hypergraph $\mathcal{G}$ we denote by $V(\mathcal{G})$ its vertex (ground) set and by $E(\mathcal{G})$ the set of it hyperedges. A *branch decomposition* of a hypergraph $\mathcal{G}$ is a pair (T, τ), where T is a tree with vertices of degree 1 or 3 and τ is a bijection from $E(\mathcal{G})$ to the set of leaves of T. The *order function* $\omega : E(T) \to 2^{V(\mathcal{G})}$ of a branch decomposition maps every edge e of T to a subset of vertices $\omega(e) \subseteq V(\mathcal{G})$ as follows. The set $\omega(e)$ consists from all vertices of $V(\mathcal{G})$ such that for every vertex $v \in \omega(e)$ there exist edges $f_1, f_2 \in E(\mathcal{G})$ such that $v \in f_1 \cap f_2$ and the leaves $\tau(f_1)$, $\tau(f_2)$ are in different components of $T - \{e\}$. The *width* of (T, τ) is equal to $\max_{e \in E(T)} |\omega(e)|$ and the *branch-width* of $\mathcal{G}$, $\mathbf{bw}(\mathcal{G})$, is the minimum width over all branch decompositions of $\mathcal{G}$.

Given an edge $e = \{x, y\}$ of a graph G, the graph G/e is obtained from G by contracting the edge e; that is, to get G/e we identify the vertices x and y and remove all loops and duplicate edges. A graph H obtained by a sequence of edge-contractions is said to be a *contraction* of G. H is a *minor* of G if H is the subgraph of a contraction of G. We use the notation $H \preceq G$ (resp. $H \preceq_c G$) for H is a minor (a contraction) of G. It is well known that $H \preceq G$ or $H \preceq_c G$ implies $\mathbf{bw}(H) \leq \mathbf{bw}(G)$. Moreover, the conditions G has a dominating set of size k and $H \preceq_c G$ imply that H has a dominating set of size $\leq k$. (Which is not true for $H \preceq G$.)

For planar graphs the branch-width can be bounded in terms of the dominating number by making use of the following deep result of Robertson, Seymour & Thomas. (Theorems (4.3) in [19] and (6.3) in [21].)

Theorem 2.1 ([21]). *Let $k \geq 1$ be an integer. Every planar graph with no (k, k)-grid as a minor has branch-width $\leq 4k - 3$.*

To give an idea on how results from Graph Minors can be used on the study of dominating sets in planar graphs, we present the following consequence of Theorem 2.1.

Lemma 2.2. *Let G be a planar graph with a dominating set of size $\leq k$. Then $\mathbf{bw}(G) \leq 12\sqrt{k} + 9$.*

Proof. Suppose that $\mathbf{bw}(G) > 12\sqrt{k} + 9$. By Theorem 2.1, there exists a sequence of edge contractions or edge/vertex removals reducing G to a (ρ, ρ)-grid where $\rho = 3\sqrt{k}+3$. We apply to G only the contractions from this sequence and call the resulting graph J. J contains a (ρ, ρ)-grid as a subgraph. As $J \preceq_c G$, J has also a dominating set D of size $\leq k$. A vertex in D cannot dominate more than 9 internal vertices of the (ρ, ρ)-grid. Therefore, $k \geq (\rho - 2)^2/9$ which implies $\rho \leq 3\sqrt{k} + 2 = \rho - 1$, a contradiction. $\square$

In the remaining part of the paper we show how the above upper bound for branch-width of a planar graph in terms of its dominating set number can be strongly improved. Our results will use as basic ingredient the following theorem that is a direct consequence of Robertson & Seymour min-max Theorem (4.3) in [19] relating tangles and branch-width and Theorem (6.6) in [20] establishing relations between tangles of dual graphs.

Theorem 2.3 ([19, 20]). *For any planar graph G of branch-width ≥ 2, the branch-width of G is equal to the branch-width of its dual.*

For our bounds we need an upper bound on the size of branch-width of a planar graph in terms of its size. The best published bound for the branch-width we were able to find in the literature, is $\mathbf{bw}(G) \leq 4\sqrt{|V(G)|} - 3$ which follows directly from Theorem 2.1. As it was noticed by Robin Thomas (in private communication), a better bound $\mathbf{bw}(G) \leq \sqrt{4.5 \cdot |V(G)|}$ can be obtained by suitably adapting the arguments from Alon, Seymour & Thomas paper [6]. Another proof of this inequality can be found in [12]. This proof is based on a relation between slopes and majorities, the two notions introduced by Robertson & Seymour in [19] and Alon, Seymour & Thomas in [6] respectively.

Theorem 2.4 ([12]). *For any planar graph G, $\mathbf{bw}(G) \leq \sqrt{4.5 \cdot |V(G)|}$.*

3 Bounding branch-width of D-dominated planar graphs

This section is devoted to the proof of the main combinatorial result of this paper: the branch-width of any planar graph with a dominated set of size k is at most $3\sqrt{4.5}\sqrt{k}$. The idea of the proof is to show that for every planar graph G with a dominated set of size k there is a graph H on $\leq k$ vertices such that $\mathbf{bw}(G) \leq 3 \leq \mathbf{bw}(H)$. Then Theorem 2.4 will do the rest of the job.

The way of constructing of the graph H and the proof of $\mathbf{bw}(G) \leq 3 \leq \mathbf{bw}(H)$ is not direct. First we prove that every planar graph with a dominating set D is a minor of a some graph with a nice structure. We call these 'structured' graphs nicely D-dominated. For a nicely D-dominated planar graph F we show how to define a graph $\mathbf{red}(F)$ on $|D|$ vertices. The most complicated part of the proof is the proof that $\mathbf{bw}(F) \leq 3 \leq \mathbf{bw}(\mathbf{red}(F))$ (clearly this implies the main combinatorial result). The proof of this inequality is based on a more general result about isomorphism of special hypergraphs obtained from F and $\mathbf{red}(F)$ (Lemma 3.16) and the structural properties of nicely D-dominated graphs.

3.1 Auxiliary results

In this section we obtain some useful technical results about branch-width.

Lemma 3.1. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be hypergraphs with one edge in common, i.e. $V(\mathcal{G}_1) \cap V(\mathcal{G}_1) = f$ and $\{f\} = E(\mathcal{G}_1) \cap E(\mathcal{G}_2)$. Then $\mathbf{bw}(\mathcal{G}_1 \cup \mathcal{G}_2) \leq \max\{\mathbf{bw}(\mathcal{G}_1), \mathbf{bw}(\mathcal{G}_1), |f|\}$. Moreover, if every vertex $v \in f$ has degree ≥ 2 in at least one of hypergraphs, (i.e. v is contained in at least two edges in $\mathcal{G}_1$ or in at least two edges in $\mathcal{G}_2$), then $\mathbf{bw}(\mathcal{G}_1 \cup \mathcal{G}_2) = \max\{\mathbf{bw}(\mathcal{G}_1), \mathbf{bw}(\mathcal{G}_1)\}$.*

Proof. Clearly, $\mathbf{bw}(\mathcal{G}_1 \cup \mathcal{G}_2) \geq \max\{\mathbf{bw}(\mathcal{G}_1), \mathbf{bw}(\mathcal{G}_1)\}$.

For $i = 1, 2$, let (T_i, τ_i) be a branch decomposition of $\mathcal{G}_i$ of width $\leq k$ and let $e_i = \{x_i, y_i\}$ be the edge of T_i having as endpoint the leaf $\tau_i(f) = x_i$. We construct tree T as follows. First we remove the vertices x_i and add edge $\{y_1, y_2\}$. Then we subdivide $\{y_1, y_2\}$ by introducing a new vertex y. Finally we add vertex x and make it adjacent to y.

We put $\tau(f) = x$. For any other edge $g \in E(\mathcal{G}_1) \cup E(\mathcal{G}_2)$ we put $\tau(g) = \tau_1(g)$ if $g \in E(\mathcal{G}_1)$ and $\tau(g) = \tau_2(g)$ otherwise.

Because $\omega(\{y_1, y\}) = \omega(\{y_2, y\}) = \omega(\{x, y\}) \leq |f|$ and for all other edges of T its order is equal to the order of the corresponding edge in one of the T_i's, we have that (T, τ) is a branch decomposition of width $\leq \max\{k, |f|\}$.

If every vertex v of f has degree ≥ 2 in one of the hypergraphs, then it is easy to see that that $|f| \leq k$. This implies that (T, τ) is a branch decomposition of width $\leq k$. $\square$

Let G be a connected Σ-plane graph where all the vertices have degree ≥ 2. For a vertex x of G and a pair (z, y) of two of its neighbors, we call (z, y) *pair of consecutive neighbors of* x if edges $\{x, z\}$, $\{x, y\}$ appear consecutively in the cyclic ordering of the edges incident to x. (Notice that if x has only two neighbors y and z, then both (y, z) and (z, y) are pairs of consecutive neighbors of x.)

Lemma 3.2. *Let G be a Σ-plane graph that is not a forest. Then G is the minor of a Σ-plane 2-connected graph H such that $\mathbf{bw}(H) = \mathbf{bw}(G)$.*

Proof. We use induction on the number of 2-connected components of G. Clearly, if G is 2-connected, the lemma follows trivially. Suppose that it is correct for every graph with $< n$ connected components. Suppose now that G is a graph with n 2-connected components. Let H_1 be one of these 2-connected components and let H_2 be the union of all the rest. W.l.o.g. we assume that H_2 is not a forest. By the induction assumption, there is a 2-connected graph H_2' such that $H_2 \preceq H_2'$ and $\mathbf{bw}(H_2') = \mathbf{bw}(H_2)$. Let $G' = H_2' \cup H_1$ and let x be the unique cutvertex of G'. Let a and b be two consecutive neighbors of x (i.e. vertices such that the edges $\{a, x\}$, $\{b, x\}$ are incident to the same region) where $a \in V(H_2')$ and $b \in V(H_1)$. We denote by G'' the graph obtained from G' by adding the edge $\{a, b\}$. Notice that G'' is 2-connected and contains G' (and therefore G) as a minor. Let G''' be the graph subgraph of G'' induced by vertices $V(H_2') \cup \{b\}$. By using Lemma 3.1 for $G'''[\{a, b, x\}]$ and H_2' we have that

$\mathbf{bw}(G''') \leq \max\{\mathbf{bw}(H_2'), 2\} = \mathbf{bw}(H_2')$ (H_2' is 2-connected and $\mathbf{bw}(H_2') \geq 2$). Applying again Lemma 3.1 for G''' and H_1, we have that $\mathbf{bw}(G'') \leq \max\{\mathbf{bw}(G'''), \mathbf{bw}(H_1)\} = \max\{\mathbf{bw}(H_2'), \mathbf{bw}(H_1)\} = \max\{\mathbf{bw}(H_2), \mathbf{bw}(H_1)\} = \mathbf{bw}(G)$. As $G \prec G''$, the lemma follows. $\square$

A graph G is *multiply triangulated* if all its regions are of length 2 or 3. A graph is $(2,3)$-regular if all its vertices have degree 2 or 3. Notice that the dual of a multiply triangulated graph is $(2,3)$-regular and vice versa.

Lemma 3.3. *Every 2-connected Σ-plane graph G has a weak triangulation H such that* $\mathbf{bw}(H) = \mathbf{bw}(G)$.

