Algebraic Transformation of Unary Partial Algebras II: Single-Pushout Approach

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The *single-pushout* approach to graph transformation is extended to the algebraic transformation of partial many-sorted unary algebras. The main result presented in this article is an algebraic characterization of the single-pushout transformation in the categories of all conformisms, all closed quomorphisms, and all closed-domain closed quomorphisms of unary partial algebras over a given signature, together with a corresponding operational characterization that may serve as a basis for implementation. Moreover, all three categories are shown to satisfy all of the HLR (*High Level Replacement*) conditions for parallelism, taking as occurrences the total morphisms in each category. Another important result presented in this article is the definition of HLR conditions for amalgamation, which are also satisfied by the categories of partial homomorphisms considered here, taking again the corresponding total morphisms as occurrences.

1 Introduction

This paper is the second in a series initiated with [6]. The global goal of this series is to study in detail the algebraic transformation of partial unary algebras, both under the double and single-pushout approach, and to apply this study to algebraic (higher-order hyper)graph transformation. See also [1], which surveys some of the main results contained in this series of papers

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(although the approach to single-pushout transformation taken therein is different from the one taken here), [15], which translates some of these results to the setting of categories of *spans*, and [14], where such categories of spans are applied to develop a single-pushout approach to hypergraph rewriting based on conformisms.

We devote this paper to single-pushout transformation systems of partial unary algebras, based on different types of partial homomorphisms. This paper should be viewed as forming a unity with [6], and we shall freely use the notations introduced therein, usually without any further notice. In particular, we assume the reader familiar with the basic language of partial algebras as introduced in Appendix A therein.

Let us recall that in the single-pushout approach to transformation in a category $\mathcal{C}$ (of partial morphisms, whatever it means), a *(single-pushout) production rule* is taken to be simply a morphism $r : L \to R$ in that category, and then the application of such a rule to an object G through an *occurrence* (a total morphism) $m : L \to G$ consists in computing (when possible) the pushout of r and m in $\mathcal{C}$

$$\begin{tikzcd} L \arrow[r, "r"] \arrow[d, "m"'] & R \arrow[d, "m'"] \\ G \arrow[r, "r'"'] & H \end{tikzcd}$$

In this case the object H is said to be *derived* from G by the application of rule $r : L \to R$ through morphism m, or that H is aobtained by means of a *direct derivation* from G by r, and it is denoted in general by $G \overset{r}{\Longrightarrow} H$.

Therefore, if $\mathcal{C}$ is a category closed under binary pushouts then every production rule can be applied to every occurrence, obtaining always a derived object unique up to isomorphism. See [9,12] for more information on single-pushout transformation.

Let now $\Gamma = (S, \Omega, \eta)$ be a graph structure (a signature with all its operations unary). In this paper we lay the basis for a single-pushout approach to partial Γ-algebras transformation, using different types of partial homomorphisms: conformisms, closed quomorphisms (c-quomorphisms, for short) and closed-domain closed quomorphisms (cdc-quomorphisms, for short). They are, respectively, closed homomorphisms from a weak, a relative and a closed subalgebra of the source partial algebra; see §2.1 for details. There is another popular type of partial homomorphism of partial algebras, the (plain) quomorphisms (a plain homomorphism from a relative subalgebra of the source), but in general the pushout of two quomorphisms does not exist, even for partial unary algebras. The case of quomorphisms is treated in [22], where the authors give, among other results, a necessary and sufficient condition on a pair of quomorphisms to have a pushout; we recall it in §2.3.

In this paper we give, for conformisms, c-quomorphisms and cdc-quomorphisms, a detailed description of the pushout of two such partial morphisms as the pushout of two total closed homomorphisms, following the spirit of [11] or [12]. Our construction of pushouts in CF-Alg$_\Gamma$, CQ-Alg$_\Gamma$ and CDCQ-Alg$_\Gamma$, $\Gamma = (S, \Omega, \eta)$ a graph structure, runs roughly as follows. Given two partial mappings of S-sets $f : K \to A$ and $g : K \to B$, let K' be the greatest subset of K such that $f^{-1}(f(K')) = K'$ and $g^{-1}(g(K')) = K'$, let $A' = (A - f(K)) \cup f(K')$ and $B' = (B - g(K)) \cup g(K')$, and let $\theta(K')$ be the greatest equivalence relation on $A \sqcup B$ (the disjoint union of A and B; we identify these sets with their images into that disjoint union) containing $(\{(f_s(k), g_s(k)) \mid k \in K'_s\})_{s \in S}$. It is well-known (see §2.2 below) that the quotient $(A \sqcup B)/_{\theta(K')}$ is the pushout object of f and g in the category of S-sets and partial mappings. Now, if $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ are partial Γ-algebras with carriers K, A and B respectively,

- in CF-Alg$_\Gamma$, the pushout object of two conformisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ is obtained as a quotient $(\mathbf{A}'_w + \mathbf{B}'_w)/_{\theta(K')}$, where $\mathbf{A}'_w$ and $\mathbf{B}'_w$ are suitable weak subalgebras of $\mathbf{A}'$ and $\mathbf{B}'$ supported on A' and B' (and $+$ stands for the disjoint sum of partial Γ-algebras);
- in CQ-Alg$_\Gamma$, the pushout object of two c-quomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ is the quotient $(\mathbf{A}' + \mathbf{B}')/_{\theta(K')}$, where $\mathbf{A}'$ and $\mathbf{B}'$ denote the relative subalgebras of $\mathbf{A}$ and $\mathbf{B}$ supported on A' and B', respectively —this construction also yields, when it exists, the pushout object of two quomorphisms in Q-Alg$_\Gamma$; cf. [22];
- in CDCQ-Alg$_\Gamma$, the pushout object of two cdc-quomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ is obtained as a fixed closed subalgebra of $(\mathbf{A}' + \mathbf{B}')/_{\theta(K')}$.

In all three cases, and contrary to the description of these pushouts given (without any detail) in the introduction [1] to this series of papers, the descriptions given here in all three cases allow a Remove-Add description of the construction of the pushout objects, and enlighten the relation between single and double-pushout transformations. The study we make of this relation sheds some light on the Open Problem 4.5.1 in [9], which asks for formal comparisons of double-pushout and single-pushout transformation systems.

The contents of this paper are briefly described as follows. For the convenience of the reader, in Section 2 we collect some preliminaries. Namely, on the one hand we recall the main concepts and results on partial homomorphisms of partial algebras used in this paper, and on the other hand we give a detailed description of the pushout of two partial mappings of S-sets as it shall be used in the remaining sections. We also recall the construction of the pushout of quomorphisms given in [22], expressing it in a language similar to the one used in our constructions.

Then, we devote a section to each type of partial homomorphism considered in this paper. The sections dealing with the different types of partial homomorphisms are ordered from the more general case to the more particular one: first conformisms, then c-quomorphisms and finally cdc-quomorphisms.

In the last section we show that the HLR conditions for parallelism introduced in [9] are satisfied for conformisms, c-quomorphisms and cdc-quomorphisms, taking as occurrences in any case the totally defined such morphisms. This gives three more examples of single-pushout transformation systems satisfying them, examples that are asked for in the Open Problem 4.5.4 in [9].

Moreover, we define HLR conditions for amalgamation and show that they are satisfied for conformisms and cdc-quomorphisms, again taking the corresponding total morphisms as occurrences. This allows to decompose non parallel-independent derivations by rules sharing a common subrule into a common derivation (by that subrule) followed by parallel-independent derivations, where such common derivations induce a global synchronization mechanism.

2 Preliminaries

2.1 Partial homomorphisms of partial algebras

For partial algebras, different notions of partial homomorphisms have been studied in the literature. In this subsection we recall the most popular ones, for which we use the names given in [8] and its "addendum" [2].

Given a signature $\Sigma = (S, \Omega, \eta)$, where $\eta(\varphi) = (w(\varphi), \sigma(\varphi))$, let $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ and $\mathbf{B} = (B, (\varphi^{\mathbf{B}})_{\varphi\in\Omega})$ be two partial Σ-algebras, and let $f : A \to B$ be a *partial mapping* of S-sets, i.e. a family of partial mappings

$$f_s : A_s \to B_s,\ \ s \in S.$$

Let $\operatorname{Dom} f_s$ denote the domain of f_s, for every $s \in S$, and let $\operatorname{Dom} f = (\operatorname{Dom} f_s)_{s\in S}$ be the domain of f.

f is a *quomorphism* when it is a homomorphism from the relative subalgebra of $\mathbf{A}$ supported on $\operatorname{Dom} f$ to $\mathbf{B}$. That is, when it satisfies the following condition: for every $\varphi \in \Omega$ and $\underline{a} \in (\operatorname{Dom} f)^{w(\varphi)}$, if $\underline{a} \in \operatorname{dom} \varphi^{\mathbf{A}}$ and $\varphi^{\mathbf{A}}(\underline{a}) \in (\operatorname{Dom} f)_{\sigma(\varphi)}$ then $f(\underline{a}) \in \operatorname{dom} \varphi^{\mathbf{B}}$ and $\varphi^{\mathbf{B}}(f(\underline{a})) = f_{\sigma(\varphi)}(\varphi^{\mathbf{A}}(\underline{a}))$.

f is a *conformism*[1] when it is a closed homomorphism from some weak subal-

[1] p-morphism, in [18]

gebra of $\mathbf{A}$ supported on $\operatorname{Dom} f$ to $\mathbf{B}$. That is, when it satisfies the following condition: for every $\varphi \in \Omega$ and $\underline{a} \in (\operatorname{Dom} f)^{w(\varphi)}$, if $f(\underline{a}) \in \operatorname{dom} \varphi^{\mathbf{B}}$ then $\underline{a} \in \operatorname{dom} \varphi^{\mathbf{A}}$, $\varphi^{\mathbf{A}}(\underline{a}) \in (\operatorname{Dom} f)_{\sigma(\varphi)}$ and $\varphi^{\mathbf{B}}(f(\underline{a})) = f_{\sigma(\varphi)}(\varphi^{\mathbf{A}}(\underline{a}))$.

f is a *closed quomorphism*[2], *c-quomorphism* for short, when it is a closed homomorphism from the relative subalgebra of $\mathbf{A}$ supported on $\operatorname{Dom} f$ to $\mathbf{B}$. That is, when it satisfies the following condition: for every $\varphi \in \Omega$ and $\underline{a} \in (\operatorname{Dom} f)^{w(\varphi)}$, one has $\underline{a} \in \operatorname{dom} \varphi^{\mathbf{A}}$ and $\varphi^{\mathbf{A}}(\underline{a}) \in (\operatorname{Dom} f)_{\sigma(\varphi)}$ iff $f(\underline{a}) \in \operatorname{dom} \varphi^{\mathbf{B}}$; and if $f(\underline{a}) \in \operatorname{dom} \varphi^{\mathbf{B}}$ then $\varphi^{\mathbf{B}}(f(\underline{a})) = f_{\sigma(\varphi)}(\varphi^{\mathbf{A}}(\underline{a}))$.

f is a *closed-domain closed quomorphism*[3], *cdc-quomorphism* for short, when it is a closed quomorphism with $\operatorname{Dom} f$ a closed subset of $\mathbf{A}$. That is, when it satisfies the following condition: for every $\varphi \in \Omega$ and $\underline{a} \in (\operatorname{Dom} f)^{w(\varphi)}$, one has $\underline{a} \in \operatorname{dom} \varphi^{\mathbf{A}}$ iff $f(\underline{a}) \in \operatorname{dom} \varphi^{\mathbf{B}}$; and if $f(\underline{a}) \in \operatorname{dom} \varphi^{\mathbf{B}}$ then $\varphi^{\mathbf{A}}(\underline{a}) \in (\operatorname{Dom} f)_{\sigma(\varphi)}$ and $\varphi^{\mathbf{B}}(f(\underline{a})) = f_{\sigma(\varphi)}(\varphi^{\mathbf{A}}(\underline{a}))$.

Other types of partial homomorphisms are considered in [2], [3, §22.1], and [7,8,10,16].

For a given signature Σ, we shall denote by Q-Alg$_\Sigma$, CF-Alg$_\Sigma$, CQ-Alg$_\Sigma$ and CDCQ-Alg$_\Sigma$ the categories whose objects are all partial Σ-algebras and whose morphisms are, respectively, the quomorphisms, conformisms, c-quomorphisms and cdc-quomorphisms. Notice in particular that the following equalities and inclusions hold:

$$\text{C-Alg}_\Sigma \subseteq \text{CDCQ-Alg}_\Sigma \subseteq \text{CQ-Alg}_\Sigma \subseteq \text{CF-Alg}_\Sigma$$
$$\text{Alg}_\Sigma \subseteq \text{Q-Alg}_\Sigma$$
$$\text{Q-Alg}_\Sigma \cap \text{CF-Alg}_\Sigma = \text{CQ-Alg}_\Sigma$$

(recall from [6] that Alg$_\Sigma$ and C-Alg$_\Sigma$ denote the categories whose objects are all partial Σ-algebras and whose morphisms are the plain homomorphisms and the closed homomorphisms, respectively). Furthermore, total (i.e., totally defined) quomorphisms are homomorphisms and total c-quomorphisms and cdc-quomorphisms are closed homomorphisms, while a total conformism $f : \mathbf{A} \to \mathbf{B}$ need not be even a homomorphism from $\mathbf{A}$ to $\mathbf{B}$. Let TCF-Alg$_\Sigma$ denote the category of all partial Σ-algebras with total conformisms as morphisms.

