Strain gradient theory of chiral Cosserat thermoelasticity without energy dissipation

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Abstract

In this paper, we use the Green-Naghdi theory of thermomechanics of continua to derive a linear strain gradient theory of Cosserat thermoelastic bodies. The theory is capable of predicting a finite speed of heat propagation and leads to a symmetric conductivity tensor. The constitutive equations for isotropic chiral thermoelastic materials are presented. In this case, in contrast with the classical Cosserat thermoelasticity, a thermal field produces a microrotation of the particles. The thermal field is influenced by the displacement and microrotation fields even in the equilibrium theory. Existence and uniqueness results are established. The theory is used to study the effects of a concentrated heat source in an unbounded homogenenous and isotropic chiral solid.

Keywords: Chiral materials; Cosserat elasticity; Strain gradient thermoelasticity; Hyperbolic heat equation; Uniqueness results; Concentrated heat source

1. Introduction

In the theory of continua with inner structure the material particles are considered geometrical points that possess properties similar to rigid particles (Cosserat continua) and deformable particles, capable of undergoing only affine deformations. In the case of Cosserat continua the degree of freedom for each material point are six: three translations and three microrotations. The domain of applicability of the theory of continua with inner structure has investigated by Fischer-Hjalmars (1982), Kunin (1983), Eringen (1999), Dyszlewicz (2004) and Chen et al. (2004). The classical theory of Cosserat elastic solids is characterized by constitutive functions which depend on the deformation gradient, microrotation vector and gradient of microrotation. Rymarz (1987) and Brulin and Hjalmars (1981) have developed a theory of Cosserat elastic solids where the second-order displacement gradient is added to the classical set of independent constitutive variables. This theory was named the grade consistent micropolar elasticity. The theory has been studied and extended in various papers (see, e.g., Scalia, 1992; Iesan, 2004; Zhang and Sharma, 2005 and references therein). In the absence of microrotation field, the theory reduces to the strain gradient theory of elasticity established by Toupin (1962, 1964) and Mindlin (1964). The coupling between the strain gradient theory and the Cosserat theory has been used in the plasticity theory. Thus, Chen and Wang (2001) investigated the deformation of thin metallic wire torsion and ultra-thin metallic beam bend. The analytical results agree well with the experimental results.

The purpose of the present paper is to present a strain gradient theory of Cosserat thermoelasticity without energy dissipation and to investigate the chiral effects. Green and Naghi (1991a, b) developed a thermomechanical theory of deformable continua that relies on an entropy balance law rather than an entropy inequality. A theory of thermoelastic bodies based on the new entropy balance law has been derived by Green and Naghdi (1993). The linearized form of this theory does not sustain energy dissipation and permits the transmission of heat as thermal waves at finite speed. Moreover, the heat flux vector is determined by the same potential function that determines the stress. The Green-Naghdi theory has been studied in various papers (see, e.g., Chandrasekharaiah 1998; Hetnarski and Ignazack 1999; Quintanilla and Straughan 2000, 2004; Quintanilla 2003; Puri and Jordan 2004; Ieşan and Quintanilla 2009; Bargmann 2012 and references therein). The gradient theories of thermomechanics have been studied in various papers (see, e.g., Ahmadi and Firoozbakhsh, 1975; Ieşan, 1983, 2004 ; Ieşan and Quintanilla, 1992; Ciarletta and Ieşan, 1993; Martinez and Quintanilla, 1998; Forest et al., 2000, 2002; Forest and Amestoy, 2008; Forest and Aifantis, 2010).

In recent years the study of chiral materials has been received a widespread attention. The mechanical behavior of chiral materials is of interest for the investigation of carbon nanotubes (Chandraseker and Mukherjee, 2006; Guz et al., 2007, Chandraseker et el., 2009), auxetic materials (Lakes, 1991, 1998; Prall and Lakes, 1997; Spadoni and Ruzzene, 2012) and bones (Lakes et al., 1983; Park and Lakes, 1986). It is known that the deformation of chiral elastic materials cannot be described within classical elasticity. Various authors have studied the behavior of chiral elastic materials by using the theory of Cosserat elasticity (see, e.g., Lakes, 2001; Park and Lakes, 1986, and references therein). The strain gradient theory of elasticity is also an adequate tool to describe the deformation of chiral elastic solids (Papanicolopulos, 2011 and references therein). In the present paper we consider the chiral effects which appear in the Cosserat theory and in the strain gradient theory of elasticity.

In the first part of the paper we derive the basic equations of the strain gradient theory of Cosserat thermoelasticity without energy dissipation and the constitutive equations of isotropic chiral materials. The field equations are expressed in terms of the displacement, microrotation and thermal fields. It is shown that, in contrast with the classical Cosserat thermoelasticity, a thermal field produces a microrotation of the particles. The thermal field is influenced by the displacement and microrotation fields even in the equilibrium theory. We establish existence and uniqueness results in the dynamic theory of thermoelasticity. The existence result is obtained by means of the semigroup theory. Then we consider the equilibrium theory and study the effects of a concentrated heat source in an unbounded chiral isotropic material. We investigate the influence of chiral coefficients on the displacements, microrotations and thermal field.

2. Basic equations

We consider a body that at time t_0 occupies the properly regular region B of Euclidean three-dimensional space and is bounded by the surface ∂B. The motion of the body is referred to a fixed system of rectangular cartesian axes Ox_i $(i = 1; 2; 3)$. We denote by n_k the outward unit normal of ∂B. We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers (1,2) whereas Latin subscripts, unless otherwise specified, are understood to range over the integers $(1, 2, 3)$, summation over repeated subscripts is implied, and subscripts preceded by a comma denote partial differentiation with respect to the corresponding cartesian coordinate. We use a superposed dot to denote partial differentiation with respect to the time. Letters in boldface stand for tensors of an order $p \geq 1$, and if $\mathbf{w}$ has order p, we write $w_{ij\ldots k}$ (p subscripts) for the rectangular Cartesian components of $\mathbf{w}$.

In what follows we study elastic media each material point of which has six degrees of freedom. We denote by u_j the displacement vector and by φ_j the microrotation vector.

Let $\mathcal{P}$ be an arbitrary material volume in the continuum, bounded by a surface $\partial\mathcal{P}$ at time t. We suppose that P is the corresponding region in the reference configuration, bounded by a surface ∂P.

Green and Naghdi (1991 a, b) have developed a theory of thermomechanics of continua which makes use the following entropy balance

$$\int_P \rho\dot{\eta}dv = \int_P \rho(s + \xi)dv + \int_{\partial P} \Phi da, \tag{1}$$

for every part P of B and every time. Here, ρ is the density in the reference configuration, η is the entropy per unit mass and unit time, s is the external rate of supply of entropy per unit mass, ξ is the internal rate of production of entropy per unit mass, and Φ is the internal flux of the entropy per unit area. From (1) we get

$$\Phi = \Phi_j n_j, \tag{2}$$

where Φ_j is the entropy flux vector and n_j is the outward unit normal at the surface ∂P. In view of (2), the balance of entropy reduces to the local form

$$\rho\dot{\eta} = \rho(s + \xi) + \Phi_{j,j}. \tag{3}$$

Let q be the heat flux across the surface $\partial\mathcal{P}$ measured per unit area of ∂P. We denote by q_j the flux of heat associated with the surfaces in the deformed body, which were originally coordinate planes perpendicular to the x_j-axes throughout the point x. We have

$$q = \theta\Phi, \quad q_j = \theta\Phi_j, \quad q = q_j n_j, \tag{4}$$

where θ is the absolute temperature. With the help of (4), the equations (3) can be presented in the form

$$\rho\theta\dot{\eta} = \rho\theta(s + \xi) + (\theta\Phi_j)_{,j} - \Phi_j\theta_{,j}. \tag{5}$$

