Single-pushout rewriting in categories of spans I: The general setting

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Abstract. A unifying view of all constructions of pushouts of partial morphisms considered so far in the literature of single-pushout transformation is given in this paper. Pushouts of partial morphisms are studied in an abstract category of *spans* formed out of two distinguished subcategories of the base category, thus generalizing previous studies in single-pushout transformation. Such spans are single pairs of morphisms, instead of equivalence classes, providing then a notion of transformation which is independent of class representatives.

A necessary and sufficient condition for the existence of the pushout of two spans is established which involves properties of the base category, from which the category of spans is derived, as well as properties of the spans themselves. Moreover, a necessary and sufficient condition for single-pushout derivations in a category of spans to subsume double-pushout derivations in the base category is established which only involves properties of the base category.

1 Introduction

The algebraic approach to graph transformation involves as fundamental rewriting step either a double pushout construction of total morphisms [7], or the construction of a single pushout of partial morphisms [10]. As soon as rewriting is studied in an arbitrary category, however, some problems arise.

A partial morphism in an arbitrary category that has all binary pullbacks has been defined in [9, 15] as an equivalence class (up to isomorphism of the central object $\mathbf{D}$) of pairs of morphisms

$$\mathbf{A} \xleftarrow{i} \mathbf{D} \xrightarrow{h} \mathbf{C}$$

with i a monomorphism. Although partial morphisms can be defined in this way for any category having all binary pullbacks, and this kind of definitions *up to isomorphism* follows the general spirit of category theory, however, partial morphisms are most often defined in practice as a concrete such a pair of morphisms but with i a monomorphism belonging to a certain class, for instance the inclusion of some kind of subobject.

Dealing with such equivalence classes and appropriately taking class representatives has proven to be misleading, at least when defining an abstract semantics of algebraic transformation, because class representatives are not, in general, independent. A similar problem has motivated the introduction of the notions of *abstract graph* and *standard graph* in the double-pushout approach to graph transformation [5], and the later focus on categories of *concrete spans* [6].

Given a category $\mathcal{C}$ and two classes of morphisms $\mathcal{M}$ and $\mathcal{H}$ satisfying suitable properties ((S0) and (S1) in Section 2), a $(\mathcal{M},\mathcal{H})$-span is a pair of morphisms $\mathbf{A} \stackrel{m}{\leftarrow} \mathbf{K} \stackrel{h}{\rightarrow} \mathbf{B}$ with $m \in \mathcal{M}$ and $h \in \mathcal{H}$. The aforementioned suitable properties on $\mathcal{M}$ and $\mathcal{H}$ make these spans to be the morphisms of a category $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$ with objects those of $\mathcal{C}$, with identities pairs of identities and with composition of spans carried out through *distinguished pullbacks* as introduced in (S1), and similar to the *choice of pullbacks* used in [6]. As a matter of fact, some categories of partial homomorphisms of partial algebras have already been understood in [2, Sect. 22.1.4] as such categories of spans.

Our interest in categories of spans comes from the observation made to us by M. Löwe in September 1994, in connection to [4], that all constructions of pushouts of partial homomorphisms considered so far in the literature of single-pushout transformation were similar, and that the reason was that all these partial homomorphisms could be understood as spans satisfying a set of suitable conditions. Our first answer to that comment was [11], where a previous, and slightly weaker, version of the main result in this paper (Corollary 11) was presented without proof; see Theorem 1.1 in *loc. cit.*[1].

In the present paper, and its sequel [12], we study in detail the aforementioned similarities. And we do so at two levels. In this paper, we consider categories of spans in an (almost) arbitrary setting: we only impose on $(\mathcal{C},\mathcal{M},\mathcal{H})$ the aforementioned conditions (S0) and (S1), which are needed to define the category of spans $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$ and to understand $\mathcal{M}$ and $\mathcal{H}$ as subcategories of $\mathcal{C}$, and two further conditions: (S2), which asks that $\mathcal{H}$ has all pushouts; and (S3), a technical condition closely related to the *existence of hereditary pushouts* condition in [9], which relates pushouts in $\mathcal{H}$ with distinguished pullbacks. Then:

- We establish a necessary and sufficient condition for the existence of the pushout of two given spans in $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$, when $(\mathcal{C},\mathcal{M},\mathcal{H})$ satisfies conditions (S0) to (S3);
- We prove that if $(\mathcal{C},\mathcal{M},\mathcal{H})$ satisfies conditions (S0) to (S2), then it also satisfies the technical condition (S3) iff the embedding $\mathcal{H} \hookrightarrow \mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$ preserves binary pushouts;
- We establish a necessary and sufficient condition for single-pushout derivations in $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$ to subsume double-pushout derivations in $(\mathcal{C},\mathcal{M},\mathcal{H})$ (see Section 4); and

[1] Actually, that result was wrong, because a certain uniqueness condition was missing in the definition of final triple diagram for two spans given in that paper; compare with Definition 4 in this paper.

– We study single-pushout HLR conditions in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$, when $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfies conditions (S0) to (S3).

And it is worthwhile to mention that, for instance, we do not need to impose that $\mathcal{M}$ be a class of monomorphisms (although, to be honest, we do not know any natural example of triple $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfying (S0) to (S3) and with $\mathcal{M}$ containing non-monomorphic morphisms), although when $\mathcal{M}$ is a class of monomorphisms then some results can be strengthened.

In the second paper [12], referred to henceforth in this paper as Part II, we consider the particular case of $(\mathcal{M}, \mathcal{H})$-spans over a category $\mathcal{C}$ with a forgetful functor over a category of S-sets, for some non-empty set S, with $\mathcal{M}$ a class of morphisms corresponding to inclusions of subobjects.

On the other hand, it must be stressed that the study of spans in $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ generalizes the study of spans in $(\mathcal{C}, \mathcal{M})$ made by [9], in the sense that it allows to take $\mathcal{H}$ a (class of morphisms inducing a) subcategory of $\mathcal{C}$, instead of the whole category, as we do in Part II to present several well-known categories of partial morphisms of total and partial unary algebras as specific examples of concrete spans in $(\mathcal{C}, \mathcal{M}, \mathcal{H})$.

In the sequel, by *pushout* we shall always understand a binary pushout.

2 Basic notions

Let $\mathcal{C}$ be a category, and let $\mathcal{M}$ and $\mathcal{H}$ be two classes of morphisms in $\mathcal{C}$ (whose elements we shall call *$\mathcal{M}$-morphisms* and *$\mathcal{H}$-morphisms*, respectively) satisfying the following condition:

(S0) $\mathcal{M}$ is closed under composition, contains all identities in $\mathcal{C}$, and $g \circ f \in \mathcal{M}$ and $g \in \mathcal{M}$ implies $f \in \mathcal{M}$, for every f, g. The same properties hold for $\mathcal{H}$.

This property allows to interpret $\mathcal{M}$ and $\mathcal{H}$ as subcategories of $\mathcal{C}$ with its same class of objects, and such that their isomorphisms are the isomorphisms in $\mathcal{C}$ belonging to them.

Definition 1. A $(\mathcal{M}, \mathcal{H})$-*span* is a pair of morphisms $\mathbf{A} \stackrel{m}{\leftarrow} \mathbf{K} \stackrel{h}{\rightarrow} \mathbf{B}$ with $m \in \mathcal{M}$ and $h \in \mathcal{H}$.

Such a $(\mathcal{M}, \mathcal{H})$-span shall be denoted by $(m, \mathbf{K}, h) : \mathbf{A} \to \mathbf{B}$, emphasizing that it is understood as a morphism from $\mathbf{A}$ to $\mathbf{B}$.

In order to ensure a well-defined composition of spans, we shall assume that the triple $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfies also the following property.

(S1) A class $\mathcal{PB}$ of pullback squares in $\mathcal{C}$

$$\begin{tikzcd} \mathbf{D} \arrow[r, "m'"] \arrow[d, "h'"'] & \mathbf{C} \arrow[d, "h"] \\ \mathbf{A} \arrow[r, "m"'] & \mathbf{B} \end{tikzcd}$$

with $m, m' \in \mathcal{M}$ and $h, h' \in \mathcal{H}$, can be fixed such that:

(S1a) For every $m : \mathbf{A} \to \mathbf{B}$ in $\mathcal{M}$ and $h : \mathbf{C} \to \mathbf{B}$ in $\mathcal{H}$ there exists one, and only one, pullback square for them in the class $\mathcal{PB}$ (referred to henceforth as their *distinguished pullback square*, and the corresponding cone $(\mathbf{D}, h' : \mathbf{D} \to \mathbf{A}, m' : \mathbf{D} \to \mathbf{C})$ as their *distinguished pullback*).

(S1b) The distinguished pullback square of any morphism $f : \mathbf{A} \to \mathbf{B}$ either in $\mathcal{M}$ or in $\mathcal{H}$, and the identity $\mathrm{Id}_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}$ (considered as a morphism in $\mathcal{H}$ or in $\mathcal{M}$, respectively) is

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\mathrm{Id}_{\mathbf{A}}"] \arrow[d, "f"'] & \mathbf{A} \arrow[d, "f"] \\ \mathbf{B} \arrow[r, "\mathrm{Id}_{\mathbf{B}}"'] & \mathbf{B} \end{tikzcd}$$

(S1c) Distinguished pullback squares satisfy the standard compositional properties of pullbacks: in every commutative diagram of the forms

$$\begin{tikzcd} \mathbf{A} \arrow[r, "h_1"] \arrow[d, "m"'] \arrow[dr, phantom, "(1)"] & \mathbf{B} \arrow[r, "h_2"] \arrow[d, "m'"] \arrow[dr, phantom, "(2)"] & \mathbf{C} \arrow[d, "m''"] \\ \mathbf{A}' \arrow[r, "h_1'"'] & \mathbf{B}' \arrow[r, "h_2'"'] & \mathbf{C}' \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} \mathbf{A} \arrow[r, "m_1"] \arrow[d, "h"'] \arrow[dr, phantom, "(1)"] & \mathbf{B} \arrow[r, "m_2"] \arrow[d, "h'"] \arrow[dr, phantom, "(2)"] & \mathbf{C} \arrow[d, "h''"] \\ \mathbf{A}' \arrow[r, "m_1'"'] & \mathbf{B}' \arrow[r, "m_2'"'] & \mathbf{C}' \end{tikzcd}$$

$(m, m', m'', m_1, m_2, m_1', m_2' \in \mathcal{M},\ h_1, h_2, h_1', h_2', h, h', h'' \in \mathcal{H})$ if (1) and (2) are distinguished pullback squares then (1)+(2) is also a distinguished pullback square, and if (1)+(2) and (2) are distinguished pullback squares then (1) is also a distinguished pullback square.

Such a family $\mathcal{PB}$ of pullbacks is called in [6] a *choice of pullbacks*.

We shall assume in the sequel, without any further mention, that $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfies properties (S0) and (S1), the latter relative to a fixed class $\mathcal{PB}$ of distinguished pullbacks.

Definition 2. Given two $(\mathcal{M}, \mathcal{H})$-spans $(m, \mathbf{K}, h) : \mathbf{A} \to \mathbf{B}$ and $(m', \mathbf{K}', h') : \mathbf{B} \to \mathbf{C}$, their *composition* $(m', \mathbf{K}', h') \circ (m, \mathbf{K}, h)$ is the $(\mathcal{M}, \mathcal{H})$-span $(m \circ \widetilde{m}', \mathbf{K}'', h' \circ \widetilde{h}) : \mathbf{A} \to \mathbf{C}$, where $(\mathbf{K}'', \widetilde{m}' : \mathbf{K}'' \to \mathbf{K}, \widetilde{h} : \mathbf{K}'' \to \mathbf{K}')$ is the distinguished pullback of $m' : \mathbf{K}' \to \mathbf{B}$ and $h : \mathbf{K} \to \mathbf{B}$.

$$\begin{tikzcd} \mathbf{A} & \mathbf{K} \arrow[l, "m"'] \arrow[d, "h"'] & \mathbf{K}'' \arrow[l, "\widetilde{m}'"'] \arrow[d, "\widetilde{h}"] \\ & \mathbf{B} & \mathbf{K}' \arrow[l, "m'"] \arrow[d, "h'"] \\ & & \mathbf{C} \end{tikzcd}$$

The objects of $\mathcal{C}$ together with the $(\mathcal{M},\mathcal{H})$-spans as morphisms form a category (taking Definition 2 for composition and $\mathbf{A} \xleftarrow{\mathrm{Id}_{\mathbf{A}}} \mathbf{A} \xrightarrow{\mathrm{Id}_{\mathbf{A}}} \mathbf{A}$ as the identity on an object $\mathbf{A}$), that henceforth shall be denoted $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$[2].

The category $\mathcal{H}$ can be embedded as a subcategory of $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$, by associating to every morphism $h : \mathbf{A} \to \mathbf{B}$ the $(\mathcal{M},\mathcal{H})$-span $\mathbf{A} \xleftarrow{\mathrm{Id}_{\mathbf{A}}} \mathbf{A} \xrightarrow{h} \mathbf{B}$.

Remark 3. Given spans $(m_1, \mathbf{K}_1, h_1) : \mathbf{A} \to \mathbf{B}$, $(m_2, \mathbf{K}_2, h_2) : \mathbf{A} \to \mathbf{C}$, $(m_3, \mathbf{K}_3, h_3) : \mathbf{B} \to \mathbf{D}$ and $(m_4, \mathbf{K}_4, h_4) : \mathbf{C} \to \mathbf{D}$, notice that $(m_3, \mathbf{K}_3, h_3) \circ (m_1, \mathbf{K}_1, h_1) = (m_4, \mathbf{K}_4, h_4) \circ (m_2, \mathbf{K}_2, h_2)$ iff there exists a diagram in $\mathcal{C}$

$$
\begin{tikzcd}
\mathbf{A} \arrow[dr, phantom, "(1)"] & \mathbf{K}_1 \arrow[l, "m_1"'] \arrow[r, "h_1"] \arrow[dr, phantom, "(2)"] & \mathbf{B} \\
\mathbf{K}_2 \arrow[u, "m_2"] \arrow[d, "h_2"'] \arrow[dr, phantom, "(3)"] & \mathbf{K}' \arrow[l, "\widetilde{m}_1"] \arrow[r, "\widetilde{h}_1"'] \arrow[u, "\widetilde{m}_2"'] \arrow[d, "\widetilde{h}_2"] \arrow[dr, phantom, "(4)"] & \mathbf{K}_3 \arrow[u, "m_3"'] \arrow[d, "h_3"] \\
\mathbf{C} & \mathbf{K}_4 \arrow[l, "m_4"] \arrow[r, "h_4"'] & \mathbf{D}
\end{tikzcd}
$$

where squares (1) and (4) commute and squares (2) and (3) are distinguished pullback squares.

This observation motivates the following definition.

Definition 4. Given two $(\mathcal{M},\mathcal{H})$-spans $(m_1, \mathbf{K}_1, h_1) : \mathbf{A} \to \mathbf{B}$ and $(m_2, \mathbf{K}_2, h_2) : \mathbf{A} \to \mathbf{C}$, a *triple diagram* for them is a diagram in $\mathcal{C}$

$$
\begin{tikzcd}
\mathbf{A} \arrow[dr, phantom, "(1)"] & \mathbf{K}_1 \arrow[l, "m_1"'] \arrow[r, "h_1"] \arrow[dr, phantom, "(2)"] & \mathbf{B} \\
\mathbf{K}_2 \arrow[u, "m_2"] \arrow[d, "h_2"'] \arrow[dr, phantom, "(3)"] & \mathbf{K}' \arrow[l, "\widetilde{m}_1"] \arrow[r, "\widetilde{h}_1"'] \arrow[u, "\widetilde{m}_2"'] \arrow[d, "\widetilde{h}_2"] & \mathbf{K}_3 \arrow[u, "m_3"'] \\
\mathbf{C} & \mathbf{K}_4 \arrow[l, "m_4"] &
\end{tikzcd}
\tag{$*$}
$$

where $m_3, m_4 \in \mathcal{M}$, square (1) is commutative, and squares (2) and (3) are distinguished pullback squares. In particular, in this diagram $\widetilde{m}_1, \widetilde{m}_2 \in \mathcal{M}$ and $\widetilde{h}_1, \widetilde{h}_2 \in \mathcal{H}$.

Such a triple diagram $(*)$ is *final* when for every other triple diagram for

[2] As a matter of fact, composition in this category (and therefore the category itself) depends on the chosen family $\mathcal{PB}$ of distinguished pullbacks, although this fact is not reflected by the notation $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$.