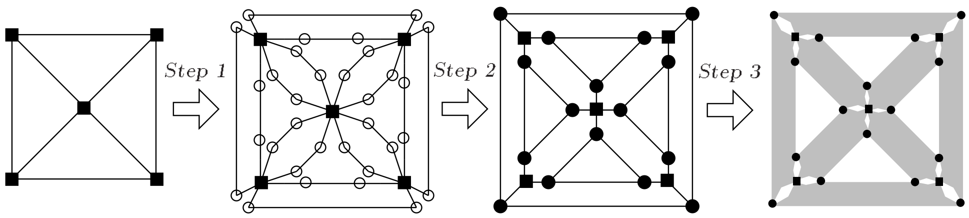
Proof. Because G is 2-connected every region of G is bounded by a cycle. Suppose that there is a region r of G bounded by a cycle $C = (x_0, \dots, x_{r-1})$, $r \geq 4$. We show that there are vertices x_i and x_j that are not adjacent in C such that the graph G' obtained from G by adding the edge $\{x_i, x_j\}$ has $\mathbf{bw}(G') = \mathbf{bw}(G)$. By applying this argument recursively, one obtains a weak triangulation of G of the same branch-width.

If there are vertices x_i and x_j that are adjacent in G and are not adjacent in C then we can draw a chord joining x_i and x_j in r. Because G is 2-connected it holds that $\mathbf{bw}(G) \geq 2$ and therefore the addition of multiple edges does not increase the branch-width. Suppose now that the cycle C is chordless. Let (T, τ) be a branch decomposition of G and let ω be its order function. It easy to check that there is an edge f of T such that one of the components of $T - \{f\}$ contains exactly two edges of C. Let e_1, e_2 be such edges. Because C is chordless and its length is at least 4, we have that $\omega(f)$ contains at least two vertices, say x_i and x_j of C that are not adjacent. Then adding edge $\{x_i, x_j\}$ does not increase the branch-width. (The decomposition can be obtained from T by subdividing f and adding the leaf corresponding to $\{x_i, x_j\}$ to the vertex subdividing f.) $\square$

In the next Lemma we use powerful duality results of Robertson & Seymour. Moreover, the implication of these results play the crucial role in our proof.

Lemma 3.4. *Every 2-connected Σ-plane graph G is the contraction of a (2,3)-regular graph H such that* $\mathbf{bw}(H) = \mathbf{bw}(G)$.

Proof. Let G^d be the dual graph of G. By Theorem 2.3, $\mathbf{bw}(G^d) = \mathbf{bw}(G)$. By Lemma 3.3,there is a weak triangulation H^d of G^d such that $\mathbf{bw}(H^d) = \mathbf{bw}(G^d)$. The dual of H^d, we denote it by H, contains G as a contraction (each edge removal in a planar graph corresponds to an edge contraction in its dual and vice versa). Applying Theorem 2.3 the second time, we obtain that $\mathbf{bw}(H) = \mathbf{bw}(H^d)$. Hence, $\mathbf{bw}(H) = \mathbf{bw}(G)$. Since H^d is multiply triangulated, we have that H is $(2,3)$-regular. $\square$



G $\quad$ G_1 $\quad$ G_2 $\quad$ $\mathbf{ext}(G)$

Figure 1: The steps 1,2, and 3 of the definition of the function **ext**.

3.2 Extensions of Σ-plane graphs

Let G be a connected Σ-plane graph where all the vertices have degree ≥ 2. We define the *exension*, $\mathbf{ext}(G)$, of G as the hypergraph obtained from G by making use of the following three steps (see Figure 1 for an example).

Step 1: *For each edge $e \in E(G)$: duplicate e and then subdivide each of its two copies 2 times. That way, each edge $e = \{x, y\}$ of G is replaced by a cycle denoted as $C_{x,y} = (x, x^+_{x,y}, y^-_{x,y}, y, y^+_{x,y}, x^-_{x,y}, x)$ (indexed in clock-wise order). Let G_1 be the resulting graph.*

Step 2: *For each vertex $x \in V(G)$ and each pair (y, z) of consecutive neighbors of x (in G), identify the edges $\{x, x^-_{x,y}\}$ and $\{x, x^+_{x,z}\}$ in G_1. Let G_2 be the resulting graph.*

Step 3: *The Hypergraph $\mathbf{ext}(G)$ is defined by setting $\mathbf{ext}(G) = (V(G_2), \{C_{x,y} \mid \{x, y\} \in E(G)\})$.*

From the above construction, if $\mathcal{H} = \mathbf{ext}(G)$ then there exists a bijection $\theta : E(G) \to E(\mathcal{H})$ mapping each edge $e = \{x, y\}$ to the hyperedge formed by the vertices of $C_{x,y}$. See Figure 1 for an example of the definition of **ext**.

Lemma 3.5. *For any $(2, 3)$-regular Σ-plane graph G, $\mathbf{bw}(\mathbf{ext}(G)) \leq 3 \cdot \mathbf{bw}(G)$.*

Proof. Let (T, τ) be a branch decomposition of G of width $\leq k$. By the definition of $\mathbf{ext}(G)$ there is a bijection $\theta : E(G) \to E(\mathbf{ext}(G))$ defining which edge of G is replaced by which hyperedge of $\mathbf{ext}(G)$. Let L be the set of leaves in T. For $\mathbf{ext}(G)$ we define branch decomposition (T, τ') with a bijection $\tau' : E(\mathbf{ext}(G)) \to L$ such that $\tau'(t) = \theta(\tau(t))$. We use the notations ω and ω' for the order functions of (T, τ) and (T, τ') respectively.

We claim that (T, τ') is a branch decomposition of $\mathbf{ext}(G)$ of width $\leq 3k$. For this, it is enough to show that for any $f \in E(T)$, $|\omega'(f)| \leq 3 \cdot |\omega(f)|$. In other words, we need to show that it is possible to define a function σ_f mapping each vertex $v \in \omega(f)$ to a set of 3 vertices of $\omega'(f)$ such that every vertex $y \in \omega'(f)$ is contained in the $\sigma_f(x)$ for some $x \in \omega(f)$.

Let T_1 and T_2 be the components of $T - \{f\}$. We construct σ_f by distinguishing two cases.

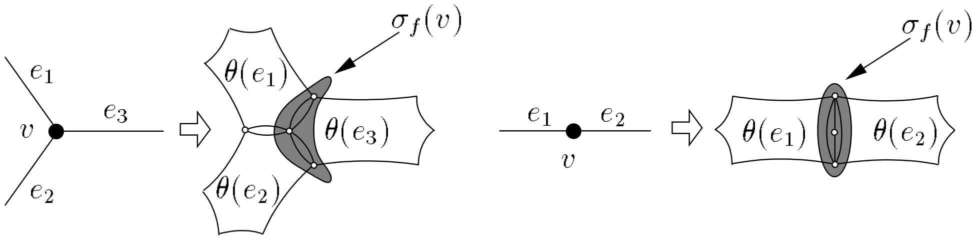


Figure 2: The construction of the value of $\sigma_f(v)$ in the proof of Lemma 3.5.

• The degree of v is 3 in G. We can assume that two, say e_1, e_2, of its incident edges are images of leaves of T_1 and one, say e_3, is an image of a leave in T_2. We define $\sigma_f(v) = (\theta(e_1) \cap \theta(e_3)) \cup (\theta(e_2) \cap \theta(e_3))$ (this process is illustrated in the left half of Figure 2).

• The degree of v is 2 in G. We can assume that one, say e_1 of its incident edges is an image of some leave of T_1 and the other, say e_2, is an image of a leave in T_2. We define $\sigma_f(v) = \theta(e_1) \cap \theta(e_2)$ (this is illustrated in the right half of Figure 2).

Notice that, in both cases $|\sigma_f(v)| = 3$. Suppose now that y is a vertex in $\omega'(f)$. Then, y should be an endpoint of at least two hyperedges α and β of $\mathbf{ext}(G)$ and w.l.o.g. we assume that $\tau'(\alpha)$ is a leaf of T_1 and $\tau'(\beta)$ is a leaf of T_2. By the definition of τ', this means that $\tau(\theta^{-1}(\alpha))$ is a leaf of T_1 and $\tau(\theta^{-1}(\beta))$ is a leaf of T_2. By the construction of $\mathbf{ext}(G)$, $\theta^{-1}(\alpha)$ and $\theta^{-1}(\beta)$ have an endpoint x in common, therefore $x \in \omega(f)$. From the definition of σ_f we get that $y \in \sigma_f(x)$. This proves the relation $|\omega'(f)| \leq 3 \cdot |\omega(f)|$ and the lemma follows. □

Let $\mathcal{H}$ be a planar hypergraph and let $E \subseteq E(\mathcal{H})$. We set $\mathbf{cl}_E(\mathcal{H}) = (V(\mathcal{H}), E_{\mathcal{H}})$ where $E_{\mathcal{H}} = E(\mathcal{H}) - E \cup \{\{x,y\} \subseteq V(\mathcal{H}) \mid \exists_{e \in E(\mathcal{H})} : \{x,y\} \in e\}$ (in other words, we replace each hyperedge $e \in E$ by a clique formed by connecting each pair of endpoints of e).

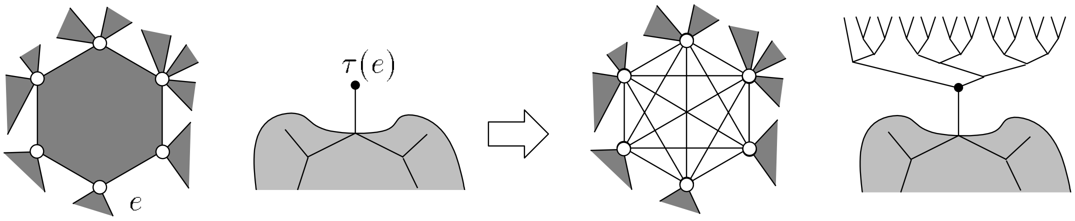


Figure 3: The construction of the branch decomposition of $\mathbf{cl}_E(H)$ in the proof of Lemma 3.6.

Lemma 3.6. *Let $\mathcal{H}$ be a hypergraph with every vertex of degree ≥ 2. Then for any $E \subseteq E(\mathcal{H})$, $\mathbf{bw}(\mathbf{cl}_E(H)) \leq \mathbf{bw}(H)$.*

Proof. If (T, τ) is a branch decomposition of H we construct a branch decomposition of $\mathbf{cl}_E(H)$ by identifying any leaf t where $\tau(t) \in E$ with the root of a binary tree T_t that has $\binom{|\tau(t)|}{2}$ leaves. The leaves of T_t are mapped to the edges of the clique made up by pairs of endpoints in $\tau(t)$ (see also Figure 3). $\square$

Lemma 3.7. *Let G and H be connected graphs, such that $G \preceq H$ and all the vertices of G have degree ≥ 2. Then* $\mathbf{bw}(\mathbf{ext}(G)) \leq \mathbf{bw}(\mathbf{ext}(H))$.