Following Toupin (1964), Green and Naghdi (1991a) and Eringen (1999) we postulate an energy balance in the form

$$\int_P (\rho\ddot{u}_j\dot{u}_j + I_{ij}\ddot{\varphi}_j\dot{\varphi}_i + \rho\dot{e})dv = \int_P \rho(f_i\dot{u}_i + g_i\dot{\varphi}_i + s\theta)dv +$$
$$+ \int_{\partial P}(t_i\dot{u}_i + m_i\dot{\varphi}_i + \mu_{ji}\dot{u}_{i,j} + \Phi\theta)da, \tag{6}$$

for all regions P of B and every time, where e is the internal energy per unit mass and I_{ij} are the components of the microinertia tensor per unit volume, f_i is the body force per unit mass, g_j is the body couple per unit mass, t_i is a part of the stress vector associated with the surface $\partial\mathcal{P}$ but measured per unit area of ∂P, m_j is the couple stress vector measured per unit area of ∂P, μ_{ij} is the dipolar surface force associated with the surface $\partial\mathcal{P}$ and measured per unit area of ∂P. We suppose that the body has arrived at a given state at a time t through some prescribed motion. As a consequence of invariance requirements under superposed rigid body motions (Green and Rivlin, 1964), from (6) we get

$$\int_P \rho\ddot{u}_j dv = \int_P \rho f_j dv + \int_{\partial P} t_j da. \tag{7}$$

Using the well-known method, from (7) we get

$$t_i = t_{ji}n_j, \tag{8}$$

where t_{ij} is the stress tensor. The local form of the relation (7) is given by

$$t_{ji,j} + \rho f_i = \rho\ddot{u}_i. \tag{9}$$

In view of (2), (8) and (9), the relation (6) reduces to

$$\int_P (I_{ij}\dot{\varphi}_i\ddot{\varphi}_j + \rho\dot{e})dv = \int_P [t_{ji}\dot{u}_{i,j} + \rho(g_i\dot{\varphi}_i + s\theta) +$$
$$+ (\Phi_i\theta)_{,i}]dv + \int_{\partial P}(m_i\dot{\varphi}_i + \mu_{ji}\dot{u}_{i,j})da, \tag{10}$$

for all regions P of B and every time. Following the method given by Green and Rivlin (1964) we consider a motion of the body which differs from the given motion only by a superposed uniform rigid body angular velocity, the body occupying the same position at time t, and let us assume that $\dot{e}, t_{ij}, g_i, s, \Phi_i, m_i$ and μ_{ij} are unaltered by such motion. From (10) we get

$$\int_P I_{ij}\ddot{\varphi}_j dv = \int_P (\rho g_i + \varepsilon_{ijk}t_{jk})dv + \int_{\partial P}(m_i + \varepsilon_{ijk}\mu_{jk})da, \tag{11}$$

where ε_{ijk} is the alternating symbol. With an argument similar to that used in obtaining (8), from (11) we find

$$m_i + \varepsilon_{ijk}\mu_{jk} = (m_{si} + \varepsilon_{ijk}\mu_{sjk})n_s, \tag{12}$$

where m_{ij} is the couple stress tensor and μ_{ijk} is the double stress tensor. In view of (12) we obtain the local from of the relation (11),

$$m_{ji,j} + \varepsilon_{irs}\tau_{rs} + \rho g_i = I_{ij}\ddot{\varphi}_j, \tag{13}$$

where we have used the notation

$$\tau_{rs} = \mu_{krs,k} + t_{rs}. \tag{14}$$

With an argument similar to that used to derive the relation (8), we find from (10) the following equation

$$(m_i - m_{si}n_s)\dot{\varphi}_i + (\mu_{ji} - \mu_{sji}n_s)\dot{u}_{i,j} = 0. \tag{15}$$

If we use the equations (13)-(15) and the divergence theorem, then the equation (10) can be written in the following local form

$$\rho\dot{e} = \tau_{ij}\dot{e}_{ij} + m_{ij}\dot{\gamma}_{ij} + \mu_{ijk}\dot{\kappa}_{ijk} + \rho s\theta + (\Phi_i\theta)_{,i}, \tag{16}$$

where

$$e_{ij} = u_{j,i} + \varepsilon_{jik}\varphi_k, \quad \gamma_{ij} = \varphi_{j,i}, \quad \kappa_{ijk} = u_{k,ij}. \tag{17}$$

With help of (14), the equations (9) become

$$\tau_{ji,j} - \mu_{kji,kj} + \rho f_i = \rho\ddot{u}_i. \tag{18}$$

If we introduce the Helmholz free-energy ψ by

$$\psi = e - \eta\theta, \tag{19}$$

then the equation (16) becomes

$$\rho\dot{\psi} = \tau_{ij}\dot{e}_{ij} + m_{ij}\dot{\gamma}_{ij} + \mu_{ijk}\dot{\kappa}_{ijk} + \Phi_i\theta_{,i} - \rho\eta\dot{\theta} - \rho\xi\theta. \tag{20}$$

Following Green and Naghdi (1993), we introduce the thermal displacement α by $\dot{\alpha} = \theta$. We require constitutive equations for $\psi, \tau_{ij}, m_{ij}, \mu_{ijk}, \eta, \Phi_i, \xi$ and assume that these are functions of the set of variables $\Gamma = (e_{ij}, \gamma_{ij}, \kappa_{ijk}, \theta, \alpha_{,j})$. For simplicity, we regard the material to be homogeneous. Introduction of constitutive equations of the form

$$\psi = \widetilde{\psi}(\Gamma), \quad \tau_{ij} = \widetilde{\tau}_{ij}(\Gamma), \ldots, \xi = \widetilde{\xi}(\Gamma),$$

into the equation (20), yields

$$\begin{aligned}\left(\frac{\partial\sigma}{\partial e_{ij}} - \tau_{ij}\right)\dot{e}_{ij} + \left(\frac{\partial\sigma}{\partial\gamma_{ij}} - m_{ij}\right)\dot{\gamma}_{ij} + \left(\frac{\partial\sigma}{\partial\kappa_{ijk}} - \mu_{ijk}\right)\dot{\kappa}_{ijk} + \\ + \left(\frac{\partial\sigma}{\partial\theta} + \rho\eta\right)\dot{\theta} + \left(\frac{\partial\sigma}{\partial a_{,i}} - \Phi_i\right)\dot{\theta}_{,i} + \rho\xi\theta = 0.\end{aligned} \tag{21}$$

Here we have used the notation $\sigma = \rho\psi$. We assume that there is no kinematical constraint. By using the procedure of Green and Naghdi we find that the necessary and sufficient conditions for the equations (21) to be satisfied under the above constitutive assumptions are

$$\begin{aligned}
&\tau_{ij} = \frac{\partial \sigma}{\partial e_{ij}}, \ \ m_{ij} = \frac{\partial \sigma}{\partial \gamma_{ij}}, \ \ \mu_{ijk} = \frac{\partial \sigma}{\partial \kappa_{ijk}}, \\
&\rho\eta = -\frac{\partial \sigma}{\partial \theta}, \ \ \Phi_k = \frac{\partial \sigma}{\partial \alpha_{,k}}, \ \ \xi = 0,
\end{aligned} \tag{22}$$

where

$$\sigma = \widetilde{\sigma}(e_{ij}, \gamma_{ij}, \kappa_{ijk}, \theta, \alpha_{,j}). \tag{23}$$

Following Green and Naghdi (1991a) we assume that there exists the reference time t_0 such that

$$\theta(x_k, t_0) = T_0, \ \ \alpha(x_k, t_0) = \alpha_0,$$

where T_0 and α_0 are constants. Let us denote

$$T = \theta - T_0, \ \ \tau = \int_{t_0}^{t} T\, dt. \tag{24}$$

We have

$$\alpha = \tau + T_0(t - t_0) + \alpha_0, \ \ \alpha_{,k} = \tau_{,k}, \ \dot{\tau} = T. \tag{25}$$