$(m_1, \mathbf{K}_1, h_1)$ and $(m_2, \mathbf{K}_2, h_2)$

$$\begin{tikzcd}
\mathbf{A} & \mathbf{K}_1 \arrow[l, "m_1"'] \arrow[r, "h_1"] & \mathbf{B} \\
\mathbf{K}_2 \arrow[u, "m_2"] \arrow[d, "h_2"'] & \widehat{\mathbf{K}}' \arrow[l, "\widehat{m}_1"] \arrow[r, "\widehat{h}_1"'] \arrow[u, "\widehat{m}_2"'] \arrow[d, "\widehat{h}_2"] & \widehat{\mathbf{K}}_3 \arrow[u, "\widehat{m}_3"'] \\
\mathbf{C} & \widehat{\mathbf{K}}_4 \arrow[l, "\widehat{m}_4"]
\end{tikzcd}$$

there exist unique $\mathcal{M}$-morphisms

$$f : \widehat{\mathbf{K}}' \to \mathbf{K}',\ f_3 : \widehat{\mathbf{K}}_3 \to \mathbf{K}_3,\ f_4 : \widehat{\mathbf{K}}_4 \to \mathbf{K}_4$$

such that

$$\begin{gathered}
\widehat{m}_1 = \widetilde{m}_1 \circ f, \quad \widehat{m}_2 = \widetilde{m}_2 \circ f, \quad \widehat{m}_3 = m_3 \circ f_3, \\
\widehat{m}_4 = m_4 \circ f_4, \quad \widetilde{h}_1 \circ f = f_3 \circ \widehat{h}_1, \quad \widetilde{h}_2 \circ f = f_4 \circ \widehat{h}_2
\end{gathered}$$

i.e., making commutative the whole diagram

$$\begin{tikzcd}
\mathbf{A} & \mathbf{K}_1 \arrow[l, "m_1"'] \arrow[rr, "h_1"] & & \mathbf{B} & \\
\mathbf{K}_2 \arrow[u, "m_2"] \arrow[dd, "h_2"'] & \mathbf{K}' \arrow[l, "\widetilde{m}_1"'] \arrow[u, "\widetilde{m}_2"] \arrow[rr, "\widetilde{h}_1"] \arrow[dd, "\widetilde{h}_2"'] & & \mathbf{K}_3 \arrow[u, "m_3"'] & \\
 & & \widehat{\mathbf{K}}' \arrow[uul, "\widehat{m}_2"'] \arrow[ull, "\widehat{m}_1"] \arrow[ul, dotted, "f"] \arrow[rr, "\widehat{h}_1"] \arrow[dd, "\widehat{h}_2"'] & & \widehat{\mathbf{K}}_3 \arrow[uul, "\widehat{m}_3"'] \arrow[ul, dotted, "f_3"] \\
\mathbf{C} & \mathbf{K}_4 \arrow[l, "m_4"] & & & \\
 & & \widehat{\mathbf{K}}_4 \arrow[ull, "\widehat{m}_4"] \arrow[ul, dotted, "f_4"] & &
\end{tikzcd}$$

Notice that (S1c) implies that the two squares in this diagram involving dotted arrows are distinguished pullback squares.

Remark 5. If

$$
\begin{tikzcd}
\mathbf{A} & \mathbf{K}_1 \arrow[l, "m_1"'] \arrow[r, "h_1"] & \mathbf{B} \\
\mathbf{K}_2 \arrow[u, "m_2"] \arrow[d, "h_2"'] \arrow[ur, phantom, "(1)"] & \mathbf{K}' \arrow[l, "\widetilde{m}_1"] \arrow[r, "\widetilde{h}_1"'] \arrow[u, "\widetilde{m}_2"'] \arrow[d, "\widetilde{h}_2"] \arrow[ur, phantom, "(2)"] & \mathbf{K}_3 \arrow[u, "m_3"'] \\
\mathbf{C} & \mathbf{K}_4 \arrow[l, "m_4"] \arrow[ul, phantom, "(3)"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{A} & \mathbf{K}_1 \arrow[l, "m_1"'] \arrow[r, "h_1"] & \mathbf{B} \\
\mathbf{K}_2 \arrow[u, "m_2"] \arrow[d, "h_2"'] & \widehat{\mathbf{K}}' \arrow[l, "\widehat{m}_1"] \arrow[r, "\widehat{h}_1"'] \arrow[u, "\widehat{m}_2"'] \arrow[d, "\widehat{h}_2"] & \widehat{\mathbf{K}}_3 \arrow[u, "\widehat{m}_3"'] \\
\mathbf{C} & \widehat{\mathbf{K}}_4 \arrow[l, "\widehat{m}_4"] &
\end{tikzcd}
$$

are two final triple diagrams for $(m_1, \mathbf{K}_1, h_1)$ and $(m_2, \mathbf{K}_2, h_2)$ then there exist three (unique) isomorphisms in $\mathcal{M}$

$$f : \widehat{\mathbf{K}}' \to \mathbf{K}', \;\; f_3 : \widehat{\mathbf{K}}_3 \to \mathbf{K}_3, \;\; f_4 : \widehat{\mathbf{K}}_4 \to \mathbf{K}_4$$

such that

$$
\begin{gathered}
\widehat{m}_1 = \widetilde{m}_1 \circ f, \quad \widehat{m}_2 = \widetilde{m}_2 \circ f, \quad \widehat{m}_3 = m_3 \circ f_3, \\
\widehat{m}_4 = m_4 \circ f_4, \quad \widetilde{h}_1 \circ f = f_3 \circ \widehat{h}_1, \quad \widetilde{h}_2 \circ f = f_4 \circ \widehat{h}_2
\end{gathered}
$$

Thus, if $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfies the following additional property

(S0m) The only isomorphisms in $\mathcal{M}$ are the identities

then every pair $(m_1, \mathbf{K}_1, h_1) : \mathbf{A} \to \mathbf{B}$ and $(m_2, \mathbf{K}_2, h_2) : \mathbf{A} \to \mathbf{C}$ of $(\mathcal{M}, \mathcal{H})$-spans has at most one final triple diagram in $\mathcal{C}$. All examples considered in Part II satisfy (S0m). □

In order to study the existence of pushouts in a category of spans, we shall use the following conditions on $(\mathcal{C}, \mathcal{M}, \mathcal{H})$.

(S2) $\mathcal{H}$ has all pushouts.

(S3) For every diagram in $\mathcal{C}$

$$
\begin{tikzcd}
\mathbf{A} \arrow[rr, "h_1"] \arrow[dd, "h_2"'] & & \mathbf{B} \arrow[dd, "h_3"] & \\
& \mathbf{A}' \arrow[ul, "m_1"] \arrow[rr, "h_1'"] \arrow[dd, "h_2'"] & & \mathbf{B}' \arrow[ul, "m_2"'] \arrow[dd, "h_3'"] \\
\mathbf{C} \arrow[rr, "h_4"'] & & \mathbf{D} & \\
& \mathbf{C}' \arrow[ul, "m_3"] \arrow[rr, "h_4'"'] & & \mathbf{D}'
\end{tikzcd}
\qquad (**)
$$

where $m_1, m_2, m_3 \in \mathcal{M}$, the back face (that is, the one with objects $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$) is a pushout square in $\mathcal{H}$, and the top and left-hand side faces are distinguished pullback squares, the following two properties are satisfied:

(S3a) If the front face in $(**)$ is a pushout square in $\mathcal{H}$ then there exist a unique isomorphism $h : \mathbf{D}' \to \widehat{\mathbf{D}}$ in $\mathcal{H}$ and a unique $\mathcal{M}$-morphism $m : \widehat{\mathbf{D}} \to \mathbf{D}$ such that the following squares are distinguished pullback squares:

$$\begin{tikzcd} \mathbf{B}' \arrow[r, "m_2"] \arrow[d, "h\circ h_3'"'] & \mathbf{B} \arrow[d, "h_3"] \\ \widehat{\mathbf{D}} \arrow[r, "m"'] & \mathbf{D} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{C}' \arrow[r, "m_3"] \arrow[d, "h\circ h_4'"'] & \mathbf{C} \arrow[d, "h_4"] \\ \widehat{\mathbf{D}} \arrow[r, "m"'] & \mathbf{D} \end{tikzcd}$$

In particular, if we add the arrow $m \circ h : \mathbf{D}' \to \mathbf{D}$ to $(**)$ then we obtain a commutative diagram.

(S3b) If there exists a $\mathcal{M}$-morphism $f : \mathbf{D}' \to \mathbf{D}$ such that adding it to $(**)$ the bottom and right-hand side faces in the resulting cube are distinguished pullback squares, then the front face in $(**)$ is a pushout square in $\mathcal{H}$.

This condition (S3) plays in this paper a role similar to that of the *existence of hereditary pushouts* condition in [9].

Remark 6. If the hypothesis of condition (S3b) holds, then the uniqueness and compositional properties of distinguished pullbacks imposed in (S1) guarantee *a priori* that the front face in $(**)$ is commutative. $\Box$

We shall discuss a set of conditions alternative to (S2) and (S3) (involving a family of distinguished pushouts, as in [11]) in Section 6.

A basic example of a category of spans based on a triple $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfying (S0) to (S3), and a paradigm of the type of categories that we shall consider in Part II, is the category of S-sets, for a non-empty set S, with partial mappings as morphisms.

Example 7. Let S-**Sets** be the category of S-sets, for some non-empty set of sorts S, let $\mathcal{I}$ denote the class of all inclusions of S-sets, and $\mathcal{H}$ the class of all mappings of S-sets (in this case, the category corresponding to $\mathcal{H}$ is S-**Sets** itself). The triple $(S\text{-}\mathbf{Sets}, \mathcal{I}, \mathcal{H})$ clearly satisfies (S0).

Given an inclusion $i : A \hookrightarrow B$ and a mapping $h : C \to B$ of S-sets, we call

$$\begin{tikzcd} h^{-1}(A) \arrow[r, "j"] \arrow[d, "h'"'] & C \arrow[d, "h"] \\ A \arrow[r, "i"'] & B \end{tikzcd}$$

their *distinguished pullback square* in S-**Sets**, where $j : h^{-1}(A) \hookrightarrow C$ stands for the inclusion of $h^{-1}(A)$ into C as a subset and $h' : h^{-1}(A) \to A$ stands for the restriction of h to $h^{-1}(A)$, considered with target set A. It is not difficult to check that $(S\text{-}\mathbf{Sets}, \mathcal{I}, \mathcal{H})$ satisfies condition (S1) relative to the class $\mathcal{PB}$ of all these distinguished pullback squares.

And given two mappings $f : K \to A$ and $g : K \to B$ of S-sets, the cocone

$$((A+B)/_{\theta(K)}, \tilde{g} : A \to (A+B)/_{\theta(K)}, \tilde{f} : B \to (A+B)/_{\theta(K)})$$

—where $A + B$ stands for the disjoint union

$$((A_s \times \{1\}) \cup (B_s \times \{2\}))_{s \in S},$$

$\theta(K)$ denotes the least equivalence relation on $A + B$ containing

$$\{((f_s(x), 1), (g_s(x), 2)) \mid x \in K_s\}, \qquad s \in S,$$

and $\tilde{g} : A \to (A+B)/_{\theta(K)}$ and $\tilde{f} : B \to (A+B)/_{\theta(K)}$ are given respectively by the mappings

$$\begin{array}{cc} \tilde{g}_s : A_s \to (A_s + B_s)/_{\theta(K)_s} & \tilde{f}_s : B_s \to (A_s + B_s)/_{\theta(K)_s} \\ a \mapsto [(a,1)]_{\theta(K)_s} & b \mapsto [(b,2)]_{\theta(K)_s} \end{array}$$

for every $s \in S$— is a pushout for them in S-**Sets**, which we shall call their *distinguished pushout*. Then, $(S\text{-}\mathbf{Sets}, \mathcal{I}, \mathcal{H})$ also satisfies (S2).

It turns out that $(S\text{-}\mathbf{Sets}, \mathcal{I}, \mathcal{H})$ satisfies also (S3). This will be a key result for Part II and its proof is quite elementary; we furnish it in an Appendix at the end of this paper.

The corresponding category of spans $\mathcal{S}(S\text{-}\mathbf{Sets}, \mathcal{I}, \mathcal{H})$ is (isomorphic to) the category of S-sets with partial mappings as morphisms. □

3 Pushouts in categories of spans

In this section we establish a necessary and sufficient condition (given by Corollary 11) for the existence of the pushout in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$ of two $(\mathcal{M}, \mathcal{H})$-spans, when $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfies conditions (S0) through (S3) in the previous section. We also discuss the relevance of (S3), the most artificial condition in (S0) to (S3), for this result.

Proposition 8. *Let $\mathcal{C}$ be a category and $\mathcal{M}$ and $\mathcal{H}$ two classes of morphisms in $\mathcal{C}$ such that $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfies conditions (S0), (S1), (S2) and (S3a).*

If two $(\mathcal{M}, \mathcal{H})$-spans $(m_1, \mathbf{K}_1, h_1) : \mathbf{A} \to \mathbf{B}$ and $(m_2, \mathbf{K}_2, h_2) : \mathbf{A} \to \mathbf{C}$ have a pushout in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$ then they have a final triple diagram in $\mathcal{C}$, and every pushout for them has the form

$$(\mathbf{D}, (m_3, \mathbf{K}_3, h_3) : \mathbf{B} \to \mathbf{D}, (m_4, \mathbf{K}_4, h_4) : \mathbf{C} \to \mathbf{D})$$

where

$$\begin{tikzcd}
\mathbf{A} \arrow[dr, phantom, "(1)"] & \mathbf{K}_1 \arrow[l, "m_1"'] \arrow[r, "h_1"] \arrow[dr, phantom, "(2)"] & \mathbf{B} \\
\mathbf{K}_2 \arrow[u, "m_2"] \arrow[d, "h_2"'] \arrow[dr, phantom, "(3)"] & \mathbf{K}' \arrow[l, "\widetilde{m}_1"] \arrow[r, "\widetilde{h}_1"'] \arrow[u, "\widetilde{m}_2"'] \arrow[d, "\widetilde{h}_2"] \arrow[dr, phantom, "(4)"] & \mathbf{K}_3 \arrow[u, "m_3"'] \arrow[d, "h_3"] \\
\mathbf{C} & \mathbf{K}_4 \arrow[l, "m_4"] \arrow[r, "h_4"'] & \mathbf{D}
\end{tikzcd}$$

is a diagram in $\mathcal{C}$ such that squares (1), (2) *and* (3) *form a final triple diagram for* $(m_1, \mathbf{K}_1, h_1)$ *and* $(m_2, \mathbf{K}_2, h_2)$, *and* (4) *is a pushout square in $\mathcal{H}$.*

Proof. Let $(\mathbf{D}, (m_3, \mathbf{K}_3, h_3) : \mathbf{B} \to \mathbf{D}, (m_4, \mathbf{K}_4, h_4) : \mathbf{C} \to \mathbf{D})$ be a pushout of $(m_1, \mathbf{K}_1, h_1) : \mathbf{A} \to \mathbf{B}$ and $(m_2, \mathbf{K}_2, h_2) : \mathbf{A} \to \mathbf{C}$ in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$, and assume that it is given by the diagram in $\mathcal{C}$

$$\begin{tikzcd}
\mathbf{A} \arrow[dr, phantom, "(1)"] & \mathbf{K}_1 \arrow[l, "m_1"'] \arrow[r, "h_1"] \arrow[dr, phantom, "(2)"] & \mathbf{B} \\
\mathbf{K}_2 \arrow[u, "m_2"] \arrow[d, "h_2"'] \arrow[dr, phantom, "(3)"] & \mathbf{K}' \arrow[l, "\widetilde{m}_1"] \arrow[r, "\widetilde{h}_1"'] \arrow[u, "\widetilde{m}_2"'] \arrow[d, "\widetilde{h}_2"] \arrow[dr, phantom, "(4)"] & \mathbf{K}_3 \arrow[u, "m_3"'] \arrow[d, "h_3"] \\
\mathbf{C} & \mathbf{K}_4 \arrow[l, "m_4"] \arrow[r, "h_4"'] & \mathbf{D}
\end{tikzcd}$$

where (2) and (3) are distingushed pullback squares and (1) and (2) are commutative squares (see Remark 3). We want to prove that (1), (2) and (3) form a final triple diagram for $(m_1, \mathbf{K}_1, h_1)$ and $(m_2, \mathbf{K}_2, h_2)$, and that (4) is a pushout square in $\mathcal{H}$.

We shall prove the latter (i.e., that (4) is a pushout square in $\mathcal{H}$) first. To do that, let

$$\begin{tikzcd}
\mathbf{K}' \arrow[r, "\widetilde{h}_1"] \arrow[d, "\widetilde{h}_2"'] & \mathbf{K}_3 \arrow[d, "h'_3"] \\
\mathbf{K}_4 \arrow[r, "h'_4"'] & \mathbf{H}
\end{tikzcd}$$

be a pushout in $\mathcal{H}$ of the $\mathcal{H}$-morphisms $\widetilde{h}_1$ and $\widetilde{h}_2$. Then on the one hand there exists a unique $\mathcal{H}$-morphism $h : \mathbf{H} \to \mathbf{D}$ such that $h_3 = h \circ h'_3$ and $h_4 = h \circ h'_4$, which entails that $(m_3, \mathbf{K}_3, h_3) = (\mathrm{Id}_{\mathbf{H}}, \mathbf{H}, h) \circ (m_3, \mathbf{K}_3, h'_3)$ and $(m_4, \mathbf{K}_4, h_4) = (\mathrm{Id}_{\mathbf{H}}, \mathbf{H}, h) \circ (m_4, \mathbf{K}_4, h'_4)$. And on the other hand there exists a unique span $(m_0, \mathbf{K}_0, h_0) : \mathbf{D} \to \mathbf{H}$ such that

$$(m_3, \mathbf{K}_3, h'_3) = (m_0, \mathbf{K}_0, h_0) \circ (m_3, \mathbf{K}_3, h_3),$$
$$(m_4, \mathbf{K}_4, h'_4) = (m_0, \mathbf{K}_0, h_0) \circ (m_4, \mathbf{K}_4, h_4)$$

By the universal property of the given pushout in $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$, it turns out that $(\mathrm{Id}_{\mathbf{H}},\mathbf{H},h)\circ(m_0,\mathbf{K}_0,h_0)=(\mathrm{Id}_{\mathbf{D}},\mathbf{D},\mathrm{Id}_{\mathbf{D}})$, and the explicit computation of this composition shows then that $\mathbf{K}_0=\mathbf{D}$, $m_0=\mathrm{Id}_{\mathbf{D}}$ and $h\circ h_0=\mathrm{Id}_{\mathbf{D}}$. But then, the way $(m_0,\mathbf{K}_0,h_0)$ was defined implies that $h_0\circ h_3=h_3'$ and $h_0\circ h_4=h_4'$, so that (by the universal property of pushouts in $\mathcal{H}$) $h_0\circ h=\mathrm{Id}_{\mathbf{H}}$. Therefore, h is an isomorphism and (4) is indeed a pushout square in $\mathcal{H}$.