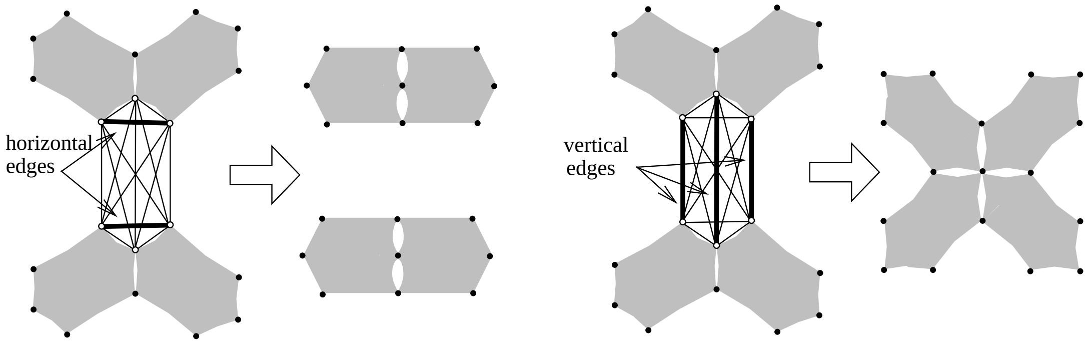


Figure 4: The construction of the branch decomposition of $\mathbf{cl}_E(H)$ in the proof of Lemma 3.7 .

Proof. Let E', (resp. E'') be the set of edges that one should contract (resp. remove) in H in order to obtain G (clearly, we can assume that $E' \cap E'' = \emptyset$). If we prove that $\mathbf{ext}(G)$ is a minor of $\mathbf{cl}_{E' \cup E''}(\mathbf{ext}(H))$ then the result will follow from Lemma 3.6. To see this, for each $e = \{x, y\} \in E'$, we distinguish the edges of the clique replacing $\theta(e) = (x, x^+_{x,y}, y^-_{x,y}, y, y^+_{x,y}, x^-_{x,y}, x)$ into two categories: we call $\{x^+_{x,y}, y^-_{x,y}\}$, $\{x, y\}$, and $\{y^+_{x,y}, x^-_{x,y}\}$ *horizontal* and we call the rest *unimportant*. Moreover, for any edge $e = \{x, y\} \in E''$, we distinguish the edges of the clique replacing $\theta(e) = (x, x^+_{x,y}, y^-_{x,y}, y, y^+_{x,y}, x^-_{x,y}, x)$ into two categories: we call $\{x^+_{x,y}, x^-_{x,y}\}$ and $\{y^+_{x,y}, y^-_{x,y}\}$ *vertical* and the rest *useless*. To obtain $\mathbf{ext}(G)$ from $\mathbf{cl}_{E'}(\mathbf{ext}(H))$ we first remove the useless and the unimportant edges and then contract all the horizontal and vertical ones (see Figure 4). $\square$

We are ready to state the main property of **ext**.

Lemma 3.8. *Let G be a connected Σ-plane graph with all vertices of degree ≥ 2. Then* $\mathbf{bw}(\mathbf{ext}(G)) \leq 3 \cdot \mathbf{bw}(G)$.

Proof. Notice that G is not a forest and by Lemma 3.2, G is the minor of a 2-connected Σ-plane graph G' such that $\mathbf{bw}(G') = \mathbf{bw}(G)$. By Lemma 3.4, G' is the minor of a $(2, 3)$-regular Σ-plane graph H where $\mathbf{bw}(H) \leq \mathbf{bw}(G')$. Notice that G is a minor of H and both G and H are connected. From Lemma 3.7, $\mathbf{bw}(\mathbf{ext}(G)) \leq \mathbf{bw}(\mathbf{ext}(H))$.

Notice that H is $(2,3)$-regular. By Lemma 3.5, $\mathbf{bw}(\mathbf{ext}(H)) \leq 3 \cdot \mathbf{bw}(H)$ and the result follows. □

3.3 Nicely D-dominated Σ-plane graphs

An important tool spanning all of our proofs is the concept of unique D-domination. We call a D-dominated graph G *uniquely dominated* if there is no path of length < 3 connecting two vertices of D. Notice that this implies that each vertex $x \in V(G) - D$ has exactly one neighbor in D (i.e. is uniquely dominated).

We call a multiple edge $\{a, b\}$ of a D-dominated Σ-plane graph G *exceptional* if its endpoints are both adjacent to a vertex in D and any pair of its copies defines a cycle containing vertices in D in both open disks it defines (for example, all the multiple edges in the graphs in Figure 5 are exceptional).

Lemma 3.9. *For every 2-connected D-dominated Σ-plane graph G without multiple edges, there exists a Σ-plane graph H such that*

(a) G is a minor of H.

(b) H is uniquely D-dominated.

(c) All multiple edges of H are exceptional.

(d) For any region r of H, $\widehat{r}$ is either a triangle or a square.

(e) If $x, y \in D$ have distance 3 in H then there exist at least two distinct (x, y)-paths in H.

(f) If a (closed) region r of H contains a vertex of D then $\widehat{r}$ is a triangle.

(g) Every square region of H contains two edges $e_i, i = 1, 2$ without common vertices such that for every $i = 1, 2$, there exists a vertex $x_i \in D$ adjacent to both endpoints of e_i.

(h) If $x, y \in D$ then every two distinct (x, y)-paths of H of length 3 are internally disjoint.

Proof. We construct a graph H, satisfying properties (a) – (f), by applying, one after the other, on G the following transformations:

• **T1**. *As long as there exists in G a vertex x with more than one neighbor y in D, subdivide the edge $\{x, y\}$.*
We call the resulting graph G_1.

As G_1 does not have multiple edges, properties (a), (c) are trivially satisfied. Moreover, notice that if G_1 is not uniquely dominated then **T1** can be further applied.

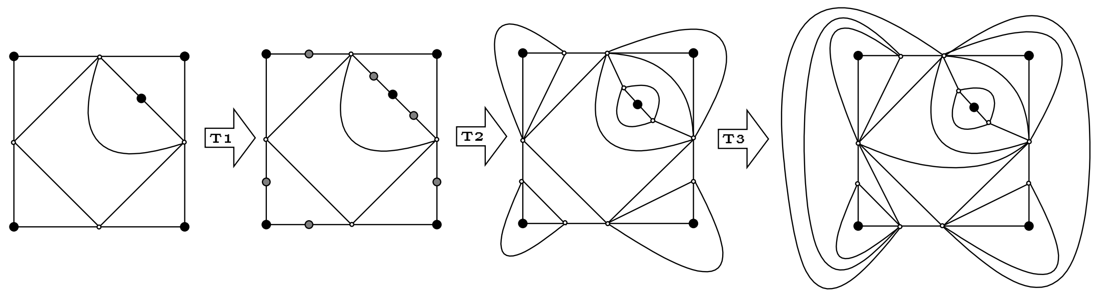


Figure 5: Example of the transformations **T1**,**T2**, and **T3** in the proof of Lemma 3.6.

Therefore, (b) holds for G_1. For an example of the application of **T1**, see the first step of Figure 5.

• **T2**. *As long as G_1 has a region r bounded by a cycle $\widehat{r} = (x_0, \ldots, x_{q-1}), q \geq 4$ and such that $x_i \in D$ for some $i, 0 \leq i \leq q-1$, then add in G_1 the edge $\{x_{i-1}, x_{i+1}\}$ (indices are taken modulo q).*
We call the resulting graph G_2.

Notice that the vertices of $\widehat{r}$ are distinct because G_2 is 2-connected. Clearly, G_2 satisfies property (a). Recall now that G_1 satisfies property (b). Therefore, if some vertex $x_i \in \widehat{r}$ is in D then its neighbors x_{i-1} and x_{i+1} (the indices are taken modulo q) are not in D. Therefore, property (b) holds also for G_2. Notice that if **T2** creates a multiple edge, then this can be only an exceptional multiple edge. Therefore (c) holds for G_2. For an example of the application of **T2**, see the second step of Figure 5.
Finally, notice that none of the vertices of D is in a region of G_2 of length ≥ 4.

We call a square region that satisfies property (g) *solid.*

• **T3**. *As long as G_2 has a region r that is not a solid square and such that $\widehat{r} = (x_0, \ldots, x_{q-1}), r \geq 4$, choose an edge in $\{\{x_1, x_3\}, \{x_0, x_2\}\}$ that is not already present in G_2 and add it to G_2.*
We call the resulting graph G_3.

The above transformation can always be applied because it is impossible that both $\{x_1, x_3\}$ and $\{x_0, x_2\}$ are in G_3. Therefore property (c) is an invariant of **T3**. Clearly, G_3 satisfies property (a). Property (b) is an invariant of **T3** as the added edge has no endpoints in D. We have that all the regions of G_3 are either triangles or solid squares and therefore G_3 also satisfies (d) and (g). For an example of the application of **T3**, see the third step of Figure 5.

• **T4**. *As long as G_3 has a unique (x, y)-path $P = (x, a, b, y)$ where $x, y \in D$, apply the first transformation of Figure 6 on P.*
We call the resulting graph G_4.

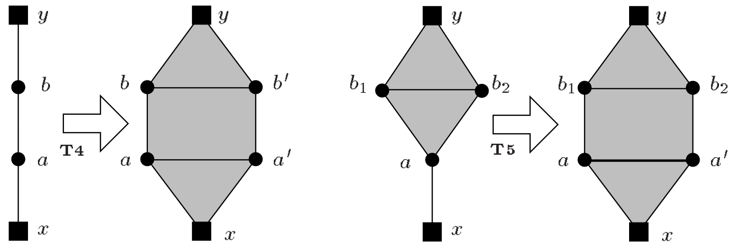


Figure 6: The transformations **T4** and **T5** in the proof of Lemma 3.6.

It is easy to verify that properties (a) – (d) are invariants of **T4**. Also, it is easy to see that the transformation of Figure 6 creates square regions with property (g) and does not alter property (g) for square regions that already have been created. Moreover, G_4 satisfies (e) because each time we apply the transformation of Figure 6 the number of pairs in D connected by unique paths decrease. Finally, none of the square regions appearing (because of **T4**) contains a vertex in D. Thus (f) holds. For an example of the application of **T4**, see Figure 7.

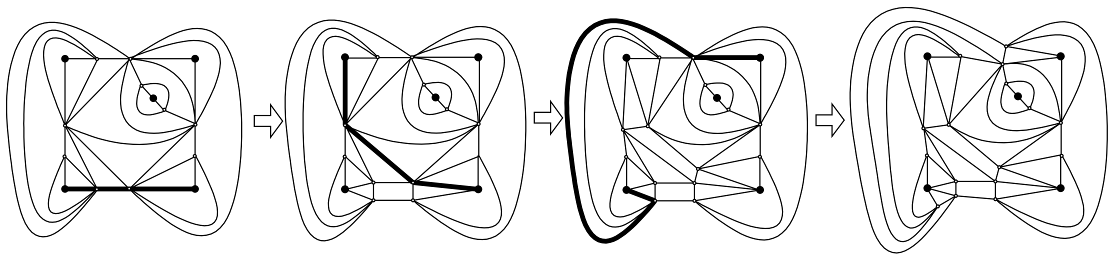


Figure 7: Example of the transformation **T4** in the proof of Lemma 3.6.

In order to give the transformation that enforces property (h) we need some definitions. Observe that if property (h) does not hold for G_4, this implies the existence of some pair of paths $P_i = (x, a, b_i, y)$, $i = 1, 2$. We call the graph O defined by this this pair (h)-obstacle and we define its *(h)-disc* as the x-avoiding closed disc Δ_O bounded by the cycle (a, b_1, y, b_2, a). Such an (h)-obstacle is *minimal* if no (x, y)-path has vertices contained in its h-disc. Notice that if G_4 has an (h)-obstacle it also has a minimal (h)-obstacle and vice versa. We call an (h)-obstacle *hollow* if its *(h)*-disc contains no neighbor of a except b_1 and b_2. Notice that a hollow (h)-obstacle is always minimal. We claim that in any hollow *(h)*-disc, vertices b_1 and b_2 are adjacent. Indeed, by property (b), a is not adjacent to y in G_4. Therefore b_1, a, b_2 are in a region of G_4 that, from property (g), cannot be a square region (otherwise, property (b) would be violated). Therefore, (b_1, a, b_2) is a triangle and the claim follows.