In what follows we restrict our attention to the linear theory and assume that $u_j = \varepsilon u'_j$, $\varphi_j = \varepsilon \varphi'_j$ and $T = \varepsilon T'$, where ε is a constant small enough for squares and higher powers to be neglected, and u'_j, φ'_j and T' are independent of ε. Assuming that the initial body is free from stress, couple stress and hyperstress and the initial heat flux vanishes, within the frames of the linear theory of anisotropic solids, we have

$$\begin{aligned}
\sigma = {} & \frac{1}{2} A_{ijrs} e_{ij} e_{rs} + B_{ijrs} e_{ij} \gamma_{rs} + \frac{1}{2} C_{ijrs} \gamma_{ij} \gamma_{rs} + \\
& + D_{ijpqr} e_{ij} \kappa_{pqr} + E_{ijpqr} \gamma_{ij} \kappa_{pqr} + \frac{1}{2} F_{ijkpqr} \kappa_{ijk} \kappa_{pqr} - \\
& - a_{ij} e_{ij} T - b_{ij} \gamma_{ij} T - c_{ijk} \kappa_{ijk} T - \frac{1}{2} a T^2 + \\
& + \frac{1}{2} K_{ij} \tau_{,i} \tau_{,j} + G_{ijk} e_{ij} \tau_{,k} + H_{ijk} \gamma_{ij} \tau_{,k} + L_{ijrs} \kappa_{ijr} \tau_{,s} - b_i T \tau_{,i}.
\end{aligned} \tag{26}$$

If we take into account that $\kappa_{ijk} = \kappa_{jik}$ and that σ is a function of class C^2 of the variables e_{ij}, γ_{ij}, κ_{ijk}, T and $\tau_{,j}$, then we find that the constitutive coefficients have the properties

$$\begin{aligned}
&A_{ijrs} = A_{rsij}, \ C_{ijrs} = C_{rsij}, \ D_{ijpqr} = D_{ijqpr}, \\
&E_{ijpqr} = E_{ijqpr}, \ F_{ijkpqr} = F_{pqrijk} = F_{jikpqr}, \\
&c_{ijk} = c_{jik}, \ K_{ij} = K_{ji}, \ L_{ijpq} = L_{jipq}.
\end{aligned} \tag{27}$$

In view of (22), (24)-(27), we obtain

$$
\begin{aligned}
&\tau_{ij} = A_{ijrs}e_{rs} + B_{ijrs}\gamma_{rs} + D_{ijpqr}\kappa_{pqr} - a_{ij}T + G_{ijk}\tau_{,k},\\
&m_{ij} = B_{rsij}e_{rs} + C_{ijrs}\gamma_{rs} + E_{ijpqr}\kappa_{pqr} - b_{ij}T + H_{ijk}\tau_{,k},\\
&\mu_{ijk} = D_{rsijk}e_{rs} + E_{rsijk}\gamma_{rs} + F_{ijkpqr}\kappa_{pqr} - c_{ijk}T + L_{ijrs}\tau_{,s},\\
&\rho\eta = a_{ij}e_{ij} + b_{ij}\gamma_{ij} + c_{ijk}\kappa_{ijk} + aT + b_j\tau_{,j},\\
&\Phi_i = G_{rsi}e_{rs} + H_{rsi}\gamma_{rs} + L_{pqri}\kappa_{pqr} - b_iT + K_{ij}\tau_{,j}.
\end{aligned} \tag{28}
$$

In the context of the linear theory, from (4) and (28) we obtain

$$
q_i = T_0\Phi_i. \tag{29}
$$

In this case the equation (5) reduces to

$$
\rho T_0\dot{\eta} = q_{j,j} + \rho S, \tag{30}
$$

where $S = \theta s$ is the external rate of supply of heat per unit mass (Green and Naghdi, 1993). We conclude that the basic equations of the linear theory are the equations of motion (13) and (18), the equation of entropy (30), the constitutive equations (28), (29), and the geometrical equations (17). The constitutive equations (28) characterize a general anisotropic solid. If the anisotropic body is achiral (centrosymmetric) then the coefficients D_{ijpqr}, G_{ijk}, E_{ijpqr}, H_{ijk}, c_{ijk} and b_i are equal to zero. In the case of isotropic chiral materials, the constitutive equations (28) become

$$
\begin{aligned}
\tau_{ij} = {} & \lambda e_{rr}\delta_{ij} + (\mu+\kappa)e_{ij} + \mu e_{ji} + C_1\gamma_{ss}\delta_{ij} + C_2\gamma_{ji} + C_3\gamma_{ij} + \\
& + d_1\varepsilon_{jmn}\kappa_{min} + d_2\varepsilon_{rij}\kappa_{rss} + d_3\varepsilon_{rij}\kappa_{ssr} - b\delta_{ij}T + b_1\varepsilon_{ijk}\tau_{,k},\\[1ex]
m_{ij} = {} & \bar{\alpha}\gamma_{ss}\delta_{ij} + \beta\gamma_{ji} + \gamma\gamma_{ij} + C_1e_{rr}\delta_{ij} + C_2e_{ji} + C_3e_{ij} + \\
& + \beta_1\varepsilon_{jmn}\kappa_{min} + \beta_2\varepsilon_{rij}\kappa_{rss} + \beta_3\varepsilon_{rij}\kappa_{ssr} - b_0\delta_{ij}T + \chi^*\varepsilon_{ijk}\tau_{,k},\\
\mu_{ijk} = {} & \frac{1}{2}d_1(\varepsilon_{rik}e_{jr} + \varepsilon_{rjk}e_{ir}) + \frac{1}{2}d_2(\varepsilon_{imn}\delta_{jk} + \varepsilon_{jmn}\delta_{ik})e_{mn} + \\
& + d_3\varepsilon_{kmn}e_{mn}\delta_{ij} + \frac{1}{2}\beta_1(\varepsilon_{rik}\gamma_{jr} + \varepsilon_{rjk}\gamma_{ir}) + \\
& + \frac{1}{2}\beta_2(\varepsilon_{imn}\delta_{jk} + \varepsilon_{jmn}\delta_{ik})\gamma_{mn} + \beta_3\varepsilon_{kmn}\gamma_{mn}\delta_{ij} + \\
& + \frac{1}{2}\alpha_1(\kappa_{rri}\delta_{jk} + 2\kappa_{krr}\delta_{ij} + \kappa_{rrj}\delta_{ik}) + \alpha_2(\kappa_{irr}\delta_{jk} + \\
& + \kappa_{jrr}\delta_{ik}) + 2\alpha_3\kappa_{rrk}\delta_{ij} + 2\alpha_4\kappa_{ijk} + \\
& + \alpha_5(\kappa_{kji} + \kappa_{kij}) + \xi_1\delta_{ij}\tau_{,k} + \xi_2(\delta_{ik}\tau_{,j} + \delta_{jk}\tau_{,i}),\\
\rho\eta = {} & be_{rr} + b_0\gamma_{ss} + aT,\\
q_i = {} & T_0(b_1\varepsilon_{rsi}e_{rs} + \chi^*\varepsilon_{rsi}\gamma_{rs} + \xi_1\kappa_{ssi} + 2\xi_2\kappa_{irr} + k\tau_{,i}),
\end{aligned} \tag{31}
$$

where δ_{ij} is the Kronecker's delta. The isotropic material moduli $\lambda, \mu, \kappa, C_j, d_j, b, b_1, \bar{\alpha}, \beta, \gamma, \beta_j, b_0, \chi^*, \alpha_r, \xi_\rho, a$ and k are functions of x for inhomogeneous materials and they are constants for homogeneous materials. In the theory of Cosserat elasticity the chiral part of constitutive equations is characterized by the coefficients C_k (Lake, 1982, 2001). The constitutive equations for chiral solids in gradient elasticity have been derived by Papanicolopulos (2011).