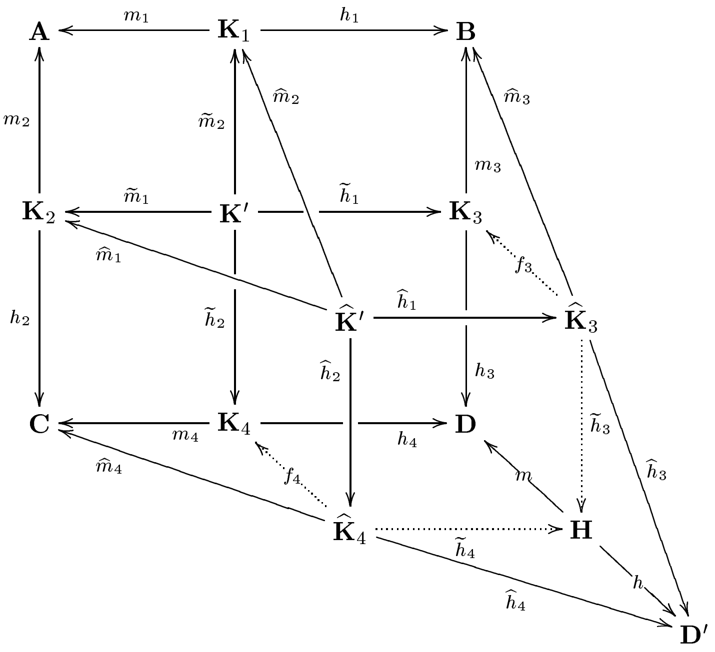
Let us prove now that (1), (2) and (3) in the diagram above form a final triple diagram. Let

$$\begin{tikzcd}
\mathbf{A} & \mathbf{K}_1 \arrow[l, "m_1"'] \arrow[r, "h_1"] & \mathbf{B} \\
\mathbf{K}_2 \arrow[u, "m_2"] \arrow[d, "h_2"'] & \widehat{\mathbf{K}}' \arrow[l, "\widehat{m}_1"] \arrow[r, "\widehat{h}_1"'] \arrow[u, "\widehat{m}_2"'] \arrow[d, "\widehat{h}_2"] & \widehat{\mathbf{K}}_3 \arrow[u, "\widehat{m}_3"'] \arrow[d, "\widehat{h}_3"] \\
\mathbf{C} & \widehat{\mathbf{K}}_4 \arrow[l, "\widehat{m}_4"] \arrow[r, "\widehat{h}_4"'] & \mathbf{D}'
\end{tikzcd}$$

be any triple diagram in $\mathcal{C}$ completed with a pushout in $\mathcal{H}$. By the universal property of pushouts in $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$, there exists a unique $(\mathcal{M},\mathcal{H})$-span $(m,\mathbf{H},h):\mathbf{D}\to\mathbf{D}'$ such that

$$(\widehat{m}_3,\widehat{\mathbf{K}}_3,\widehat{h}_3)=(m,\mathbf{H},h)\circ(m_3,\mathbf{K}_3,h_3),\quad (\widehat{m}_4,\widehat{\mathbf{K}}_4,\widehat{h}_4)=(m,\mathbf{H},h)\circ(m_4,\mathbf{K}_4,h_4)$$

and therefore there exist morphisms $f_3:\widehat{\mathbf{K}}_3\to\mathbf{K}_3$, $f_4:\widehat{\mathbf{K}}_4\to\mathbf{K}_4$ in $\mathcal{M}$ and $\widetilde{h}_3:\widehat{\mathbf{K}}_3\to\mathbf{H}$, $\widetilde{h}_4:\widehat{\mathbf{K}}_4\to\mathbf{H}$ in $\mathcal{H}$ making commutative the diagrams in the bottom and right-hand side faces in the following big diagram, and the squares (involving dotted arrows) in these faces to be distinguished pullback squares.



By the universal property of the two distinguished pullbacks in the back face of this diagram (those having $\mathbf{K}'$ as pullback object), there exist $\mathcal{M}$-morphisms $f_1, f_1' : \widehat{\mathbf{K}}' \to \mathbf{K}'$ such that, respectively, $\widehat{m}_2 = \widetilde{m}_2 \circ f_1$, $f_3 \circ \widehat{h}_1 = \widetilde{h}_1 \circ f_1$, and $\widehat{m}_1 = \widetilde{m}_1 \circ f_1'$ and $f_4 \circ \widehat{h}_2 = \widetilde{h}_2 \circ f_1'$. Let us prove that $f_1 = f_1'$. Consider the following commutative diagram:

$$\begin{tikzcd}
\widehat{\mathbf{K}}' \arrow[r, "\widehat{h}_1"] \arrow[d, "f_1"'] \arrow[dr, phantom, "(1)"] & \widehat{\mathbf{K}}_3 \arrow[r, "\widetilde{h}_3"] \arrow[d, "f_3"] \arrow[dr, phantom, "(3)"] & \mathbf{H} \arrow[d, "m"] \\
\mathbf{K}' \arrow[r, "\widetilde{h}_1"'] \arrow[d, "\widetilde{m}_2"'] \arrow[dr, phantom, "(2)"] & \mathbf{K}_3 \arrow[r, "h_3"'] \arrow[d, "m_3"] & \mathbf{D} \\
\mathbf{K}_1 \arrow[r, "h_1"'] & \mathbf{B} &
\end{tikzcd}$$

In it, $(1)+(2)$, (2) and (3) are distinguished pullback squares, and therefore, by (S1c), $(1)+(3)$ is also a distinguished pullback square. A similar argument shows that

$$\begin{tikzcd}
\widehat{\mathbf{K}}' \arrow[r, "\widetilde{h}_4 \circ \widehat{h}_2"] \arrow[d, "f_1'"] & \mathbf{H} \arrow[d, "m"] \\
\mathbf{K}' \arrow[r, "h_4 \circ \widetilde{h}_2"'] & \mathbf{D}
\end{tikzcd}$$

is also a distinguished pullback square. Since condition (S1) ensures uniqueness of the distinguished pullback of m and $h_3 \circ \widetilde{h}_1 = h_4 \circ \widetilde{h}_2$, it must happen $f_1' = f_1$ (and $\widetilde{h}_3 \circ \widehat{h}_1 = \widetilde{h}_4 \circ \widehat{h}_2$).

We have proved so far that there exist $\mathcal{M}$-morphisms $f_1 : \widehat{\mathbf{K}}' \to \mathbf{K}'$, $f_3 : \widehat{\mathbf{K}}_3 \to \mathbf{K}_3$ and $f_4 : \widehat{\mathbf{K}}_4 \to \mathbf{K}_4$ such that

$$\widehat{m}_1 = \widetilde{m}_1 \circ f_1, \quad \widehat{m}_2 = \widetilde{m}_2 \circ f_1, \quad \widehat{m}_3 = m_3 \circ f_3, \quad \widehat{m}_4 = m_4 \circ f_4$$
$$\widetilde{h}_1 \circ f_1 = f_3 \circ \widehat{h}_1, \quad \widetilde{h}_2 \circ f_1 = f_4 \circ \widehat{h}_2$$

We must prove now that they are unique with this property. Assume that there exist some other $\mathcal{M}$-morphisms $f_1' : \widehat{\mathbf{K}}' \to \mathbf{K}'$, $f_3' : \widehat{\mathbf{K}}_3 \to \mathbf{K}_3$ and $f_4' : \widehat{\mathbf{K}}_4 \to \mathbf{K}_4$ such that $\widehat{m}_1 = \widetilde{m}_1 \circ f_1'$, $\widehat{m}_2 = \widetilde{m}_2 \circ f_1'$, $\widehat{m}_3 = m_3 \circ f_3'$, $\widehat{m}_4 = m_4 \circ f_4'$, $\widetilde{h}_1 \circ f_1' = f_3' \circ \widehat{h}_1$, and $\widetilde{h}_2 \circ f_1' = f_4' \circ \widehat{h}_2$. By (S3a), there exist an isomorphism $h_0 : \mathbf{D}' \to \mathbf{D}_0$ in $\mathcal{H}$ and a $\mathcal{M}$-morphism $m_0 : \mathbf{D}_0 \to \mathbf{D}$ such that

$$\begin{tikzcd} \widehat{\mathbf{K}}_3 \arrow[r, "f_3'"] \arrow[d, "h_0 \circ \widehat{h}_3"'] & \mathbf{K}_3 \arrow[d, "h_3"] \\ \mathbf{D}_0 \arrow[r, "m_0"'] & \mathbf{D} \end{tikzcd} \qquad \begin{tikzcd} \widehat{\mathbf{K}}_4 \arrow[r, "f_4'"] \arrow[d, "h_0 \circ \widehat{h}_4"'] & \mathbf{K}_4 \arrow[d, "h_4"] \\ \mathbf{D}_0 \arrow[r, "m_0"'] & \mathbf{D} \end{tikzcd}$$

are distinguished pullback squares. But then

$$(m_0, \mathbf{D}_0, h_0^{-1}) \circ (m_3, \mathbf{K}_3, h_3) = (\widehat{m}_3, \widehat{\mathbf{K}}_3, \widehat{h}_3),$$
$$(m_0, \mathbf{D}_0, h_0^{-1}) \circ (m_4, \mathbf{K}_4, h_4) = (\widehat{m}_4, \widehat{\mathbf{K}}_4, \widehat{h}_4)$$

and by the uniqueness of such a span, we have $(m_0, \mathbf{D}_0, h_0^{-1}) = (m, \mathbf{H}, h)$. In particular, $\mathbf{D}_0 = \mathbf{H}$ and $m_0 = m$ and then, by the uniqueness of distinguished pullbacks, $f_3' = f_3$, $f_4' = f_4$ and (as before) $f_1' = f_1$. $\Box$

Notice that (S3a) is used in the previous proof only to prove the uniqueness of the $\mathcal{M}$-morphisms f_1, f_2, f_3. If m_3 and m_4 are monomorphisms in $\mathcal{M}$, then this uniqueness follows without using (S3a), and in particular the following result also holds.

Corollary 9. *Let $\mathcal{C}$ be a category and $\mathcal{M}$ and $\mathcal{H}$ two classes of morphisms in $\mathcal{C}$ such that $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfies conditions (S0), (S1) and (S2). Assume moreover that all morphisms in $\mathcal{M}$ are monomorphisms in it.*

Then the thesis of Proposition 8 is true: If two $(\mathcal{M}, \mathcal{H})$-spans $(m_1, \mathbf{K}_1, h_1) : \mathbf{A} \to \mathbf{B}$ and $(m_2, \mathbf{K}_2, h_2) : \mathbf{A} \to \mathbf{C}$ have a pushout $(\mathbf{D}, (m_3, \mathbf{K}_3, h_3) : \mathbf{B} \to \mathbf{D}, (m_4, \mathbf{K}_4, h_4) : \mathbf{C} \to \mathbf{D})$ in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$ then it is obtained by completing a final triple diagram for $(m_1, \mathbf{K}_1, h_1)$ and $(m_2, \mathbf{K}_2, h_2)$ with a pushout in $\mathcal{H}$. $\Box$

Next result yields the converse implication for Proposition 8.

Proposition 10. *Let $\mathcal{C}$ be a category and $\mathcal{M}$ and $\mathcal{H}$ two classes of morphisms in $\mathcal{C}$ such that $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfies conditions (S0) to (S3).*

If two $(\mathcal{M},\mathcal{H})$-spans $(m_1,\mathbf{K}_1,h_1):\mathbf{A}\to\mathbf{B}$ and $(m_2,\mathbf{K}_2,h_2):\mathbf{A}\to\mathbf{C}$ have a final triple diagram in $\mathcal{C}$ then they have a pushout in $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$, obtained by completing this final triple diagram with a pushout in $\mathcal{H}$.

Proof. Consider the diagram

$$\begin{tikzcd}
\mathbf{A} & \mathbf{K}_1 \arrow[l, "m_1"'] \arrow[r, "h_1"] & \mathbf{B} \\
\mathbf{K}_2 \arrow[u, "m_2"] \arrow[d, "h_2"'] \arrow[ur, phantom, "(1)"] \arrow[dr, phantom, "(3)"] & \mathbf{K}' \arrow[l, "\widetilde{m}_1"] \arrow[r, "\widetilde{h}_1"'] \arrow[u, "\widetilde{m}_2"'] \arrow[d, "\widetilde{h}_2"] \arrow[ur, phantom, "(2)"] \arrow[dr, phantom, "(4)"] & \mathbf{K}_3 \arrow[u, "m_3"'] \arrow[d, "h_3"] \\
\mathbf{C} & \mathbf{K}_4 \arrow[l, "m_4"] \arrow[r, "h_4"'] & \mathbf{D}
\end{tikzcd}$$

where (1), (2) and (3) form a final triple diagram for $(m_1,\mathbf{K}_1,h_1):\mathbf{A}\to\mathbf{B}$ and $(m_2,\mathbf{K}_2,h_2):\mathbf{A}\to\mathbf{C}$ and (4) is a pushout square in $\mathcal{H}$: in particular $(m_3,\mathbf{K}_3,h_3)\circ(m_1,\mathbf{K}_1,h_1)=(m_4,\mathbf{K}_4,h_4)\circ(m_2,\mathbf{K}_2,h_2)$. We shall prove that this diagram yields a pushout square of $(m_1,\mathbf{K}_1,h_1)$ and $(m_2,\mathbf{K}_2,h_2)$ in $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$.

Let $(\widehat{m}_3,\widehat{\mathbf{K}}_3,\widehat{h}_3):\mathbf{B}\to\mathbf{D}'$ and $(\widehat{m}_4,\widehat{\mathbf{K}}_4,\widehat{h}_4):\mathbf{C}\to\mathbf{D}'$ be two $(\mathcal{M},\mathcal{H})$-spans such that $(\widehat{m}_3,\widehat{\mathbf{K}}_3,\widehat{h}_3)\circ(m_1,\mathbf{K}_1,h_1)=(\widehat{m}_4,\widehat{\mathbf{K}}_4,\widehat{h}_4)\circ(m_2,\mathbf{K}_2,h_2)$. There is then a diagram

$$\begin{tikzcd}
\mathbf{A} & \mathbf{K}_1 \arrow[l, "m_1"'] \arrow[r, "h_1"] & \mathbf{B} \\
\mathbf{K}_2 \arrow[u, "m_2"] \arrow[d, "h_2"'] \arrow[ur, phantom, "(1')"] \arrow[dr, phantom, "(3')"] & \widehat{\mathbf{K}}' \arrow[l, "\widehat{m}_1"] \arrow[r, "\widehat{h}_1"'] \arrow[u, "\widehat{m}_2"'] \arrow[d, "\widehat{h}_2"] \arrow[ur, phantom, "(2')"] \arrow[dr, phantom, "(4')"] & \widehat{\mathbf{K}}_3 \arrow[u, "\widehat{m}_3"'] \arrow[d, "\widehat{h}_3"] \\
\mathbf{C} & \widehat{\mathbf{K}}_4 \arrow[l, "\widehat{m}_4"] \arrow[r, "\widehat{h}_4"'] & \mathbf{D}'
\end{tikzcd}$$

where squares $(1')$ and $(4')$ commute and $(2')$ and $(3')$ are distinguished pullback squares. In particular, squares $(1')$, $(2')$ and $(3')$ form a triple diagram. By the universal property of the final triple diagram, there exist three unique $\mathcal{M}$-morphisms $f_1:\widehat{\mathbf{K}}'\to\mathbf{K}'$, $f_3:\widehat{\mathbf{K}}_3\to\mathbf{K}_3$ and $f_4:\widehat{\mathbf{K}}_4\to\mathbf{K}_4$ making commute the diagram

$$
\begin{tikzcd}
\mathbf{A} & \mathbf{K}_1 \arrow[l, "m_1"'] \arrow[rr, "h_1"] & & \mathbf{B} & \\
\mathbf{K}_2 \arrow[u, "m_2"] \arrow[ddd, "h_2"'] & \mathbf{K}' \arrow[u, "\widetilde{m}_2"] \arrow[l, "\widetilde{m}_1"'] \arrow[rr, "\widetilde{h}_1"] \arrow[ddd, "\widetilde{h}_2"'] & & \mathbf{K}_3 \arrow[u, "m_3"'] \arrow[ddd, "h_3"] & \\
 & & \widehat{\mathbf{K}}' \arrow[ull, "\widehat{m}_1"] \arrow[uul, "\widehat{m}_2"'] \arrow[ul, dotted, "f_1"] \arrow[rr, "\widehat{h}_1"] \arrow[dd, "\widehat{h}_2"'] & & \widehat{\mathbf{K}}_3 \arrow[uul, "\widehat{m}_3"'] \arrow[ul, dotted, "f_3"] \arrow[dd, "\widehat{h}_3"] \\
\mathbf{C} & \mathbf{K}_4 \arrow[l, "m_4"] \arrow[rr, "h_4"'] & & \mathbf{D} & \\
 & & \widehat{\mathbf{K}}_4 \arrow[ull, "\widehat{m}_4"] \arrow[ul, dotted, "f_4"] \arrow[rr, "\widehat{h}_4"'] & & \mathbf{D}'
\end{tikzcd}
$$

Condition (S1c) ensures that the new squares formed with these morphisms are distinguished pullback squares.