The categories CF-Alg$_\Sigma$, Q-Alg$_\Sigma$, CQ-Alg$_\Sigma$, CDCQ-Alg$_\Sigma$ and TCF-Alg$_\Sigma$ are thoroughly studied in [2,5,7,8,13,16]. In particular:

– Q-Alg$_\Sigma$ is never complete or cocomplete if $\Omega \neq \emptyset$ [8].

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[2] quomorphic conformism, in [1]
[3] closed-domain quomorphic conformism, in [1]

– For every one of the categories CF-Alg_Σ [8], CQ-Alg_Σ [8], CDCQ-Alg_Σ [2] and TCF-Alg_Σ [16], one has that it is complete and cocomplete iff Σ is a graph structure.

We shall denote by P-TAlg_Σ the category of total Σ-algebras with partial homomorphisms in the sense of [12] [4]: homomorphisms from subalgebras of the source algebra. Since the subalgebras of a total algebra are exactly its closed subalgebras, and since for total algebras the concepts of homomorphism (in the total algebras sense) and closed homomorphism are exactly the same, it turns out that P-TAlg_Σ is a full subcategory of CDCQ-Alg_Σ. It is proved in [12] that P-TAlg_Σ is cocomplete iff Σ is a graph structure [5].

2.2 Pushouts of partial mappings

We recall here a description of the pushout of two partial mappings of S-sets that lies at the basis of all constructions of pushouts that appear in the next sections. We shall freely use in the rest of this paper the notations introduced in this subsection, usually without any further notice.

Given a non-empty set S (of sorts), let S-**Set** and S-**PSet** denote the categories of S-sets with total and partial mappings, respectively, as morphisms.

Let $f : K \to A$ and $g : K \to B$ be two partial mappings of S-sets. Let $\theta(K)'$ be the least equivalence relation on the disjoint union $A \sqcup B$ such that it contains [6]

$$\Big(\Big\{(f_s(x), g_s(x)) \in A_s \times B_s \subseteq (A_s \sqcup B_s)^2 \mid x \in \mathrm{Dom}\, f_s \cap \mathrm{Dom}\, g_s\Big\}\Big)_{s\in S}$$

Definition 1 *The* gluing set *of f and g is the greatest subset $K' = (K'_s)_{s\in S}$ of K compatible with $\theta(K)'$; i.e.*

$$\begin{aligned}K'_s = \Big\{x \in \mathrm{Dom}\, f_s \cap \mathrm{Dom}\, g_s \mid\ & f_s^{-1}([f_s(x)]_{\theta(K)'_s}) \cup g_s^{-1}([g_s(x)]_{\theta(K)'_s}) \\ & \subseteq \mathrm{Dom}\, f_s \cap \mathrm{Dom}\, g_s\}\end{aligned}$$

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[4] It is denoted $\mathrm{Alg}^{\mathrm{P}}(\Sigma)$ in *loc. cit.*

[5] Actually, it is proved in [12] that P-TAlg_Σ is *finitely* cocomplete iff Σ is a graph structure, but the construction of a binary pushout of partial homomorphisms of unary algebras given therein can be easily generalized to the construction of an arbitrary pushout.

[6] As in [6], for the sake of simplicity we shall always identify an element of a given set with its image in the disjoint union of this set with other sets.

The gluing set of two partial mappings can also be characterized by means of the following property, whose easy proof we leave to the reader.

Lemma 2 *The gluing set K' of f and g is the greatest subset of K such that $f^{-1}(f(K')) = K'$ and $g^{-1}(g(K')) = K'$.* $\square$

Set now $A' = (A - f(K)) \cup f(K')$ and $B' = (B - g(K)) \cup g(K')$ and consider the restrictions

$$f|_{K'} : K' \to A', \quad g|_{K'} : K' \to B'$$

of f and g to K'. Set $H = (A' \sqcup B')/_{\theta(K')}$, where $\theta(K')$ stands for the least equivalence relation on $A' \sqcup B'$ containing

$$(\{(f_s(x), g_s(x)) \mid x \in K'_s\})_{s \in S},$$

and let $\tilde{f} : B' \to H$ and $\tilde{g} : A' \to H$ denote the restrictions to B' and A', respectively, of the quotient map $A' \sqcup B' \to (A' \sqcup B')/_{\theta(K')}$. Then the set H, together with the mappings $\tilde{f} : B' \to H$ and $\tilde{g} : A' \to H$, is a pushout of $f|_{K'} : K' \to A'$ and $g|_{K'} : K' \to B'$ in S-**Set**; we shall refer to this construction as the *usual pushout* of $f|_{K'}$ and $g|_{K'}$ in S-**Set**.

Let $\tilde{g} : A \to H$ and $\tilde{f} : B \to H$ also denote the partial mappings corresponding to the homonymous total mappings in the corresponding square, with domains $\mathrm{Dom}\,\tilde{g} = A'$ and $\mathrm{Dom}\,\tilde{f} = B'$ respectively.

Proposition 3 *The square*

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "\tilde{g}"] \\ B \arrow[r, "\tilde{f}"'] & H \end{tikzcd}$$

*is a pushout square in S-**PSet**.*

PROOF. If Γ is a graph structure with set of sorts S and empty set of operations, then the pushout of two partial homomorphisms of total Γ-algebras (i.e., two partial mappings of S-sets) described in [12, Cons. 2.6] (cf. [19, Prop. 3]) amounts to the construction summarized in this proposition. $\square$

From this description of the pushout of two partial mappings of S-sets the following properties, which shall be used later, are easily deduced; cf. [12, Cor. 2.8].

Corollary 4 *Let H, together with $\tilde{g} : A \to H$ and $\tilde{f} : B \to H$ be a pushout in S-**PSet** of $f : K \to A$ and $g : K \to B$.*

i) The partial mappings $\tilde{g}$ and $\tilde{f}$ are jointly surjective ($\tilde{g}(A) \cup \tilde{f}(B) = H$).

ii) The partial mapping $\tilde{g}$ is total iff $g^{-1}(g(\mathrm{Dom}\, f)) = \mathrm{Dom}\, f$.

iii) If f and g are total then $\tilde{g}$ and $\tilde{f}$ are also total. $\Box$

2.3 Pushouts of quomorphisms

It is well known [8,22] that for every signature Σ with non-empty set of operations (even a graph structure), there exist quomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ of partial Σ-algebras whose pushout in Q-Alg_Σ does not exist. A. Wagner and M. Gogolla give in [22] a characterization of those pairs of quomorphisms that have a pushout. Although we shall not use it in this paper, we recall this characterization here because of its close relation with our constructions.

Proposition 5 [22, Thm. 7] *Let $\Sigma = (S, \Omega, \eta)$ be a signature with $\Omega \neq \emptyset$, and let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two quomorphisms of partial Σ-algebras. Let H, together with $\tilde{f} : B \to H$ and $\tilde{g} : A \to H$, be the pushout of f and g in S-**PSet** described in Proposition 3, and let $\mathbf{A}'$ and $\mathbf{B}'$ denote respectively the relative subalgebras of $\mathbf{A}$ and $\mathbf{B}$ supported on the domains A' and B' of $\tilde{g}$ and $\tilde{f}$.*

Then, the quomorphisms f and g have pushout in Q-Alg_Σ iff for every $\varphi \in \Omega$, the following description of $\varphi^{\mathbf{H}}$ yields a well defined partial mapping $\varphi^{\mathbf{H}} : H^{w(\varphi)} \to H_{\sigma(\varphi)}$: the domain of $\varphi^{\mathbf{H}}$ is

$$\mathrm{dom}\, \varphi^{\mathbf{H}} = \left\{ \tilde{f}(\underline{b}) \mid \underline{b} \in \mathrm{dom}\, \varphi^{\mathbf{B}'} \right\} \cup \left\{ \tilde{g}(\underline{a}) \mid \underline{a} \in \mathrm{dom}\, \varphi^{\mathbf{A}'} \right\}$$

and if $\underline{b} \in \mathrm{dom}\, \varphi^{\mathbf{B}'}$ (resp. $\underline{a} \in \mathrm{dom}\, \varphi^{\mathbf{A}'}$) then

$$\varphi^{\mathbf{H}}(\tilde{f}(\underline{b})) = \tilde{f}_{\sigma(\varphi)}(\varphi^{\mathbf{B}'}(\underline{b})) \quad (\textit{resp. } \varphi^{\mathbf{H}}(\tilde{g}(\underline{a})) = \tilde{g}_{\sigma(\varphi)}(\varphi^{\mathbf{A}'}(\underline{a}))).$$

And when it exists, the pushout of f and g in Q-Alg_Σ is given by $\mathbf{H} = (H, (\varphi^{\mathbf{H}})_{\varphi \in \Omega})$, together with the quomorphisms $\tilde{f} : \mathbf{B} \to \mathbf{H}$ and $\tilde{g} : \mathbf{A} \to \mathbf{H}$. $\Box$

This proposition can be rephrased as follows.

Proposition 6 *With the notations of the previous proposition, $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ have pushout in Q-Alg_Σ iff $\theta(K')$ (see §2.2) is a congruence on the coproduct $\mathbf{A}' + \mathbf{B}'$. And when f and g have pushout, it is $\mathbf{H} = (\mathbf{A}' + \mathbf{B}')/_{\theta(K')}$ together with the quomorphisms $\tilde{f} : \mathbf{B} \to \mathbf{H}$ and*

$\tilde{g} : \mathbf{A} \to \mathbf{H}$ *given respectively by the compositions*

$$\mathbf{B}' \hookrightarrow \mathbf{A}' + \mathbf{B}' \stackrel{\mathrm{nat}_{\theta(K')}}{\to} (\mathbf{A}' + \mathbf{B}')/_{\theta(K')}, \quad \mathbf{A}' \hookrightarrow \mathbf{A}' + \mathbf{B}' \stackrel{\mathrm{nat}_{\theta(K')}}{\to} (\mathbf{A}' + \mathbf{B}')/_{\theta(K')}.$$

□

3 Single-pushout transformation in CF-Alg$_\Gamma$

Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two conformisms of partial Γ-algebras. In order to compute their pushout in CF-Alg$_\Gamma$, we shall define suitable weak subalgebras $\mathbf{K}'_w$, $\mathbf{A}'_w$ and $\mathbf{B}'_w$ of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ supported respectively on the sets K', A' and B' defined in §2.2, in such a way that $f|_{K'}$ and $g|_{K'}$ become closed homomorphisms between them and their pushout in C-Alg$_\Gamma$ turns out to yield the desired pushout of f and g in CF-Alg$_\Gamma$. Such $\mathbf{K}'_w$, $\mathbf{A}'_w$ and $\mathbf{B}'_w$ will actually be the greatest weak subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ supported on K', A' and B', respectively, such that $f|_{K'}$ and $g|_{K'}$ are closed homomorphisms between them.

Specifically, let $\mathbf{K}'_w$, $\mathbf{A}'_w$ and $\mathbf{B}'_w$ be the following weak subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$:

- $\mathbf{K}'_w = (K', (\varphi^{\mathbf{K}'_w})_{\varphi\in\Omega})$ is the weak subalgebra of $\mathbf{K}$ supported on K', whose operations $\varphi^{\mathbf{K}'_w}$, for every $\varphi \in \Omega$ with $\eta(\varphi) = (s, s')$, have their domain given by
$$\operatorname{dom}\varphi^{\mathbf{K}'_w} = \Big\{x \in K'_s \mid f_s(x) \in \operatorname{dom}\varphi^{\mathbf{A}}, g_s(x) \in \operatorname{dom}\varphi^{\mathbf{B}} \text{ and } \varphi^{\mathbf{K}}(x) \in K'_{s'}\Big\}$$
(recall that, since f and g are conformisms, if $f_s(x) \in \operatorname{dom}\varphi^{\mathbf{A}}$ or $g_s(x) \in \operatorname{dom}\varphi^{\mathbf{B}}$ then $x \in \operatorname{dom}\varphi^{\mathbf{K}}$).
- $\mathbf{A}'_w = (A', (\varphi^{\mathbf{A}'_w})_{\varphi\in\Omega})$ is the weak subalgebra of $\mathbf{A}$ supported on A', whose operations $\varphi^{\mathbf{A}'_w}$, for every $\varphi \in \Omega$ with $\eta(\varphi) = (s, s')$, have their domain given by
$$\begin{aligned}\operatorname{dom}\varphi^{\mathbf{A}'_w} = {} & \Big\{a \in A_s - f_s(K_s) \mid a \in \operatorname{dom}\varphi^{\mathbf{A}} \text{ and } \varphi^{\mathbf{A}}(a) \in A'_{s'}\Big\} \cup \\ & \Big\{f_s(x) \in f_s(K'_s) \mid x \in \operatorname{dom}\varphi^{\mathbf{K}'_w}\Big\}\end{aligned}$$
- In a similar way, $\mathbf{B}'_w = (B', (\varphi^{\mathbf{B}'_w})_{\varphi\in\Omega})$ is the weak subalgebra of $\mathbf{B}$ supported on B', whose operations $\varphi^{\mathbf{B}'_w}$, for every $\varphi \in \Omega$ with $\eta(\varphi) = (s, s')$, have their domain given by
$$\begin{aligned}\operatorname{dom}\varphi^{\mathbf{B}'_w} = {} & \Big\{b \in B_s - g_s(K_s) \mid b \in \operatorname{dom}\varphi^{\mathbf{B}} \text{ and } \varphi^{\mathbf{B}}(b) \in B'_{s'}\Big\} \cup \\ & \Big\{g_s(x) \in g_s(K'_s) \mid x \in \operatorname{dom}\varphi^{\mathbf{K}'_w}\Big\}\end{aligned}$$

Lemma 7 *Both $f|_{K'}$ and $g|_{K'}$ are closed homomorphisms from $\mathbf{K}'_w$ to $\mathbf{A}'_w$ and $\mathbf{B}'_w$, respectively.*

Furthermore, if $\mathbf{K}''$, $\mathbf{A}''$ and $\mathbf{B}''$ are any weak subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ supported on K', A' and B', respectively, such that $f|_{K'}$ and $g|_{K'}$ are closed homomorphisms between them, then they are also weak subalgebras of $\mathbf{K}'_w$, $\mathbf{A}'_w$ and $\mathbf{B}'_w$, respectively.