For isotropic bodies we have $I_{ij} = I\delta_{ij}$. From (13), (17), (18) and (31) we find the field equations expressed in terms of the functions u_j, φ_j and τ,

$$\begin{aligned}
&(\mu+\kappa-\nu_1\Delta)\Delta\boldsymbol{u}+(\lambda+\mu-\nu_2\Delta)\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u}-2\kappa_1\Delta\mathrm{curl}\boldsymbol{u}+\\
&+(C_3+\kappa_3)\Delta\boldsymbol{\varphi}+(C_1+C_2+\kappa_2)\mathrm{grad}\,\mathrm{div}\boldsymbol{\varphi}+(\kappa-\bar{\zeta}_1\Delta)\mathrm{curl}\,\boldsymbol{\varphi}-\\
&-b\,\mathrm{grad}\,\dot{\tau}-\bar{\zeta}_2\Delta\mathrm{grad}\,\tau+\rho\boldsymbol{f}=\rho\ddot{\boldsymbol{u}}, \qquad (32)\\
&(C_3+\kappa_1)\Delta\boldsymbol{u}+(C_1+C_2+\kappa_2)\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u}+(\kappa-\bar{\zeta}_1\Delta)\mathrm{curl}\,\boldsymbol{u}+\\
&+(\gamma\Delta-2\kappa)\boldsymbol{\varphi}+(\bar{\alpha}+\beta)\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\varphi}-b_0\mathrm{grad}\,\dot{\tau}+\\
&+2b_1\mathrm{grad}\,\tau+\rho\boldsymbol{g}=I\ddot{\boldsymbol{\varphi}},\\
&k\Delta\tau+\bar{\zeta}_2\Delta\mathrm{div}\boldsymbol{u}-b\mathrm{div}\,\dot{\boldsymbol{u}}-b_0\mathrm{div}\,\dot{\boldsymbol{\varphi}}-2b_1\mathrm{div}\,\boldsymbol{\varphi}-a\ddot{\tau}=-\frac{1}{T_0}\rho S,
\end{aligned}$$

where Δ is the Laplacian and we have used the notations

$$\begin{aligned}
&\nu_1=2(\alpha_3+\alpha_4),\nu_2=2(\alpha_1+\alpha_2+\alpha_5),\ \bar{\zeta}_1=\beta_3-\beta_1,\ \bar{\zeta}_2=\xi_1+2\xi_2,\\
&\kappa_1=d_3-d_1,\kappa_2=d_1+2d_2,\ \kappa_3=2d_3-d_1. \qquad (33)
\end{aligned}$$

It follows from (32) that the thermal field is influenced by mechanical fields even in the equilibrium theory. In the case of an achiral material the coefficients C_k, d_k, b_0 and b_1 are equal to zero.

For a given deformation, $\dot{u}_{i,j}$ and $\dot{\varphi}_k$ in (15) may by chosen arbitrary so that, on the basis of the constitutive equations we obtain

$$m_i = m_{ji}n_j,\ \ \mu_{ji} = \mu_{sji}n_s. \qquad (34)$$

We assume that the boundary ∂B consists in the union of a finite number of smooth surfaces, smooth curves (edges) and points (corners). Let C be the union of edges. Following Toupin (1964) and Mindlin (1964) we get

$$\int_{\partial B}(t_i\dot{u}_i+\mu_{ji}\dot{u}_{i,j})da=\int_{\partial B}(P_i\dot{u}_i+R_iD\dot{u}_i)da+\int_C Q_i\dot{u}_i dl, \qquad (35)$$

where

$$\begin{aligned}
&P_i=(\tau_{ki}-\mu_{ski,s})n_k-D_j(n_r\mu_{rji})+(D_kn_k)n_sn_p\mu_{spi},\\
&R_i=\mu_{rsi}n_rn_s,\ Q_i=<\mu_{pji}n_pn_q>\varepsilon_{jrq}s_r,\ Df=f_{,k}n_k. \qquad (36)
\end{aligned}$$

Here, D_i are the components of the surface gradients, $D_i = (\delta_{ij} - n_in_k)\partial/\partial x_k$, s_j are the components of the unit vector tangent to C, and $< g >$ denotes the difference of limits of g from both sides of C. Let $\Sigma_k, (k = 1, 2, \ldots, 8)$ be subsets of ∂B such that $\overline{\Sigma}_1 \cup \Sigma_2 = \overline{\Sigma}_3 \cup \Sigma_4 = \overline{\Sigma}_5 \cup \Sigma_6 = \overline{\Sigma}_7 \cup \Sigma_8 = \partial B$, $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \Sigma_5 \cap \Sigma_6 = \Sigma_7 \cap \Sigma_8 = \emptyset$. We consider the boundary conditions

$$u_i = \widetilde{u}_i \text{ on } \overline{\Sigma}_1 \times (t_0, t_1), P_i = \widetilde{P}_i \text{ on } \Sigma_2 \times (t_0, t_1), Du_i = \widetilde{d}_i \text{ on } \overline{\Sigma}_3 \times (t_0, t_1),$$
$$R_i = \widetilde{R}_i \text{ on } \Sigma_4 \times (t_0, t_1), \varphi_j = \widetilde{\varphi}_j \text{ on } \overline{\Sigma}_5 \times (t_0, t_1), m_{ji} n_j = \widetilde{m}_i \text{ on } \Sigma_6 \times (t_0, t_1), \tag{37}$$
$$\tau = \widetilde{\tau} \text{ on } \overline{\Sigma}_7 \times (t_0, t_1), \ q_j n_j = \widetilde{q} \text{ on } \Sigma_8 \times (t_0, t_1), Q_j = \widetilde{Q}_j \text{ on } C \times (t_0, t_1),$$

where $\widetilde{u}_i, \widetilde{d}_i, \widetilde{\varphi}_j, \widetilde{\tau}, \widetilde{P}_j, \widetilde{R}_j, \widetilde{m}_j, \widetilde{q}$ and $\widetilde{Q}_j$ are prescribed functions. The initial conditions are given by

$$\begin{aligned} &u_i(x,0) = u_i^0(x), \dot{u}_i(x,0) = v_i^0(x), \varphi_j(x,0) = \varphi_j^0(x), \\ &\dot{\varphi}_j(x,0) = \psi_j^0(x), \tau(x,0) = \tau^0(x), \ \dot{\tau}(x,0) = \vartheta^0(x), \quad x \in \overline{B}, \end{aligned} \tag{38}$$

where the functions $u_i^0, v_i^0, \varphi_j^0, \psi_j^0, \tau^0$ and ϑ^0 are prescribed.

3. Uniqueness

In this section we establish a uniqueness theorem in the dynamic theory of thermoelasticity. We denote

$$\begin{aligned} 2W = {} & A_{ijrs} e_{ij} e_{rs} + 2B_{ijrs} e_{ij} \gamma_{rs} + C_{ijrs} \gamma_{ij} \gamma_{rs} + \\ & + 2D_{ijpqr} e_{ij} \kappa_{pqr} + 2E_{ijpqr} \gamma_{ij} \kappa_{pqr} + \\ & + F_{ijkpqr} \kappa_{ijk} \kappa_{pqr} + K_{ij} \tau_{,i} \tau_{,j} + 2G_{ijk} e_{ij} \tau_{,k} + \\ & + 2H_{ijk} \gamma_{ij} \tau_{,k} + 2L_{ijrs} \kappa_{ijr} \tau_{,s}, \\ 2E = {} & \int_B (\rho \dot{u}_k \dot{u}_k + I_{ij} \dot{\varphi}_i \dot{\varphi}_j + 2W + aT^2) dv. \end{aligned} \tag{39}$$

Theorem 1. *Assume that*
(i) W is a positive semi-definite form;
(ii) ρ and a are strictly positive;
(iii) I_{ij} is a positive definite tensor;
(iv) the constitutive coefficients satisfy the relations (27).
Proof. We introduce the notation

$$U = \tau_{ij} \dot{e}_{ij} + m_{ij} \dot{\gamma}_{ij} + \mu_{ijk} \dot{\kappa}_{ijk} + \rho \dot{\eta} T + \Phi_k T_{,k}. \tag{40}$$