Let now $(\mathbf{H}, \widetilde{h}_3 : \widehat{\mathbf{K}}_3 \to \mathbf{H}, \widetilde{h}_4 : \widehat{\mathbf{K}}_4 \to \mathbf{H})$ be a pushout of $\widehat{h}_1$ and $\widehat{h}_2$ in $\mathcal{H}$. On the one hand, by (S3a) there exist a unique isomorphism $h : \mathbf{H} \to \widehat{\mathbf{H}}$ in $\mathcal{H}$ and a unique $\mathcal{M}$-morphism $m : \widehat{\mathbf{H}} \to \mathbf{D}$ such that the following squares are distinguished pullback squares

$$
\begin{tikzcd}
\widehat{\mathbf{K}}_3 \arrow[r, "f_3"] \arrow[d, "h \circ \widetilde{h}_3"'] & \mathbf{K}_3 \arrow[d, "h_3"] \\
\widehat{\mathbf{H}} \arrow[r, "m"'] & \mathbf{D}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\widehat{\mathbf{K}}_4 \arrow[r, "f_4"] \arrow[d, "h \circ \widetilde{h}_4"'] & \mathbf{K}_4 \arrow[d, "h_4"] \\
\widehat{\mathbf{H}} \arrow[r, "m"'] & \mathbf{D}
\end{tikzcd}
$$

And on the other hand, by the universal property of pushouts in $\mathcal{H}$, there exists a $\mathcal{H}$-morphism $h_0 : \mathbf{H} \to \mathbf{D}'$ such that $\widehat{h}_3 = h_0 \circ \widetilde{h}_3$ and $\widehat{h}_4 = h_0 \circ \widetilde{h}_4$. Then, the $(\mathcal{M}, \mathcal{H})$-span $(m, \widehat{\mathbf{H}}, h_0 \circ h^{-1}) : \mathbf{D} \to \mathbf{D}'$ is such that

$$
\begin{aligned}
(\widehat{m}_3, \widehat{\mathbf{K}}_3, \widehat{h}_3) &= (m, \widehat{\mathbf{H}}, h_0 \circ h^{-1}) \circ (m_3, \mathbf{K}_3, h_3), \\
(\widehat{m}_4, \widehat{\mathbf{K}}_4, \widehat{h}_4) &= (m, \widehat{\mathbf{H}}, h_0 \circ h^{-1}) \circ (m_4, \mathbf{K}_4, h_4)
\end{aligned}
$$

Suppose now that there is another $(\mathcal{M}, \mathcal{H})$-span $(m', \mathbf{H}', h') : \mathbf{D} \to \mathbf{D}'$ such that

$$
\begin{aligned}
(m', \mathbf{H}', h') \circ (m_3, \mathbf{K}_3, h_3) &= (\widehat{m}_3, \widehat{\mathbf{K}}_3, \widehat{h}_3), \\
(m', \mathbf{H}', h') \circ (m_4, \mathbf{K}_4, h_4) &= (\widehat{m}_4, \widehat{\mathbf{K}}_4, \widehat{h}_4)
\end{aligned}
$$

Then there are two commutative diagrams

$$
\begin{tikzcd}
\mathbf{B} & & \\
\mathbf{K}_3 \arrow[u, "m_3"] \arrow[d, "h_3"'] & \widehat{\mathbf{K}}_3 \arrow[l, "f_3"'] \arrow[ul, "\widehat{m}_3"'] \arrow[d, "\widetilde{h}'_3"] \arrow[dr, "\widehat{h}_3"] & \\
\mathbf{D} & \mathbf{H}' \arrow[l, "m'"] \arrow[r, "h'"'] & \mathbf{D}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{C} & \mathbf{K}_4 \arrow[l, "m_4"'] \arrow[r, "h_4"] & \mathbf{D} \\
& \widehat{\mathbf{K}}_4 \arrow[u, "f_4"] \arrow[ul, "\widehat{m}'_4"] \arrow[r, "\widetilde{h}'_4"'] \arrow[dr, "\widehat{h}_4"'] & \mathbf{H}' \arrow[u, "m'"'] \arrow[d, "h'"'] \\
& & \mathbf{D}'
\end{tikzcd}
$$

where the two squares are distinguished pullback squares; f_3 and f_4 appear in these diagrams by the uniqueness of f_1, f_3 and f_4 in the definition of a final triple diagram. Then, in the cubic diagram

$$
\begin{tikzcd}
\mathbf{K}' \arrow[rr, "\widetilde{h}_1"] \arrow[dd, "\widetilde{h}_2"'] & & \mathbf{K}_3 \arrow[dd, "h_3"] & \\
& \widehat{\mathbf{K}}' \arrow[ul, "f_1"] \arrow[rr, "\widehat{h}_1"] \arrow[dd, "\widehat{h}_2"] & & \widehat{\mathbf{K}}_3 \arrow[ul, "f_3"'] \arrow[dd, "\widetilde{h}'_3"] \\
\mathbf{K}_4 \arrow[rr, "h_4"'] & & \mathbf{D} & \\
& \widehat{\mathbf{K}}_4 \arrow[ul, "f_4"] \arrow[rr, "\widetilde{h}'_4"'] & & \mathbf{H}' \arrow[ul, "m'"]
\end{tikzcd}
$$

the back side is a pushout square in $\mathcal{H}$, the top, bottom, left-hand and right-hand sides are distinguished pullback squares and $m' \in \mathcal{M}$. By condition (S3b), the front side is a pushout square, and therefore there exists an isomorphism $\hat{h} : \mathbf{H} \to \mathbf{H}'$ in $\mathcal{H}$ such that $\widetilde{h}'_3 = \hat{h} \circ \widetilde{h}_3$ and $\widetilde{h}'_4 = \hat{h} \circ \widetilde{h}_4$. But then $\hat{h} : \mathbf{H} \to \mathbf{H}'$ and $m' : \mathbf{H}' \to \mathbf{D}$ also satisfy the conditions of (S3a), and therefore $\mathbf{H}' = \widehat{\mathbf{H}}$, $h = \hat{h}$ and $m = m'$. And then, finally, $h' = h_0 \circ h^{-1}$, because of the universal property of pushouts. $\square$

Propositions 8 and 10 can be summarized in the following result.

Corollary 11. *Let $\mathcal{C}$ be a category and $\mathcal{M}$ and $\mathcal{H}$ two classes of morphisms in $\mathcal{C}$ such that $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfies conditions (S0) to (S3). Let $(m_1, \mathbf{K}_1, h_1) : \mathbf{A} \to \mathbf{B}$ and $(m_2, \mathbf{K}_2, h_2) : \mathbf{A} \to \mathbf{C}$ be two $(\mathcal{M}, \mathcal{H})$-spans. Then, $(\mathbf{D}, (m_3, \mathbf{K}_3, h_3) : \mathbf{B} \to \mathbf{D}, (m_4, \mathbf{K}_4, h_4) : \mathbf{C} \to \mathbf{D})$ is a pushout of $(m_1, \mathbf{K}_1, h_1)$ and $(m_2, \mathbf{K}_2, h_2)$ in $\mathcal{S}(\mathcal{C},$*

$\mathcal{M}, \mathcal{H})$ *iff there exists a commutative diagram in* $\mathcal{C}$

$$\begin{tikzcd}
\mathbf{A} & \mathbf{K}_1 \arrow[l, "m_1"'] \arrow[r, "h_1"] & \mathbf{B} \\
\mathbf{K}_2 \arrow[u, "m_2"] \arrow[d, "h_2"'] \arrow[ur, phantom, "(1)"] & \mathbf{K}' \arrow[l, "\widetilde{m}_1"] \arrow[r, "\widetilde{h}_1"'] \arrow[u, "\widetilde{m}_2"'] \arrow[d, "\widetilde{h}_2"] \arrow[ur, phantom, "(2)"] \arrow[dl, phantom, "(3)"] \arrow[dr, phantom, "(4)"] & \mathbf{K}_3 \arrow[u, "m_3"'] \arrow[d, "h_3"] \\
\mathbf{C} & \mathbf{K}_4 \arrow[l, "m_4"] \arrow[r, "h_4"'] & \mathbf{D}
\end{tikzcd}$$

where (1), (2) and (3) form a final triple diagram for $(m_1, \mathbf{K}_1, h_1)$ *and* $(m_2, \mathbf{K}_2, h_2)$*, and (4) is a pushout square in* $\mathcal{H}$*.*

In particular, $(m_1, \mathbf{K}_1, h_1)$ *and* $(m_2, \mathbf{K}_2, h_2)$ *have a pushout in* $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$ *iff they have a final triple diagram in* $\mathcal{C}$*.* $\square$

A close analysis of the proof of Proposition 10 shows that (S3) is necessary for the thesis to hold.

Proposition 12. *Assume* $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ *satisfies (S0), (S1) and (S2). If the thesis of Proposition 10 is true (i.e., if completing with a pushout in* $\mathcal{H}$ *a final triple diagram in* $\mathcal{C}$ *for a pair of spans always yields a pushout in* $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$ *of them) then* $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ *satisfies (S3).*

Proof. Let $h_1 : \mathbf{A} \to \mathbf{B}$ and $h_2 : \mathbf{A} \to \mathbf{C}$ be two $\mathcal{H}$-morphisms and let $\bar{h}_1 := (\mathrm{Id}_\mathbf{A}, \mathbf{A}, h_1) : \mathbf{A} \to \mathbf{B}$ and $\bar{h}_2 := (\mathrm{Id}_\mathbf{A}, \mathbf{A}, h_2) : \mathbf{A} \to \mathbf{C}$ be the corresponding $(\mathcal{M}, \mathcal{H})$-spans. Consider the diagram

$$\begin{tikzcd}
\mathbf{A} & \mathbf{A} \arrow[l, "\mathrm{Id}_\mathbf{A}"'] \arrow[r, "h_1"] & \mathbf{B} \\
\mathbf{A} \arrow[u, "\mathrm{Id}_\mathbf{A}"] \arrow[d, "h_2"'] \arrow[ur, phantom, "(1)"] & \mathbf{A} \arrow[l, "\mathrm{Id}_\mathbf{A}"] \arrow[r, "h_1"'] \arrow[u, "\mathrm{Id}_\mathbf{A}"'] \arrow[d, "h_2"] \arrow[ur, phantom, "(2)"] \arrow[dl, phantom, "(3)"] \arrow[dr, phantom, "(4)"] & \mathbf{B} \arrow[u, "\mathrm{Id}_\mathbf{B}"'] \arrow[d, "h_3"] \\
\mathbf{C} & \mathbf{C} \arrow[l, "\mathrm{Id}_\mathbf{C}"] \arrow[r, "h_4"'] & \mathbf{D}
\end{tikzcd}$$

where (4) is a pushout in $\mathcal{H}$; it is clear that (1), (2) and (3) form a final triple diagram for $\bar{h}_1$ and $\bar{h}_2$. If the thesis of Proposition 10 is true, then this diagram gives a pushout in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$ of $\bar{h}_1$ and $\bar{h}_2$. We shall prove that if the latter is true then (S3) is satisfied for every cube having (4) as back face.

So, assume in the sequel that this diagram corresponds to a pushout square in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$. For strategy reasons, we prove first (S3b) and then (S3a).

(S3b) Consider a cube

$$
\begin{tikzcd}
\mathbf{A} \arrow[rr, "h_1"] \arrow[dd, "h_2"'] & & \mathbf{B} \arrow[dd, "h_3"] & \\
& \mathbf{A}' \arrow[ul, "m_1"] \arrow[rr, "h_1'"] \arrow[dd, "h_2'"] & & \mathbf{B}' \arrow[ul, "m_2"] \arrow[dd, "h_3'"] \\
\mathbf{C} \arrow[rr, "h_4"'] & & \mathbf{D} & \\
& \mathbf{C}' \arrow[ul, "m_3"] \arrow[rr, "h_4'"'] & & \mathbf{D}' \arrow[ul, "m"]
\end{tikzcd}
$$

where $m_1, m_2, m_3, m \in \mathcal{M}$, and the top, bottom, right-hand side and left-hand side faces are distinguished pullback squares. We must prove then that its front face is a pushout square in $\mathcal{H}$.

Notice first that $(m, \mathbf{D}', \mathrm{Id}_{\mathbf{D}'}) : \mathbf{D} \to \mathbf{D}'$ satisfies that

$$
\begin{aligned}
(m, \mathbf{D}', \mathrm{Id}_{\mathbf{D}'}) \circ (\mathrm{Id}_{\mathbf{B}}, \mathbf{B}, h_3) = (m_2, \mathbf{B}', h_3') \\
(m, \mathbf{D}', \mathrm{Id}_{\mathbf{D}'}) \circ (\mathrm{Id}_{\mathbf{C}}, \mathbf{C}, h_4) = (m_3, \mathbf{C}', h_4')
\end{aligned}
$$

and it is the only such span because of the universal property of pushouts in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$.

Now, let $(\mathbf{H}, \widetilde{h}_3 : \mathbf{B}' \to \mathbf{H}, \widetilde{h}_4 : \mathbf{C}' \to \mathbf{H})$ be a pushout of h_1' and h_2' in $\mathcal{H}$. Then, on the one hand there exists a unique $(\mathcal{M}, \mathcal{H})$-span $(\widetilde{m}_0, \widetilde{\mathbf{K}}_0, \widetilde{h}_0) : \mathbf{D} \to \mathbf{H}$ such that

$$
\begin{aligned}
(\widetilde{m}_0, \widetilde{\mathbf{K}}_0, \widetilde{h}_0) \circ (\mathrm{Id}_{\mathbf{B}}, \mathbf{B}, h_3) = (m_2, \mathbf{B}', \widetilde{h}_3) \\
(\widetilde{m}_0, \widetilde{\mathbf{K}}_0, \widetilde{h}_0) \circ (\mathrm{Id}_{\mathbf{C}}, \mathbf{C}, h_4) = (m_3, \mathbf{C}', \widetilde{h}_4)
\end{aligned}
$$

And on the other hand, since $h_3' \circ h_1' = h_4' \circ h_2'$ by Remark 6, there exists a unique $\widetilde{h} : \mathbf{H} \to \mathbf{D}'$ such that $\widetilde{h} \circ \widetilde{h}_3 = h_3'$ and $\widetilde{h} \circ \widetilde{h}_4 = h_4'$. Then

$$
\begin{aligned}
(\widetilde{m}_0, \widetilde{\mathbf{K}}_0, \widetilde{h} \circ \widetilde{h}_0) \circ (\mathrm{Id}_{\mathbf{B}}, \mathbf{B}, h_3) = (m_2, \mathbf{B}', h_3'), \\
(\widetilde{m}_0, \widetilde{\mathbf{K}}_0, \widetilde{h} \circ \widetilde{h}_0) \circ (\mathrm{Id}_{\mathbf{C}}, \mathbf{C}, h_4) = (m_3, \mathbf{C}', h_4')
\end{aligned}
$$

from where we obtain $\widetilde{\mathbf{K}}_0 = \mathbf{D}'$, $m = \widetilde{m}_0$ and $\mathrm{Id}_{\mathbf{D}'} = \widetilde{h} \circ \widetilde{h}_0$. Moreover, the universal property of pushouts and the unicity of the distinguished pullbacks entail that $\widetilde{h}_0 \circ \widetilde{h} = \mathrm{Id}_{\mathbf{H}}$. Therefore, $\widetilde{h}$ is an isomorphism and the front face in the cube is a pushout square.

(S3a) Consider now a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd}
\mathbf{A} \arrow[rr, "h_1"] \arrow[dd, "h_2"'] & & \mathbf{B} \arrow[dd, "h_3"] & \\
& \mathbf{A'} \arrow[ul, "m_1"] \arrow[rr, "h'_1"] \arrow[dd, "h'_2"'] & & \mathbf{B'} \arrow[ul, "m_2"'] \arrow[dd, "h'_3"] \\
\mathbf{C} \arrow[rr, "h_4"'] & & \mathbf{D} & \\
& \mathbf{C'} \arrow[ul, "m_3"] \arrow[rr, "h'_4"'] & & \mathbf{D'}
\end{tikzcd}$$

where $m_1, m_2, m_3 \in \mathcal{M}$, the top and left-hand side squares are distinguished pullbacks and the front square is a pushout in $\mathcal{H}$.

Then, there exists a unique $(\mathcal{M}, \mathcal{H})$-span $(m_0, \mathbf{K}_0, h_0) : \mathbf{D} \to \mathbf{D'}$ such that

$$\begin{aligned}
(m_0, \mathbf{K}_0, h_0) \circ (\mathrm{Id}_{\mathbf{B}}, \mathbf{B}, h_3) = (m_2, \mathbf{B'}, h'_3) \\
(m_0, \mathbf{K}_0, h_0) \circ (\mathrm{Id}_{\mathbf{C}}, \mathbf{C}, h_4) = (m_3, \mathbf{C'}, h'_4)
\end{aligned}$$

This entails the existence of the following diagram, where the bottom and right-hand side squares involving m_0 are distinguished pullback squares

$$\begin{tikzcd}
\mathbf{A} \arrow[rr, "h_1"] \arrow[dd, "h_2"'] & & \mathbf{B} \arrow[dd, "h_3"] & & \\
& \mathbf{A'} \arrow[ul, "m_1"] \arrow[rr, "h'_1"] \arrow[dd, "h'_2"'] & & \mathbf{B'} \arrow[ul, "m_2"'] \arrow[dd, "\widetilde{h}_3"] \arrow[ddr, "h'_3"] & \\
\mathbf{C} \arrow[rr, "h_4"'] & & \mathbf{D} & & \\
& \mathbf{C'} \arrow[ul, "m_3"] \arrow[rr, "\widetilde{h}_4"'] \arrow[rrrd, "h'_4"'] & & \mathbf{K}_0 \arrow[ul, dotted, "m_0"] \arrow[dr, dotted, "h_0"] & \\
& & & & \mathbf{D'}
\end{tikzcd}$$

The cube formed with m_0 in this diagram satisfies the conditions of (S3b), and therefore its front face is a pushout square. Thus, $h_0 : \mathbf{K}_0 \to \mathbf{D'}$ is the unique isomorphism in $\mathcal{H}$ such that $h'_3 = h_0 \circ \widetilde{h}_3$ and $h'_4 = h_0 \circ \widetilde{h}_4$.