PROOF. The first assertion is clear from the definitions. As far as the second assertion goes, let $\mathbf{K}''$, $\mathbf{A}''$ and $\mathbf{B}''$ be any weak subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ supported on K', A' and B', respectively, such that $f|_{K'} : \mathbf{K}'' \to \mathbf{A}''$ and $g|_{K'} : \mathbf{K}'' \to \mathbf{B}''$ are closed homomorphisms.

Let $\varphi \in \Omega$ be an operation symbol with $\eta(\varphi) = (s, s')$. We have to show that $\operatorname{dom} \varphi^{\mathbf{K}''} \subseteq \operatorname{dom} \varphi^{\mathbf{K}'_w}$, $\operatorname{dom} \varphi^{\mathbf{A}''} \subseteq \operatorname{dom} \varphi^{\mathbf{A}'_w}$ and $\operatorname{dom} \varphi^{\mathbf{B}''} \subseteq \operatorname{dom} \varphi^{\mathbf{B}'_w}$.

If $x \in \operatorname{dom} \varphi^{\mathbf{K}''}$ then $f_s(x) \in \operatorname{dom} \varphi^{\mathbf{A}''} \subseteq \operatorname{dom} \varphi^{\mathbf{A}}$ and $g_s(x) \in \operatorname{dom} \varphi^{\mathbf{B}''} \subseteq \operatorname{dom} \varphi^{\mathbf{B}}$, and moreover $\varphi^{\mathbf{K}}(x) = \varphi^{\mathbf{K}''}(x) \in K'_{s'}$, and therefore $x \in \operatorname{dom} \varphi^{\mathbf{K}'_w}$ by definition. This proves $\operatorname{dom} \varphi^{\mathbf{K}''} \subseteq \operatorname{dom} \varphi^{\mathbf{K}'_w}$.

If $a \in \operatorname{dom} \varphi^{\mathbf{A}''}$ then we must distinguish two cases. If $a \in A_s - f_s(K_s)$ then $a \in \operatorname{dom} \varphi^{\mathbf{A}''} \subseteq \operatorname{dom} \varphi^{\mathbf{A}}$ and $\varphi^{\mathbf{A}}(a) = \varphi^{\mathbf{A}''}(a) \in A'_{s'}$ imply $a \in \operatorname{dom} \varphi^{\mathbf{A}'_w}$. And if $a = f_s(z)$ with $z \in K'_s$ then $z \in \operatorname{dom} \varphi^{\mathbf{K}''} \subseteq \operatorname{dom} \varphi^{\mathbf{K}'_w}$ because $f|_{K'} : \mathbf{K}'' \to \mathbf{A}''$ is closed, and therefore $a \in \operatorname{dom} \varphi^{\mathbf{A}'_w}$. This proves that $\operatorname{dom} \varphi^{\mathbf{A}''} \subseteq \operatorname{dom} \varphi^{\mathbf{A}'_w}$; a similar argument proves that $\operatorname{dom} \varphi^{\mathbf{B}''} \subseteq \operatorname{dom} \varphi^{\mathbf{B}'_w}$. $\square$

Let now

$$\begin{tikzcd} \mathbf{K}'_w \arrow[r, "f|_{K'}"] \arrow[d, "g|_{K'}"'] & \mathbf{A}'_w \arrow[d, "\tilde{g}"] \\ \mathbf{B}'_w \arrow[r, "\tilde{f}"'] & \mathbf{H} \end{tikzcd}$$

be the pushout of $f|_{K'} : \mathbf{K}'_w \to \mathbf{A}'_w$ and $g|_{K'} : \mathbf{K}'_w \to \mathbf{B}'_w$ in C-Alg$_\Gamma$ described in [6, Prop. 19 and Lem. 20]. From *loc. cit.* we know that it is also their pushout in Alg$_\Gamma$, and that the underlying commutative square of mappings of S-sets is the usual pushout of $f|_{K'}$ and $g|_{K'}$ in S-**Set**, described in §2.2. Let $\tilde{g} : \mathbf{A} \to \mathbf{H}$ and $\tilde{f} : \mathbf{B} \to \mathbf{H}$ also denote the conformisms corresponding to the closed homomorphisms $\tilde{g} : \mathbf{A}'_w \to \mathbf{H}$ and $\tilde{f} : \mathbf{B}'_w \to \mathbf{H}$.

Now we have the following result.

Proposition 8 *The commutative square*

$$\begin{tikzcd} \mathbf{K} \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{A} \arrow[d, "\tilde{g}"] \\ \mathbf{B} \arrow[r, "\tilde{f}"'] & \mathbf{H} \end{tikzcd}$$

is a pushout square in CF-Alg$_\Gamma$.

PROOF. Let $p : \mathbf{A} \to \mathbf{C}$ and $q : \mathbf{B} \to \mathbf{C}$ be two conformisms of partial Γ-algebras such that $p \circ f = q \circ g$. We know from Proposition 3 that there exists a unique partial mapping $h : H \to C$ such that $h \circ \tilde{g} = p$ and $h \circ \tilde{f} = q$. It is enough to check that h is a conformism.

Notice first that the equalities $h \circ \tilde{g} = p$ and $h \circ \tilde{f} = q$ imply in particular that $\mathrm{Dom}\, p \subseteq \mathrm{Dom}\, \tilde{g} = A'$ and $\mathrm{Dom}\, q \subseteq \mathrm{Dom}\, \tilde{f} = B'$, and then $\mathrm{Dom}\, p \circ f = f^{-1}(\mathrm{Dom}\, p) \subseteq f^{-1}(A') = K'$, and in a similar way $\mathrm{Dom}\, q \circ g \subseteq K'$.

Let now $\varphi \in \Omega$ be any operation symbol with $\eta(\varphi) = (s, s')$, and let $x \in \mathrm{Dom}\, h_s$ such that $h_s(x) \in \mathrm{dom}\, \varphi^{\mathbf{C}}$. Since $\tilde{f}$ and $\tilde{g}$ are jointly surjective, we can assume without any loss of generality that $x = \tilde{g}_s(a)$ for some $a \in A'_s$. Then, $a \in \mathrm{Dom}\, h_s \circ \tilde{g}_s = \mathrm{Dom}\, p_s$ and $p_s(a) = h_s(x) \in \mathrm{dom}\, \varphi^{\mathbf{C}}$ and therefore, p being a conformism, $a \in \mathrm{dom}\, \varphi^{\mathbf{A}}$, $\varphi^{\mathbf{A}}(a) \in \mathrm{Dom}\, p_{s'}$ and $p_{s'}(\varphi^{\mathbf{A}}(a)) = \varphi^{\mathbf{C}}(p_s(a)) = \varphi^{\mathbf{C}}(h_s(x))$.

There are two possibilities for such $a \in A'_s$:

- If $a \in A_s - f_s(K_s)$ then $a \in \mathrm{dom}\, \varphi^{\mathbf{A}}$ and $\varphi^{\mathbf{A}}(a) \in \mathrm{Dom}\, p_{s'} \subseteq A'_{s'}$ imply that $a \in \mathrm{dom}\, \varphi^{\mathbf{A}'_w}$.
- If $a \in f_s(K'_s)$, say $a = f_s(z)$ with $z \in K'_s$, then $g_s(z) \in \mathrm{dom}\, \varphi^{\mathbf{B}}$ (because $q_s(g_s(z)) = p_s(f_s(z)) = p_s(a) \in \mathrm{dom}\, \varphi^{\mathbf{C}}$ and q is a conformism) and $\varphi^{\mathbf{K}}(z) \in K'_{s'}$ (because $p_s(f_s(z)) \in \mathrm{dom}\, \varphi^{\mathbf{C}}$ and therefore, $p \circ f$ being a conformism, $z \in \mathrm{dom}\, \varphi^{\mathbf{K}}$ and $\varphi^{\mathbf{K}}(z) \in \mathrm{Dom}\,(p_{s'} \circ f_{s'}) \subseteq K'_{s'}$). Since moreover $f_s(z) \in \mathrm{dom}\, \varphi^{\mathbf{A}}$, by the definition of $\mathbf{K}'_w$ we conclude that $z \in \mathrm{dom}\, \varphi^{\mathbf{K}'_w}$ and thus, by the definition of $\mathbf{A}'_w$, that $a \in \mathrm{dom}\, \varphi^{\mathbf{A}'_w}$.

Therefore $a \in \mathrm{dom}\, \varphi^{\mathbf{A}'_w}$, and then $x = \tilde{g}_s(a) \in \mathrm{dom}\, \varphi^{\mathbf{H}}$, $\varphi^{\mathbf{H}}(x) = \tilde{g}_{s'}(\varphi^{\mathbf{A}}(a)) \in \mathrm{Dom}\, h_{s'}$ and

$$h_{s'}(\varphi^{\mathbf{H}}(x)) = h_{s'}(\tilde{g}_{s'}(\varphi^{\mathbf{A}}(a))) = p_{s'}(\varphi^{\mathbf{A}}(a)) = \varphi^{\mathbf{C}}(h_s(x)).$$

The symbol φ standing for any operation symbol in the signature, this shows that h is a conformism. $\square$

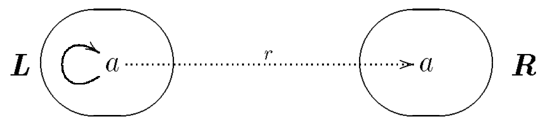
From this proposition we deduce that the derived partial Γ-algebra $\mathbf{H}$ of a partial Γ-algebra $\mathbf{G}$ by the application of a production rule $r : \mathbf{L} \to \mathbf{R}$ in CF-Alg$_\Gamma$ (i.e. a conformism of partial Γ-algebras) through a total conformism $m : \mathbf{L} \to \mathbf{G}$ can be obtained (up to isomorphism) as follows:

(i) Compute the *gluing set* L' of $r : L \to R$ and $m : L \to G$ (Definition 1).
(ii) *Remove* from R and G the elements of $r(L) - r(L')$ and $m(L) - m(L')$, respectively. Define on the resulting subsets of R and G the weak subalgebras $\mathbf{R}'_w$ and $\mathbf{G}'_w$ of $\mathbf{R}$ and $\mathbf{G}$, respectively, described above.
(iii) *Add* $\mathbf{R}'_w$ to $\mathbf{G}'_w$, by first forming their coproduct (disjoint union) $\mathbf{G}'_w + \mathbf{R}'_w$ and then identifying in this coproduct all images of elements of L' by r and m (by means of the least equivalence relation containing all pairs $(r_s(z), m_s(z))$, for every $s \in S$ and $z \in L'_s$).

In the sequel we give two examples of application of single-pushout transformation using conformisms, to show some features of this approach as well as its differences with the classical approach to single-pushout transformation using partial homomorphisms of total algebras.

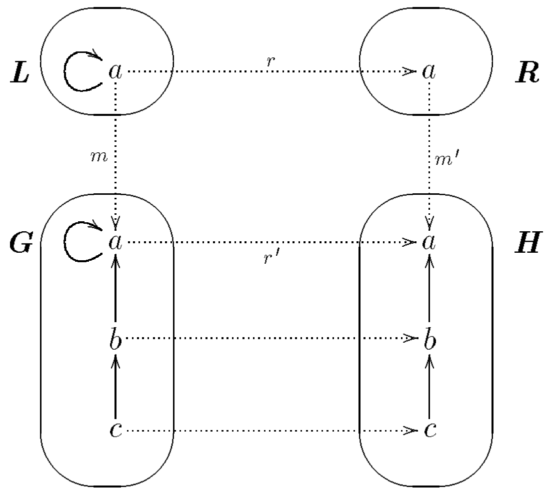
Example 9 *Let Γ be a graph structure with a single sort (which we shall omit as a subscript in practice) and a single unary operation φ. Suppose we want to perform on partial Γ-algebras such a simple operation as removing all φ-loops in them without touching anything else, and in particular without deleting the point where the loop is defined.*

In CF-Alg$_\Gamma$ *it can be done by simply applying the single-pushout rule $r : \mathbf{L} \to \mathbf{R}$ where $\mathbf{L}$ is a Γ-algebra with a single point a and a loop $\varphi^{\mathbf{L}}(a) = a$, $\mathbf{R}$ is a discrete Γ-algebra supported on $\{a\}$ and r is given by $r(a) = a$.*



Indeed, if there exists some total conformism $m : \mathbf{L} \to \mathbf{G}$ to some Γ-algebra $\mathbf{G}$, then either $m(a) \notin \operatorname{dom} \varphi^{\mathbf{G}}$ or $m(a) \in \operatorname{dom} \varphi^{\mathbf{G}}$ and $\varphi^{\mathbf{G}}(m(a)) = m(a)$. In the first case, the application of r to $\mathbf{G}$ through m produces $\mathbf{G}$ again, without any change[7]*, while in the second case the application of r to $\mathbf{G}$ through m produces the weak subalgebra $\mathbf{H}$ of $\mathbf{G}$ with its same universe and with* $\operatorname{dom} \varphi^{\mathbf{H}} = \operatorname{dom} \varphi^{\mathbf{G}} - \{m(a)\}$.