It follows from (27), (28) and (39) that

$$U = \frac{\partial}{\partial t} (W + \frac{1}{2} a T^2). \tag{41}$$

On the other hand, in view of (13), (17), (18), (29) and (30) we get

$$\begin{aligned} U = {} & [(\tau_{jk} - \mu_{jik,i}) \dot{u}_k + m_{jk} \dot{\varphi}_k + \mu_{ijk} \dot{u}_{k,i} + T \Phi_j]_{,j} + \\ & + \rho (f_i \dot{u}_i + g_i \dot{\varphi}_i + \frac{1}{T_0} TS) - \rho \ddot{u}_i \dot{u}_i - I_{ij} \ddot{\varphi}_j \dot{\varphi}_i. \end{aligned} \tag{42}$$

From (4), (8), (14), (34), (35), (41) and (42) we obtain

$$\dot{E} = \int_{\partial B} (P_i \dot{u}_i + R_i D\dot{u}_i + m_j \dot{\varphi}_j + \frac{1}{T_0} qT) da + \\ + \int_C Q_i \dot{u}_i dl + \int_B \rho (f_i \dot{u}_i + g_i \dot{\varphi}_i + \frac{1}{T_0} TS) dv. \tag{43}$$

Let us assume that there are two solutions. Then their difference $(u_j^*, \varphi_j^*, \tau^*)$ corresponds to null data. In view of hypotheses (i)-(iv), from (43) we get $u_j^* = 0$, $\varphi_j^* = 0$ and $\tau^* = 0$.□

4. An existence theorem

In this section we consider a semigroup approach (see Goldstein, 1985) to obtain an existence result in the dynamical theory, with suitable initial and boundary conditions. In this section, we assume that the boundary ∂B is smooth and consider the following homogeneous boundary conditions

$$u_i = Du_i = \varphi_i = \tau = 0 \text{ on } \partial B.$$

In addition to the assumptions (i)-(iv) made previously in the Theorem 1, we should toughen condition (i). We assume that

(i') W is a positive definite form.

Let $W_0^{2,2}$, $W^{4,2}$ and L^2 be the usual Hilbert spaces and denote

$$\mathcal{Z} = \{(\mathbf{u}, \mathbf{v}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \tau, \theta), \mathbf{u} \in \mathbf{W}_0^{2,2}(B), \\ \mathbf{v}, \boldsymbol{\zeta} \in \mathbf{L}^2(B),\ \boldsymbol{\varphi} \in \mathbf{W}_0^{1,2}(B),\ \tau \in W_0^{1,2}, T \in L^2(B)\},$$

where $\mathbf{W}_0^{2,2} = [W_0^{2,2}]^3$, $\mathbf{W}_0^{1,2} = [W_0^{1,2}]^3$ and $\mathbf{L}^2 = [L^2]^3$.

We introduce the operators

$$M_i \mathbf{u} = \rho^{-1} \left(A_{jirs} u_{s,r} + D_{jipqr} \kappa_{pqr} \right)_{,j} - \rho^{-1} (D_{rskji} u_{s,r} + F_{kjipqr} \kappa_{pqr})_{,jk},$$
$$N_i \boldsymbol{\varphi} = \rho^{-1} \left(A_{jirs} \varepsilon_{srk} \varphi_k + B_{jirs} \gamma_{rs} \right)_{,j} - \rho^{-1} (D_{rskji} \varepsilon_{srl} \varphi_l + E_{rskji} \gamma_{rs})_{,jk},$$
$$N_i^* \tau = \rho^{-1} \left(G_{jik} \tau_{,k} \right)_{,j} - \rho^{-1} (L_{kjis} \tau_{,s})_{,jk},\ P_i^* T = \rho^{-1} \left(-a_{ji} T \right)_{,j} + \rho^{-1} (c_{kji} T)_{,jk},$$
$$R_m^* \mathbf{u} = I_{mi}^* (B_{rsji} u_{s,r} + E_{jipqr} \kappa_{pqr})_{,j} + I_{mi}^* \varepsilon_{irs} (A_{rslj} u_{j,l} + D_{rspjk} \kappa_{pjk}),$$
$$S_m \boldsymbol{\varphi} = I_{mi}^* [(C_{jirs} \gamma_{rs})_{,j} + \varepsilon_{irs} B_{rskj} \gamma_{kj}] + I_{mi}^* [(B_{rsji} \varepsilon_{srl} \varphi_l)_{,j} + \varepsilon_{irs} \varepsilon_{nlk} A_{rsln} \varphi_k],$$
$$T_m \tau = I_{mi}^* [(H_{jik} \tau_{,k})_{,j} + \varepsilon_{irs} G_{rsk} \tau_{,k}], U_m T = -I_{mi}^* [(b_{ji} T)_{,j} - \varepsilon_{irs} a_{rs} T],$$
$$\bar{V} \mathbf{u} = a^{-1} (G_{rsi} u_{s,r} + L_{pqri} \kappa_{pqr})_{,i},$$
$$\bar{W} \boldsymbol{\varphi} = a^{-1} [(H_{rsi} \gamma_{rs})_{,i} + (G_{rsi} \varepsilon_{srj} \varphi_j)_{,i}],\ X\tau = a^{-1} (K_{ij} \tau_{,j})_{,i},$$
$$YT = -a^{-1} (b_j T_{,j} + (b_i T)_{,i}),\ \bar{Q} \mathbf{v} = a^{-1} \left(-a_{ij} v_{j,i} - c_{ijk} \xi_{ijk} \right),$$
$$L \boldsymbol{\zeta} = -a^{-1} (a_{ij} \varepsilon_{jik} \zeta_k - b_{ij} \chi_{ij}).$$

Let us consider the matrix operator $\mathcal{A}$ defined on $\mathcal{Z}$ by

$$\begin{pmatrix} \mathbf{0} & \mathbf{Id} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{M} & \mathbf{0} & \mathbf{N} & \mathbf{0} & \mathbf{N}^* & \mathbf{P}^* \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Id} & \mathbf{0} & \mathbf{0} \\ \mathbf{R}^* & \mathbf{0} & \mathbf{S} & \mathbf{0} & \mathbf{T} & \mathbf{U} \\ 0 & 0 & 0 & 0 & 0 & Id \\ \bar{V} & \bar{Q} & \bar{W} & L & X & Y \end{pmatrix}, \tag{44}$$

where (I^*_{mi}) is the inverse of (I_{mi}), $\xi_{ijk} = v_{k,ij}$, $\chi_{ij} = \zeta_{j,i}$, $\mathbf{M} = (M_i), \mathbf{N} = (N_i), \mathbf{N}^* = (N^*_i)$, $\mathbf{R}^* = (R^*_i)$, $\mathbf{S} = (S_i)$, $\mathbf{T} = (T_i)$, $\mathbf{U} = (U_i)$ and $\mathbf{P}^* = (P^*_i)$.

The domain $\mathcal{D}$ of the operator $\mathcal{A}$ is the set

$$\{(\mathbf{u}, \mathbf{v}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \tau, T), \text{ such that } \mathcal{A} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \boldsymbol{\varphi} \\ \boldsymbol{\zeta} \\ \tau \\ T \end{pmatrix} \in \mathcal{Z}\}.$$

We note that

$$\mathbf{W}^{2,2}_0 \cap \mathbf{W}^{4,2} \times \mathbf{W}^{2,2}_0 \times \mathbf{W}^{1,2}_0 \cap \mathbf{W}^{2,2} \times \mathbf{W}^{1,2}_0 \times W^{1,2}_0 \cap W^{2,2} \times W^{1,2}_0$$

is a dense subspace of the Hilbert space $\mathcal{Z}$ which is contained in $\mathcal{D}$. Therefore the domain of the operator is dense.