We have found an isomorphism $h_0^{-1} : \mathbf{D'} \to \mathbf{K}_0$ and a $\mathcal{M}$-morphism $m_0 : \mathbf{K}_0 \to \mathbf{D}$ satisfying the properties asked for in (S3a). And this pair of morphisms is unique with these properties, because any other such pair $\widetilde{h}_0 : \mathbf{D'} \to \widetilde{\mathbf{K}}_0$ and $\widetilde{m}_0 : \widetilde{\mathbf{K}}_0 \to \mathbf{D}$ would yield another span $(\widetilde{m}_0, \widetilde{\mathbf{K}}_0, \widetilde{h}_0^{-1}) : \mathbf{D} \to \mathbf{D'}$ such that $(\widetilde{m}_0, \widetilde{\mathbf{K}}_0, \widetilde{h}_0^{-1}) \circ (\mathrm{Id}_{\mathbf{B}}, \mathbf{B}, h_3) = (m_2, \mathbf{B'}, h'_3)$ and $(\widetilde{m}_0, \widetilde{\mathbf{K}}_0, \widetilde{h}_0^{-1}) \circ (\mathrm{Id}_{\mathbf{C}}, \mathbf{C}, h_4) = (m_3, \mathbf{C'}, h'_4)$. $\square$

Actually, the proof of the previous proposition also proves that, under conditions (S0) to (S2), if the embedding $\mathcal{H} \hookrightarrow \mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$ preserves all pushouts then $(\mathcal{C},\mathcal{M},\mathcal{H})$ satisfies (S3). Combining it with Proposition 10 we obtain the following corollary.

Corollary 13. *Assume that $(\mathcal{C},\mathcal{M},\mathcal{H})$ satisfies (S0), (S1) and (S2). The embedding $\mathcal{H} \hookrightarrow \mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$ preserves all pushouts iff $(\mathcal{C},\mathcal{M},\mathcal{H})$ also satisfies (S3).* $\square$

And using Corollary 9 we also have the following result.

Corollary 14. *Assume that $(\mathcal{C},\mathcal{M},\mathcal{H})$ satisfies (S0), (S1) and (S2), and assume moreover that all morphisms in $\mathcal{M}$ are monomorphisms in it and that all isomorphisms in $\mathcal{M}$ belong to $\mathcal{H}$. Then $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$ has all pushouts iff $(\mathcal{C},\mathcal{M},\mathcal{H})$ satisfies (S3) and every pair of $(\mathcal{M},\mathcal{H})$-spans with the same source object has a final triple diagram in $\mathcal{C}$.*

Proof. The "if" implication is given by Proposition 10 (actually, without any assumption on $\mathcal{M}$ besides (S0) to (S2)). As far as the "only if" implication goes, Corollary 9 entails that if $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$ has all pushouts then every pair of $(\mathcal{M},\mathcal{H})$-spans with the same source object has a final triple diagram in $\mathcal{C}$. And if all isomorphisms in $\mathcal{M}$ belong to $\mathcal{H}$ then, by Remark 5, two final triple diagrams for a pair of spans are isomorphic in $\mathcal{H}$, which implies that completing with a pushout in $\mathcal{H}$ *any* final triple diagram for a pair of spans yields a pushout for them in $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$. In particular, then, the embedding $\mathcal{H} \hookrightarrow \mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$ preserves all pushouts (see the proof of Proposition 12) and therefore $(\mathcal{C},\mathcal{M},\mathcal{H})$ also satisfies (S3). $\square$

Remark 15. The condition that $\mathcal{H}$ contains all isomorphisms in $\mathcal{M}$ will appear in other contexts in this paper, and we shall refer henceforth to it as condition (S0i). Notice in particular that condition (S0m) in Remark 5 implies (S0i). $\square$

The last two corollaries show that, under conditions (S0) to (S2), the technical condition (S3) is necessary for the well-behavior of $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$ as far as the existence of pushouts goes. On the other hand, notice that condition (S2) is not necessary for the existence of pushouts in $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$, even when $\mathcal{M}$ is a class of monomorphisms and $(\mathcal{C},\mathcal{M},\mathcal{H})$ satisfies condition (S0i), as the following example shows.

Example 16. Consider a category $\mathcal{C}$ consisting only of the following objects and morphisms

$$\begin{tikzcd}
\mathbf{A} & \mathbf{A} \arrow[l, "\mathrm{Id}_{\mathbf{A}}"'] \arrow[r, "h_1"] & \mathbf{B} \\
\mathbf{A} \arrow[u, "\mathrm{Id}_{\mathbf{A}}"] \arrow[d, "h_2"'] & \mathbf{K} \arrow[l, "m"] \arrow[u, "m"'] \arrow[r, "h'_1"'] \arrow[d, "h'_2"] \arrow[ur, phantom, "(2)"] \arrow[dl, phantom, "(3)"] \arrow[dr, phantom, "(4)"] & \mathbf{K}_3 \arrow[u, "m_3"'] \arrow[d, "h_3"] \\
\mathbf{C} & \mathbf{K}_4 \arrow[l, "m_4"] \arrow[r, "h_4"'] & \mathbf{D}
\end{tikzcd}$$

plus the identites of all objects (we have added the identity on $\mathbf{A}$ to make the argument clearer). Assume moreover that, beside the identities, the classes $\mathcal{M}$ and $\mathcal{H}$ are given by

$$\mathcal{M} = \{m, m_3, m_4\}, \mathcal{H} = \{h_1, h_1', h_2, h_2', h_3, h_4\}$$

and that (4) is a commutative square, and (2) and (3) are distinguished pullback squares (thus, the class $\mathcal{PB}$ of distinguished pullback squares is formed by these two squares together with all those squares required by condition (S1b)).

Then $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfies conditions (S0), (S1) and (S0i), but not (S2), since $h_1 : \mathbf{A} \to \mathbf{B}$ and $h_2 : \mathbf{A} \to \mathbf{C}$ do not have a pushout in $\mathcal{H}$. And this diagram clearly yields a pushout in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$ for $(\mathrm{Id}_{\mathbf{A}}, \mathbf{A}, h_1) : \mathbf{A} \to \mathbf{B}$ and $(\mathrm{Id}_{\mathbf{A}}, \mathbf{A}, h_2) : \mathbf{A} \to \mathbf{C}$; as a matter of fact, it is not difficult to check that, actually, $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$ has all pushouts. □

Example 17. Let $(i_1, K_1, f_1) : A \to B$ and $(i_2, K_2, f_2) : A \to C$ be two $(\mathcal{I}, \mathcal{H})$-spans in S-**Sets** (see Example 7). Their pushout in $\mathcal{S}(S\text{-}\mathbf{Sets}, \mathcal{I}, \mathcal{H})$ is given then by the diagram

$$\begin{tikzcd}
A \arrow[dr, phantom, "(1)"] & K_1 \arrow[l, "i_1"'] \arrow[r, "f_1"] \arrow[dr, phantom, "(2)"] & B \\
K_2 \arrow[u, "i_2"] \arrow[d, "f_2"'] \arrow[dr, phantom, "(3)"] & K' \arrow[l, "j_1"] \arrow[r, "\widetilde{f}_1"'] \arrow[u, "j_2"'] \arrow[d, "\widetilde{f}_2"] \arrow[dr, phantom, "(4)"] & K_3 \arrow[u, "i_3"'] \arrow[d, "f_3"] \\
C & K_4 \arrow[l, "i_4"] \arrow[r, "f_4"'] & D
\end{tikzcd}$$

where

- K' is the greatest subset of A that is simultaneously the preimage under f_1 and f_2 of a subset of B and of C, respectively —it can also be described as the greatest subset K' of $K_1 \cap K_2$ such that $f_1^{-1}(f_1(K')) = K'$ and $f_2^{-1}(f_2(K')) = K'$— and $j_2 : K' \to K_1$, $j_1 : K' \to K_2$ are the corresponding inclusions;
- K_3 is the greatest subset of B such that $f_1^{-1}(K_3) = K'$, i.e., $K_3 = (B \setminus f_1(K_1)) \cup f_1(K')$, $i_3 : K_3 \to B$ is the corresponding inclusion, and $\widetilde{f}_1 : K' \to K_3$ the restriction of f_1 to K' and K_3;
- K_4 is the greatest subset of C such that $f_2^{-1}(K_4) = K'$, i.e., $K_4 = (C \setminus f_2(K_2)) \cup f_2(K')$, $i_4 : K_4 \to C$ is the corresponding inclusion, and $\widetilde{f}_2 : K' \to K_4$ the restriction of f_2 to K' and K_4;
- (4) is a pushout square in S-**Sets** for $\widetilde{f}_1 : K' \to K_3$ and $\widetilde{f}_2 : K' \to K_4$.

Indeed, the diagram (1)+(2)+(3) defined in this way is a final triple diagram for (i_1, K_1, f_1) and (i_2, K_2, f_2) in S-**Sets** (actually, their only final triple diagram, since $\mathcal{I}$ clearly satisfies (S0m) in Remark 5), and then Proposition 10 applies.

Recall that the pushout of two partial mappings of S-sets described in [10, Cons. 2.6] (see also [14, Prop. 3]) is obtained exactly in this way. □

4 Comparison with double-pushout rewriting

Double-pushout rewriting in $\mathcal{C}$ has been compared in [9] with single-pushout rewriting of corresponding spans in $(\mathcal{C}, \mathcal{M})$, considered as equivalence classes of pairs of morphisms, in two particular categories: hypergraphs and term hypergraphs, also called *jungles*. In this section, we generalize this comparison in two ways. On the one hand, we study spans in $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ instead of spans in $(\mathcal{C}, \mathcal{M})$, as already mentioned in Section 1, and we do it for an arbitrary category. On the other hand, we find a necessary and sufficient condition for double-pushout derivations in $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ to be *subsumed* (Definition 20) by single-pushout derivations in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$.

Let $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ be in the sequel a triple satisfying conditions (S0) to (S3), relative to a given class $\mathcal{PB}$ of distinguished pullback squares.

Definition 18. Given a $\mathcal{M}$-morphism $m : \mathbf{K} \to \mathbf{L}$ and a $\mathcal{H}$-morphism $h : \mathbf{L} \to \mathbf{G}$, $(\mathbf{D}, h' : \mathbf{K} \to \mathbf{D}, m' : \mathbf{D} \to \mathbf{G})$ is a $(\mathcal{M}, \mathcal{H})$*-pushout complement* of them when $h' \in \mathcal{H}$, $m' \in \mathcal{M}$, and $(\mathbf{G}, h : \mathbf{L} \to \mathbf{G}, m' : \mathbf{D} \to \mathbf{G})$ is a pushout in $\mathcal{C}$ of $m : \mathbf{K} \to \mathbf{L}$ and $h' : \mathbf{K} \to \mathbf{D}$.

Such a $(\mathcal{M}, \mathcal{H})$-pushout complement of m and h is *final* when for every other $(\mathcal{M}, \mathcal{H})$-pushout complement $(\mathbf{D}_0, h_0 : \mathbf{K} \to \mathbf{D}_0, m_0 : \mathbf{D}_0 \to \mathbf{G})$ of m and h there exists a unique $\mathcal{M}$-morphism $\widehat{m} : \mathbf{D}_0 \to \mathbf{D}$ such that $m' \circ \widehat{m} = m_0$ and $\widehat{m} \circ h_0 = h'$.

$$\begin{tikzcd}
\mathbf{L} \arrow[d, "h"'] & \mathbf{K} \arrow[l, "m"'] \arrow[d, "h'"] \arrow[dd, bend left=60, "h_0"] \\
\mathbf{G} & \mathbf{D} \arrow[l, "m'"] \\
 & \mathbf{D}_0 \arrow[ul, "m_0"] \arrow[u, dotted, "\widehat{m}"']
\end{tikzcd}$$

Notice that given two such final $(\mathcal{M}, \mathcal{H})$-pushout complements $(\mathbf{D}, h' : \mathbf{K} \to \mathbf{D}, m' : \mathbf{D} \to \mathbf{G})$ and $(\mathbf{D}_0, h_0 : \mathbf{K} \to \mathbf{D}_0, m_0 : \mathbf{D}_0 \to \mathbf{G})$ of m and h, there exists an isomorphism $\widehat{m} : \mathbf{D}_0 \to \mathbf{D}$ in $\mathcal{M}$ such that $m' \circ \widehat{m} = m_0$ and $\widehat{m} \circ h_0 = h'$.

A *double-pushout transformation* $(\mathcal{M}, \mathcal{H})$*-rule* (a *DPO rule*, for short) in $\mathcal{C}$ is a pair of morphisms $P = (\mathbf{L} \overset{m}{\leftarrow} \mathbf{K} \overset{r}{\to} \mathbf{R})$ with $m : \mathbf{K} \to \mathbf{L}$ in $\mathcal{M}$ and $r : \mathbf{K} \to \mathbf{R}$ in $\mathcal{H}$: i.e., a $(\mathcal{M}, \mathcal{H})$-span. It can be *applied* to an object $\mathbf{G}$ through a $\mathcal{H}$-morphism $h : \mathbf{L} \to \mathbf{G}$ when there exists a final $(\mathcal{M}, \mathcal{H})$-pushout complement $(\mathbf{D}, h' : \mathbf{K} \to \mathbf{D}, m' : \mathbf{D} \to \mathbf{G})$ of m and h. And, if this is the case, we call any pushout object $\mathbf{H}$ in $\mathcal{H}$ of $r : \mathbf{K} \to \mathbf{R}$ and $h' : \mathbf{K} \to \mathbf{D}$ a *derived object in* $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ of $\mathbf{G}$ by the application of P through h; cf. [3].

$$\begin{tikzcd}
\mathbf{L} \arrow[d, "h"'] & \mathbf{K} \arrow[l, "m"'] \arrow[r, "r"] \arrow[d, "h'"] & \mathbf{R} \arrow[d, "h''"] \\
\mathbf{G} & \mathbf{D} \arrow[l, "m'"] \arrow[r, "r'"'] & \mathbf{H}
\end{tikzcd}$$

Remark 19. Under (S0) to (S3) alone, two derived objects of $\mathbf{G}$ in $(\mathcal{C},\mathcal{M},\mathcal{H})$ by the application of a DPO rule $P=(\mathbf{L}\overset{m}{\leftarrow}\mathbf{K}\overset{r}{\to}\mathbf{R})$ through $h:\mathbf{L}\to\mathbf{G}$ need not be isomorphic (even not in $\mathcal{C}$).

But if $(\mathcal{C},\mathcal{M},\mathcal{H})$ satisfies condition (S0i) in Remark 15 then the isomorphism in $\mathcal{M}$ (and therefore in $\mathcal{H}$) of two final $(\mathcal{M},\mathcal{H})$-pushout complements of m and h, together with the universal property of pushouts in $\mathcal{H}$, implies that any two such derived objects in $(\mathcal{C},\mathcal{M},\mathcal{H})$ are isomorphic in $\mathcal{H}$. □

Now, recall that a *single-pushout transformation rule* (a *SPO rule*, for short) in $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$ is again nothing but a $(\mathcal{M},\mathcal{H})$-span $\mathbf{L}\overset{l}{\leftarrow}\mathbf{K}\overset{r}{\to}\mathbf{R}$. It can be *applied* to an object $\mathbf{G}$ through a $(\mathcal{M},\mathcal{H})$-span $\mathbf{L}\overset{n}{\leftarrow}\mathbf{E}\overset{h}{\to}\mathbf{G}$ (which is assumed to belong to a fixed class $\mathcal{O}$ of *occurrences*) when $(l,\mathbf{K},r)$ and $(n,\mathbf{E},h)$ have a pushout in $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$. In that case, the *derived object* $\mathbf{H}'$ in $\mathcal{H}$ is the pushout object of $(l,\mathbf{K},r)$ and $(n,\mathbf{E},h)$ in $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$; it is unique up to an isomorphism in $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$.

In practice, the class $\mathcal{O}$ of occurrences is taken to be the class of all "total" morphisms in the category of "partial" morphisms where single-pushout transformation is carried out. Here we shall understand as "total" a $(\mathcal{M},\mathcal{H})$-span with left-hand side $\mathcal{M}$-morphism an identity, and consequently we shall take as occurrences the $(\mathcal{M},\mathcal{H})$-spans of the form $\mathbf{L}\overset{\mathrm{Id}_\mathbf{L}}{\leftarrow}\mathbf{L}\overset{h}{\to}\mathbf{G}$, i.e., the $\mathcal{H}$-morphisms (understood as $(\mathcal{M},\mathcal{H})$-spans through the embedding $\mathcal{H}\hookrightarrow\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$).

In this section we want to compare double and single-pushout derivations by the application of the same $(\mathcal{M},\mathcal{H})$-span (considered as a DPO or as a SPO rule) to the same object through the same $\mathcal{H}$-morphism. This comparison will come through the notion of *subsumption* introduced in the next definition.

Definition 20. Let $\mathbf{L}\overset{m}{\leftarrow}\mathbf{K}\overset{r}{\to}\mathbf{R}$ be a $(\mathcal{M},\mathcal{H})$-span, and $h:\mathbf{L}\to\mathbf{G}$ a $\mathcal{H}$-morphism.