[7] If one wants to get rid of this possibility, it is enough to impose derivations to be made through closed homomorphisms, instead of through general total conformisms

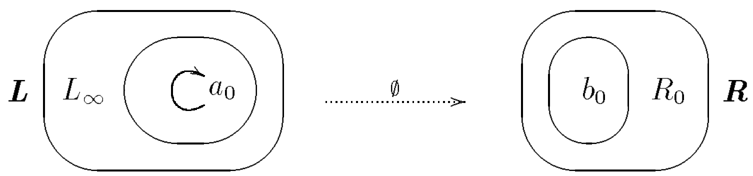


There is also a possibility to do this by first replacing the partial Γ-algebras by total $\widetilde{\Gamma}$-algebras, where

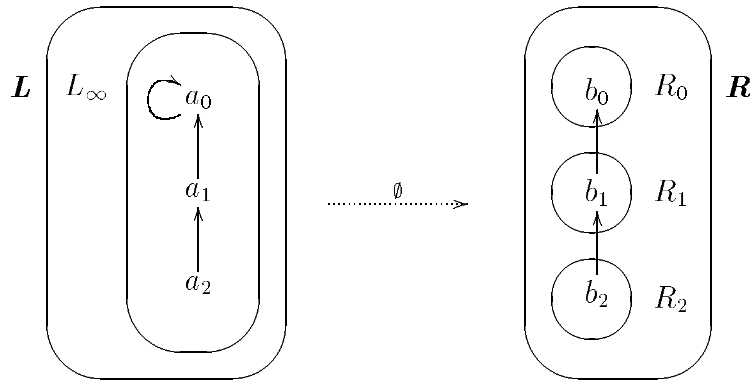
$$\widetilde{\Gamma} = (\mathbb{N} \cup \{\infty\}, (\varphi_n)_{n \in \mathbb{N} \cup \{\infty\}}, \eta)$$

is a graph structure with $\eta(\varphi_\infty) = (\infty, \infty)$ and $\eta(\varphi_n) = (n+1, n)$ for every $n \in \mathbb{N}$ (to simplify the notation, we shall overload the symbol φ and denote all operations φ_n, $n \in \mathbb{N} \cup \{\infty\}$, by simply φ), as follows: given a partial Γ-algebra $\mathbf{A} = (A, \varphi^{\mathbf{A}})$, we define the total $\widetilde{\Gamma}$-algebra $\widetilde{\mathbf{A}} = \left((A_n)_{n \in \mathbb{N} \cup \{\infty\}}, \varphi^{\widetilde{\mathbf{A}}}\right)$ by splitting its universe into layers given by the domains of the Γ-terms (i.e., $A_0 = \{a \mid a \notin \operatorname{dom} \varphi^{\mathbf{A}}\}$, $A_n = \operatorname{dom} \varphi^{n\mathbf{A}} - \operatorname{dom} \varphi^{n+1\mathbf{A}}$ for every $n \in \mathbb{N} - \{0\}$, and $A_\infty = \bigcap_{n \in \mathbb{N}} \operatorname{dom} \varphi^{n\mathbf{A}}$), and breaking the partial operation $\varphi^{\mathbf{A}}$ into total operations $\varphi^{\widetilde{\mathbf{A}}} : A_n \to A_{n-1}$ $(n \in \mathbb{N} - \{0\})$ and $\varphi^{\widetilde{\mathbf{A}}} : A_\infty \to A_\infty$. This construction gives an isomorphism between C-Alg$_\Gamma$ *and the category* TAlg$_{\widetilde{\Gamma}}$ *of total $\widetilde{\Gamma}$-algebras with usual homomorphisms.*

Then, one could try to remove all loops in a partial Γ-algebra $\mathbf{A}$ by using single-pushout transformation of total unary algebras on $\widetilde{\mathbf{A}}$ in the sense of [12]. The idea would be to use an (empty) single-pushout rule in P-TAlg$_{\widetilde{\Gamma}}$ *to remove a point in A_∞ having a loop defined on it and to add a new point in A_0.*



The main drawback with this procedure is that it not only removes the loop without removing the point, but it also removes all points in A_∞ (different from the point where the loop was defined) having some term defined on them yielding the point where the loop was defined. If we want to get rid of this *problem, we need infinitely many production rules: roughly speaking, for every sequence* $a_n, \ldots, a_0 \in A_\infty$ *(n can be infinite, here) such that* $\varphi^{\widetilde{\mathbf{A}}}(a_k) = a_{k-1}$ *for* $k \geq 1$ *and* $\varphi^{\widetilde{\mathbf{A}}}(a_0) = a_0$*, we need an (empty) production rule that removes all points* $a_n, \ldots, a_0$ *and adds new points* $b_i \in A_i$, $i = 0, \ldots, n$ *such that* $\varphi^{\widetilde{\mathbf{A}}}(b_k) = b_{k-1}$ *for* $k \geq 1$*.*



And moreover, we need application conditions that make such a rule be applied through a homomorphism m only when $m_\infty(a_n)$ *is not in the image of* $\varphi^{\widetilde{\mathbf{A}}}$*.*

Example 10 *In this example we show how to use single-pushout transformation based on conformisms to produce the hypergraph underlying a hierarchic higher-order hypergraph [1]. As in the previous example, this could also be achieved by means of 'classical' single-pushout transformation based on partial homomorphisms of total unary algebras, but again at an extremely high price: we would need a non-countably infinite number of rules to take care of all possibilities (just as in the previous example we needed countably many rules to handle all sequences of operations in* A_∞*), as well as application conditions.*

Let $\Sigma_H = (\{V, E\}, \Omega_H, \eta)$ *be a graph structure with*

$$\Omega_H = \Big\{s_i \mid i \in \mathbb{Z}^+\} \cup \{t_i \mid i \in \mathbb{Z}^+\Big\} \cup \{a_i \mid i \in \mathbb{Z}^+\}$$

and the arity function η *defined by* $\eta(s_i) = \eta(t_i) = (E, V)$ *and* $\eta(a_i) = (E, E)$, $i \in \mathbb{Z}^+$*. Recall from [6, Ex. 3] that a* higher-order hypergraph *can be understood then as a finite partial* Σ_H*-algebra* $\mathbf{G}$ *satisfying the following conditions:*

- $\operatorname{dom} s^{\mathbf{G}}_{i+1} \subseteq \operatorname{dom} s^{\mathbf{G}}_i$, $\operatorname{dom} t^{\mathbf{G}}_{i+1} \subseteq \operatorname{dom} t^{\mathbf{G}}_i$ *and* $\operatorname{dom} a^{\mathbf{G}}_{i+1} \subseteq \operatorname{dom} a^{\mathbf{G}}_i$*, for all* $i \in \mathbb{Z}^+$,
- *there exists an* $n(\mathbf{G}) \geq 0$ *such that* $\operatorname{dom} s^{\mathbf{G}}_{n(\mathbf{G})} = \operatorname{dom} t^{\mathbf{G}}_{n(\mathbf{G})} = \operatorname{dom} a^{\mathbf{G}}_{n(\mathbf{G})} = \emptyset$.

The elements of G_V *and* G_E *are called, respectively, the* nodes *and the* arcs *of* $\mathbf{G}$*, and the operations* $s^{\mathbf{G}}_i$, $t^{\mathbf{G}}_i$ *and* $a^{\mathbf{G}}_i$ *are called, respectively, the ith* source, target *and* abstraction *operations on* $\mathbf{G}$*.*

For every $e \in G_E$*, we shall denote by* $n_s(e)$ *(resp.* $n_t(e)$, $n_a(e)$*) the greatest*

$i \geq 1$ such that $e \in \operatorname{dom} s_i^{\mathbf{G}}$ (resp. $e \in \operatorname{dom} t_i^{\mathbf{G}}$, $e \in \operatorname{dom} a_i^{\mathbf{G}}$) or 0 if $e \notin \operatorname{dom} s_1^{\mathbf{G}}$ (resp. $e \notin \operatorname{dom} t_1^{\mathbf{G}}$, $e \notin \operatorname{dom} a_1^{\mathbf{G}}$).

We shall say that such a higher-order hypergraph $\mathbf{G}$ is hierarchic *when there is no sequence of abstraction operations $a_{i_1}^{\mathbf{G}}, \ldots, a_{i_n}^{\mathbf{G}}$, $n \geq 1$, such that*

$$a_{i_1}^{\mathbf{G}}(\ldots(a_{i_n}^{\mathbf{G}}(e))\ldots) = e$$

for some arc $e \in G_E$ (i.e. such that $a_{i_1}^{\mathbf{G}}(\ldots(a_{i_n}^{\mathbf{G}}(e))\ldots)$ is defined and yields the same e). For instance, the higher-order hypergraphs involved in the methods of knowledge base verification using higher-order hypergraph transformation developed in [20,21] are always hierarchic.

We show in the sequel how to compute the hypergraph underlying *a hierarchic higher-order hypergraph (i.e., its weak subalgebra with the same sets of nodes and arcs and the same source and target operations, but no abstraction operation defined in it) by means of the application of suitable production rules in* CF-Alg_{Σ_H}.

For every $n \geq 1$ let $\mathbf{G}(n)$ be the partial Σ_H-algebra with $G(n)_E = \{e, e_1, \ldots, e_n\}$ and

$$G(n)_V = \{v_j \mid 1 \leq j \leq 2n\} \cup \{w_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq 2n\}$$

and with operations defined as follows:

$$\begin{array}{l} a_j^{\mathbf{G}(n)}(e) = e_j, \quad s_j^{\mathbf{G}(n)}(e) = v_j, \quad t_j^{\mathbf{G}(n)}(e) = v_{n+j}, \qquad j = 1, \ldots, n \\ s_j^{\mathbf{G}(n)}(e_i) = w_{i,j}, \quad t_j^{\mathbf{G}(n)}(e_i) = w_{i,n+j}, \qquad i,j = 1, \ldots, n \end{array}$$

and let $\widetilde{\mathbf{G}}(n)$ be the weak subalgebra of $\mathbf{G}(n)$ with the same carrier and the same source and target operations, but all abstraction operations discrete. Let $r_n : \mathbf{G}(n) \to \widetilde{\mathbf{G}}(n)$ be the conformism given by the identity mapping.