The boundary-initial-value problem can be transformed into the following abstract equation in the space $\mathcal{Z}$

$$\frac{d\omega}{dt} = \mathcal{A}\omega + G(t), \quad \omega(0) = \omega_0, \tag{45}$$

where $G(t) = (\mathbf{0}, \mathbf{f}, \mathbf{0}, I^*_{mi}(\rho g_i), 0, a^{-1}T_0^{-1}S)$, $\omega_0 = (\mathbf{u}^0, \mathbf{v}^0, \boldsymbol{\varphi}^0, \boldsymbol{\psi}^0, \tau^0, \vartheta^0)$. Let $\omega = (\mathbf{u}, \mathbf{v}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \tau, T)$ and $\omega' = (\mathbf{u}', \mathbf{v}', \boldsymbol{\varphi}', \boldsymbol{\zeta}', \tau', T')$. We introduce the inner product

$$< \omega, \omega' >= \int_B \left(\rho v_i v'_i + I_{ij}\zeta_i\zeta'_j + aTT' + 2W^* \right) dv, \tag{46}$$

where

$$2W^* = A_{ijrs}e_{ij}e'_{rs} + B_{ijrs}(e_{ij}\gamma'_{rs} + e'_{ij}\gamma_{rs}) + C_{ijrs}\gamma_{ij}\gamma'_{rs} + D_{ijpqr}(e_{ij}\kappa'_{pqr} + e'_{ij}\kappa_{pqr})$$
$$+ E_{ijpqr}(\gamma_{ij}\kappa'_{pqr} + \gamma'_{ij}\kappa_{pqr}) + F_{ijkpqr}\kappa_{ijk}\kappa'_{pqr} + K_{ij}\tau_{,i}\tau'_{,j} + G_{ijk}(e'_{ij}\tau_{,k}$$
$$+ e_{ij}\tau'_{,k}) + H_{ijk}(\gamma'_{ij}\tau_{,k} + \gamma_{ij}\tau'_{,k}) + L_{ijrs}(\kappa'_{ijr}\tau_{,s} + \kappa_{ijr}\tau'_{,s})$$

It is worth noting that this inner product defines the norm

$$||\omega||^2 = \int_B \left(\rho v_i v_i + I_{ij}\zeta_i\zeta_j + aT^2 + 2W \right) dv, \tag{47}$$

where $2W$ is defined by (39).

This norm is equivalent to the usual norm in $\mathcal{Z}$. We also note that for every $\omega \in \mathcal{D}$, we have

$$< \mathcal{A}\omega, \omega >= 0. \tag{48}$$

Lemma 1. *Suppose that hypotheses (i'), (ii)-(iv) hold. Let $\rho^*(\mathcal{A})$ be the resolvent of $\mathcal{A}$. Then, $0 \in \rho^*(\mathcal{A})$.*

Proof. Let us show that we can find $\omega = (\mathbf{u}, \mathbf{v}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \tau, T) \in \mathcal{D}$ such that

$$\mathcal{A}\omega = \mathcal{F}, \tag{49}$$

for any $\mathcal{F} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4, f_5, f_6) \in \mathcal{Z}$. In terms of the components we get

$$\mathbf{v} = \mathbf{f}_1, \ \ \mathbf{M}\mathbf{u} + \mathbf{N}\boldsymbol{\varphi} + \mathbf{N}^*\tau + \mathbf{P}^*T = \mathbf{f}_2, \boldsymbol{\zeta} = \mathbf{f}_3, \ \ \mathbf{R}^*\mathbf{u} + \mathbf{S}\boldsymbol{\varphi} + \mathbf{T}\tau + \mathbf{U}\theta = \mathbf{f_4} \tag{50}$$

$$T = f_5, \ \ \bar{V}\mathbf{u} + \bar{W}\boldsymbol{\varphi} + X\tau + \bar{Q}\mathbf{v} + L\boldsymbol{\zeta} + YT = f_6. \tag{51}$$

From these equation we see that $\mathbf{v} \in \mathbf{W}_0^{2,2}$, $\boldsymbol{\zeta} \in \mathbf{W}_0^{1,2}$ and $T \in W_0^{1,2}$ and we can write the system

$$\mathbf{M}\mathbf{u} + \mathbf{N}\boldsymbol{\varphi} + \mathbf{N}^*\tau = \mathbf{f}_2 - \mathbf{P}^* f_5, \mathbf{R}^*\mathbf{u} + \mathbf{S}\boldsymbol{\varphi} + \mathbf{T}\tau = \mathbf{f}_4 - \mathbf{U} f_5, \tag{52}$$

$$\bar{V}\mathbf{u} + \bar{W}\boldsymbol{\varphi} + X\tau = f_6 - \bar{Q}\mathbf{f}_1 - L\mathbf{f}_3 - Y f_5. \tag{53}$$

To study this system we define the bilinear form:

$$\mathcal{B}[(\mathbf{u}, \boldsymbol{\varphi}, \tau), (\mathbf{u}^*, \boldsymbol{\varphi}^*, \tau^*)] = \int_B I^* dv,$$

where

$$I^* = \rho(M_i\mathbf{u} + N_i\boldsymbol{\varphi} + N_i^*\tau)u_i^* + I_{ij}(R_i^*\mathbf{u} + S_i\boldsymbol{\varphi} + T_i\tau)\boldsymbol{\varphi}_j^* + a(\bar{V}\mathbf{u} + \bar{W}\boldsymbol{\varphi} + X\tau)\tau^*.$$

After the use of the divergence theorem we see that this is a bounded bilinear form defined in $\mathbf{W}_0^{2,2} \times \mathbf{W}_0^{1,2} \times W_0^{1,2}$. In view of the condition (i') it is coercive. The right-hand side belongs to $\mathbf{W}^{-2,2} \times \mathbf{W}^{-1,2} \times W^{-1,2}$. The solution of this system is guaranteed on the basis of the Lax-Milgram theorem (see Gilbarg and Trudinger, 1985). Consequently, there exists $(\mathbf{u}, \boldsymbol{\varphi}, \tau) \in \mathbf{W}_0^{2,2} \times \mathbf{W}_0^{1,2} \times W_0^{1,2}$ satisfying the system (52), (53). Thus, we conclude that the equation (49) has a solution in the domain $\mathcal{D}$ and the theorem is proved. $\Box$

Theorem 2. *Suppose that hypotheses (i'), (ii)-(iv) hold. Then the operator $\mathcal{A}$ is the generator of a C^0-semigroup of contractions in the Hilbert space $\mathcal{Z}$.*

Proof. The proof is a direct consequence of the Lumer-Phillips theorem, since the operator $\mathcal{A}$ is dissipative, with a dense domain and $0 \in \rho^*(\mathcal{A})$ (see Liu and Zheng, 1999).

Now, we can state the main result of this section.

Theorem 3. *Suppose that hypotheses (i'), (ii)-(iv) hold. Let $G(t) \in C^1(\mathbb{R}^+, \mathcal{Z}) \cap C^0(\mathbb{R}^+, \mathcal{D})$ and $\omega_0 \in \mathcal{D}$. Then, there exists a unique solution $\omega(t) \in C^1(\mathbb{R}^+, \mathcal{Z}) \cap C^0(\mathbb{R}^+, \mathcal{D})$ to the problem (45).*

Remark. The existence of a C^0-semigroup implies the continuous dependence with respect initial data and supply terms. As we have presented the existence, uniqueness and continuous dependence of solutions, we conclude that the problem is well posed in the sense of Hadamard. Furthermore, as the internal energy is positive we see that

$$||\omega(t)||^2 \leq E(0),$$

for every t. This gives the stability of the solutions if the hypotheses (i'), (ii)-(iv) hold.

5. The effects of a concentrated heat source

Mindlin (1964) established a general solution of the displacement equations in gradient elastostatics and used it to derive the solution to the problem of a concentrated force acting in an infinite region.

In this section we study the problem of a concentrated heat source acting in an unbounded isotropic chiral solid. We consider the case of the equilibrium theory and assume that the body force and the body couple are absent. In the context of the linear theory of Cosserat thermoelastostatics the problem of a concentrated heat source has been studied by Dyszlewicz (2004).