Let $\mathbf{H}$ be a derived object of $\mathbf{G}$ in $(\mathcal{C},\mathcal{M},\mathcal{H})$ by the application of the DPO rule $\mathbf{L}\overset{m}{\leftarrow}\mathbf{K}\overset{r}{\to}\mathbf{R}$ through $h:\mathbf{L}\to\mathbf{G}$; assume that this derivation is given by the diagram in $\mathcal{C}$

$$\begin{tikzcd}
\mathbf{L} \arrow[d, "h"'] \arrow[dr, phantom, "(1)"] & \mathbf{K} \arrow[l, "m"'] \arrow[r, "r"] \arrow[d, "h'"] \arrow[dr, phantom, "(2)"] & \mathbf{R} \arrow[d, "h''"] \\
\mathbf{G} & \mathbf{D} \arrow[l, "m'"] \arrow[r, "r'"'] & \mathbf{H}
\end{tikzcd}$$

where $(\mathbf{D},h':\mathbf{K}\to\mathbf{D},m':\mathbf{D}\to\mathbf{G})$ in square (1) is a final $(\mathcal{M},\mathcal{H})$-pushout complement of m and h and (2) is a pushout square in $\mathcal{H}$.

We shall say that this DPO derivation is *subsumed* by single-pushout transformation in $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$ when

$$\begin{tikzcd}
\mathbf{L} \arrow[r, "{(m,\mathbf{K},r)}"] \arrow[d, "{(\mathrm{Id}_\mathbf{L},\mathbf{L},h)}"'] & \mathbf{R} \arrow[d, "{(\mathrm{Id}_\mathbf{R},\mathbf{R},h'')}"] \\
\mathbf{G} \arrow[r, "{(m',\mathbf{D},r')}"'] & \mathbf{H}
\end{tikzcd}$$

is a pushout square in $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$ (and therefore it corresponds to an application of the SPO rule $\mathbf{L} \stackrel{m}{\leftarrow} \mathbf{K} \stackrel{r}{\rightarrow} \mathbf{R}$ to $\mathbf{G}$ through the occurrence $\mathbf{L} \stackrel{\mathrm{Id}_{\mathbf{L}}}{\leftarrow} \mathbf{L} \stackrel{h}{\rightarrow} \mathbf{G}$).

And we shall say that single-pushout transformation in $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$ *subsumes* double-pushout transformation in $(\mathcal{C},\mathcal{M},\mathcal{H})$ when for every $(\mathcal{M},\mathcal{H})$-span $\mathbf{L} \stackrel{m}{\leftarrow} \mathbf{K} \stackrel{r}{\rightarrow} \mathbf{R}$ and for every $\mathcal{H}$-morphism $h : \mathbf{L} \to \mathbf{G}$, every derivation of $\mathbf{G}$ in $(\mathcal{C},\mathcal{M},\mathcal{H})$ by the application of the DPO rule $\mathbf{L} \stackrel{m}{\leftarrow} \mathbf{K} \stackrel{r}{\rightarrow} \mathbf{R}$ through h is subsumed by single-pushout derivation in $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$.

We provide in this section a simple necessary and sufficient condition on $(\mathcal{C},\mathcal{M},\mathcal{H})$ for this subsumption. This condition consists, roughly, in the equivalence of final $(\mathcal{M},\mathcal{H})$-pushout complements and final distinguished pullback complements (defined next).

Definition 21. Given a $\mathcal{M}$-morphism $m : \mathbf{K} \to \mathbf{L}$ and a $\mathcal{H}$-morphism $h : \mathbf{L} \to \mathbf{G}$, we say that $(\mathbf{D}, m' : \mathbf{D} \to \mathbf{G}, h' : \mathbf{K} \to \mathbf{D})$, with m' in $\mathcal{M}$ and h' in $\mathcal{H}$, is a *distinguished pullback complement* of m and h when $(\mathbf{K}, m : \mathbf{K} \to \mathbf{L}, h' : \mathbf{K} \to \mathbf{D})$ is a distinguished pullback of $h : \mathbf{L} \to \mathbf{G}$ and $m' : \mathbf{D} \to \mathbf{G}$.

Such a distinguished pullback complement is *final* when for every $\mathcal{M}$-morphism $m_0 : \mathbf{K}_0 \to \mathbf{K}$ and for every distinguished pullback complement $(\mathbf{D}_0, m_0' : \mathbf{D}_0 \to \mathbf{G}, h'' : \mathbf{K}_0 \to \mathbf{D}_0)$ of $m_1 = m \circ m_0 : \mathbf{K}_0 \to \mathbf{L}$ and $h : \mathbf{L} \to \mathbf{G}$, there exists a unique $\mathcal{M}$-morphism $m_0'' : \mathbf{D}_0 \to \mathbf{D}$ such that $m_0' = m' \circ m_0''$ and $h' \circ m_0 = m_0'' \circ h''$.

$$\begin{tikzcd}
\mathbf{K}_0 \arrow[r, "m_0"] \arrow[rr, bend left, "m_1"] \arrow[d, "h''"'] & \mathbf{K} \arrow[r, "m"] \arrow[d, "h'"] & \mathbf{L} \arrow[d, "h"] \\
\mathbf{D}_0 \arrow[r, dotted, "m_0''"'] \arrow[rr, bend right, "m_0'"'] & \mathbf{D} \arrow[r, "m'"'] & \mathbf{G}
\end{tikzcd}$$

Example 22. With the definition of distinguished pullback in S-**Sets** given in Example 7, an inclusion $i : K \hookrightarrow L$ and a mapping $h : L \to G$ of S-sets have a distinguished pullback complement iff there exists a subset G' of G such that $h^{-1}(G') = K$; i.e., iff $h^{-1}(h(K)) = K$.

And when $h^{-1}(h(K)) = K$, a final distinguished pullback complement of $i : K \hookrightarrow L$ and $h : L \to G$ is given by

$$D = (G \setminus h(L)) \cup h(K)$$

together with the inclusion $j : D \hookrightarrow G$ and the restriction $h' : K \to D$ of h to K (understood with target set D, which contains $h(K)$). Indeed, on the one hand it is clear $h^{-1}(D) = h^{-1}(h(K)) = K$, so that

$$\begin{tikzcd}
K \arrow[r, "i"] \arrow[d, "h'"'] & L \arrow[d, "h"] \\
D \arrow[r, "j"'] & G
\end{tikzcd}$$

is a distinguished pullback square in S-**Sets**. And, on the other hand, given an inclusion $i_0 : K_0 \hookrightarrow K$ such that $h^{-1}(h(K_0)) = K_0$ and given any distinguished pullback complement $(D_0, j_0 : D_0 \hookrightarrow G, h'_0 : K_0 \to D_0)$ of $i \circ i_0$ and h, since $h^{-1}(D_0) = K_0$, it must be $D_0 \subseteq (G \setminus h(L)) \cup h(K_0) \subseteq D$, and therefore there exists a (unique) inclusion $j'_0 : D_0 \hookrightarrow D$ such that $j_0 = j \circ j'_0$ and $j'_0 \circ h'_0 = h' \circ i_0$. $\square$

Proposition 23. *Single-pushout transformation in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$ subsumes double-pushout transformation in $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ iff the following condition holds:*

> *Every final $(\mathcal{M}, \mathcal{H})$-pushout complement of a $\mathcal{M}$-morphism $m : \mathbf{K} \to \mathbf{L}$ and a $\mathcal{H}$-morphism $h : \mathbf{L} \to \mathbf{G}$ is also a final distinguished pullback complement of them.*

Proof. Assume first that every final $(\mathcal{M}, \mathcal{H})$-pushout complement of a $\mathcal{M}$-morphism $m : \mathbf{K} \to \mathbf{L}$ and a $\mathcal{H}$-morphism $h : \mathbf{L} \to \mathbf{G}$ is also a final distinguished pullback complement of them. Let $\mathbf{L} \stackrel{m}{\leftarrow} \mathbf{K} \stackrel{r}{\to} \mathbf{R}$ be a $(\mathcal{M}, \mathcal{H})$-span and $h : \mathbf{L} \to \mathbf{G}$ a $\mathcal{H}$-morphism, and assume that the DPO rule $\mathbf{L} \stackrel{m}{\leftarrow} \mathbf{K} \stackrel{r}{\to} \mathbf{R}$ can be applied to $\mathbf{G}$ through h. Let

$$\begin{tikzcd}
\mathbf{L} \arrow[d, "h"'] \arrow[dr, phantom, "(1)"] & \mathbf{K} \arrow[l, "m"'] \arrow[r, "r"] \arrow[d, "h'"] \arrow[dr, phantom, "(2)"] & \mathbf{R} \arrow[d, "h''"] \\
\mathbf{G} & \mathbf{D} \arrow[l, "m'"] \arrow[r, "r'"'] & \mathbf{H}
\end{tikzcd}$$

be a diagram in $\mathcal{C}$ corresponding to such a derivation in $(\mathcal{C}, \mathcal{M}, \mathcal{H})$, with square (1) corresponding to a final pushout complement and (2) a pushout square in $\mathcal{H}$. In particular, $(\mathbf{D}, h' : \mathbf{K} \to \mathbf{D}, m' : \mathbf{D} \to \mathbf{G})$ is a final distinguished pullback of m and h.

It is easy to prove then that (1), (2) and (3) in the following diagram form a final triple diagram in $\mathcal{C}$, and therefore that this diagram yields a pushout in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$ of $\mathbf{L} \stackrel{m}{\leftarrow} \mathbf{K} \stackrel{r}{\to} \mathbf{R}$ and $\mathbf{L} \stackrel{\mathrm{Id}_{\mathbf{L}}}{\leftarrow} \mathbf{L} \stackrel{h}{\to} \mathbf{G}$:

$$\begin{tikzcd}
\mathbf{L} & \mathbf{K} \arrow[l, "m"'] \arrow[r, "r"] & \mathbf{R} \\
\mathbf{L} \arrow[u, "\mathrm{Id}_{\mathbf{L}}"] \arrow[d, "h"'] \arrow[ur, phantom, "(1)"] \arrow[dr, phantom, "(3)"] & \mathbf{K} \arrow[l, "m"'] \arrow[r, "r"] \arrow[u, "\mathrm{Id}_{\mathbf{K}}"] \arrow[d, "h'"] \arrow[ur, phantom, "(2)"] & \mathbf{R} \arrow[u, "\mathrm{Id}_{\mathbf{R}}"'] \arrow[d, "h''"] \\
\mathbf{G} & \mathbf{D} \arrow[l, "m'"] \arrow[r, "r'"'] & \mathbf{H}
\end{tikzcd}$$

So, this (arbitrary) double-pushout derivation of $\mathbf{G}$ in $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ is subsumed by single-pushout transformation in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$. This entails that single-pushout transformation in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$ subsumes double-pushout transformation in $(\mathcal{C}, \mathcal{M}, \mathcal{H})$, which proves the "if" implication.

As far as the "only if" implication goes, assume that single-pushout transformation in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$ subsumes double-pushout transformation in $(\mathcal{C}, \mathcal{M}, \mathcal{H})$. Let $m : \mathbf{K} \to \mathbf{L}$ and $h : \mathbf{L} \to \mathbf{G}$ be a $\mathcal{M}$-morphism and a $\mathcal{H}$-morphism, respectively, and let $(\mathbf{D}, m' : \mathbf{D} \to \mathbf{G}, h' : \mathbf{K} \to \mathbf{D})$ be a final $(\mathcal{M}, \mathcal{H})$-pushout complement of them. We want to prove that it is also a final distinguished pullback complement of them.

Consider the $(\mathcal{M}, \mathcal{H})$-span $\mathbf{L} \overset{m}{\leftarrow} \mathbf{K} \overset{\mathrm{Id}_\mathbf{K}}{\rightarrow} \mathbf{K}$. The existence of a final $(\mathcal{M}, \mathcal{H})$-pushout complement of m and h implies that this DPO rule can be applied to $\mathbf{G}$ through $h : \mathbf{L} \to \mathbf{G}$. Take *any* derivation of $\mathbf{G}$ in $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ by the application of $\mathbf{L} \overset{m}{\leftarrow} \mathbf{K} \overset{\mathrm{Id}_\mathbf{K}}{\rightarrow} \mathbf{K}$ through $h : \mathbf{L} \to \mathbf{G}$ using this final $(\mathcal{M}, \mathcal{H})$-pushout complement of m and h. It will be given by a diagram in $\mathcal{C}$

$$\begin{tikzcd}
\mathbf{L} \arrow[d, "h"'] & \mathbf{K} \arrow[l, "m"'] \arrow[r, "\mathrm{Id}_\mathbf{K}"] \arrow[d, "h'"] & \mathbf{K} \arrow[d, "h_0 \circ h'"] \\
\mathbf{G} & \mathbf{D} \arrow[l, "m'"] \arrow[r, "h_0"'] & \mathbf{H}
\end{tikzcd}$$

where $h_0 : \mathbf{D} \to \mathbf{H}$ is any isomorphism in $\mathcal{H}$.

Since by assumption single-pushout transformation in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$ subsumes double-pushout transformation in $(\mathcal{C}, \mathcal{M}, \mathcal{H})$,

$$(\mathbf{H}, (\mathrm{Id}_\mathbf{K}, \mathbf{K}, h_0 \circ h') : \mathbf{K} \to \mathbf{H}, (m', \mathbf{D}, h_0) : \mathbf{G} \to \mathbf{H})$$

is a pushout in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$ of $(m, \mathbf{K}, \mathrm{Id}_\mathbf{K}) : \mathbf{L} \to \mathbf{K}$ and $(\mathrm{Id}_\mathbf{L}, \mathbf{L}, h) : \mathbf{L} \to \mathbf{G}$. This implies the existence in $\mathcal{C}$ of a diagram

$$\begin{tikzcd}
\mathbf{L} & \mathbf{K} \arrow[l, "m"'] \arrow[r, "\mathrm{Id}_\mathbf{K}"] & \mathbf{K} \\
\mathbf{L} \arrow[u, "\mathrm{Id}_\mathbf{L}"] \arrow[d, "h"'] \arrow[ur, phantom, "(1)"] & \mathbf{K}_1 \arrow[l, "m_2"'] \arrow[r, "h_1"] \arrow[u, "m_1"'] \arrow[d, "h_2"] \arrow[ur, phantom, "(2)"] \arrow[dr, phantom, "(4)"] & \mathbf{K} \arrow[u, "\mathrm{Id}_\mathbf{K}"'] \arrow[d, "h_0 \circ h'"] \\
\mathbf{G} \arrow[ur, phantom, "(3)"] & \mathbf{D} \arrow[l, "m'"] \arrow[r, "h_0"'] & \mathbf{H}
\end{tikzcd}$$

where (1), (2) and (3) form a final triple diagram and (4) is a pushout square in $\mathcal{H}$. Then, since (2) is a distinguished pullback square, $\mathbf{K}_1 = \mathbf{K}$, $m_1 = h_1 = \mathrm{Id}_\mathbf{K}$, and then a simple diagram chasing shows that $m_2 = m$ and $h_2 = h'$, so that this diagram is actually:

$$
\begin{tikzcd}
\mathbf{L} & \mathbf{K} \arrow[l, "m"'] \arrow[r, "\mathrm{Id}_{\mathbf{K}}"] & \mathbf{K} \\
\mathbf{L} \arrow[u, "\mathrm{Id}_{\mathbf{L}}"] \arrow[d, "h"'] \arrow[r, phantom, "(1)"] \arrow[dr, phantom, "(3)"] & \mathbf{K} \arrow[l, "m"'] \arrow[r, "\mathrm{Id}_{\mathbf{K}}"] \arrow[u, "\mathrm{Id}_{\mathbf{K}}"] \arrow[d, "h'"] \arrow[r, phantom, "(2)"] \arrow[dr, phantom, "(4)"] & \mathbf{K} \arrow[u, "\mathrm{Id}_{\mathbf{K}}"'] \arrow[d, "h_0 \circ h'"] \\
\mathbf{G} & \mathbf{D} \arrow[l, "m'"] \arrow[r, "h_0"'] & \mathbf{H}
\end{tikzcd}
$$

And then a simple argument shows that if (1), (2) and (3) in this diagram form a final triple diagram then $(\mathbf{D}, m' : \mathbf{D} \to \mathbf{G}, h' : \mathbf{K} \to \mathbf{D})$ is a final distinguished pullback complement of m and h, as we wanted to prove. $\Box$

Example 24. We continue with our running exemple of $(\mathcal{I}, \mathcal{H})$-spans in S-**Sets** (see Examples 7, 17 and 22), which correspond to partial mappings of S-sets.

It is well known (see, for instance, [7]), that there exists a pushout complement of an inclusion $i : K \hookrightarrow L$ and a mapping of S-sets $h : L \to G$ iff $h(K) \cap h(L \setminus K) = \emptyset$ and the restriction of h to $L \setminus K$ is injective (this is known as the *identification condition* in the language of algebraic graph transformation), and that when this condition is satisfied then the only (and therefore final) $(\mathcal{I}, \mathcal{H})$-pushout complement of i and h is given by $D = (G \setminus h(L)) \cup h(K)$ together with the inclusion $j : D \hookrightarrow G$ and the restriction $h' : K \to D$ of h to K (understood with target set D, which contains $h(K)$). By Example 22, this final pushout complement of i and h is also a final distinguished pullback complement of them.

Therefore, single-pushout transformation in $\mathcal{S}(S\text{-}\mathbf{Sets}, \mathcal{I}, \mathcal{H})$ subsumes double-pushout transformation in $(S\text{-}\mathbf{Sets}, \mathcal{I}, \mathcal{H})$, as it is on the other hand well known (see [10] or [4] for the more general case of total and partial unary algebras, respectively). $\Box$

5 HLR conditions

HLR conditions are category-theoretical simple conditions that measure the "goodness" of algebraic transformation systems, by implying abstract properties of the latter, related to parallelism and concurrency. Contrary to the case of double-pushout algebraic transformation, where many HLR conditions for many results have been introduced (see [13] for a survey), the field of HLR conditions in single-pushout algebraic transformation is at its beginnings. In particular, and to our knowledge, only six HLR conditions have been introduced so far in the literature, to study three properties; cf. [8, 4].