• **T5**. *As long as G_4 has a hollow (h)-obstacle O, apply the second transformation of Figure 6 on edge $\{a, x\}$ and the region bounded by (b_1, b_2, a).*

We call the resulting graph G_5.

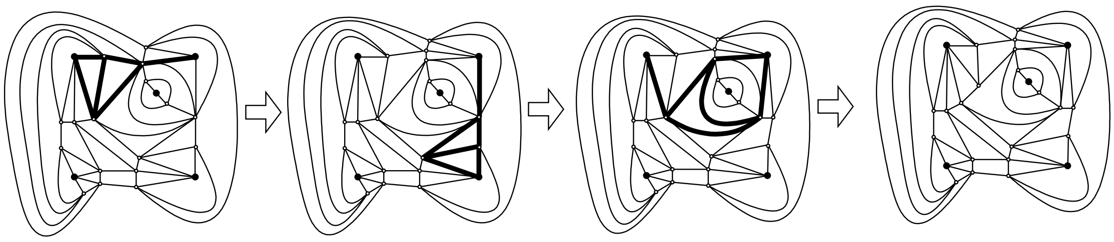
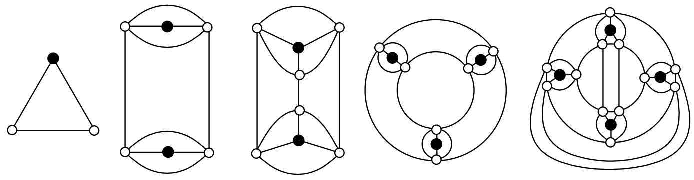


Figure 8: Example of the transformation **T5** in the proof of Lemma 3.6.

Notice that after **T5** none of the properties (a) – (g) is altered by the application of **T5** (the arguments are the same as those used for the previous transformations). Moreover, each time the second transformation of Figure 6 is applied, the number of hollow minimal (h)-obstacles decreases and no new non-hollow (h)-obstacles appear. For an example of the application of **T5**, see Figure 8. To finish the proof, is enough to show that **T5** is able to eliminate all the (h)-obstacles. For this, it remains to prove the following claim.

Claim: If a 2-connected D-dominated Σ-plane graph satisfies properties (b) – (g) and contains a minimal (h)-obstacle then it also contains a hollow (h)-obstacle.
Proof of Claim: Let $O = (P_1, P_2)$, be a minimal non-hollow (h)-obstacle with (h)-disk Δ_O and let $\mathcal{O}$ be the set containing O along with of all the minimal (h)-obstacles that contain the edge $\{a, x\}$ and whose (h)-disk is a subset of Δ_O. If $O_1, O_2 \in \mathcal{O}$ and $\Delta_{O_1} \subset \Delta_{O_2}$ then we say that $O_1 < O_2$ (clearly, for any $O' \in \mathcal{O} - \{O\}$, $O' < O$). Notice that relation "$<$" is a partial order on $\mathcal{O}$ and that all its minimal elements are hollow (h)-obstacles. The claim follows and thus **T5** is able to enforce property (h). $\square$

Let G be a connected D-dominated Σ-plane graph satisfying properties (b) – (h) of Lemma 3.9. We call such graphs *nicely* D-dominated Σ-plane graphs. Notice that the graphs of Figure 9 and the last graph in Figure 8 are nicely D-dominated Σ-plane graphs (see also Figure 10 and all the graphs of Figure 11).



Σ_1 Σ_2^2 Σ_2^3 Σ_3 Σ_4

Figure 9: Simple examples of nicely D-dominated Σ-plane graphs.

Given a nicely D-dominated Σ-plane graph G we define $\mathcal{T}(G)$ as the set of all the triangles (cycles of length 3) containing a vertex of D. By property (f), for every region r with $\widehat{r} \cap D \neq \emptyset$, $\widehat{r} \in \mathcal{T}(G)$ (the inverse is not always correct, i.e. not every triangle in $\mathcal{T}(G)$ bounds a region). We call the triangles in $\mathcal{T}(G)$, D-triangles. We also define $\mathcal{C}(G)$ as the set of all the cycles containing exactly two vertices of D. From properties (b) and (h) of nicely dominated graphs we have that each cycle C in $\mathcal{C}(G)$ is of length 6 and is the union of two length-3 paths connecting its two dominating vertices. We call the cycles in $\mathcal{C}(G)$ *D-hexagons*. The *poles* of a cycle $C \in \mathcal{C}(G)$ are the vertices in $D \cap C$. We call a D-triangle T (D-hexagon C) *empty* if one of the open discs bounded in Σ by T (C) does not contain vertices of G. Notice that all empty D-triangles are boundaries of regions of G. For some examples of the above definitions see Figure 10.

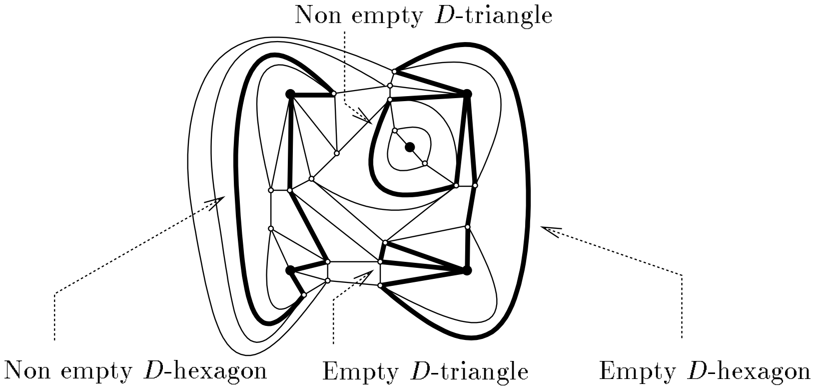


Figure 10: D-triangles and D-hexagons of the last graph of Figure 8.

3.4 Decomposing nicely D-dominated Σ-planar graphs

We continue by showing how nicely D-dominated planar graphs can be simplified. The idea is based on the structure imposed by properties (b) – (h): any nicely D-dominated planar graph can be seen as the result of gluing together two simpler structures of the same type. This is described by the following two lemmata.

Lemma 3.10. *Let G be a nicely D-dominated Σ-plane graph G and let $T \in \mathcal{T}(G)$ be a non-empty D-triangle bounding the closed discs Δ_1, Δ_2. Then, for $i = 1, 2$, the subgraph of G induced by $V(G) \cap \Delta_i$ is a nicely D_i-dominated graph that has less vertices than G and where $D_i \subseteq D$.*

Proof. Let $G_i = G[\Delta_i \cap V(G)]$ and let $D_i = D \cap \Delta_i$, $i = 1, 2$. Clearly, $D_i \subseteq D$. Moreover, as T is non-empty, we have that $|V(G_i)| < |V(G)|$. Let us verify that properties (b)–(h) hold for G_i, $i = 1, 2$.

To prove property (b), we show first that G_i is D_i-dominated. For sake of contradiction, suppose that there exists a vertex $a \in V(G_i)$ that is not dominated by D_i.

As property (b) holds for G there exists a vertex $w \in D - D_i$ so that a is uniquely dominated by w in G. This means that $w \in \Sigma - \Delta_i$ and $a \in \Delta_i$. Therefore, a is a vertex of T. Because T is a D-triangle, there is $x \in D \cap T$. Since a is adjacent in G_i to x and $x \neq w$, we have a contradiction to the property (b) on G. Now it remains to prove that G_i is properly D'-dominated and this is a direct consequence of the fact that G_i is an induced subgraph of G.

Property (c) is invariant under taking of induced subgraphs.

For (d), notice that all the regions of G_i that are in Δ_i are also the regions of G. Therefore, property (d) holds for all these regions. Also, it holds for the unique new region $r = \Sigma - \Delta_i$ of G_i because $\widehat{r}$ is a triangle.

For property (e), let x, y be two vertices in D_i of distance 3 in G_i. Let P_i^1 and P_i^2 be two internally disjoint paths connecting x and y in G (these paths exist because of properties (e) and (h) in G). Notice that (e) holds if we prove that both P_i^j, $j = 1, 2$ are paths of G_i, $i = 1, 2$, as well. Suppose in contrary that one, say $P_i^1 = (x, a, b, y)$, of P_i^j, $j = 1, 2$, is not a path in G_i. This means that at least one of a, b is in $(\Sigma - \Delta_i) \cap V(G)$. It follows that two non-consecutive vertices of P_i^1 are vertices of T. Therefore, the distance between x and y in G is ≤ 2, a contradiction to property (b) for G.

Suppose now that (f) does not hold for G_i. As (d) holds for G_i we have that there exists a square in G_i containing a vertex of D. As G_i is an induced subgraph of G, this square should exist also in G, a contradiction to (f) for G.

To prove (g) suppose that (a, b, c, d) is a square of G_i. As G_i is an induced subgraph of G, (a, b, c, d) is also a square of G, therefore, we may assume that there are vertices $z, w \in D$ where (z, a, b) and (w, c, d) are triangles of G. It is enough to prove that $\{z, a\}, \{z, b\}, \{w, c\}$ and $\{w, d\}$ are edges of G_i. Suppose, in contrary, that one, say $\{a, z\}$, of them is not an edge of G_i. As G_i is an induced subgraph of G this means that $z \notin V(G_i)$. In other words, we have that (z, a, b) is a triangle of G where $z \in (\Sigma - \Delta_i) \cap V(G)$ and $\{a, b\} \in \Delta_i \cap V(G)$. If this is true, then a, b should be vertices of T, therefore z and the dominating vertex belonging in T are at distance ≤ 2 in G, a contradiction to property (b).

Finally, if there exist two paths violating (h) in G_i the same should happen also in G as G_i is an induced subgraph of G. □

For an example of the application of Lemma 3.10, see the second step of Figure 11.

Lemma 3.11. *Let G be a nicely D-dominated Σ-plane graph G and let $C = (x, a, b, y, c, d, x)$ be a non-empty D-hexagon with poles x, y bounding the closed discs Δ_1, Δ_2. Then, for $i = 1, 2$, the subgraph of G induced by $V(G) \cap \Delta_i$ and extended by adding the edges $\{b, c\}$ and $\{a, d\}$, is a nicely D_i-dominated graph graph that has less vertices than G for some $D_i \subseteq D$.*

Proof. Let $G_i^- = G[\Delta_i \cap V(G)]$ and let $D_i = D \cap \Delta_i, i = 1, 2$. Let also G_i be the

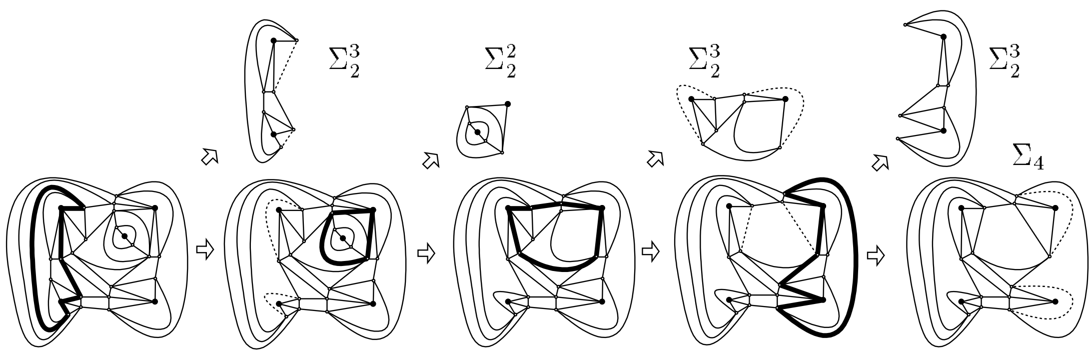


Figure 11: Examples of the application of Lemmata 3.10 and 3.11.

graph with $V(G_i) = V(G_i^-)$ and $E(G_i) = E(G_i^-) \cup \{\{b, c\} \cup \{a, d\}\}$. As in the proof of Lemma 3.10, we have that $D_i \subseteq D$ and $|V(G_i)| < |V(G)|$. Let us verify properties (b)–(h) for G_i, $i = 1, 2$.