Let now $\mathbf{G}$ be a higher-order hypergraph and let $m : \mathbf{G}(n) \to \mathbf{G}$ be a total conformism. Let $m_E(e) = e'$, $m_E(e_i) = e'_i$, $m_V(v_j) = v'_j$ and $m_V(w_{i,j}) = w'_{i,j}$ ($i = 1, \ldots, n$, $j = 1, \ldots, 2n$). Since m is a conformism, the following conditions are satisfied:

- *If $e' \in \operatorname{dom} a_i^{\mathbf{G}}$ then $i \leq n$ and $a_i^{\mathbf{G}}(e') = e'_i$.*
- *$e'_i \notin \operatorname{dom} a_1^{\mathbf{G}}$, $i = 1, \ldots, n$.*
- *If $e' \in \operatorname{dom} s_j^{\mathbf{G}}$ then $j \leq n$ and $s_j^{\mathbf{G}}(e') = v'_j$. Let $n_s = n_s(e') \leq n$.*
- *If $e'_i \in \operatorname{dom} s_j^{\mathbf{G}}$ then $j \leq n$ and $s_j^{\mathbf{G}}(e'_i) = w'_{i,j}$. Let $n_{s,i} = n_s(e'_i) \leq n$.*
- *If $e' \in \operatorname{dom} t_j^{\mathbf{G}}$ then $j \leq n$ and $t_j^{\mathbf{G}}(e') = v'_{n+j}$. Let $n_t = n_t(e') \leq n$.*
- *If $e'_i \in \operatorname{dom} t_j^{\mathbf{G}}$ then $j \leq n$ and $t_j^{\mathbf{G}}(e'_i) = w'_{i,n+j}$. Let $n_{t,i} = n_t(e'_i) \leq n$.*

The pushout of $r_n : \mathbf{G}(n) \to \widetilde{\mathbf{G}}(n)$ and $m : \mathbf{G}(n) \to \mathbf{G}$ in CF-Alg_{Σ_H} *is then* *given by the pushout of the closed homomorphisms* $r_n : \mathbf{G}(n)_w \to \widetilde{\mathbf{G}}(n)_w$ *and* $m : \mathbf{G}(n)_w \to \mathbf{G}_w$*, where*

- $\mathbf{G}(n)_w$ *is the weak subalgebra of* $\mathbf{G}(n)$ *with its same carrier and such that* $n_a(e) = 0$*,* $n_s(e) = n_s$*,* $n_t(e) = n_t$*,* $n_s(e_i) = n_{s,i}$ *and* $n_t(e_i) = n_{t,i}$*,* $i = 1, \ldots, n$*,*
- $\widetilde{\mathbf{G}}(n)_w = \mathbf{G}(n)_w$*,*
- $\mathbf{G}_w$ *is the weak subalgebra of* $\mathbf{G}$ *with its same carrier and such that* $n_a(e') = 0$*, the rest being as in* $\mathbf{G}$*.*

Since $r_n : \mathbf{G}(n)_w \to \widetilde{\mathbf{G}}(n)_w$ *is an identity in* CF-Alg$_{\Sigma_G}$*, the derived partial* Σ_H*-algebra of* $\mathbf{G}$ *by the application of rule* r_n *through* m *is simply* $\mathbf{G}_w$*, obtained from* $\mathbf{G}$ *by deleting all abstraction operations defined on* $m_E(e)$ *and not touching anything else.*

Finally, let $\mathbf{G}$ *be a hierarchic higher-order hypergraph, and let* n_0 *be an upper bound for* $\{n_s(e), n_t(e), n_a(e) \mid e \in G_E\}$*. The argument developed above shows that successive applications of the production rule* $r_{n_0} : \mathbf{G}(n_0) \to \widetilde{\mathbf{G}}(n_0)$ *through total conformisms end up by deleting all abstraction operations in* $\mathbf{G}$ *without modifying either its carrier or its source and target operations. In other words, they end up by computing the hypergraph underlying* $\mathbf{G}$*.*

If $\mathbf{G}$ *is not hierarchic then this procedure does not delete all abstraction operations. Actually, it may happen that there does not exist any total conformism* $m : \mathbf{G}(n) \to \mathbf{G}$ *and therefore that we could never apply any rule* r_n *to* $\mathbf{G}$*.*

The description of the pushout of two conformisms as the pushout of two closed homomorphisms given in Proposition 8, similar in spirit to the description of the pushout of two partial homomorphisms in P-TAlg$_\Gamma$ given in [12], motivates to investigate whether there is some relation between single-pushout transformation in CF-Alg$_\Gamma$ and double-pushout transformation in Alg$_\Gamma$, as it is the case for single-pushout transformation in P-TAlg$_\Gamma$ [12, §3.3] and double-pushout transformation of total unary algebras. This relation should come through the following notion of a (double-pushout) production rule in Alg$_\Gamma$ associated to a (single-pushout) production rule in CF-Alg$_\Gamma$ (cf. [12, §3.3]).

Definition 11 *Given a conformism of partial* Γ*-algebras* $r : \mathbf{L} \to \mathbf{R}$*, let* $\mathbf{L}_r$ *be the weak subalgebra of* $\mathbf{L}$ *supported on* $\operatorname{Dom} r$ *such that* r *is a closed homomorphism from it, let* $i : \mathbf{L}_r \to \mathbf{L}$ *be the corresponding embedding and let* $r : \mathbf{L}_r \to \mathbf{R}$ *denote also the corresponding closed homomorphism. Then the* production rule $P(r)$ associated to r *is the production rule in* Alg$_\Gamma$

$$P(r) = (\mathbf{L} \xleftarrow{i} \mathbf{L}_r \xrightarrow{r} \mathbf{R})$$

The question we ask is, then, whether or not the derived partial algebra of

$\mathbf{G}$ by the production rule r in CF-Alg$_\Gamma$ and by the production rule $P(r)$ in Alg$_\Gamma$, through a total conformism that is also a homomorphism, i.e. a closed homomorphism, $m : \mathbf{L} \to \mathbf{G}$ satisfying the gluing condition [6, Def. 8] w.r.t. i, are the same (up to isomorphism). The answer is, in general, no, as the following example shows.

Example 12 *Let Γ be a graph structure with a single sort (which we shall omit as a subscript in practice) and a single unary operation φ. Let $\mathbf{L}$ be a Γ-algebra with $L = \{a, b\}$ and $\varphi^{\mathbf{L}}(a) = \varphi^{\mathbf{L}}(b) = b$, and let $\mathbf{R}$ be the discrete relative subalgebra of $\mathbf{L}$ supported on $R = \{a\}$. Let $r : \mathbf{L} \to \mathbf{R}$ be the conformism (actually, a c-quomorphism) given by the identity on R.*

Let now $\mathbf{G}$ be equal to $\mathbf{L}$, and let $m : \mathbf{L} \to \mathbf{G}$ be the identity mapping, which is a closed homomorphism. The gluing set of r and m is $L' = \{a\}$, the weak subalgebras $\mathbf{R}'_w$ and $\mathbf{G}'_w$ obtained in the Remove *step above are discrete with universe $\{a\}$ and therefore the derived partial Γ-algebra of $\mathbf{G}$ by the application of r through m is isomorphic to $\mathbf{R}$.*

On the other hand, in this case the production rule $P(r)$ would be $(\mathbf{L} \xleftarrow{i} \mathbf{L}_r \xrightarrow{r} \mathbf{R})$, with $\mathbf{L}_r$ the relative (discrete) subalgebra of $\mathbf{L}$ supported on $\{a\}$. It is clear that m satisfies the identification, dangling and relative closedness conditions in [6, Def. 8] w.r.t. i (for instance, because $\mathbf{L} = C_{\mathbf{L}}(\{a\})$). And a simple computation shows that the derived partial Γ-algebra of $\mathbf{G}$ through g by the application of rule $P(r)$ in Alg$_\Gamma$ *[6, Def. 15] is isomorphic to $\mathbf{G}$, and therefore clearly non-isomorphic to $\mathbf{R}$.*

Another example is given by the c-quomorphism $r : \mathbf{L} \to \mathbf{R}$ and the closed homomorphism $g : \mathbf{L} \to \mathbf{G}'$, with $\mathbf{G}'$ a total Γ-algebra with carrier $G' = \{a'\}$ and $\varphi^{\mathbf{G}'}(a') = a'$, given by $g(a) = g(b) = a'$; again, g satisfies the gluing condition w.r.t. r. In this case, it is easy to check that the derived partial Γ-algebra of $\mathbf{G}'$ by the application of r through g is empty, while there is no double-pushout diagram

$$\begin{tikzcd}
\mathbf{L} \arrow[d, "g"'] & \mathbf{L}_r \arrow[l, "i"'] \arrow[r, "r"] \arrow[d, "g'"] & \mathbf{R} \arrow[d, "g''"] \\
\mathbf{G}' & \mathbf{D} \arrow[l, "i'"] \arrow[r, "r'"'] & \mathbf{H}
\end{tikzcd}$$

yielding an empty algebra $\mathbf{H}$ as an algebra derived from $\mathbf{G}'$ by the application of $P(r)$ through g.

This example also shows that, in general, the derivation of a partial Γ-algebra $\mathbf{G}$ by the application of a rule $r : \mathbf{L} \to \mathbf{R}$ in CF-Alg$_\Gamma$ through a closed homomorphism $m : \mathbf{L} \to \mathbf{G}$ fits better into our intuitive expectations than the corresponding derivation by the application of $P(r)$ in Alg$_\Gamma$. For instance, it is clear from the intuitive notion of transformation that the application of a rule through the identity on its left-hand side object should give its right-hand side object. And it happens in this way with the application of a single-pushout production rule $r : \mathbf{L} \to \mathbf{R}$ in CF-Alg$_\Gamma$, because (by category theoretical reasons, but also by Proposition 8) the pushout algebra of a conformism $r : \mathbf{L} \to \mathbf{R}$ and the identity $\mathrm{Id}_\mathbf{L} : \mathbf{L} \to \mathbf{L}$ is $\mathbf{R}$. But it does not happen in general in this way with the application of the double-pushout production rule $P(r) = (\mathbf{L} \xleftarrow{i} \mathbf{L}_r \xrightarrow{r} \mathbf{R})$ in Alg$_\Gamma$, because in this case the algebra derived from $\mathbf{G}$ by the application of $P(r)$ through $\mathrm{Id}_\mathbf{L} : \mathbf{L} \to \mathbf{L}$ (which always satisfies the gluing condition w.r.t. $i : \mathbf{L}_r \to \mathbf{L}$) is, according to [6, Def. 15], the pushout algebra of $r : \mathbf{L}_r \to \mathbf{R}$ and $i : \mathbf{L}_r \to \mathbf{C}_\mathbf{L}(L_r)$ (the inclusion of $\mathbf{L}_r$ in the closed subalgebra generated by it), and it need not be isomorphic to $\mathbf{R}$ (see for instance the first case in the last example).

To close this section, we show that the pushout of two total conformisms in CF-Alg$_\Gamma$ is also their pushout in TCF-Alg$_\Gamma$. We shall use it in §6.

Proposition 13 *Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two total conformisms of partial Γ algebras, and let*

$$\begin{tikzcd} \mathbf{K} \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{A} \arrow[d, "\tilde{g}"] \\ \mathbf{B} \arrow[r, "\tilde{f}"'] & \mathbf{H} \end{tikzcd}$$

be their pushout in CF-Alg$_\Gamma$ *described in Proposition 8. Then it is also their pushout in* TCF-Alg$_\Gamma$. □

PROOF. If f and g are total conformisms then $\tilde{f}$ and $\tilde{g}$ are also total conformisms by Corollary 4.(iii). And if $p : \mathbf{A} \to \mathbf{C}$ and $q : \mathbf{B} \to \mathbf{C}$ are two total conformisms such that $p \circ f = q \circ g$ then the unique conformism $h : \mathbf{H} \to \mathbf{C}$ such that $h \circ \tilde{f} = p$ and $h \circ \tilde{g} = q$ is total because $\tilde{f}$ and $\tilde{g}$ are jointly surjective by Corollary 4.(i). □

4 Single-pushout transformation in CQ-Alg$_\Gamma$

Let Γ be a graph structure and let now $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two c-quomorphisms of partial Γ-algebras. In this section we prove that their pushout in CF-Alg$_\Gamma$ is also their pushout in CQ-Alg$_\Gamma$. We shall freely use the notations introduced in the previous sections.

Let $\mathbf{K}'$, $\mathbf{A}'$ and $\mathbf{B}'$ be the relative subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ supported respectively on the sets K', A' and B' defined in §2.2.

Lemma 14 $\mathbf{K}' = \mathbf{K}'_w$, $\mathbf{A}' = \mathbf{A}'_w$ *and* $\mathbf{B}' = \mathbf{B}'_w$.

PROOF. By Lemma 7, and since $\mathbf{K}'_w$, $\mathbf{A}'_w$ and $\mathbf{B}'_w$ are weak subalgebras of $\mathbf{K}'$, $\mathbf{A}'$ and $\mathbf{B}'$, respectively, it is enough to check that $f|_{K'} : \mathbf{K}' \to \mathbf{A}'$ and $g|_{K'} : \mathbf{K}' \to \mathbf{B}'$ are closed homomorphisms.

Let $\varphi \in \Omega$ be an operation symbol with $\eta(\varphi) = (s, s')$. If $x \in \operatorname{dom} \varphi^{\mathbf{K}'}$ then $x \in (\operatorname{Dom} f_s) \cap \operatorname{dom} \varphi^{\mathbf{K}}$ and $\varphi^{\mathbf{K}}(x) \in K'_{s'} \subseteq \operatorname{Dom} f_{s'}$. Being f a quomorphism, this implies that $f_s(x) \in \operatorname{dom} \varphi^{\mathbf{A}} \cap A'_s$ and $\varphi^{\mathbf{A}}(f_s(x)) = f_{s'}(\varphi^{\mathbf{K}}(x)) \in f_{s'}(K'_{s'}) \subseteq A'_{s'}$, i.e. $f_s(x) \in \operatorname{dom} \varphi^{\mathbf{A}'}$ and $\varphi^{\mathbf{A}'}(f_s(x)) = f_{s'}(\varphi^{\mathbf{K}}(x))$.

Now let $x \in K'_s$ such that $f_s(x) \in \operatorname{dom} \varphi^{\mathbf{A}'}$. Then $f_s(x) \in \operatorname{dom} \varphi^{\mathbf{A}}$ and $\varphi^{\mathbf{A}}(f_s(x)) \in A'_{s'}$. Since $f : \mathbf{K} \to \mathbf{A}$ is a c-quomorphism, $f_s(x) \in \operatorname{dom} \varphi^{\mathbf{A}}$ implies that $x \in \operatorname{dom} \varphi^{\mathbf{K}}$, $\varphi^{\mathbf{K}}(x) \in \operatorname{Dom} f_{s'}$ and $f_{s'}(\varphi^{\mathbf{K}}(x)) = \varphi^{\mathbf{A}}(f_s(x))$. But since the latter belongs to $A'_{s'} \cap f_{s'}(K_{s'}) = f_{s'}(K'_{s'})$, we deduce from Lemma 2 that $\varphi^{\mathbf{K}}(x) \in K'_{s'}$ and therefore that $x \in \operatorname{dom} \varphi^{\mathbf{K}'}$.