Following Mindlin (1964) we first establish a solution of the field equations. We introduce the notations

$$\begin{aligned}
&\eta_1 = C_3 + \kappa_3, \eta_2 = C_1 + C_2 + \kappa_2,\ \eta_3 = C_3 + \kappa_1,\\
&\Lambda_1 = (\mu + \kappa + \nu_1\Delta)\Delta,\ \Lambda_2 = \lambda + \mu - \nu_2\Delta,\ \Lambda_3 = \kappa - \bar{\zeta}_1\Delta,\\
&\Lambda_4 = \gamma\Delta - 2\kappa,\ \Lambda_5 = [\lambda + 2\mu + \kappa - (\nu_1 + \nu_2)\Delta]\Delta,\\
&\zeta_0 = 2b_1\bar{\zeta}_2(\eta_1 + 2\eta_2 + \eta_3) - k(\eta_1 + \eta_2)(\eta_2 + \eta_3).
\end{aligned} \tag{54}$$

The equations (32) reduce to

$$\begin{aligned}
&\Lambda_1\boldsymbol{u} + \Lambda_2\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} - 2\kappa_1\Delta\mathrm{curl}\,\boldsymbol{u} + \eta_1\Delta\boldsymbol{\varphi} +\\
&+ \eta_2\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\varphi} + \Lambda_3\mathrm{curl}\,\boldsymbol{\varphi} - \bar{\zeta}_2\Delta\mathrm{grad}\,\tau = \mathbf{0},\\
&\eta_3\Delta\boldsymbol{u} + \eta_2\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} + \Lambda_3\mathrm{curl}\boldsymbol{u} + \Lambda_4\boldsymbol{\varphi} +\\
&+ (\bar{\alpha} + \beta)\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\varphi} + 2b_1\mathrm{grad}\,\tau = 0,\\
&k\Delta\tau + \bar{\zeta}_2\Delta\mathrm{div}\,\boldsymbol{u} + 2b_1\mathrm{div}\,\boldsymbol{\varphi} = -Q,
\end{aligned} \tag{55}$$

where $Q = \rho S/T_0$. We consider a thermoelastic body that occupies the entire three-dimensional Euclidean space. We assume that

$$\boldsymbol{u} = \mathrm{grad}\,F,\ \boldsymbol{\varphi} = \mathrm{grad}\,\Psi,\ \tau = \chi, \tag{56}$$

where F, Ψ and χ are unknown functions. The equations (55) are satisfied if the functions F, Ψ and χ satisfy the equations

$$\begin{aligned}
&\Pi\Delta F + (\eta_1 + \eta_2)\Delta\Psi - \bar{\zeta}_2\Delta\chi = 0,\\
&(\eta_2 + \eta_3)\Delta F + \Omega\Psi + 2b_1\chi = 0,\\
&k\Delta\chi + \bar{\zeta}_2\Delta\Delta F - 2b_1\Delta\Psi = -Q,
\end{aligned} \tag{57}$$

where

$$\Pi = \lambda + 2\mu + \kappa - (\nu_1 + \nu_2)\Delta,\ \Omega = (\bar{\alpha} + \beta + \gamma)\Delta - 2\kappa. \tag{58}$$

It is easy to see that a solution of the system (57) can be expressed in the form

$$\begin{aligned} F &= [\bar{\zeta}_2\Omega + 2b_1(\eta_1 + \eta_2)]V, \\ \Psi &= -[2b_1\Pi + \bar{\zeta}_2(\eta_2 + \eta_4)\Delta]V, \\ \chi &= [\Pi\Omega - (\eta_1 + \eta_2)(\eta_2 + \eta_3)\Delta]V, \end{aligned} \tag{59}$$

where the function V satisfies the equation

$$[(k\Pi + \bar{\zeta}_2^2\Delta)\Omega + 4b_1^2\Pi + \zeta_0\Delta]\Delta V = -Q. \tag{60}$$

Let us denote

$$\lambda_0 = \lambda + 2\mu + \kappa,\ \bar{\alpha}_0 = \bar{\alpha} + \beta + \gamma,\ \ \nu_0 = \nu_1 + \nu_2. \tag{61}$$

The equation (60) can be written in the form

$$\bar{\alpha}_0(\bar{\zeta}_2^2 - k\nu_0)\Delta(\Delta - k_1^2)(\Delta - k_2^2)V = -Q, \tag{62}$$

where k_1^2 and k_2^2 are the roots of the equation

$$\begin{aligned} &\bar{\alpha}_0(\bar{\zeta}_2^2 - k\nu_0)z^4 + [k(\bar{\alpha}_0\lambda_0 + 2\kappa\nu_0) + \zeta_0 - 2\kappa\bar{\zeta}_2^2 - 4b_1^2\nu_0]z^2 + \\ &+ 4b_1^2\lambda_0 - 2\kappa k\lambda_0 = 0. \end{aligned} \tag{63}$$

The function V satisfies the equation

$$(\Delta - k_1^2)(\Delta - k_2^2)\Delta V = -dQ, \tag{64}$$

where $d = [\bar{\alpha}_0(\bar{\zeta}_2^2 - k\nu_0)]^{-1}$. In what follows we assume that k_1 and k_2 are distinct positive constants. The other cases can be studied in a similar way. Let V_j be functions that satisfy the equations

$$(\Delta - k_1^2)V_1 = -dQ,\ \ (\Delta - k_2^2)V_2 = -dQ,\ \Delta V_3 = -dQ. \tag{65}$$

The solution of the equation (64) can be written in the form

$$V = \sum_{j=1}^{3} p_j V_j, \tag{66}$$

where

$$p_1^{-1} = k_1^2(k_1^2 - k_2^2),\ p_2^{-1} = k_2^2(k_2^2 - k_1^2),\ p_3^{-1} = k_1^2 k_2^2. \tag{67}$$

Let us assume that a concentrated heat source is acting at the point $\boldsymbol{y}(y_j)$. In this case we have $Q = \delta(\boldsymbol{x} - \boldsymbol{y})$ where $\delta(\cdot)$ is the Dirac delta. We consider the following conditions at infinity

$$u_i \to 0,\ u_{i,j} \to 0,\ u_{i,jk} \to 0,\ \varphi_i \to 0,\ \varphi_{i,j} \to 0,\ \tau \to 0, \tau_{,i} \to 0 \text{ for } r \to \infty,$$

where $r = [(x_j - y_j)(x_j - y_j)]^{1/2}$. From (65) we find that the functions V_j, which vanish at infinity, are given by

$$V_\alpha = \frac{d}{4\pi r}\exp(-k_\alpha r), \quad V_3 = \frac{d}{4\pi r}. \tag{68}$$

Thus, we obtain

$$V = \frac{d}{4\pi r}[p_1 \exp(-k_1 r) + p_2 \exp(-k_2 r) + 1]. \tag{69}$$

It follows from (56), (59) and (69) that the functions u_i, φ_i and τ are given by

$$\begin{aligned}
u_i &= -\frac{d(x-y_i)}{4\pi r^2}\left[\left(\frac{1}{r} - k_1\right) K_1 \exp(-k_1 r)\right.\\
&\quad \left. + \left(\frac{1}{r} - k_2\right) K_2 \exp(-k_2 r) + 2p_3 A_1 r^{-1}\right],\\
\varphi_i &= -\frac{d(x_i-y_i)}{4\pi r^2}\left[\left(\frac{1}{r} - k_1\right) H_1 \exp(-k_1 r)\right.\\
&\quad \left. + \left(\frac{1}{r} - k_2\right) H_2 \exp(-k_2 r) - 2b_1 p_3 \lambda_0 r^{-1}\right],\\
\tau &= \frac{d}{4\pi r}[\Lambda_1^* \exp(-k_1 r) + \Lambda_2^* \exp(-k_2 r) - 2\kappa\lambda_0 p_3].
\end{aligned} \tag{70}$$