To prove (b) we first claim that G_i is D_i-dominated. If some vertex $\alpha \in V(G_i) - D_i$ is not dominated by D_i then it is dominated by some vertex $w \in D - D_i$ (property (b) for G). This means that $w \in \Sigma - \Delta_i$ implying $\alpha \in C$. Thus $\alpha \in \{a, b, c, d\}$. But this means that the distance between $w, x \in D$ or the distance between $w, y \in D$ in G is ≤ 2 that also violates (b) for G. Therefore G_i is D_i-dominated. Clearly, as G_i is an induced subgraph of G, G_i should be uniquely dominated and (b) holds for G_i.

Property (c) holds for G_i^- because it is invariant under taking induced subgraphs. As the edges $\{b, c\}$ and $\{a, d\}$ are missing from G_i^-, (c) holds also for G_i.

Notice that all the regions of G_i that are included in Δ_i are also regions of G_i. The boundaries of the new regions are the cycles (y, a, b), (a, b, c, d), and (x, c, b) that are all either triangles or squares. Therefore, (d) holds for G_i.

If property (e) holds for G_i^- then it also holds for G_i. Let P be a (w, v)-path in G_i^- of length 3. Property (e) holds trivially for G_i^- if $\{w, v\} = \{x, y\}$. So suppose that it is violated for some pair $\{w, v\} \neq \{x, y\}$. Because (e) holds for G we can find a $\{w, v\}$-path $P' = (w, \alpha, \beta, v)$ of length 3 in G that is not a path in G_i^-. As $\{w, v\} \neq \{x, y\}$, only one, say α, of α, β can be outside Δ_i. This means that w and β are vertices of C. Since $\beta \in \{a, b, c, d\}$, we have that v is adjacent in G to a vertex in $\{a, b, c, d\}$. This contradicts to property (b) for G, as it implies the existence of a path of length ≤ 2 connecting $v \in D$ and one of the vertices $x, y \in D$.

It is easy to verify (f) for the new regions (x, a, d), (a, b, c, d) and (y, c, d) of G_i. Suppose now that (f) is violated for some region of G_i that is also a region of G. As (d) holds for G_i we have that there exist a square in G_i containing a vertex of G_i. As G_i is an induced subgraph of G, this square should exist also in G, a contradiction to (f) for G.

(g) is trivial for the new square region of G_i bounded by (a, b, c, d). Let us prove that (g) holds also for all the square regions of G_i^-. Let $\widehat{r} = (\alpha, \beta, \gamma, \delta)$ be the boundary of some square region r of G_i^-. As G_i^- is an induced subgraph of G, $(\alpha, \beta, \gamma, \delta)$ is also the boundary of some square region of G, therefore, we may assume that there are vertices $z, w \in D$ where (z, α, β) and (w, γ, δ) are triangles of G. It is enough to prove that $\{z, \alpha\}, \{z, \beta\}, \{w, \gamma\}$ and $\{w, \delta\}$ are all edges of G_i^-. Suppose, in contrary, that one of them, say $\{a, z\}$, is not an edge of G_i^-. As G_i^- is an induced subgraph of G this means that $z \notin V(G_i^-)$. In other words we have that (z, α, β) is a triangle of G where $z \in (\Sigma - \Delta_i) \cap V(G)$ and $\{\alpha, \beta\} \in \Delta_i \cap V(G)$. Then α, β should be vertices of C different form x and y. Therefore, either z, x or z, y are at distance ≤ 2 in G contradicting property (b).

For (h), notice that no path of length 3 in G_i connecting two vertices of D can use the edges $\{a, d\}$ and $\{b, c\}$ in G_i. Indeed, if this is possible for one, say $\{a, d\}$ of the edges $\{a, d\}$ and $\{b, c\}$ then such a path would have extremes in distance 2 from x, a contradiction to property (c) for G_i. Therefore if there exist two paths violating (h) in G_i, they should be paths of G_i^- and also paths of G as G_i^- is an induced subgraph of G, a contradiction to property (b). $\square$

For an example of the application of Lemma 3.10, see steps 1,3, and 4 of Figure 11.

3.5 Prime D-dominated Σ-plane graphs

Given a nicely D-dominated Σ-plane graph G, we define its *reduced* graph, $\mathbf{red}(G)$, as the graph with vertex set D and where two vertices $x, y \in D$ are adjacent in $\mathbf{red}(G)$ if and only if the distance x and y in G is 3. Notice that $\mathbf{red}(G)$ is a connected graph. The main idea of our proof is that $\mathbf{red}(G)$ express a "good" part of the structure of a nicely D-dominated graph G.

A nicely D-dominated Σ-plane graph G is *prime* if all its D-triangles and D-hexagons are empty. Notice that al the graphs in Figure 9 are prime.

Lemma 3.12. *Let G be a prime D-dominated Σ-plane graph. If G contains two vertices $x, y \in D$ connected by 3 paths of length 3 then $V(P_1) \cup V(P_2) \cup V(P_3) = V(G)$.*

Proof. By property (h), the paths P_i, $i = 1, 2, 3$, are mutually internally disjoint. Then $\Sigma - (P_1 \cup P_2 \cup P_3)$ contains 3 connected components that are open discs. We call them $\Delta_{1,2}$, $\Delta_{2,3}$, and $\Delta_{1,3}$ assuming that they do not contain vertices of P_3, P_1, and P_2 respectively. Let i, j, h be any three distinct indices of $\{1, 2, 3\}$. As $P_i \cup P_j$ form an empty D-hexagon, all the vertices of G should be contained in one, say Δ of the closed discs bounded by the cycle $P_i \cup P_j$. Notice that P_h should be entirely included in $\overline{\Delta}_{i,j}$ because of its internal vertices. Therefore, $\Delta = \overline{\Delta}_{i,j}$. Resuming we have that $V(G) \subseteq V(G) \cap (\overline{\Delta}_{1,2} \cap \overline{\Delta}_{2,3} \cap \overline{\Delta}_{1,3}) = V(P_1) \cup V(P_2) \cup V(P_3)$ and the lemma follows. $\square$

Notice the graph of Lemma 3.12 is the graph Σ_2^3 of Figure 11.

An important relation of a prime graph and its reduced graph is provided by the following lemma.

Lemma 3.13. *Let G be a prime D-dominated Σ-plane graph with $|D| \geq 3$. Then the mapping*

$$\phi : E(\mathbf{red}(G)) \to \mathcal{C}(G) \text{ where } \phi(e) = C \text{ if and only if the endpoints of } e \text{ are in } D \cap C$$

is a bijection.

Proof. Clearly, any D-hexagon C with poles x and y implies the existence of a (x, y)-path in G and therefore C is the image of $\{x, y\} \in E(\mathbf{red}(G))$. In order to show that ϕ is a bijection, we have to show that for every $e = \{x, y\} \in E(\mathbf{red}(G))$, there exist a *unique* D-hexagon C with poles x and y. By (e), there are at least two internally disjoint paths connecting x and y. Suppose in contrary that G has at least 3 (x, y)-paths P_1, P_2, P_3. As $|D| \geq 3$, G contains vertices that are not in $V(P_1) \cup V(P_2) \cup V(P_3)$, a contradiction to Lemma 3.12. □

Let G be a prime D-dominated Σ-plane graph with $|D| \geq 3$ and let ϕ be the bijection defined in Lemma 3.13. For every edge $e = \{x, y\} \in E(\mathbf{red}(G))$ we choose a vertex $w \in D - \{x\} - \{y\}$ and define $\Delta(e)$ as the w-avoiding open disc bounded by $\phi(e)$ (because G is prime, the definition does not depend on the choice of w). Observe that for any two different $e_1, e_2 \in E(\mathbf{red}(G))$, its holds that $\Delta(e_1) \cap \Delta(e_2) = \emptyset$.

Some of the properties of prime D-dominated Σ-plane graphs are given by the next two lemmata.

Lemma 3.14. *Let G be a prime D-dominated Σ-plane graph with $|D| \geq 2$. For any D-triangle $T = (x, a, b)$ with $x \in D$, the edges $\{x, a\}$ and $\{x, b\}$ are also the edges of some D-hexagon of G with poles x and y. Moreover, if $|D| \geq 3$, the edge $\{a, b\}$ is in $\Delta(\{x, y\})$.*

Proof. Because G is a prime graph, one of the open discs bounded by T is the region of G. Let r_x, $\widehat{r}_x = T = (x, a, b)$, be such a region. Let r, $r \neq r_x$, be the (unique) region incident to $\{a, b\}$, i.e. $\{a, b\} \subseteq \widehat{r}$. By (d), r is either a triangle or a square region.

We claim that it is a square region. Suppose in contrary that $\widehat{r} = (a, b, c)$. Then, from property (b), $c \notin D$. Let $y \in D$ be the unique vertex dominating c. We distinguish two cases:

Case 1. $x = y$. In this case all vertices in $V(G) - \{x, a, b, c\}$ are covered (in Σ) by 4 open discs bounded by triangles $(x, a, b), (x, a, c), (x, b, c)$ and (a, b, c). Since D-triangles $(x, a, b), (x, a, c), (x, b, c)$ are empty, we have that all vertices in $V(G) - \{x, a, b, c\}$ are in the x-avoiding open disc Δ bounded by (a, b, c). As $\Delta = r$ a the region of G, we have that $V(G) - \{x, a, b, c\} = \emptyset$, a contradiction to the fact that $|D| \geq 2$.

Case 2. $x \neq y$. Then G contains the paths (x, a, c, y) and (x, b, c, y), a contradiction to property (h) and the claim holds.

As r is a square region, we assume that $\widehat{r} = (a, b, c, d)$. Property (g) together with the fact that a, b are adjacent to x, implies that either all vertices a, b, c, d are adjacent to x, or there is $y \in D$, $y \neq x$, that is adjacent to c and d.

We claim that the first case is impossible. Indeed, if a, b, c, d are adjacent to x then all the vertices in $V(G) - \{x, a, b, c, d\}$ should be included in the five open discs bounded by triangles (x, a, b), (x, a, c), (x, b, d), (c, d, x) and square (a, b, c, d). Four discs bounded by D-triangles are regions of G (G is prime), thus all the vertices of $V(G) - \{x, a, b, c, d\}$ are in the x-avoiding open disc r bounded by (a, b, c, d). Because r is the region of G, we conclude that $V(G) - \{x, a, b, c, d\} = \emptyset$. Since by property (b), $a, b, c, d \notin D$, we have a contradiction to the fact that $|D| \geq 2$ and the claim holds.