Since φ stands for any operation symbol in the signature, this shows that $f|_{K'} : \mathbf{K}' \to \mathbf{A}'$ is a closed homomorphism. By symmetry, $g|_{K'} : \mathbf{K}' \to \mathbf{B}'$ is also a closed homomorphism. $\square$

Let now

$$\begin{array}{ccc} \mathbf{K}' & \xrightarrow{f|_{K'}} & \mathbf{A}' \\ {\scriptstyle g|_{K'}}\downarrow & & \downarrow{\scriptstyle \tilde{g}} \\ \mathbf{B}' & \xrightarrow[\tilde{f}]{} & \mathbf{H} \end{array}$$

be the pushout of $f|_{K'} : \mathbf{K}' \to \mathbf{A}'$ and $g|_{K'} : \mathbf{K}' \to \mathbf{B}'$ in C-Alg$_\Gamma$. As in the case of conformisms, let $\tilde{g} : \mathbf{A} \to \mathbf{H}$ and $\tilde{f} : \mathbf{B} \to \mathbf{H}$ also denote the c-quomorphisms corresponding to (the closed homomorphisms) $\tilde{g}$ and $\tilde{f}$ respectively.

In this way, $\mathbf{H}$ together with $\tilde{g} : \mathbf{A} \to \mathbf{H}$ and $\tilde{f} : \mathbf{B} \to \mathbf{H}$, is the pushout of f and g in CF-Alg$_\Gamma$ by Lemma 14 and Proposition 8. And it turns out that it is also their pushout in CQ-Alg$_\Gamma$.

Proposition 15 *Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two c-quomorphisms of partial Γ-algebras, and let*

$$\begin{array}{ccc} \mathbf{K} & \xrightarrow{f} & \mathbf{A} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \tilde{g}} \\ \mathbf{B} & \xrightarrow[\tilde{f}]{} & \mathbf{H} \end{array}$$

be their pushout in CF-Alg$_\Gamma$. *Then it is also their pushout in* CQ-Alg$_\Gamma$.

PROOF. Let $p : \mathbf{A} \to \mathbf{C}$ and $q : \mathbf{B} \to \mathbf{C}$ be two c-quomorphisms of partial Γ-algebras such that $p \circ f = q \circ g$. We know from Proposition 8 that there exists a unique conformism $h : \mathbf{H} \to \mathbf{C}$ such that $h \circ \tilde{g} = p$ and $h \circ \tilde{f} = q$. We must check that h is also a quomorphism.

So, let $\varphi \in \Omega$, say with $\eta(\varphi) = (s, s')$, and let $x \in \operatorname{Dom} h_s \cap \operatorname{dom} \varphi^{\mathbf{H}}$ such that $\varphi^{\mathbf{H}}(x) \in \operatorname{Dom} h_{s'}$. Since $\tilde{f}$ and $\tilde{g}$ are jointly surjective, we can assume without any loss of generality that $x = \tilde{g}(a)$ for some $a \in A'_s$, and since $\tilde{g} : \mathbf{A}' \to \mathbf{H}$ is a closed homomorphism, we have that $a \in \operatorname{dom} \varphi^{\mathbf{A}'}$ and $\tilde{g}_{s'}(\varphi^{\mathbf{A}'}(a)) = \varphi^{\mathbf{H}}(x)$. Therefore $a \in \operatorname{dom} \varphi^{\mathbf{A}} \cap \operatorname{Dom} p_s$ and $\varphi^{\mathbf{A}}(a) \in \operatorname{Dom} p_{s'}$, and then ($p$ being a quomorphism) $h_s(x) = p_s(a) \in \operatorname{dom} \varphi^{\mathbf{C}}$ and

$$h_{s'}(\varphi^{\mathbf{H}}(x)) = p_{s'}(\varphi^{\mathbf{A}}(a)) = \varphi^{\mathbf{C}}(p_s(a)) = \varphi^{\mathbf{C}}(h_s(x))$$

The symbol φ standing for any operation symbol in the signature, this shows that h is a quomorphism, and therefore also a c-quomorphism. $\square$

Corollary 16 *Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two c-quomorphisms of partial Γ-algebras. Then their pushout in* CF-Alg$_\Gamma$ *is also their pushout in* CQ-Alg$_\Gamma$. $\square$

So, single-pushout transformation using c-quomorphisms is a special case of single-pushout transformation using conformisms (in the sense that the rules —c-quomorphisms— and the occurrences —closed homomorphisms— to be used in CQ-Alg$_\Gamma$ are special cases of those used in CF-Alg$_\Gamma$), and therefore it actually does not yield a new approach to transformation of partial unary algebras. Moreover, Example 12 entails that, in general, there is no relation between single-pushout transformation in CQ-Alg$_\Gamma$ and double-pushout transformation in Alg$_\Gamma$, in the sense explained therein (Definition 11).

To close this section, notice that arguing as in Proposition 13 we obtain the following result, which we shall use later.

Proposition 17 *Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two closed homomorphisms of partial Γ-algebras, and let*

$$\begin{tikzcd} \mathbf{K} \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{A} \arrow[d, "\tilde{g}"] \\ \mathbf{B} \arrow[r, "\tilde{f}"'] & \mathbf{H} \end{tikzcd}$$

be their pushout in CQ-Alg$_\Gamma$. *Then it is also their pushout in* C-Alg$_\Gamma$. $\square$

5 Single-pushout transformation in CDCQ-Alg$_\Gamma$

In the previous section we have seen that the pushout of two c-quomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ in CQ-Alg$_\Gamma$ is the same as their pushout in CF-Alg$_\Gamma$, i.e. their pushout as conformisms. This is no longer true for pushouts of cdc-quomorphisms in CDCQ-Alg$_\Gamma$ and in CF-Alg$_\Gamma$, as the following example shows.

Example 18 *Let Γ be a graph structure with a single sort and a single unary operation φ. Let $\mathbf{K}$ be a partial Γ-algebra with carrier $K = \{a, b\}$ and with $\varphi^{\mathbf{K}}$ total and given by $\varphi^{\mathbf{K}}(a) = \varphi^{\mathbf{K}}(b) = b$. Let $\mathbf{A}$ and $\mathbf{B}$ be both equal to $\mathbf{K}$.*

Let $f : \mathbf{K} \to \mathbf{A}$ be the cdc-quomorphism with domain $\operatorname{Dom} f = \{b\}$ *and $f(b) = b$. Let $m : \mathbf{K} \to \mathbf{B}$ be the closed homomorphism with $m(a) = m(b) = b$.*

It turns out that $K' = \emptyset$ and then $A' = B' = \{a\}$, so that the pushout $\mathbf{H}$ of f and m in CF-Alg$_\Gamma$ *is a discrete Γ-algebra with two elements. But then the c-quomorphisms $\tilde{f} : \mathbf{B} \to \mathbf{H}$ and $\tilde{g} : \mathbf{A} \to \mathbf{H}$ are not cdc-quomorphisms, because A' and B' are not closed subsets of $\mathbf{A}$ and $\mathbf{B}$. Therefore, the pushout of f and g in* CQ-Alg$_\Gamma$ *is not their pushout in* CDCQ-Alg$_\Gamma$. *And, in fact, a simple argument shows that the pushout algebra of f and g in* CDCQ-Alg$_\Gamma$ *is the empty algebra (see also Proposition 22 below).*

The drawback found in this example is general. Given two cdc-quomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ of partial Γ-algebras, the sets A' and B' defined in §2.2 need not be closed subsets of their respective home algebras. In this section we shall show that the naïve modification to the construction given in Section 4, consisting in replacing A' and B' by the greatest closed subsets of $\mathbf{A}$ and $\mathbf{B}$ contained in them (which exist, because all operations in the signature are unary), works, so that one gets in this way a pushout for f and g in CDCQ-Alg$_\Gamma$.

So, let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two cdc-quomorphisms of partial Γ-algebras, and let

$$\begin{tikzcd} \mathbf{K} \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{A} \arrow[d, "\tilde{g}"] \\ \mathbf{B} \arrow[r, "\tilde{f}"'] & \mathbf{H} \end{tikzcd}$$

be their pushout in CQ-Alg$_\Gamma$ (or CF-Alg$_\Gamma$) as described in Section 4. Let A^c and B^c be the greatest closed subsets of $\mathbf{A}$ and $\mathbf{B}$ contained in A' and B', respectively. To compute the subsets A^c and B^c, we can use the following well known description of the greatest closed subset of a partial Γ-algebra contained in a given subset.

Lemma 19 *Let $\mathbf{A}$ be a partial Γ-algebra, and let $X \subseteq A$. The greatest closed subset $X^c = (X^c_s)_{s \in S}$ of $\mathbf{A}$ contained in X is given by*

$$X^c_s = \{x \in X_s \mid \textit{for every } \Gamma\textit{-term } \mathbf{t} \textit{ of sort } s' \in S, \textit{ if } x \in \operatorname{dom} \mathbf{t}^{\mathbf{A}} \\ \textit{then } \mathbf{t}^{\mathbf{A}}(x) \in X_{s'}\Big\}$$

Alternatively, it is also given by

$$X^c = \bigcup \{C_{\mathbf{A}}(\{x\}) \mid x \in \bigcup_{s\in S} X_s \text{ and } C_{\mathbf{A}}(\{x\}) \subseteq X\}$$

where $C_{\mathbf{A}}(\{x\})$ denotes the closed subset of $\mathbf{A}$ generated by the S-set with all its carriers of all sorts empty, except the one of the sort corresponding to x, which is $\{x\}$. $\square$

Let now $\mathbf{H}^c$ be the relative subalgebra of $\mathbf{H}$ supported on $H^c = \tilde{g}(A^c) \cup \tilde{f}(B^c)$. Since A^c and B^c are closed subsets of A' and B', respectively, and $\tilde{g} : \mathbf{A}' \to \mathbf{H}$ and $\tilde{f} : \mathbf{B}' \to \mathbf{H}$ are closed homomorphisms, we have that $\mathbf{H}^c$ is a closed subalgebra of $\mathbf{H}$.

Lemma 20 *Let $\hat{f} : B \to H^c$ and $\hat{g} : A \to H^c$ be the restrictions of $\tilde{f}$ and $\tilde{g}$ to B^c and A^c (considered with target set H^c), respectively. Then $\hat{g}$ and $\hat{f}$ are cdc-quomorphisms $\hat{g} : \mathbf{A} \to \mathbf{H}^c$ and $\hat{f} : \mathbf{B} \to \mathbf{H}^c$.*

PROOF. A^c is a closed subset of $\mathbf{A}$ and $\tilde{g}|_{A^c} : \mathbf{A}^c \to \mathbf{H}$ is the restriction of the closed homomorphism $\tilde{g} : \mathbf{A}' \to \mathbf{H}$ to a closed subalgebra of its domain, and therefore a closed homomorphism [3, Prop. 3.1.11.(ii)]. $\mathbf{H}^c$ is a relative subalgebra of $\mathbf{H}$ containing the closed subset $\tilde{g}(A^c)$ of $\mathbf{H}$. Then $\hat{g} = \tilde{g}|_{A^c} : \mathbf{A}^c \to \mathbf{H}^c$ is still a closed homomorphism. By a similar reason, $\hat{f} = \tilde{f}|_{B^c} : \mathbf{B}^c \to \mathbf{H}^c$ is also a closed homomorphism. $\square$

It turns out that $\mathbf{H}^c$, together with these cdc-quomorphisms $\hat{g} : \mathbf{A} \to \mathbf{H}^c$ and $\hat{f} : \mathbf{B} \to \mathbf{H}^c$, is indeed the pushout of f and g in CDCQ-Alg$_\Gamma$. The following lemma will be used to prove it.

Lemma 21 *Let K^c be the greatest closed subset of $\mathbf{K}$ contained in K'. Then* $\mathrm{Dom}\,(\hat{g} \circ f) = f^{-1}(A^c)$ *and* $\mathrm{Dom}\,(\hat{f} \circ g) = g^{-1}(B^c)$ *are equal to K^c.*

PROOF. On the one hand, $\mathrm{Dom}\,(\hat{g} \circ f)$ is closed, because it is the domain of a cdc-quomorphism, and it is contained in K', because $A^c \subseteq A'$ and therefore $\mathrm{Dom}\,(\hat{g} \circ f) = f^{-1}(A^c) \subseteq f^{-1}(A') = K'$. Thus, $\mathrm{Dom}\,(\hat{g} \circ f) \subseteq K^c$. By symmetry, $\mathrm{Dom}\,(\hat{f} \circ g) \subseteq K^c$.

On the other hand, since $f : \mathbf{Dom}\, f \to \mathbf{A}$ is a closed homomorphism and $K^c \subseteq K' \subseteq \mathrm{Dom}\, f$, we have that $f(K^c)$ is a closed subset of $\mathbf{A}$ (by [3, Prop. 3.1.9]) contained in $f(K') \subseteq A'$ and therefore $f(K^c) \subseteq A^c$, which implies $K^c \subseteq f^{-1}(A^c)$; a similar argument shows that $K^c \subseteq g^{-1}(B^c)$. $\square$

Proposition 22 *The square*

$$\begin{tikzcd} \mathbf{K} \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{A} \arrow[d, "\hat{g}"] \\ \mathbf{B} \arrow[r, "\hat{f}"'] & \mathbf{H}^c \end{tikzcd}$$

is a pushout square in CDCQ-Alg$_\Gamma$.