These HLR conditions for single-pushout algebraic transformation (SPO-HLR conditions, for short) refer to pairs $(\mathcal{R}, \mathcal{O})$, where $\mathcal{R}$ is a (general) category where single-pushout transformation is carried out, whose morphisms are called *rules* (and they are usually understood as partial morphisms, in some sense), and $\mathcal{O}$ is a subcategory of $\mathcal{R}$ with its same class of objects, whose morphisms are called *occurrences* (and, as we mentioned in the previous section, they are usually understood as total morphisms in $\mathcal{R}$).

The SPO-HLR conditions introduced so far in the literature are the following ones.

(HLR1) There exists a pushout in $\mathcal{R}$ of every rule $r : \mathbf{L} \to \mathbf{R}$ and every occurrence $f : \mathbf{L} \to \mathbf{G}$.

(HLR2) $\mathcal{O}$ has all binary coproducts, which are preserved by the inclusion functor $\mathcal{O} \hookrightarrow \mathcal{R}$.

(HLR3) $\mathcal{O}$ has all binary pushouts, which are preserved by the inclusion functor $\mathcal{O} \hookrightarrow \mathcal{R}$.

(HLR4) If $f, g \in \mathcal{R}$ and $f \circ g \in \mathcal{O}$ then $g \in \mathcal{O}$.

(HLR5) There exists a rule $r : \mathbf{A} \to \mathbf{B}$ in $\mathcal{R}$ for every two objects $\mathbf{A}, \mathbf{B}$ in $\mathcal{R}$.

(HLR6) If f is a clean occurrence for $r : \mathbf{L} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{S}$ is an occurrence, then f is also clean for $g \circ r$.

Recall that an occurrence $f : \mathbf{L} \to \mathbf{G}$ is *clean* for a rule $r : \mathbf{L} \to \mathbf{R}$ if whenever r is factorized into two rules, $r = q \circ p$ with $\mathbf{L} \xrightarrow{p} \mathbf{S} \xrightarrow{q} \mathbf{R}$, and if the square (1) in

$$\begin{tikzcd}
\mathbf{L} \arrow[r, "p"] \arrow[rr, bend left, "r"] \arrow[d, "f"'] \arrow[dr, phantom, "(1)"] & \mathbf{S} \arrow[r, "q"] \arrow[d, "f'"] & \mathbf{R} \\
\mathbf{G} \arrow[r, "p'"'] & \mathbf{H} &
\end{tikzcd}$$

is a pushout square in $\mathcal{R}$ then the morphism f' is again an occurrence.

Now, it turns out that:

– If the pair $(\mathcal{R}, \mathcal{O})$ satisfies (HLR1) then single-pushout algebraic transformation in $\mathcal{R}$ through occurrences in $\mathcal{O}$ satisfies the *Local Church-Rosser Theorem*; see [8, Th. 3.5].

– If the pair $(\mathcal{R}, \mathcal{O})$ satisfies (HLR1), (HLR2), (HLR4) and (HLR5) then single-pushout algebraic transformation in $\mathcal{R}$ through clean occurrences in $\mathcal{O}$ satisfies the *Parallelism Theorem*; see [8, Th. 3.12] and [4, Prop. 31].

– If the pair $(\mathcal{R}, \mathcal{O})$ satisfies (HLR1), (HLR3), (HLR4) and (HLR6) then single-pushout algebraic transformation in $\mathcal{R}$ through clean occurrences in $\mathcal{O}$ satisfies the *Amalgamation Theorem*; see [4, Props. 46, 47].

In the rest of this section we study the HLR conditions introduced above in the case when $\mathcal{R}$ is a category of spans $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$, for a triple $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfying (S0) to (S3), and $\mathcal{O}$ is (as in the previous section) the image in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$

of $\mathcal{H}$, i.e., the class of all $(\mathcal{M},\mathcal{H})$-spans of the form $(\mathrm{Id}_{\mathbf{A}},\mathbf{A},h):\mathbf{A}\to\mathbf{B}$; for simplicity, we shall make the abuse of notation of denoting this class of occurrences by $\mathcal{H}$. This study consists mainly in translating these conditions to conditions on the triple $(\mathcal{C},\mathcal{M},\mathcal{H})$. So, *we assume in the sequel that the triple* $(\mathcal{C},\mathcal{M},\mathcal{H})$ *satisfies conditions (S0) to (S3)*, relative to a given class $\mathcal{PB}$ of distinguished pullback squares.

In particular, it turns out that in this case some of these HLR conditions are always satisfied. For instance, from Corollary 13 we know that if $(\mathcal{C},\mathcal{M},\mathcal{H})$ satisfies (S0) to (S3) then $(\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H}),\mathcal{H})$ satisfies (HLR3) (and that under (S0) and (S1), conditions (S2) and (S3) are equivalent to (HLR3)).

Notice also that, according to Corollary 11, condition (HLR1) amounts to the existence of a final triple diagram for every span $(m_1,\mathbf{K}_1,h_1):\mathbf{A}\to\mathbf{B}$ and every $\mathcal{H}$-morphism $(\mathrm{Id}_{\mathbf{A}},\mathbf{A},h_2):\mathbf{A}\to\mathbf{C}$. Since it is the basic SPO-HLR condition, in order to simplify some arguments *we assume in the sequel that the pair* $(\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H}),\mathcal{H})$ *satisfies condition (HLR1).*

We first characterize the clean occurrences.

Proposition 25. *An occurrence* $(\mathrm{Id}_{\mathbf{L}},\mathbf{L},f):\mathbf{L}\to\mathbf{G}$ *is clean for a span* $(m,\mathbf{K},h):\mathbf{L}\to\mathbf{R}$ *iff whenever* m *is decomposed in* $\mathcal{M}$ *as* $\mathbf{K}\overset{m_2}{\to}\mathbf{K}'\overset{m_1}{\to}\mathbf{L}$, *there exists a final distinguished pullback complement for* m_1 *and* f *(Definition 21).*

$$
\begin{tikzcd}
\mathbf{L} \arrow[d, "f"'] & \mathbf{K}' \arrow[l, "m_1"] \arrow[d, "f'"] & \mathbf{K} \arrow[l, "m_2"] \arrow[ll, bend right, "m"'] \arrow[r, "h"] & \mathbf{R} \\
\mathbf{G} & \mathbf{D} \arrow[l, "m_1'"] & &
\end{tikzcd}
$$

Proof. Assume first that $(\mathrm{Id}_{\mathbf{L}},\mathbf{L},f):\mathbf{L}\to\mathbf{G}$ is clean for a span $(m,\mathbf{K},h):\mathbf{L}\to\mathbf{R}$, and let $m=m_1\circ m_2$, with $\mathbf{K}\overset{m_2}{\to}\mathbf{K}'\overset{m_1}{\to}\mathbf{L}$. Then $(m,\mathbf{K},h)=(m_2,\mathbf{K},h)\circ(m_1,\mathbf{K}',\mathrm{Id}_{\mathbf{K}'})$ and therefore the pushout in $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$ of $(m_1,\mathbf{K}',\mathrm{Id}_{\mathbf{K}'}):\mathbf{L}\to\mathbf{K}'$ and $(\mathrm{Id}_{\mathbf{L}},\mathbf{L},f):\mathbf{L}\to\mathbf{G}$ must have (up to isomorphism in $\mathcal{S}(\mathcal{C},\mathcal{M},\mathcal{H})$) the form

$$
\begin{tikzcd}
\mathbf{L} & \mathbf{K}' \arrow[l, "m_1"'] \arrow[r, "\mathrm{Id}_{\mathbf{K}'}"] & \mathbf{K}' \\
\mathbf{L} \arrow[u, "\mathrm{Id}_{\mathbf{L}}"] \arrow[d, "f"'] \arrow[ur, phantom, "(1)"] & \mathbf{K}' \arrow[l, "m_1"] \arrow[r, "\mathrm{Id}_{\mathbf{K}'}"'] \arrow[u, "\mathrm{Id}_{\mathbf{K}'}"] \arrow[d, "f'"] \arrow[ur, phantom, "(2)"] & \mathbf{K}' \arrow[u, "\mathrm{Id}_{\mathbf{K}'}"'] \arrow[d, "f'"] \\
\mathbf{G} \arrow[ur, phantom, "(3)"] & \mathbf{D} \arrow[l, "m_1'"] \arrow[r, "\mathrm{Id}_{\mathbf{D}}"'] & \mathbf{D}
\end{tikzcd}
$$

And as we already found in the proof of Proposition 23, if squares (1), (2) and (3) form a final triple diagram for $(m_1,\mathbf{K}',\mathrm{Id}_{\mathbf{K}'})$ and $(\mathrm{Id}_{\mathbf{L}},\mathbf{L},f)$ then $(\mathbf{D},m_1':\mathbf{D}\to\mathbf{G},f':\mathbf{K}'\to\mathbf{D})$ is a final distinguished pullback complement for m_1 and f. This proves the "only if" implication.

As far as the "if" implication goes, if $(m, \mathbf{K}, h) = (m_2, \mathbf{K}'', h_2) \circ (m_1, \mathbf{K}', h_1)$ then there exists a $\mathcal{M}$-morphism $m' : \mathbf{K} \to \mathbf{K}'$ such that $m = m_1 \circ m'$. And then, the existence of a final distinguished pullback complement $(\mathbf{D}, m_1' : \mathbf{D} \to \mathbf{G}, f' : \mathbf{K}' \to \mathbf{D})$ for f and m_1 implies that

$$\begin{tikzcd}
\mathbf{L} & \mathbf{K}' \arrow[l, "m_1"'] \arrow[r, "h_1"] & \mathbf{S} \\
\mathbf{L} \arrow[u, "\mathrm{Id}_{\mathbf{L}}"] \arrow[d, "f"'] & \mathbf{K}' \arrow[l, "m_1"] \arrow[r, "h_1"'] \arrow[u, "\mathrm{Id}_{\mathbf{K}'}"] \arrow[d, "f'"] & \mathbf{S} \arrow[u, "\mathrm{Id}_{\mathbf{S}}"'] \\
\mathbf{G} & \mathbf{D} \arrow[l, "m_1'"] &
\end{tikzcd}$$

is a final triple diagram for $(m_1, \mathbf{K}', h_1)$ and $(\mathrm{Id}_{\mathbf{L}}, \mathbf{L}, f)$, and therefore that a pushout of these spans is obtained by completing it with a pushout in $\mathcal{H}$ of h_1 and f'. $\square$

Example 26. In the case of partial mappings of sets, this proposition says that a total mapping $(\mathrm{Id}_L, L, f) : L \to G$ is a clean occurrence for a partial mapping $r = (i, K, h) : L \to R$ iff for every set K' such that $K \subseteq K' \subseteq L$, we have that $f^{-1}(f(K')) = K'$ (see Example 22). This condition is clearly equivalent to the d-injectivity of f w.r.t. r [10, Def. 3.10.(2)], which is the well-known characterization of cleanness of f for r in this case; see for instance Prop. 36 and the Remark after Prop. 39 in [4]. $\square$

With this characterization of clean occurrences we have the following result.

Corollary 27. *The pair $(\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H}), \mathcal{H})$ always satisfies (HLR6).*

Proof. The description of clean occurrences given in Proposition 25 entails that $(\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H}), \mathcal{H})$ always satisfies (HLR6), because $(\mathrm{Id}_{\mathbf{R}}, \mathbf{R}, g) \circ (m, \mathbf{K}, h) = (m, \mathbf{K}, g \circ h)$, and cleanness only depends on the left-hand side $\mathcal{M}$-morphism of a span. $\square$

It remains to discuss conditions (HLR2), (HLR4) and (HLR5).

Condition (HLR5) is so simple that it cannot be greatly improved.

As to (HLR2), arguments similar to those leading to Corollary 13 (scattered along the proofs of Propositions 8, 10 and 12) prove the following result, whose details we leave to the reader.

Proposition 28. *The pair $(\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H}), \mathcal{H})$ satisfies condition (HLR2) iff the following two assertions hold:*

i) $\mathcal{H}$ has all coproducts;

ii) For every diagram in $\mathcal{C}$

$$
\begin{tikzcd}
 & & \mathbf{B} \arrow[dd, "h_1"'] & \\
 & & & \mathbf{B}' \arrow[ul, "m_1"'] \arrow[dd, "h'_1"] \\
\mathbf{C} \arrow[rr, "h_2"] & & \mathbf{D} & \\
 & \mathbf{C}' \arrow[ul, "m_2"] \arrow[rr, "h'_2"'] & & \mathbf{D}'
\end{tikzcd}
\qquad (***)
$$

where $m_1, m_2 \in \mathcal{M}$ and $(\mathbf{D}, h_1 : \mathbf{B} \to \mathbf{D}, h_2 : \mathbf{C} \to \mathbf{D})$ is a coproduct in $\mathcal{H}$ of $\mathbf{B}$ and $\mathbf{C}$, the following two properties are satisfied:

a) If $(\mathbf{D}', h'_1 : \mathbf{B}' \to \mathbf{D}', h'_2 : \mathbf{C}' \to \mathbf{D}')$ is a coproduct in $\mathcal{H}$ of $\mathbf{B}'$ and $\mathbf{C}'$ then there exist a unique isomorphism $h : \mathbf{D}' \to \widehat{\mathbf{D}}$ in $\mathcal{H}$ and a unique $\mathcal{M}$-morphism $m : \widehat{\mathbf{D}} \to \mathbf{D}$ such that the following squares are distinguished pullback squares.

$$
\begin{tikzcd}
\mathbf{B}' \arrow[r, "m_1"] \arrow[d, "h \circ h'_1"'] & \mathbf{B} \arrow[d, "h_1"] \\
\widehat{\mathbf{D}} \arrow[r, "m"'] & \mathbf{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{C}' \arrow[r, "m_2"] \arrow[d, "h \circ h'_2"'] & \mathbf{C} \arrow[d, "h_2"] \\
\widehat{\mathbf{D}} \arrow[r, "m"'] & \mathbf{D}
\end{tikzcd}
$$

b) If there exists a $\mathcal{M}$-morphism $m_0 : \mathbf{D}' \to \mathbf{D}$ such that

$$
\begin{tikzcd}
\mathbf{B}' \arrow[r, "m_1"] \arrow[d, "h'_1"'] & \mathbf{B} \arrow[d, "h_1"] \\
\mathbf{D}' \arrow[r, "m_0"'] & \mathbf{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{C}' \arrow[r, "m_2"] \arrow[d, "h'_2"'] & \mathbf{C} \arrow[d, "h_2"] \\
\mathbf{D}' \arrow[r, "m_0"'] & \mathbf{D}
\end{tikzcd}
$$

are distinguished pullback squares, then $(\mathbf{D}', h'_1 : \mathbf{B}' \to \mathbf{D}', h'_2 : \mathbf{C}' \to \mathbf{D}')$ is a coproduct of $\mathbf{B}'$ and $\mathbf{C}'$ in $\mathcal{H}$. $\Box$

Remark 29. Since $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfies (S3), simple category-theoretical considerations show that if $\mathcal{H}$ has an initial object $\mathbf{I}$ such that if $m : \mathbf{A} \to \mathbf{I}$ is a $\mathcal{M}$-morphism then $\mathbf{A} = \mathbf{I}$ (and $m = \mathrm{Id}_{\mathbf{I}}$), then $(\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H}), \mathcal{H})$ satisfies (HLR2):

just translate diagram $(***)$ into

$$\begin{tikzcd}
\mathbf{I} \arrow[rr] \arrow[dd] & & \mathbf{B} \arrow[dd, "h_1"] & \\
& \mathbf{I} \arrow[ul, "\mathrm{Id}_{\mathbf{I}}"'] \arrow[rr] \arrow[dd] & & \mathbf{B}' \arrow[ul, "m_1"'] \arrow[dd, "h'_1"] \\
\mathbf{C} \arrow[rr, "h_2"'] & & \mathbf{D} & \\
& \mathbf{C}' \arrow[ul, "m_2"] \arrow[rr, "h'_2"'] & & \mathbf{D}'
\end{tikzcd}$$

where the arrows with source object $\mathbf{I}$ correspond to the unique morphisms given by the initiality of $\mathbf{I}$.

This condition is, of course, not necessary. □

As to (HLR4), we have the following very simple characterization.

Proposition 30. *The pair $(\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H}), \mathcal{H})$ satisfies condition (HLR4) iff the only one-side invertible $\mathcal{M}$-morphisms are the identities.*

Proof. Assume $(\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H}), \mathcal{H})$ satisfies condition (HLR4), and let $m : \mathbf{A} \to \mathbf{B}$, $m' : \mathbf{B} \to \mathbf{A}$ such that $m' \circ m = \mathrm{Id}_{\mathbf{A}}$. Then $(m, \mathbf{A}, \mathrm{Id}_{\mathbf{A}}) \circ (m', \mathbf{B}, \mathrm{Id}_{\mathbf{B}}) = (\mathrm{Id}_{\mathbf{A}}, \mathbf{A}, \mathrm{Id}_{\mathbf{A}})$ is an occurrence, and therefore $(m', \mathbf{B}, \mathrm{Id}_{\mathbf{B}})$ must be an occurrence: m' must be an identity.

Conversely, if the only one-side invertible $\mathcal{M}$-morphisms are the identities, and $(m_2, \mathbf{K}_2, h_2) \circ (m_1, \mathbf{K}_1, h_1) = (\mathrm{Id}_{\mathbf{A}}, \mathbf{A}, h)$ then there exists some $m'_2 \in \mathcal{M}$ such that $m_1 \circ m'_2 = \mathrm{Id}_{\mathbf{A}}$, and therefore $m_1 = \mathrm{Id}_{\mathbf{A}}$: i.e., $(m_1, \mathbf{K}_1, h_1)$ is an occurrence. □

Example 31. The triple $(S\text{-}\mathbf{Sets}, \mathcal{I}, \mathcal{H})$ satisfies (HLR1) —see Example 17—, (HLR2) —because the initial object $\emptyset$ of S-**Sets** satisfies that if $m : A \to \emptyset$ is an inclusion then $A = \emptyset$, and then Remark 29 applies—, (HLR4) —because every one-side invertible inclusion is an identity—, and (HLR5) —for every sets A and B, we always have the empty partial mapping from A to B—, as well as (HLR3) and (HLR6).