In (70) we have used the notations

$$\begin{aligned}
&K_\alpha = (\bar\zeta_2 \bar\alpha_0 k_\alpha^2 + 2A_1)p_\alpha, \quad A_1 = b_1(\eta_1 + \eta_2) - \bar\zeta_2 \kappa,\\
&H_\alpha = (k_\alpha^2 A_2 - 2b_1\lambda_0)p_\alpha, \; A_2 = 2b_1\nu_0 - (\eta_2 + \eta_3)\bar\zeta_2,\\
&\Lambda_\alpha^* = (k_\alpha^2 A_3 - \bar\alpha_0 \nu_0 k_\alpha^4 - 2\kappa\lambda_0)p_\alpha, \; A_3 = \bar\alpha_0\lambda_0 + 2\kappa\nu_0 - (\eta_1+\eta_2)(\eta_2+\eta_3).
\end{aligned} \tag{71}$$

From (70) we see that the conditions at infinity are satisfied. In the classical thermoelasticity the solution of the problem of a concentrated heat source is given by (Nowacki, 1960). In this case the displacements are given by

$$u_i = \frac{b(x_i - y_i)}{8\pi k(\lambda + 2\mu)r}, \tag{72}$$

and the thermal field is $(4\pi k r)^{-1}$. In contrast with the theory of centrosymmetric materials, the thermal field produces a microrotation of particles. We note that η_j and b_1 are chiral coefficients. Let us assume that the b_1 and η_j change the sign. Then, from (63) we see that the roots k_j and the functions V, F and χ are invariant but the function Ψ changes the sign. This result is in accordance with the fact that $\boldsymbol{\varphi}$ is a pseudovector. In the case of achiral materials we have $b_1 = 0$ and $\eta_j = 0$, and from (71) we find that $B = 0$, $A_{2\alpha} = 0$. It follows from (56) and (70) that in this case the microrotation vector is equal to zero.

6. Conclusions

The results established in this paper can be summarized as follows:

(a) We establish a strain gradient theory of Cosserat thermoelasticity without energy dissipation.The theory is capable of predicting a finite speed of heat propagation.

(b) We derive the constitutive equations for isotropic chiral materials. In this theory, in contrast with the classical Cosserat thermoelasticity, a thermal field produces a microrotation of the particles. The thermal field is influenced by the displacement and microrotation fields even in the equilibrium theory.

(c)We present existence and uniqueness results in the dynamic theory.

(d) In context of the equilibrium theory we study the effects of a concentrated heat source in an unbounded chiral isotropic material.We investigate the influence of chiral thermoelastic coefficients on the displacement, microrotation and thermal fields.

Acknowledgments

We express our gratitude to the referees for their helpful suggestions.

R.Q. is supported by the Project "Análisis Matemático de las Ecuaciones en Derivada Parciales de la Termomecánica" (MTM2013-42004-P) of the Spanish Ministry of Economy and Competitiveness. This work was initiated meanwhile the first author D. I. visited the Department of Mathematics of the Polytechnic University of Catalonia.

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List of Symbols

A_{ijkl}, B_{ijkl}, C_{ijkl}......................Micropolar constitutive coefficients
$\mathcal{A}$..Matrix operator of the field equations
B..Body
∂B..Boundary of B

C_j .... Chiral constitutive coefficients
C .... Union of edges
D_{rsijk}, E_{rsijk}, F_{rsijk} .... Dipolar constitutive coefficients
D_j .... Surface gradient
D .... Normal derivative
$\mathcal{D}$ .... Domain of the operator $\mathcal{A}$
E .... Total energy
F, Ψ, χ .... Scalar fields in the representation of solution
G .... The matrix of body sources
G_{ijk}, a_{ij} .... Stress-temperature tensors
H_{ijk}, b_{ij} .... Couple stress-temperature tensors
I_{jk} .... Microinertia tensor
I .... Coefficient of inertia
Id .... Unit tensor
K_{ij}, b_j .... Conductivity coefficients for anisotropic solids
L_{ijkl}, c_{ijk} .... Dipolar stress-temperature tensors
P .... Part of B
P_j .... Surface traction
$\tilde{P}_j$ .... Prescribed surface traction
Q_j .... Surface traction associated to C
$\tilde{Q}_j$ .... Prescribed traction on edge
QT_0 .... Heat supply per unit of initial volume
R_j .... Dipolar surface traction
$\tilde{R}_j$ .... Prescribed dipolar surface traction
S .... Heat supply
T .... Temperature difference
T_0 .... Reference temperature
V .... Scalar field in the representation of solution
W .... Quadratic part of the internal energy
$W_0^{2,2}$, $W^{4,2}$, L^2, $\mathcal{Z}$ .... Hilbert spaces
M_j, N_j, P_j^*, R_j^*, S_j, T_j, U_j
$\bar{V}$, $\bar{Q}$, $\bar{W}$, L, X, Y .... Elements of the matrix $\mathcal{A}$
aT_0 .... Specific heat
b .... Stress-temperature modulus
b_0, b_1, d_j .... Constitutive coefficients for isotropic solids
$\tilde{d}_j$ .... Prescribed normal derivative of displacement
e .... Internal energy
e_{ij} .... Strain tensor
f_j .... Body force
g_j .... Body couple
k, ξ_1, ξ_2 .... Conductivity coefficients for isotropic solids
m_j .... Surface couple
$\tilde{m}_j$ .... Prescribed surface couple
m_{ji} .... Couple stress tensor
n_j .... Outward unit normal vector on ∂B

q..Heat flux
q_j..Heat flux vector
$\tilde{q}$..Prescribed heat flux on boundary
r..Distance between the points $\mathbf{x}$ and $\mathbf{y}$
s..External rate of supply of entropy
s_j..Unit vector tangent to C
t..Time
t_j..Classical stress vector
t_{ji}..Stress tensor
u_j..Displacement vector
$\tilde{u}_j$..Prescribed displacement on boundary
u_j^0..Initial displacement
v_j..Velocity
v_j^0..Initial velocity
$\mathbf{x}$, $\mathbf{y}$..Points in space
Λ_r, Π, Ω..Differential operators
Σ_j..Subsets of ∂B
Φ..Internal flux of entropy per unit area
Φ_j..Entropy flux vector
α..Thermal displacement
α_0..Initial thermal displacement
$\bar{\alpha}$, β, γ, κ..Micropolar constitutive coefficients
α_j, β_j..Constitutive coefficients in gradient elasticity
χ^*..Couple-stress temperature modulus
δ_{ij}..Kronecker's delta
ε_{ijk}..Permutation symbol
γ_{ij}..Micropolar strain tensor
η..Entropy density per unit mass
φ_j..Microrotation vector
$\tilde{\varphi}_j$..Prescribed microrotation on boundary
φ_j^0..Initial microrotation vector
κ_{ijk}..Second gradient of displacement
λ, μ..Elastic moduli
μ_{jk}..Double force per unit area
μ_{ijk}..Double stress tensor
ν_1, ν_2, $\bar{\zeta}_1$, $\bar{\zeta}_2$, κ_j, η_j
$\bar{\alpha}_0, \lambda_0, \nu_0, \zeta_0$..Constants dependent on constitutive coefficients
θ..Absolute temperature
ρ..Density in the reference configuration
$\rho^*(\mathcal{A})$..Resolvent of $\mathcal{A}$
σ..Free energy per unit of initial volume
τ..Thermal displacement difference field
τ^0..Initial thermal displacement difference field
$\tilde{\tau}$..Prescribed thermal displacement on boundary
τ_{jk}..Nonlocal stress tensor
ϑ^0..Initial time rate of thermal displacement field

ξ..Internal rate of supply of entropy
ψ..Free energy per unit mass
ψ_j^0..Initial microgyration vector
ω...Element of $\mathcal{Z}$
ω_0..Initial data
ζ_j..Microgyration vector
$\triangle$..Laplacian
grad...Gradient
curl..Curl
div...Divergence
da..Element of area
dv..Element of volume