PROOF. This square commutes because $\mathrm{Dom}\,(\hat{g}\circ f) = \mathrm{Dom}\,(\hat{f}\circ g)$ by the previous lemma, and $\tilde{g}\circ f = \tilde{f}\circ g$.

Let now $p : \mathbf{A} \to \mathbf{C}$ and $q : \mathbf{B} \to \mathbf{C}$ be two cdc-quomorphisms of partial Γ-algebras such that $p\circ f = q\circ g$. We must prove that there exists a unique cdc-quomorphism $h : \mathbf{H}^c \to \mathbf{C}$ such that $h\circ\hat{g} = p$ and $h\circ\hat{f} = q$.

We know from Proposition 15 that there exists a unique c-quomorphism $h : \mathbf{H} \to \mathbf{C}$ such that $h\circ\tilde{g} = p$ and $h\circ\tilde{f} = q$. By construction, this c-quomorphism is a closed homomorphism from the relative subalgebra $\widetilde{\mathbf{H}}$ of $\mathbf{H}$ supported on $\widetilde{H} = \tilde{f}(\mathrm{Dom}\, q)\cup\tilde{g}(\mathrm{Dom}\, p)$. If $\widetilde{\mathbf{H}}$ is a closed subalgebra of $\mathbf{H}^c$ then this c-quomorphism $h : \mathbf{H} \to \mathbf{C}$ will yield a cdc-quomorphism $h : \mathbf{H}^c \to \mathbf{C}$ such that $h\circ\hat{g} = p$ and $h\circ\hat{f} = q$. And it will be unique with this property, because of the uniqueness of the c-quomorphism $h : \mathbf{H} \to \mathbf{C}$.

So, we must prove that $\widetilde{H}$ is a closed subset of $\mathbf{H}^c$, and, to do that, it is enough to prove that it is a subset of H^c that is closed in $\mathbf{H}$.

Since $\mathrm{Dom}\, p$ and $\mathrm{Dom}\, q$ are closed subsets of $\mathbf{A}$ and $\mathbf{B}$ contained in A' and B', respectively, we have that $\mathrm{Dom}\, p \subseteq A^c$ and $\mathrm{Dom}\, q\subseteq B^c$ and then

$$\widetilde{H} = \tilde{f}(\mathrm{Dom}\, q)\cup\tilde{g}(\mathrm{Dom}\, p)\subseteq \tilde{f}(B^c)\cup\tilde{g}(A^c) = H^c.$$

As to the closedness of $\widetilde{H}$ in $\mathbf{H}$, let $\varphi\in\Omega$ be an operation symbol with $\eta(\varphi) = (s, s')$ and let $x\in\widetilde{H}_s\cap\mathrm{dom}\,\varphi^{\mathbf{H}}$. Since $\widetilde{H} = \tilde{f}(\mathrm{Dom}\, q)\cup\tilde{g}(\mathrm{Dom}\, p)$, we can assume without any loss of generality that $x = \tilde{g}_s(a)$ for some $a\in\mathrm{Dom}\, p_s$. Then, $\tilde{g}$ being a c-quomorphism, $x\in\mathrm{dom}\,\varphi^{\mathbf{H}}$ implies that $a\in\mathrm{dom}\,\varphi^{\mathbf{A}}$, $\varphi^{\mathbf{A}}(a)\in\mathrm{Dom}\,\tilde{g}_{s'}$ and $\tilde{g}_{s'}(\varphi^{\mathbf{A}}(a)) = \varphi^{\mathbf{H}}(x)$. And $\mathrm{Dom}\, p$ being a closed subset of $\mathbf{A}$, we have that $\varphi^{\mathbf{A}}(a)\in\mathrm{Dom}\, p_{s'}$, so that $\varphi^{\mathbf{H}}(x)\in\tilde{g}_{s'}(\mathrm{Dom}\, p_{s'})\subseteq\widetilde{H}_{s'}$. Since φ stands for any operation symbol in Ω, this proves that $\widetilde{H}$ is a closed subset of $\mathbf{H}$. $\Box$

So, the pushout algebra in CDCQ-Alg$_\Gamma$ of two cdc-quomorphisms of partial Γ-algebras is a (strict, in general) closed subalgebra of their pushout algebra in CF-Alg$_\Gamma$.

According to this proposition, the derived partial Γ-algebra of a partial Γ-algebra $\mathbf{G}$ by the application of a production rule $r : \mathbf{L} \to \mathbf{R}$ in CDCQ-Alg$_\Gamma$ (i.e. a cdc-quomorphism of partial Γ-algebras) through a closed homomorphism $m : \mathbf{L} \to \mathbf{G}$ is obtained (up to isomorphism) as follows:

(i) Compute the *gluing set* L' of $r : L \to R$ and $m : L \to G$.
(ii) *Remove* from R and G the elements of $r(L) - r(L')$ and $m(L) - m(L')$, respectively. Let $\mathbf{R}'$ and $\mathbf{G}'$ be the relative subalgebras of $\mathbf{R}$ and $\mathbf{G}$ supported on the resulting subsets of R and G, respectively.
(iii) *Remove* from R' and G' those elements for which there exists some term (in $\mathbf{R}$ or $\mathbf{G}$, respectively) that applied to them gives an element not in R' or G', respectively. Let R^c and G^c be the corresponding closed subsets of $\mathbf{R}$ and $\mathbf{G}$ obtained in this way.
(iv) *Add* $\mathbf{R}'$ to $\mathbf{G}'$, by first forming the coproduct $\mathbf{G}' + \mathbf{R}'$ and then identifying in this coproduct all images of elements of L' by r and m.
(v) Finally, *remove* again from the result of the last step those elements not coming from R^c or G^c.

We want to point out another description of the pushout of two cdc-quomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ of partial Γ-algebras, which we give in the following proposition.

Proposition 23 *Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two cdc-quomorphisms of partial Γ-algebras. As before, let K^c be the greatest closed subset of $\mathbf{K}$ contained in the gluing set K' of f and g, let $\mathbf{K}^c$ be the closed subalgebra of $\mathbf{K}$ supported on K^c, let A^c and B^c be the greatest closed subsets of $\mathbf{A}$ and $\mathbf{B}$ contained respectively in A' and B', and let $\mathbf{A}^c$ and $\mathbf{B}^c$ be the corresponding closed subalgebras of $\mathbf{A}$ and $\mathbf{B}$ supported on A^c and B^c, respectively.*

a) $f|_{K^c} : \mathbf{K}^c \to \mathbf{A}^c$ and $g|_{K^c} : \mathbf{K}^c \to \mathbf{B}^c$ are closed homomorphisms.

Let now

$$\begin{tikzcd} \mathbf{K}^c \arrow[r, "f|_{K^c}"] \arrow[d, "g|_{K^c}"'] & \mathbf{A}^c \arrow[d, "\bar{g}"] \\ \mathbf{B}^c \arrow[r, "\bar{f}"'] & \mathbf{H}_0 \end{tikzcd}$$

be the pushout of $f|_{K^c}$ and $g|_{K^c}$ in C-Alg$_\Gamma$, and let also $\bar{g} : \mathbf{A} \to \mathbf{H}_0$ and $\bar{f} : \mathbf{B} \to \mathbf{H}_0$ denote the cdc-quomorphisms corresponding to the homonymous closed homomorphisms.

b) The square

$$\begin{tikzcd} \mathbf{K} \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{A} \arrow[d, "\bar{g}"] \\ \mathbf{B} \arrow[r, "\bar{f}"'] & \mathbf{H}_0 \end{tikzcd}$$

is a pushout square in CDCQ-Alg_Γ.

PROOF. a) To prove that $f|_{K^c} : \mathbf{K}^c \to \mathbf{A}^c$ is closed, notice that $f|_{K^c} : \mathbf{K}^c \to \mathbf{A}$ is the restriction of the closed homomorphism $f : \mathbf{Dom}\, f \to \mathbf{A}$ to the closed subalgebra $\mathbf{K}^c$ and therefore it is a closed homomorphism, and that $f(K^c) \subseteq A^c$, as we have seen in the proof of Lemma 21. A similar proof applies to $g|_{K^c} : \mathbf{K}^c \to \mathbf{B}^c$.

b) Let $\mathbf{H}^c$, together with cdc-quomorphisms $\hat{g} : \mathbf{A} \to \mathbf{H}^c$ and $\hat{f} : \mathbf{B} \to \mathbf{H}^c$, be the pushout of f and g described in Proposition 22. In particular, $\hat{g}$ and $\hat{f}$ are the cdc-quomorphisms obtained by restricting the c-quomorphisms $\tilde{g} : \mathbf{A}' \to \mathbf{H}'$ and $\tilde{f} : \mathbf{B}' \to \mathbf{H}'$ (appearing in the pushout of f and g in CQ-Alg_Γ described in Proposition 15) to $\mathbf{A}^c$ and $\mathbf{B}^c$, respectively. From the first part of the proof of Proposition 22 (and the reasoning before Lemma 21) we know that $\hat{g} : \mathbf{A}^c \to \mathbf{H}^c$ and $\hat{f} : \mathbf{B}^c \to \mathbf{H}^c$ are closed homomorphisms such that $\hat{g} \circ f|_{K^c} = \hat{f} \circ g|_{K^c}$. Therefore, by the universal property of pushouts, there exists a unique closed homomorphism $h : \mathbf{H}_0 \to \mathbf{H}^c$ such that $h \circ \bar{f} = \hat{f}$ and $h \circ \bar{g} = \hat{g}$. Such h is clearly surjective, because $\hat{f}$ and $\hat{g}$ are jointly surjective by the construction of $\mathbf{H}^c$. It is enough to prove that h is also injective and total (and therefore an isomorphism).

That h is total follows from the facts that $\mathrm{Dom}\, \bar{f} = \mathrm{Dom}\, \hat{f} = \mathbf{B}^c$ and $\mathrm{Dom}\, \bar{g} = \mathrm{Dom}\, \hat{g} = \mathbf{A}^c$, and that $\bar{f}$ and $\bar{g}$ are jointly surjective.

Notice now that this closed homomorphism is nothing but the mapping

$$h : H_0 = (A^c \sqcup B^c)/_{\theta(K^c)} \longrightarrow H = (A' \sqcup B')/_{\theta(K')}$$

induced by the inclusion $A^c \sqcup B^c \hookrightarrow A' \sqcup B'$, and considered as a mapping with target $H^c = \mathrm{Im}\, h$. To show that this mapping is injective, we shall prove that

$$\theta(K') \cap ((A^c \sqcup B^c) \times (A'_s \sqcup B'_s)) = \theta(K^c)$$

(this equality implies that h is not only injective, but actually the inclusion of a subset).

So, let $s \in S$ and let $(z, z') \in \theta(K')_s$ with $z \in A^c_s \sqcup B^c_s$ and $z' \in A'_s \sqcup B'_s$. Since the four cases to be considered ($z \in A^c_s$ and $z' \in A'_s$, $z \in A^c_s$ and $z' \in B'_s$, $z \in B^c_s$ and $z' \in A'_s$, $z \in B^c_s$ and $z' \in B'_s$) are similar, we shall only prove it in a sample case, namely the first one, leaving the others to the reader. So, let $(z, z') \in \theta(K')_s \cap (A^c_s \times A'_s)$. Then there exist $x_0, \ldots, x_{2n} \in K'_s$ such that

$$f_s(x_0) = z,\ f_s(x_{2n}) = z',$$
$$g_s(x_{2k-2}) = g_s(x_{2k-1}),\ f_s(x_{2k-1}) = f_s(x_{2k}), \quad k = 1, \ldots, n$$

Using that $K_s^c = f_s^{-1}(A_s^c) = g_s^{-1}(B_s^c)$ (see Lemma 21) and starting with $f_s(x_0) = z \in A_s^c$, a simple argument by induction on k shows that $x_{2k}, x_{2k+1} \in K_s^c$ for every k. In this way we finally obtain on the one hand that $x_{2n} \in K_s^c$, and therefore $z' = f_s(x_{2n}) \in A_s^c$, and on the other hand that $(z, z') \in \theta(K^c)$, as desired. □

Corollary 24 *Given two partial homomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ of total Γ-algebras, their pushout in* CDCQ-Alg$_\Gamma$ *is also their pushout in* P-TAlg$_\Gamma$.

PROOF. Comparing the construction of the pushout of two cdc-quomorphisms in CDCQ-Alg$_\Gamma$ given in Proposition 23 with the construction of the pushout of two partial homomorphisms in P-TAlg$_\Gamma$ given in [12, Constr. 2.6], one easily sees that both constructions, when applied to two partial homomorphisms of total algebras, are exactly the same, provided that the following two assertions hold.

- K^c is the greatest closed subset of K such that $f^{-1}(f(K^c)) = K^c$ and $g^{-1}(g(K^c)) = K^c$.
- The greatest closed subset A^c of $\mathbf{A}$ contained in A' is also the greatest closed subset of $\mathbf{A}$ contained in $(A - f(K)) \cup f(K^c)$.