6 Conclusions

This is the first of a series of two papers on pushouts of partial homomorphisms, and their application in single-pushout algebraic transformation. In this paper we start the study of pushouts in categories of spans, with the ultimate goal of understanding the similarities in the construction of pushouts in different categories of partial homomorphisms and, in general, in the main features of the single-pushout transformation systems based on them. The analysis of specific categories of spans, corresponding to well-known categories of partial homomorphisms of algebras and hypergraphs, will be carried out in Part II [12], while in this paper we deal with general categories of spans.

Corollary 11 states that in certain categories of spans, namely those categories $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$ such that the original triple $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfies conditions (S0) to (S3) in Section 2, the existence of a pushout of two given spans is equivalent to the existence of a certain final diagram for them in the original category $\mathcal{C}$. Among conditions (S0) to (S3), conditions (S0) and (S1) are used to define the category $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$ itself, (S2) asserts that the subcategory $\mathcal{H}$ of $\mathcal{C}$ has all pushouts, and (S3) is a technical condition that, for triples $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfying (S0) to (S2), turns out to be equivalent to the preservation of all pushouts by the embedding functor $\mathcal{H} \hookrightarrow \mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$.

In [11], a completely different approach to the conditions implying the existence of pushouts of spans was undertaken. It was based on a condition stronger than the following one, which replaces (S2) and (S3) in Section 2.

(S2-alt) A class $\mathcal{PO}$ of pushout squares in $\mathcal{H}$

$$\begin{tikzcd} \mathbf{A} \arrow[r, "h_1"] \arrow[d, "h_2"'] & \mathbf{B} \arrow[d, "h_3"] \\ \mathbf{C} \arrow[r, "h_4"'] & \mathbf{D} \end{tikzcd}$$

can be fixed such that:

(S2-alt.a) For every pair of $\mathcal{H}$-morphisms $h_1 : \mathbf{A} \to \mathbf{B}$ and $h_2 : \mathbf{A} \to \mathbf{C}$ there exists one, and only one, pushout square for them in the class $\mathcal{PO}$ (referred to henceforth as their *distinguished pushout square*).

(S2-alt.b) In every cubic diagram in $\mathcal{C}$

$$\begin{tikzcd} \mathbf{A} \arrow[rr, "h_1"] \arrow[dd, "h_2"'] & & \mathbf{B} \arrow[dd, "h_3"] & \\ & \mathbf{A}' \arrow[ul, "m_1"] \arrow[rr, "h_1'"] \arrow[dd, "h_2'"] & & \mathbf{B}' \arrow[ul, "m_2"'] \arrow[dd, "h_3'"] \\ \mathbf{C} \arrow[rr, "h_4"'] & & \mathbf{D} & \\ & \mathbf{C}' \arrow[ul, "m_3"] \arrow[rr, "h_4'"'] & & \mathbf{H} \end{tikzcd}$$

where $m_1, m_2, m_3 \in \mathcal{M}$, the back face is a distinguished pushout square, and the top and left-hand side faces are distinguished pullback squares, the following condition holds:

the front face is a distinguished pushout square iff there exists a $\mathcal{M}$-morphism $m_4 : \mathbf{H} \to \mathbf{D}$ such that, when added to this cubic diagram, the bottom and right-hand side faces formed in this way are distinguished pullback squares.

And it turns out that Corollary 11 is still valid for spans in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$ when $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfies conditions (S0), (S1) and (S2-alt).

This last approach has the advantage that, once fixed a class $\mathcal{PO}$ of distinguished pushouts in $\mathcal{H}$, if $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfies condition (S0m) then, taking always as the pushout in $\mathcal{H}$ completing a final triple diagram the corresponding distinguished pushout we obtain a uniquely defined pushout of two spans, which may serve as a basis for an abstract semantics of single-pushout transformation. But, of course, different classes of distinguished pushouts may be fixed in $\mathcal{H}$, yielding different classes of pushouts of spans in $\mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$. And furthermore, it can be difficult to decide whether such a family of distinguished pushouts does exist or not.

On the other hand, when $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfies conditions (S0) and (S1), it turns out that (S2-alt) implies (S2) and (S3). Indeed, (S2-alt.a) clearly implies (S2), and it turns out that if $(\mathcal{C}, \mathcal{M}, \mathcal{H})$ satisfies (S0), (S1) and (S2-alt) then $\mathcal{H} \hookrightarrow \mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$ preserves all pushouts, and therefore (S3) must also be satisfied. However, it is not clear to us the necessity of (S2-alt.b) for the preservation of pushouts by $\mathcal{H} \hookrightarrow \mathcal{S}(\mathcal{C}, \mathcal{M}, \mathcal{H})$.

In Part II we shall consider some well-known categories of partial homomorphisms having all pushouts, and we shall study all of them simultaneously as categories of spans.

References

1. R. Alberich, P. Burmeister, F. Rosselló, G. Valiente, B. Wojdyło, "A Partial Algebras Approach to Graph Transformation." *Proc. 5th Int. Workshop on Graph Grammars and their Applications to Computer Science*, Lect. Notes in Comp. Sc. 1073 (1996), 1–15.
2. P. Burmeister, *A Model Theoretic Oriented Approach to Partial Algebras.* Mathematical Research vol. 32, Akademie-Verlag (1986).
3. P. Burmeister, F. Rosselló, J. Torrens, G. Valiente, "Algebraic Transformation of Unary Partial Algebras I: Double-Pushout Approach". To appear in *Theoretical Computer Science*, 1997. Preprint available as Research Report LSI-96-16-R, Technical University of Catalonia, at http://www-lsi.upc.es/dept/techreps/ps/R96-16.ps.gz.
4. P. Burmeister, M. Monserrat, F. Rosselló, G. Valiente, "Algebraic Transformation of Unary Partial Algebras II: Single-Pushout Approach". To appear in *Theoretical Computer Science*, 1998. Preprint available at http://www-lsi.upc.es/~valiente/atupa2.ps.gz. Preliminary version also available as Research Report LSI-96-36-R, Technical University of Catalonia, at http://www-lsi.upc.es/dept/techreps/ps/R96-36.ps.gz.

5. A. Corradini, H. Ehrig, M. Löwe, U. Montanari, F. Rossi, "Abstract Graph Derivations in the Double-Pushout Approach." *Proc. Dagstuhl Workshop on Graph Transformations in Computer Science*, Springer-Verlag, Lect. Notes in Comp. Sc. 776 (1994), pp. 86–103.
6. A. Corradini, H. Ehrig, M. Löwe, U. Montanari, J. Padberg, "Typed Graph Grammars and their Adjunction with Categories of Derivations." *Proc. 5th Int. Workshop on Graph Grammars and their Applications to Computer Science*, Lect. Notes in Comp. Sc. 1073 (1996), 56–74.
7. H. Ehrig, "Introduction to the Algebraic Theory of Graph Grammars." *Proc. 1st Int. Workshop on Graph Grammars and their Applications to Computer Science*, Springer-Verlag, Lect. Notes in Comp. Sc. 73 (1979), pp. 1–69.
8. H. Ehrig and M. Löwe. "Categorical principles, techniques and results for high-level-replacement systems in computer science." *Applied Categorical Structures*, 1 (1993), 21–50.
9. R. Kennaway, "Graph rewriting in some categories of partial morphisms." *Proc. 4th Int. Workshop on Graph Grammars and their Applications to Computer Science*, Springer-Verlag, Lect. Notes in Comp. Sc. 532 (1991), pp. 490–504.
10. M. Löwe, "Algebraic approach to single-pushout graph transformation." *Theoretical Computer Science* 109 (1993), pp. 181–224.
11. M. Monserrat, F. Rosselló, J. Torrens, G. Valiente, "Hypergraph Rewriting using Conformisms." Electr. Notes in Th. Comp. Sc. 2 (1995), pp. 105–112.
12. M. Monserrat, F. Rosselló, J. Torrens, G. Valiente, "Rewriting in categories of spans II: Some concrete examples." In preparation.
13. J. Padberg, "Survey of High-Level Replacement Systems." Technical Report 93-8, Technical University of Berlin (1993).
14. C. Raoult, "On graph rewritings." Theoretical Computer Science 32 (1984), 1–24.
15. E. Robinson, G. Rosolini, "Categories of partial maps." *Information and Computation* 79 (1988), pp. 95–130.

Appendix: Proof of (S3) for $(S\text{-}\mathbf{Sets}, \mathcal{I}, \mathcal{H})$ Let

$$\begin{tikzcd}
A \arrow[rr, "h_1"] \arrow[dd, "h_2"'] & & B \arrow[dd, "h_3"] & \\
& A' \arrow[ul, "i_1"'] \arrow[rr, "h'_1"] \arrow[dd, "h'_2"] & & B' \arrow[ul, "i_2"'] \arrow[dd, "h'_3"] \\
C \arrow[rr, "h_4"'] & & D & \\
& C' \arrow[ul, "i_3"] \arrow[rr, "h'_4"'] & & D'
\end{tikzcd}$$

be a cubic diagram in S-**Sets** where i_1, i_2, i_3 are inclusions of S-sets, the back square is a pushout, the top and left-hand side squares are distinguished pullbacks and the bottom and right-hand side squares commute.

(S3a) Assume first that the front and the back faces in this cube are distinguished pushouts. Then we have $D' = (B' + C')\big/\theta(A')$, and by hypothesis $D = (B + C)\big/\theta(A)$.[3] Moreover, there exists a unique mapping $f : D' \to D$ that, when added to the previous diagram makes it commutative; such f is defined as follows: $f_s([x']_{\theta(A')}) = [x']_{\theta(A)}$, for every $x' \in B'_s + C'_s$, $s \in S$.

We shall prove now that:

$$\text{If } x' \in B'_s + C'_s \text{ then } [x']_{\theta(A')_s} = [x']_{\theta(A)_s},\ s \in S. \qquad (*)$$

This clearly would entail that f is an inclusion.

Since $\theta(A') \subseteq \theta(A)$, it is enough to prove that if $x' \in B'_s + C'_s$ then $[x']_{\theta(A)_s} \subseteq [x']_{\theta(A)_s}$. So, let $x' \in B'_s + C'_s$, say $x' \in B'_s$ (by symmetry, the case $x' \in C'_s$ will also hold), and let $x \in [x']_{\theta(A)}$, say $x \in C_s$ (a similar argument will prove the case $x \in B_s$). Then, there exist $n \geq 0$ and $a_0, \ldots, a_{2n} \in A_s$ such that $x' = h_{1s}(a_0), x = h_{2s}(a_{2n})$ and

$$h_{1s}(a_{2k}) = h_{1s}(a_{2k+1}), h_{2s}(a_{2k+1}) = h_{2s}(a_{2k+2}) \quad \text{for all } k = 0, \ldots, n-1.$$

And now, a simple argument shows (by induction on k) that $(x, h_{2s}(a_{2k})) \in \theta(A')_s$ for every $k = 0, \ldots, n$. Taking $k = n$, this finally implies $(x, x') \in \theta(A')_s$, as we wanted to prove.

Moreover, if $b' \in B'_s$ then $h_{3s}(b') = [b']_{\theta(A)_s} = [b']_{\theta(A')_s} \in D'_s$, so that $b' \in h_{3s}^{-1}(D'_s)$. And if $b \in h_{3s}^{-1}(D'_s)$ then $h_{3s}(b) = [b]_{\theta(A)_s} \in D'_s$, i.e., $[b]_{\theta(A)_s} = [x']_{\theta(A')_s}$ for some $x' \in B'_s + C'_s$: by (*) above, this implies $b \in B'_s$. Therefore, $B'_s = h_{3s}^{-1}(D'_s)$ and right-hand side square in the cube is a distinguished pullback. A similar argument proves that the bottom square is also a distinguished pullback.

Now consider the general case, when the front and back faces are pushout squares, but not necessarily distinguished. There exists still a unique mapping $f : D' \to D$ that, when added to the previous cubic diagram, makes it commutative. Let

$$(H, g_3 : B \to H, g_4 : C \to H), \qquad (H', g'_3 : B' \to H', g'_4 : C' \to H')$$

be the distinguished pushouts of h_1 and h_2, and of h'_1 and h'_2, respectively. We have just proved that there exists a unique inclusion $i : H' \to H$ such that

$$\begin{tikzcd} B' \arrow[r, "i_2"] \arrow[d, "g'_3"'] & B \arrow[d, "g_3"] \\ H' \arrow[r, "i"'] & H \end{tikzcd} \qquad \begin{tikzcd} C' \arrow[r, "i_3"] \arrow[d, "g'_4"'] & C \arrow[d, "g_4"] \\ H' \arrow[r, "i"'] & H \end{tikzcd}$$

are distinguished pullbacks; and there exist bijections $h' : D' \to H'$ and $h : H \to D$ such that $h' \circ h'_i = g'_i$ and $h \circ g_i = h_i$ for $i = 3, 4$. Let $j : h(H') \to D$ be the inclusion of $h(H')$ into D, and let $\widetilde{h} : H' \to h(H')$ be the restriction of h to H', understood with target set $h(H')$; it is a bijection, and $(H', i : H' \to H, \widetilde{h} : H' \to h(H'))$ is the distinguished pullback of j and h. Then $f = j \circ (\widetilde{h} \circ h')$, where j is an inclusion and $\widetilde{h} \circ h'$ a bijection, and by the compositional properties of distinguished pullbacks we

[3] To simplify the notations, in this proof we shall identify $B_s \times \{1\}$ with B_s, $B'_s \times \{1\}$ with B'_s, $C_s \times \{2\}$ with C_s and $C'_s \times \{2\}$ with C'_s, $s \in S$.

have that

$$\begin{tikzcd} B' \arrow[r, "i_2"] \arrow[d, "(\tilde{h}\circ h')\circ h'_3"'] & B \arrow[d, "h_3"] \\ h(H') \arrow[r, "j"'] & D \end{tikzcd} \qquad \begin{tikzcd} C' \arrow[r, "i_3"] \arrow[d, "(\tilde{h}\circ h')\circ h'_4"'] & C \arrow[d, "h_4"] \\ h(H') \arrow[r, "j"'] & D \end{tikzcd}$$

are distinguished pullbacks. Since the decomposition of a mapping of S-sets as a bijection followed by an inclusion is unique, this finishes the proof of (S3a).

(S3b) Assume now that we have a cube

$$\begin{tikzcd} A \arrow[rr, "h_1"] \arrow[dd, "h_2"'] & & B \arrow[dd, "h_3"] & \\ & A' \arrow[ul, "i_1"] \arrow[rr, "h'_1"] \arrow[dd, "h'_2"'] & & B' \arrow[ul, "i_2"'] \arrow[dd, "h'_3"] \\ C \arrow[rr, "h_4"'] & & D & \\ & C' \arrow[ul, "i_3"] \arrow[rr, "h'_4"'] & & D' \arrow[ul, "i"'] \end{tikzcd}$$

where $i_1, i_2, i_3, i \in \mathcal{I}$, the back face is a pushout square, and the top, bottom, left-hand side and right-hand side faces are distinguished pullbacks; by Remark 6, the front square is commutative, and we want to prove that it is a pushout square.

Consider first the case when the back face is a distinguished pushout square. Reasoning as in the previous implication, we have $[x']_{\theta(A')_s} = [x']_{\theta(A)_s}$ for any $x' \in B'_s + C'_s$; we shall denote it simply by $[x']$. Now let us prove from this statement that $D'_s = (B'_s + C'_s)\big/_{\theta(A')_s}$. Actually, the homomorphisms h'_i are the restrictions of the corresponding homomorphisms h_i for $i = 1, 2, 3, 4$ and then any element $d \in D$ is in D' iff $[d] = [b]_{\theta(A)_s}$ with $h_{3s}(b) \in D'_s$ or $[d] = [c]_{\theta(A)_s}$ with $h_{4s}(c) \in D'_s$. But this holds iff $b \in h_{3s}^{-1}(D'_s) = B'_s$ or $c \in h_{4s}^{-1}(D'_s) = C'_s$, which is equivalent to $[d] \in (B'_s + C'_s)\big/_{\theta(A')_s}$, as we wanted to prove. Finally, from the equality $D' = (B' + C')\big/_{\theta(A')}$ and the commutativity of the diagram, it is clear that the front square in a distinguished pushout.

Now, if the back face is a pushout square, but not necessarily distinguished, then let $(H, g_3 : B \to H, g_4 : C \to H)$ be the distinguished pushout of h_1 and h_2, and let $h : H \to D$ be the bijection such that $h \circ g_3 = h_3$ and $h \circ g_4 = h_4$. Let $(H', h' : H' \to D', j : H' \to H)$ be the distinguished pullback of i and h. Then h' is an isomorphism, and

$$\begin{tikzcd} B' \arrow[r, "i_2"] \arrow[d, "h'^{-1}\circ h'_3"'] & B \arrow[d, "g_3"] \\ H' \arrow[r, "j"'] & H \end{tikzcd} \qquad \begin{tikzcd} C' \arrow[r, "i_3"] \arrow[d, "h'^{-1}\circ h'_4"'] & C \arrow[d, "g_4"] \\ H' \arrow[r, "j"'] & H \end{tikzcd}$$

are distinguished pullbacks. Then, by the case with a distinguished pushout back square already considered, $(H', h'^{-1} \circ h'_3 : B' \to H', h'^{-1} \circ h'_4 : C' \to H')$ is the distinguished pushout of h_3' and h_4', and therefore the front face in the cube above is a pushout square.