Therefore, there is $y \in D$, $y \neq x$, and y is adjacent to c and d. Because (y, c, d) is D-triangle in prime graph, one of the discs r_y bounded by (y, c, d) is the region of G. Hence $C = (x, a, c, y, d, b, x)$ is a D-hexagon containing edges $\{x, a\}$ and $\{x, b\}$, as required. Notice now that $\Delta = r_x \cup \{a, b\} \cup r \cup \{c, d\} \cup r_y$ is one of the open discs bounded by C (here an edge represents an open set). As $V(G) \cap \Delta = \emptyset$, we have that $\Delta(\{x, y\}) = \Delta$ and thus the edge $\{a, b\}$ is contained in $\Delta(\{x, y\})$. $\square$

Lemma 3.15. *Let G be a prime D-dominated Σ-plane graph with $|D| \geq 2$. Then the endpoints of each edge of G are the vertices of some D-hexagon.*

Proof. Let $e = \{x, y\}$ be an edge of G.

Case 1. $\{x, y\} \cap D = \{x\}$ (notice that from property (b), $|\{x, y\} \cap D| \leq 1$). Let r be the region of G incident to $e = \{x, y\}$. From from property (f), r is a D-triangle and the result follows from 3.14

Case 2. $\{x, y\} \cap D = \emptyset$. Let d_x and d_y be the vertices of D dominating x and y respectively. If $d_x = d_y$ then e is incident to the D-triangle (d_x, x, y), and the result follows from 3.14. Suppose now that $d_x \neq d_y$. Then (d_x, x, y, d_y) is the path connecting two vertices in D. From properties (e) and (g) we have that x, y should be an edge of some D-hexagon and the lemma follows. $\square$

3.6 On the structure of nicely D-dominated Σ-plane graphs

For a given nicely D-dominated Σ-plane graph G, we define hypergraph $\mathcal{G}^*$ with the vertex set $V(\mathcal{G}^*) = V(G)$ and edge set $E(\mathcal{G}^*) = E(G) \cup \mathcal{T}(G) \cup \mathcal{C}(G)$, i.e. $\mathcal{G}^*$ is obtained from G by adding all D-triangles and D-hexagons as hyperedges. We also define hypergaph $\mathcal{G}^h$ with the vertex set $V(\mathcal{G}^h) = V(G)$ and the edge set $E(\mathcal{G}^h) = \mathcal{C}(G)$, i.e. $\mathcal{G}^h$ has the vertices of G as vertices and each of its hyperedges contains the vertices of some D-hexagon of G. Observe that $\mathcal{G}^h$ can be obtained from $\mathcal{G}^*$ by removing all the (hyper)edges of size 2 or 3.

Lemma 3.16. *For any prime D-dominated Σ-plane graph G with $|D| \geq 2$, $\mathbf{bw}(\mathcal{G}^*) \leq \mathbf{bw}(\mathcal{G}^h)$.*

Proof. From Lemmata 3.14 and 3.15 we have that for each hyperedge in $\mathcal{G}^*$ there exists some D-hexagon containing all its endpoints. In other words, each hyperedge of $\mathcal{G}^*$ is a subset of some hyperedge of $\mathcal{G}^h$. Applying Lemma 3.1 recursively for every hyperedge of $\mathcal{G}^*$ we have the required. $\square$

The following structural result will serve as a base for the recursive application of Lemmata 3.10 and 3.11 in the proof of Lemma 3.21.

Lemma 3.17. *Let G be a prime D-dominated Σ-plane graph with $|D| \geq 3$. Then $\mathbf{red}(G)$ is a connected Σ-plane graph, all vertices of G have degree ≥ 2 and $\mathcal{G}^h$ is isomorphic to $\mathbf{ext}(\mathbf{red}(G))$.*

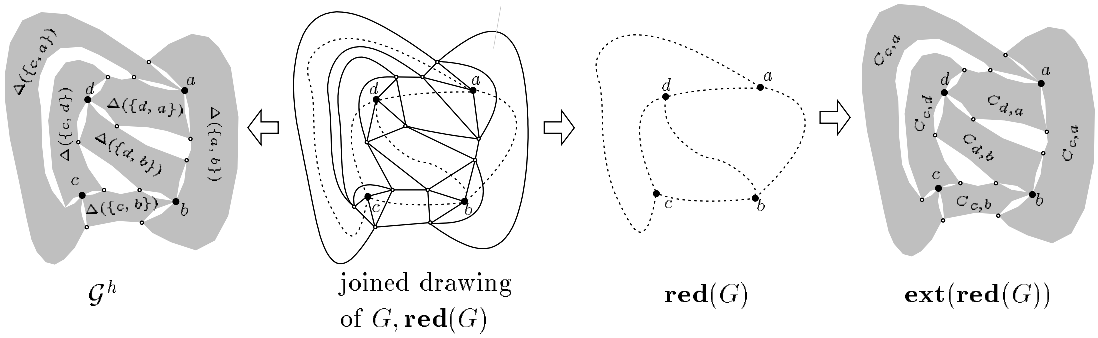


Figure 12: An example of the proof of Lemma 3.17.

Proof. We define the *joined drawing of G and $\mathbf{red}(G)$ in Σ* as follows (see Figure 12):

> *Take a drawing of G on Σ and draw the vertices of $\mathbf{red}(G)$ identically to the vertices of G. For any edge $e_i \in E(\mathbf{red}(G))$ we draw $\{x, y\}$ as an I-arc connecting x and y and contained in $\Delta(e)$.*

The following three auxiliary propositions are used in the proof of the lemma.

Proposition 3.18. *If G is a prime D-dominated Σ-plane graph, then $\mathbf{red}(G)$ is a Σ-plane graph.*

Proof. Take the joined drawing of G and $\mathbf{red}(G)$ in Σ. Observe that, for any pair of edges $e_i, e_i \in E(\mathbf{red}(G))$, $\Delta(e_i) \cap \Delta(e_i) = \emptyset$. Therefore if in this drawing we delete all the points that are not points of vertices or edge of $\mathbf{red}(G)$, what remains is a planar drawing of $\mathbf{red}(G)$. $\square$

The following easy fact is a consequence of Lemma 3.14 and the definition of the joined drawing of G and $\mathbf{red}(G)$.

Proposition 3.19. *Let G be a nicely D-dominated Σ-plane graph where $|D| \geq 3$ and let ϕ be the bijection defined in Lemma 3.13. In the joined drawing of G and $\mathbf{red}(G)$ in Σ, for any vertex $x \in D$, with degree ≥ 3, two edges $\{x, y\}$ and $\{x, z\}$ are consecutive iff the D-hexagons $\phi(\{x, y\})$ and $\phi(\{x, z\})$ have exactly one edge in common. In the special case where $x \in D$ has degree 2, the D-hexagons $\phi(\{x, y\})$ and $\phi(\{x, z\})$ have exactly two edges in common.*

Proposition 3.20. *Let G be a prime D-dominated Σ-plane graph where $|D| \geq 3$. Then all vertices of $\mathbf{red}(G)$ have degree ≥ 2.*

Proof. Let $x \in D$ be a vertex of G incident to a region r. By property (f), the boundary of r is a triangle $\widehat{r} = (x, a_1, a_2)$. By Lemma 3.14, the edges $\{x, a_1\}$ and $\{x, a_2\}$ are also the edges of some D-hexagon with poles x and y. We distinguish the following cases:

Case 1. x has a neighbor a_3, distinct from a_1 and a_2. We choose a_3 so that a_2 and a_3 are consecutive in the cyclic ordering of the neighbors of x. Notice that the unique region whose boundary contains x, a_2 and a_3, should be a triangle (otherwise we have a contradiction to property (g)). By Lemma 3.14, the edges $\{x, a_2\}$ and $\{x, a_3\}$ are contained in some D-hexagon with poles x and w. Clearly $w \neq y$, otherwise x and y are connected by 3 internally disjoint paths and from Lemma 3.12 we have that $|D| = 2$, a contradiction. We conclude that $\{x, w\}$ is an edge of $\mathbf{red}(G)$, different than $\{x, y\}$.

Case 2. The only neighbors of x are the vertices a_1 and a_2. This means that $\{a_1, a_2\}$ is an exceptional edge, i.e. there are two multiple edges e^1 and e^2 with endpoints $\{a_1, a_2\}$. Let T^1, T^2 be the triangles containing x and edges e^1 and e^2 respectively. For $i = 1, 2$, we apply Lemma 3.14 for T^i and have that both $\{x, a_i\}, i = 1, 2$ belong to some D-hexagon C^i of G with poles x and y_i. Moreover, as $|D| \geq 3$, the edge e^i is contained in $\Delta(\{x, y_i\})$. Therefore, for the case $y_1 = y_2$ we have that both edges e^1, e^2 are in $\Delta(\{x, y_i\})$, which is impossible. So, x has two neighbors in $\mathbf{red}(G)$. $\square$

Now we continue the proof of the Lemma 3.17.

By Proposition 3.18, G is a Σ-plane graph. By Proposition 3.20, all vertices of $\mathbf{red}(G)$ have degree ≥ 2. Therefore, the three transformation steps of $\mathbf{ext}$ can be applied on $\mathbf{red}(G)$. Consider now the joint drawing of G and $\mathbf{red}(G)$ in Σ. For each edge $e = \{x, y\} \in E(\mathbf{red}(G))$ we use the notation $\phi(x, y) = (x, x^+_{x,y}, y^-_{x,y}, y, x^+_{x,y}, x^-_{x,y}, x)$ (the ordering is clockwise). Apply Steps 1 and 2 of the definition of $\mathbf{ext}$ on $\mathbf{red}(G)$. During Step 2, identify vertices $x^-_{x,y}$, $x^+_{x,z}$ with the vertices of G that are denoted the in the same way. This is possible because of Proposition 3.19 and the graph G_2 created after Step 2 has exactly the same vertex set as the graph G. Let us remind, that there exists a bijection $\theta : E(G) \rightarrow E(\mathbf{ext}(G))$ mapping each edge $e = \{x, y\}$ to the hyperedge formed by the vertices of $C_{x,y}$. Moreover, for any edge $e = \{x, y\} \in E(\mathbf{red}(G))$, the cycle $\theta(x, y) = C_{x,y}$ is identical to the D-hexagon $\phi(x, y)$. Notice now that the application of Step 3 of the definition of **red** on G_2, ignores the edges of G_2 and add as edges all the cycles $\phi(e), e \in E(\mathbf{red}(G))$. As these cycles are exactly those added towards constructing $\mathcal{G}^h$, the graph $\mathcal{G}^h$ is also identical to the result of Step 3. Therefore $\mathcal{G}^h$ is isomorphic to $\mathbf{ext}(\mathbf{red}(G))$. □

3.7 Main combinatorial result

Lemma 3.21. *For any nicely D-dominated Σ-plane graph G,* $\mathbf{bw}(G) \leq 3 \cdot \sqrt{4.5 \cdot |D|}$.

Proof. For $|D| = 1$, $G - D$ is outerplanar. It is well known that the branch-width of an outerplanar graph is at most 2 implying $\mathbf{bw}(G) \leq 3$.

Suppose that $|D| \geq 2$. Clearly, $\mathbf{bw}(G) \leq \mathbf{bw}(\mathcal{G}^*)$ and to prove the lemma we show that $\mathbf{bw}(\mathcal{G}^*) \leq 3 \cdot \sqrt{4.5 \cdot |D|}$.

Prime case. We first examine the special case where G is prime D-dominated Σ-plane graph. There are two subcases:

• If $|D| = 2$, then, we set $D = \{x, y\}$. Notice that if there are only 2 (x, y)-paths in G then $G = \Sigma_2^2$. If there are 3 (x, y)-paths in G then $G = \Sigma_2^3$ (see Figure 9). Moreover, G cannot contain more than 3 (x, y)-paths, otherwise it would not be prime. Therefore, $|V(G)| \leq 8$ and thus $\mathbf{bw}(\mathcal{G}^*) \leq 8 \leq 3 \cdot \sqrt{4.5 \cdot 2} = 9$.

• Suppose now that G is a prime D-dominated Σ-plane graph and $|D| \geq 3$. By Theorem 2.4, $\mathbf{bw}(\mathbf{red}(G)) \leq \sqrt{4.5 \cdot |D|}$. By Lemma 3.17, all the vertices $\mathbf{red}(G)$ have degree ≥ 2. Therefore we can apply Lemma 3.8 on $\mathbf{red}(G)$ (recall that $\mathbf{red}(G)$ is connected) and get $\mathbf{bw}(\mathbf{ext}(\mathbf{red}(G))) \leq 3 \cdot \mathbf{bw}(\mathbf{red}(G))$. By Lemma 3.17, $\mathbf{bw}(\mathcal{G}^h) = \mathbf{bw}(\mathbf{ext}(\mathbf{red}(G)))$ and by Lemma 3.16, $\mathbf{bw}(\mathcal{G}^*) \leq \mathbf{bw}(\mathcal{G}^h)$. Resuming, we conclude that, if G is prime then $\mathbf{bw}(\mathcal{G}^*) \leq 3 \cdot \sqrt{4.5 \cdot |D|}$.

General case: Suppose that G is a nicely D-dominated Σ-plane graph. We use induction on the number of vertices of G. If $|V(G)| = 3$, G is a triangle (the graph Σ_1 of Figure 9) and $\mathbf{bw}(\mathcal{G}^*) = 3 \leq 3 \cdot \sqrt{4.5}$. Suppose that $\mathbf{bw}(\mathcal{G}^*) \leq 3 \cdot \sqrt{4.5 \cdot |D|}$ for every nicely D-dominated graph on $< n$ vertices. Let G be a nicely D-dominated Σ-plane graph where $|V(G)| = n$ and let q be a non-empty D-triangle or D-hexagon (if q does not exist then induction step follows by the prime case above). By Lemmata 3.10 and 3.11, we have that if Δ_1, Δ_2 are the discs bounded by q then, for $i = 1, 2$, $G_i = G[V(G) \cap \Delta_i]$ is a subgraph of a nicely D_i-dominated Σ-plane graph for some $D_i \subseteq D$, $i = 1, 2$, and that $|V(G_i)| < n$ (we use the expression "subgraph" in order to capture the case when q is a D-hexagon). Applying the induction hypothesis we have that $\mathbf{bw}(G_i^*) \leq 3 \cdot \sqrt{4.5 \cdot |D_i|}, i = 1, 2$. Notice also that $\mathcal{G}^* = \mathcal{G}_1^* \cup \mathcal{G}_2^*$ and that $V(\mathcal{G}_1^*) \cap V(\mathcal{G}_2^*) = q \in E(\mathcal{G}_1^*) \cap E(\mathcal{G}_2^*)$. Therefore, we can apply Lemma 3.1 and we get $\mathbf{bw}(\mathcal{G}^*) \leq 3 \cdot \sqrt{4.5 \cdot |D_i|}$. □

For an example of the induction of the general case in the proof of Lemma 3.21, see Figure 11.

Here is the main combinatorial result of this paper.

Theorem 3.22. *Let G be a D-dominated Σ-plane graph. Then* $\mathbf{bw}(G) \le 3\sqrt{4.5 \cdot |D|}$.

Proof. Let A be the set of articulation points of G. Let G_i be 2-connected components of G, $D_i = D \cap V(G_i)$, and $A_i = A \cap V(G_i)$, $1 \le i \le r$. Let also N_i be the vertices of G_i that are not dominated by D_i, $1 \le i \le r$.

Notice that each vertex of N_i is dominated in G by some vertex from $V(G) - V(G_i)$. Moreover, a vertex from $V(G) - V(G_i)$ cannot dominate more than one vertex in G_i. Therefore, $|N_i| \le |D - D_i|$. Thus for $D'_i = N_i \cup D_i$, we have that G_i is D'_i-dominated and $|D'_i| \le |D|$.

G_i is a D'_i-dominated 2-connected planar graph. We take a drawing of this graph in a sphere Σ and apply Lemma 3.9. That way we construct a nicely D'_i-dominated Σ-plane graph H_i containing (property (a)) G_i as a minor. By Lemma 3.21, $\mathbf{bw}(H_i) \le 3 \cdot \sqrt{4.5 \cdot |D'_i|}$. Since G_i is a minor of H_i, we have that $\mathbf{bw}(G_i) \le 3\sqrt{4.5 \cdot |D'_i|} \le 3\sqrt{4.5 \cdot |D|}$, $1 \le i \le r$.

Each graph G_i can be treated as a hypergraph with the ground set $V(G_i)$ and the edge set $E(G_i) \cup V(G_i)$. As hypergraphs, graphs G_i have at most one edge (edge consisting of one vertex) in common and applying Lemma 3.1 recursively we obtain that $\mathbf{bw}(G) \le \max\{1, \max_{1 \le i \le r} \mathbf{bw}(G_i)\} \le 3\sqrt{4.5 \cdot |D|}$. □

4 Algorithmic consequences

In this section we discuss an algorithm that, given a planar graph G on n vertices and an integer k, either computes a dominating set of size $\le k$, or concludes that there is no such a dominating set. The algorithm uses $O(2^{12.75\sqrt{k}}k + n^3 + k^4)$ time and works in 3 phases as follows.

Phase 1: We use the known reduction of PLANAR DOMINATING SET problem to a linear problem kernel as a preprocessing procedure. Alber, Fellows & Niedermeier [3] design a procedure that for a given integer k and planar graph G on n vertices outputs a planar graph H on $\le 335k$ vertices such that G has a dominating set of size $\le k$ if and only if H has a dominating set of size $\le k$. This reduction can be performed in $O(n^3)$ time.

Phase 2: We compute an optimal branch-decomposition of a graph H. For this step, one can use the algorithms due to Seymour & Thomas (algorithms (7.3) and (9.1) of Sections 7 and 9 of [22] — for an implementation, see the results of Hicks [15]). These algorithms need $O(n^2)$ steps for checking and $O(n^4)$ steps for constructing the branch decomposition for graphs on n vertices. And what is important for practical applications, there is no *large hidden constants* in the running time of these algorithms.

Thus branch-decomposition of H can be constructed in $O(k^4)$ steps. Check whether $\mathbf{bw}(H) \leq (3\sqrt{4.5})\sqrt{k} < 12.75/2\sqrt{k}$. If the answer is "no", then by Theorem 3.7 we conclude that there is no dominating set of size k in G. If the answer is "yes" we proceed with the next phase.

Phase 3: Here we use a dynamic programming approach to solve the PLANAR DOMINATING SET problem on graph H. Alber et al. [1] suggested a dynamic programming algorithm based on the so-called "monotonicity" property of domination problem. For a graph G on n vertices with a given tree-decomposition of width ℓ, the algorithm of Alber et al. can be implemented in $O(2^{2\ell}n)$ steps. There is a well known transformation due to Robertson & Seymour [19] that given a branch decomposition of width $\leq \ell$ of a graph with m edges, constructs a tree decomposition of width $\leq (3/2)\ell$ in $O(m^2)$ steps. Thus the result of Alber et al. immediately implies that the DOMINATING SET problem on graphs with n vertices and m edges and of branch-width $\leq \ell$ can be solved in $O(2^{3\ell}n + m^2)$ steps. Notice now that, for planar graphs $m = O(n)$. As $3 \cdot 3\sqrt{4.5} \leq 19.1$, this phase requires $O(2^{19.1\sqrt{k}}k + k^2)$ steps and implies a $O(2^{19.1\sqrt{k}}k + n^3 + k^4)$ time algorithm that finds, if exists, a dominating set of any planar graph with size at most k. However, in the next subsection (Theorem 4.1) we construct a dynamic programming algorithm solving the DOMINATING SET problem on graphs of branch-width $\leq \ell$ in $O(2^{(3 \cdot \log_4 3)\ell}m)$ steps, where m is the number of edges in a graph. Because $(3 \cdot \log_4 3) \cdot 3\sqrt{4.5} < 15.13$ and $m = O(n)$ we can reduce the cost of this phase to $O(2^{15.13\sqrt{k}})$ steps and conclude with a $O(2^{15.13\sqrt{k}}k + n^3 + k^4)$ time algorithm.

4.1 Dynamic programming on graphs of bounded branch-width

Let (T', τ) be a branch decomposition of a graph G with m edges and let $\omega' : E(T') \to 2^{V(\mathcal{G})}$ be the order function of (T', τ). We choose an edge $\{x, y\}$ in T', put a new vertex v of degree 2 on this edge and make v adjacent to a new vertex r. By choosing r as a root in the new tree $T = T' \cup \{v, r\}$ we turn T into a rooted tree. For every edge of $f \in E(T) \cap E(T')$ we put $\omega(f) = \omega'(f)$. Also we put $\omega(\{x, v\}) = \omega(\{v, y\}) = \omega'(\{x, y\})$ and $\omega(\{r, v\}) = \emptyset$.

For an edge f of T we define E_f (V_f) as the set of edges (vertices) that are "below" f, i.e. the set of all edges (vertices) g such that every path containing g and $\{v, r\}$ in T contains f. With such a notation, $E(T) = E_{\{v,r\}}$ and $V(T) = V_{\{v,r\}}$. Every edge f of T that is not incident to a leaf has two children that are the edges of E_f incident to f.

For every edge f of T we color the vertices of $\omega(f)$ in three colors

black (represented by 1, meaning that the vertex is in the dominating set),

white (represented by 0, meaning that the vertex is dominated at the current step of the algorithm and is not in dominating set), and

grey (represented by $\hat{0}$, meaning that at the current step of the algorithm we still have not decided to color this vertex white or black.)

For every edge f of T we use mapping

$$A_f\colon \{0, \hat{0}, 1\}^{|\omega(f)|} \to \mathbb{N} \cup \{+\infty\}.$$

For a coloring $c \in \{0, \hat{0}, 1\}^{|\omega(f)|}$, the value $A_f(c)$ stores how many vertices are needed for a minimum dominating set in the subgraph G_f of G that is defined by inducing the edge set

$$\{\tau^{-1}(x)\colon x \in V_f \wedge (\text{ } x \text{ is a leaf of } T')\}$$

and removing all grey vertices, subject to the condition that for every $u \in \omega(f)$ the color assigned to u is $c(u)$. In other words, $A_f(c)$ stores the minimum cardinality of a set $D_f(c)$ such that

- Every vertex of $V(G_f) \setminus \omega(f)$ is adjacent to a vertex of $D_f(c)$,
- For every vertex $u \in \omega(f)$: $c(u) = 1 \Rightarrow u \in D_f(c)$ and $c(u) = 0 \Rightarrow (u \notin D_f(c)$ and u is adjacent to a vertex from $D_f(c))$.

We put $A_f(c) = +\infty$ if there is no such a set $D_f(c)$. Because $\omega(\{r, v\}) = \emptyset$ and $G_{\{r,v\}} = G$, we have that $A_{\{r,v\}}(c)$ is the smallest size of a dominating set in G.

Let f be a non leaf edge of T and let f_1, f_2 be the children of f. Define $X_1 = \omega(f) - \omega(f_2)$, $X_2 = \omega(f) - \omega(f_1)$, $X_3 = \omega(f) \cap (\omega(f_1) \cap \omega(f_2))$, and $X_4 = (\omega(f_1) \cup \omega(f_2)) - \omega(f)$.

Notice that $X_i \cap X_j \neq \emptyset$, $1 \leq i \neq j \leq 4$ and

$$\omega(f) = X_1 \cup X_2 \cup X_3. \tag{1}$$

Notice now that by the definition of ω it is impossible that a vertex belongs in exactly one of $\omega(f), \omega(f_1), \omega(f_2)$. Therefore, condition $u \in X_4$ implies that $u \in \omega(f_1) \cap \omega(f_2)$. Therefore,

$$\omega(f_1) = X_1 \cup X_3 \cup X_4, \tag{2}$$

and

$$\omega(f_2) = X_2 \cup X_3 \cup X_4. \tag{3}$$

We say that a coloring c of $\omega(f)$ is *formed* from coloring c_1 of $\omega(f_1)$ and coloring c_2 of $\omega(f_2)$ if

[F1] For every $u \in X_1$, $c(u) = c_1(u)$;

[F2] For every $u \in X_2$, $c(u) = c_2(u)$;

[F3] For every $u \in X_3$, $(c(u) \in \{\hat{0}, 1\} \Rightarrow c(u) = c_1(u) = c_2(u))$, and $(c(u) = 0 \Rightarrow [c_1(u), c_2(u) \in \{\hat{0}, 0\} \wedge (c_1(u) = 0 \vee c_2(u) = 0)])$. (The color 1 ($\hat{0}$) can appear only if both colors in c_1 and c_2 are 1 ($\hat{0}$). The color 0 appears when both colors in c_1, c_2 are not 1 and at least one of them is 0.);

[F4] For every $u \in X_4$, $(c_1(u) = c_2(u) = 1) \vee$ $(c_1(u) = c_2(u) = 0) \vee$ $(c_1(u) = 0 \wedge c_2(u) = \hat{0}) \vee$ $(c_1(u) = \hat{0} \wedge c_2(u) = 0)$. This property says that every vertex u of $\omega(f_1)$ and $\omega(f_2)$ that do not appear in $\omega(f)$ (and hence do not appear further) should be finally colored either by 1 (if both colors of u in c_1 and c_2 were 1), or 0 (0 can appear if both colors of u in c_1 and c_2 are not 1 and at least one color is 0).

Notice that every coloring of f is formed from some colorings of its children f_1 and f_2. We start computations of functions A_f from leaves of T. Brute force algorithm takes $O(m)$ time for this step.

Then we compute the values of the corresponding functions by in bottom-up fashion. The main observation here is that if f_1 and f_2 are the children of f, then the vertex sets $\omega(f_1)$ $\omega(f_2)$ 'separate' subgraphs G_1 and G_2, so the value $A_f(c)$ can be obtained from the information on colorings of $\omega(f_1)$ and $\omega(f_2)$. More precisely, let $\#_1(X_i, c)$, $1 \le i \le 4$, be the number of vertices in X_i colored by color 1 in coloring c. For a coloring c we assign

$$A_f(c) = \min\{A_{f_1}(c_1) + A_{f_2}(c_2) - \#_1(X_3, c_1) - \#_1(X_4, c_1) | c_1, c_2 \text{ forms } c\}. \tag{4}$$

(Every 1 from X_3 and X_4 is counted in $A_{f_1}(c_1) + A_{f_2}(c_2)$ twice and $X_3 \cap X_4 = \emptyset$.) The number of steps to compute the minimum in (4) is given by

$$O(\sum_c |\{c_1, c_2\}\colon c_1, c_2 \text{ forms } c|).$$

Let $x_i = |X_i|$, $1 \le i \le 4$. For every vertex $u \in X_1$ the color $c(u) = c_1(u) = c_2(u)$, i.e. there are three possibilities to 'form' this vertex. Thus when a coloring of all vertices from X_2, X_3, X_4 is fixed, there are at most 3^{x_1} pairs c_1 and c_2 forming all colorings c. The same is true for the set X_2, so when a coloring of all vertices from X_3, X_4 is fixed, there are at most $3^{x_1+x_2}$ pairs c_1 and c_2 forming all colorings c.

There are five ways of 'forming' a color of $u \in X_3$: Every 1 ($\hat{0}$) of a vertex $u \in X_3$ is 'formed' by 1 ($\hat{0}$) from c_1 and 1 ($\hat{0}$) from c_2. Every 0 of a vertex $u \in X_3$ can be 'formed' in three ways, from $\hat{0}$ in c_1 and 0 in c_2, from 0 in c_1 and 0 in c_2, and from 0 in c_1 and $\hat{0}$ in c_2. Thus when a coloring of all vertices from X_1, X_2, X_4 is fixed, there are at most 5^{x_3} pairs c_1 and c_2 forming all colorings c and

A color of $u \in X_4$ can be obtained in four ways: 1 from 1 in c_1 and 1 in c_2, 0 from 0 in c_1 and 0 in c_2, 0 from 0 in c_1 and $\hat{0}$ in c_2, and 0 from $\hat{0}$ in c_1 and 0 in c_2. Thus the number of steps needed to evaluate $A_f(c)$ for all possible colorings c of $\omega(f)$ is

$$3^{x_1+x_2}5^{x_3}4^{x_4}.$$

The obtained bound can be reduced by using the trick due to Alber et al. [1]. The trick is based on the following observation. If for some coloring c of f we replace a color of a vertex u from $\hat{0}$ to 0, then for the new coloring c', $A_f(c) \leq A_f(c')$. Thus in (4) we can replace "c_1 and c_2 forms c" to "c_1 and c_2 satisfies **[F1]**,**[F2]**,**[F3']**, and **[F4']**", where **[F3']** and **[F4']** are as follows

[F3'] For every $u \in X_3$, $(c(u) \in \{\hat{0}, 1\} \Rightarrow c(u) = c_1(u) = c_2(u))$, and $(c(u) = 0 \Rightarrow [c_1(u), c_2(u) \in \{\hat{0}, 0\} \wedge (c_1(u) \neq c_2(u))]$.

[F4'] For every $u \in X_4$, $(c_1(u) = c_2(u) = 1) \vee [c_1(u), c_2(u) \in \{\hat{0}, 0\} \wedge (c_1(u) \neq c_2(u))]$

Now the number of possibilities of forming a color for a vertex from X_3 is reduced to 4 and for a vertex from X_4 to 3. Thus the number of steps for evaluating $A_f(c)$ is bounded by

$$3^{x_1+x_2}4^{x_3}3^{x_4}.$$

Let ℓ be the branch-width of G. By (1), (2) and (3),

$$\begin{aligned} x_1 + x_2 + x_3 &\leq \ell \\ x_1 + x_3 + x_4 &\leq \ell \\ x_2 + x_3 + x_4 &\leq \ell. \end{aligned} \tag{5}$$

The maximum value of the linear function $\log_4 3 \cdot (x_1 + x_2 + x_4) + x_3$ subject to constrains (5) is $\frac{3 \log_4 3}{2}\ell$ (This is because the value of the corresponding LP achieves maximum in $x_1 = x_2 = x_4 = 0.5\ell$, $x_3 = 0$.) Thus

$$3^{x_1+x_2}4^{x_3}3^{x_4} \leq 4^{\frac{3\log_4 3}{2}\ell} = 2^{3 \log_4 3 \cdot \ell}.$$

It is easy to check that the number of edges in T is $O(m)$ and the number of steps needed to evaluate $A_{\{r,v\}}(c)$ is $O(2^{3 \log_4 3 \cdot \ell} m)$. Summarizing, we get the following theorem.

Theorem 4.1. *For a graph G on m edges and with given a branch-decomposition of width $\leq \ell$, the dominating set of G can be computed in $O(2^{3 \log_4 3 \cdot \ell} m)$.*

5 Concluding remarks

We have proved that for any planar graph with a dominating set of size $\leq k$, $\mathbf{bw}(G) \leq 3\sqrt{4.5 \cdot k} = 6.364\sqrt{k}$. One of the multiplicative factors "3" follows from our results on the structure of planar graphs with a given dominating sets and the second factor "$\sqrt{4.5} \approx 2.121$" follows from the bound on branch-width of planar graphs (Theorem 2.4). Improvement of any of these two factors immediately implies improvement of our results and further speed-up of fixed-parameter algorithms for dominating set. However, our approach can not be strongly improved because upper bound of Theorem 3.22 is not far from the optimal.

Lemma 5.1. *There exist planar graphs with a dominating set of size $\leq k$ and with branch-width $> 3\sqrt{k}$.*

Proof. Let G be a $(3n+2, 3n+2)$-grid for any $n \geq 1$. Let V' be the vertices of G of degree < 4. We define D as the unique $S \subseteq V(G) - V'$ where $|S| = n^2$ and such that the distance of all pairs $v, u \in D$ in G is a multiple of 3. Then for any vertex $v \in D$, and for any possible cycle (square) (v, x, y, z, v) add the edge $\{x, z\}$. The construction is completed by connecting all the vertices in V' with a new vertex v_{new}. (See Fig. 13.) We call the resulting graph J_n. Clearly, $D \cup \{v_{\text{new}}\}$ is a dominating set of J_n of size $k = n^2 + 1$. As the $(3n+2, 3n+2)$-grid is a subgraph of J_n we have that $\mathbf{bw}(J_n) \geq 3n + 2 \geq 3\sqrt{k-1} + 2 > 3\sqrt{k}$ (from [19], the (ρ, ρ)-grid has branch-width ρ). $\square$

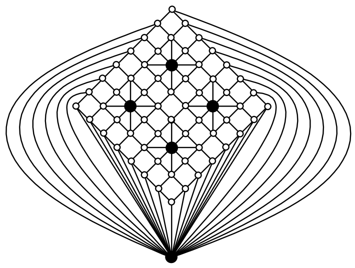


Figure 13: An example of the proof of Theorem 5.1.

Finally let us note that similar approach can be applied for a wide number of problems related to the Planar Dominating Set problem. Phase 3 in Section 4 is adapted in each problem in the same fashion as it is done in [1, 4, 8, 10, 18]. When a reduction to a linear kernel is not possible (Phase 1) our approach provides algorithms of running time $O(2^{c\sqrt{k}}n + n^4)$. (Here constant c depends from the type of the problem.) That way, our upper bound implies the construction of faster algorithms for a series of problems when their inputs are restricted to planar graphs. As a sample we mention the following: Independent Dominating Set, Perfect Dominating Set, Perfect Code, Weighted Dominating Set, Total Dominating Set, Edge Dominating Set, Face Cover, Vertex Feedback Set, Vertex Cover, Minimum Maximal Matching, Clique Transversal Set, Disjoint Cycles, and Digraph Kernel.

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