The first assertion turns out to be true. Indeed, on the one hand by Lemma 2 we have that $f^{-1}(f(K^c)) \subseteq f^{-1}(f(K')) = K'$, and on the other hand, f being a closed homomorphism from its domain $\mathbf{Dom}\, f$, which is closed in $\mathbf{K}$, and K^c being a closed subset of this domain, by [3, Prop. 3.1.9] we have that $f^{-1}(f(K^c))$ is a closed subset of $\mathbf{K}$. Then, K^c being the greatest closed subset of $\mathbf{K}$ contained in K', we deduce that $f^{-1}(f(K^c)) \subseteq K^c$, and this inclusion becomes an equality because $K^c \subseteq \mathrm{Dom}\, f$. A similar argument shows that $g^{-1}(g(K^c)) = K^c$. And if K^* is some closed subset of K such that $f^{-1}(f(K^*)) = K^*$ and $g^{-1}(g(K^*)) = K^*$ then, by Lemma 2 again, $K^* \subseteq K'$ and therefore $K^* \subseteq K^c$.

And the second assertion is also true, because by Lemma 21 we have that $f^{-1}(A^c) = K^c$ and therefore $A^c \subseteq (A - f(K)) \cup f(K^c)$. □

This entails that the current approach to single-pushout transformation of total unary algebras [12] is a special case of the single-pushout transformation in CDCQ-Alg$_\Gamma$ (but not of the single-pushout transformation in CQ-Alg$_\Gamma$ or CF-Alg$_\Gamma$).

Notice that Proposition 23 and the proof of Corollary 24 yield another way of obtaining the derived partial Γ-algebra of $\mathbf{G}$ by the application of a production rule $r : \mathbf{L} \to \mathbf{R}$ in CDCQ-Alg$_\Gamma$ through a closed homomorphism $m : \mathbf{L} \to \mathbf{G}$.

This method is formally similar to the aforementioned Construction 2.6 in [12].

(i) Compute the gluing set L' of r and m.

(ii) *Remove* from L' those elements for which there exists some term that applied to them gives an element not in L'. Let L^c be the subset of L' obtained in this way.

(iii) Also, *remove* from R and G those elements for which there exists some term that applied to them gives an element in $r(L) - r(L^c)$ and $m(L) - m(L^c)$, respectively. Let $\mathbf{R}^c$ and $\mathbf{G}^c$ be the corresponding closed subalgebras of $\mathbf{R}$ and $\mathbf{G}$ supported on the sets obtained in this way.

(iv) *Add* $\mathbf{R}^c$ to $\mathbf{G}^c$, by first forming the coproduct $\mathbf{G}^c+\mathbf{R}^c$ and then identifying in this coproduct all images of elements of L^c by r and m.

Another important feature of single-pushout transformation of cdc-quomorphisms is that there is a relation between single-pushout transformation in CDCQ-Alg$_\Gamma$ and double-pushout transformation in C-Alg$_\Gamma$, similar to the relation between single-pushout and double-pushout transformation of total unary algebras shown in [12, §3.3].

Definition 25 *Given a cdc-quomorphism of partial* Γ*-algebras* $r : \mathbf{L} \to \mathbf{R}$*, let* $\mathbf{L}_r$ *be the closed subalgebra of* $\mathbf{L}$ *supported on* $\operatorname{Dom} r$*. Let* $i : \mathbf{L}_r \to \mathbf{L}$ *be the corresponding (closed) embedding and let* $r : \mathbf{L}_r \to \mathbf{R}$ *denote also the corresponding closed homomorphism. We shall say that the production rule in* C-Alg$_\Gamma$

$$P(r) = (\mathbf{L} \xleftarrow{i} \mathbf{L}_r \xrightarrow{r} \mathbf{R})$$

is the production rule associated to r.

Let us recall from [6, Def. 21] that, given two closed homomorphisms of partial Γ-algebras $r : \mathbf{K} \to \mathbf{L}$ and $m : \mathbf{L} \to \mathbf{G}$, we say that m satisfies the gluing condition w.r.t. r when $m(r(K)) \cap m(L - r(K)) = \emptyset$ and the restriction of m to $L - r(K)$ is injective (*identification condition*), and $m(L - r(K))$ is an initial segment of $\mathbf{G}$ (*dangling condition*).

Proposition 26 *Let* $r : \mathbf{L} \to \mathbf{R}$ *be a cdc-quomorphism of partial* Γ*-algebras, and let* $m : \mathbf{L} \to \mathbf{G}$ *be a closed homomorphism satisfying the gluing condition w.r.t.* $i : \mathbf{L}_r \to \mathbf{L}$*.*

Then the derived partial Γ*-algebras of* $\mathbf{G}$ *through* m *by the production rule* $P(r)$ *in* C-Alg$_\Gamma$ *[6, Def. 20] and by the production rule* r *in* CDCQ-Alg$_\Gamma$ *are the same (i.e., isomorphic).*

PROOF. Let us follow the construction given in Proposition 23 of the derived partial Γ-algebra of $\mathbf{G}$ through m by means of the production rule r.

We must compute first the greatest closed subset L^c of $\mathbf{L}$ contained in the gluing set L' of r and m. To do this, notice that $r^{-1}(r(L_r)) = L_r$ (because $L_r = \operatorname{Dom} r$) and $m^{-1}(m(L_r)) = L_r$ (because m is total and it satisfies the gluing condition w.r.t. i). Therefore, by Lemma 2, $L' = L_r$. And since L_r is a closed subset of $\mathbf{L}$, because r is a cdc-quomorphism, we also have that $L^c = L_r$.

Now, R^c is the greatest closed subset of $(R - r(L)) \cup r(L_r) = R$, i.e. R itself, and then $\mathbf{R}^c = \mathbf{R}$. And G^c is the greatest closed subset of $\mathbf{G}$ contained in $G' = (G - m(L)) \cup m(L_r)$, but by the gluing condition this set G' is closed (cf. the proof of [6, Prop. 10]), and therefore $G^c = G'$. This implies that $\mathbf{G}^c = \mathbf{G}'$, which is the context algebra of the application of rule $P(r)$ to $\mathbf{G}$ through m [6, Def. 15].

Then, the pushout algebra of $r : \mathbf{L} \to \mathbf{R}$ and $m : \mathbf{L} \to \mathbf{G}$ in CDCQ-Alg$_\Gamma$ is obtained by Proposition 23 as the pushout of the closed homomorphisms $r : \mathbf{L}_r \to \mathbf{R}$ and $m|_{L_r} : \mathbf{L}_r \to \mathbf{G}^c$, and the derived partial Γ-algebra of $\mathbf{G}$ through m by the application of rule $P(r)$ in C-Alg$_\Gamma$ is obtained by [6, Def. 15] exactly in the same way. $\square$

The converse assertion also holds.

Proposition 27 *Let* $P = (\mathbf{L} \xleftarrow{l} \mathbf{K} \xrightarrow{r} \mathbf{R})$ *be a production rule in* C-Alg$_\Gamma$ *with* l *injective, and let* $m : \mathbf{L} \to \mathbf{G}$ *be a closed homomorphism satisfying the gluing condition w.r.t.* l. *Let* $\tilde{r} : \mathbf{L} \to \mathbf{R}$ *be the cdc-quomorphism corresponding to* r *considered as a closed homomorphism from the closed subalgebra of* $\mathbf{L}$ *supported on* $l(K)$.

Then the derived partial Γ*-algebras of* $\mathbf{G}$ *through* m *by the production rule* P *in* C-Alg$_\Gamma$ *and by the production rule* $\tilde{r}$ *in* CDCQ-Alg$_\Gamma$ *are the same.*

PROOF. Notice that P and $P(\tilde{r})$ are the same rule, up to an isomorphism $\mathbf{K} \cong \mathbf{L}_{\tilde{r}}$ compatible with the left and right-hand side morphisms of both rules, so that the previous proposition can be applied. We leave the details to the reader. $\square$

Therefore, and in a similar way to what happens with double-pushout transformation of total unary algebras, double-pushout transformation of unary partial algebras using rules in C-Alg$_\Gamma$ with left-hand side homomorphism injective can be seen as a particular case of single-pushout transformation in CDCQ-Alg$_\Gamma$.

Example 28 *The cdc-quomorphisms between (higher-order) hypergraphs (considered as special partial* Σ_H*-algebras) correspond to the usual partial mor-*

phisms between them; see [1]. And, as it was already mentioned in [6, Ex. 3], (higher-order) hypergraphs considered as partial algebras are closed under push-outs w.r.t. closed homomorphisms, which are the usual morphisms of (higher-order) hypergraphs. Therefore, usual single-pushout transformations of (higher-order) hypergraphs are examples of transformations in CDCQ-Alg_{Σ_H}.

As an example, we shall show how to delete from (finite) hierarchic higher-order hypergraphs all arcs where some abstraction operation is defined. The argument will be similar to the one in Example 10, but it is worth pointing out that using closed homomorphisms as redices instead of total conformisms (as we could do therein) makes us to need a specific rule to delete each arc (a rule that will depend on "how many operations" are defined on it), instead of a single rule for all arcs.

For every $n \geq 1$, *let* $\mathbf{G}(\underline{n})$, *with* $\underline{n} = (n_1, n_2, n, (m_{i,j})_{i=1,\dots,n,j=1,2})$, *be the partial* Σ_H*-algebra with* $G(\underline{n})_E = \{e, e_1, \dots, e_n\}$ *and*

$$G(\underline{n})_V = \{v_j \mid 1 \leq j \leq n_1 + n_2\} \cup \{w_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq m_{i,1} + m_{i,2}\}$$

and with operations defined as follows:

$$\begin{aligned}
&a_j^{\mathbf{G}(\underline{n})}(e) = e_j, \quad j = 1, \dots, n \\
&s_j^{\mathbf{G}(\underline{n})}(e) = v_j, \quad j = 1, \dots, n_1 \\
&t_j^{\mathbf{G}(\underline{n})}(e) = v_{n_1+j}, \quad j = 1, \dots, n_2 \\
&s_j^{\mathbf{G}(\underline{n})}(e_i) = w_{i,j}, \quad i = 1, \dots, n,\ j = 1, \dots, m_{i,1} \\
&t_j^{\mathbf{G}(\underline{n})}(e_i) = w_{i,m_{i,1}+j}, \quad i = 1, \dots, n,\ j = 1, \dots, m_{i,2}
\end{aligned}$$

and let $\widehat{\mathbf{G}}(\underline{n})$ *be the relative subalgebra of* $\mathbf{G}(\underline{n})$ *obtained by removing* e *from* $G(\underline{n})_E$. *Let* $r_{\underline{n}} : \mathbf{G}(\underline{n}) \to \widehat{\mathbf{G}}(\underline{n})$ *be the cdc-quomorphism given by the identity on the carrier of* $\widehat{\mathbf{G}}(\underline{n})$.

Let now $\mathbf{G}$ *be a higher-order hypergraph and let* $m : \mathbf{G}(\underline{n}) \to \mathbf{G}$ *be a closed homomorphism. Let* $m_E(e) = e'$, $m_E(e_i) = e'_i$, $m_V(v_j) = v'_j$ *and* $m_V(w_{i,j}) = w'_{i,j}$. *Since* m *is a closed homomorphism, the following conditions hold:*

- $n_a(e') = n$, *and* $a_i^{\mathbf{G}}(e') = e'_i$ *for every* $i \leq n$.
- $n_a(e'_i) = 0$.
- $n_s(e') = n_1$, *and* $s_j^{\mathbf{G}}(e') = v'_j$ *for every* $j \leq n_1$.
- $n_t(e') = n_2$, *and* $t_j^{\mathbf{G}}(e') = v'_j$ *for every* $n_1 < j \leq n_1 + n_2$.
- $n_s(e'_i) = m_{i,1}$, *and* $s_j^{\mathbf{G}}(e'_i) = w'_{i,j}$, *for every* $i \leq n$ *and* $j \leq m_{i,1}$.
- $n_t(e'_i) = m_{i,2}$, *and* $t_j^{\mathbf{G}}(e'_i) = w'_{i,m_{i,1}+j}$ *for every* $i \leq n$ *and* $m_{i,1} < j \leq m_{i,1} + m_{i,2}$.

Arguing as in Example 10, it is easy to see that the pushout algebra of $r_{\underline{n}}$: $\mathbf{G}(\underline{n}) \to \widehat{\mathbf{G}}(\underline{n})$ and $m : \mathbf{G}(\underline{n}) \to \mathbf{G}$ in CDCQ-Alg$_{\Sigma_H}$ *is given by the relative subalgebra of $\mathbf{G}$ supported on $(G_E - \{e'\}, G_V)$.*

Finally, let $\mathbf{G}$ be a hierarchic (see Example 10) higher-order hypergraph. Let $e' \in G_E$ be an arc such that $n_a(e') \geq 1$, but $n_a(a_i^{\mathbf{G}}(e')) = 0$ for every $i = 1, \ldots, n_a(e')$. Then there is one and only one higher-order hypergraph $\mathbf{G}(\underline{n})$ with a closed homomorphism $m : \mathbf{G}(\underline{n}) \to \mathbf{G}$ such that $m_E(e) = e'$. And the application of rule $r_{\underline{n}}$ to $\mathbf{G}$ through such m removes e' (and all operations defined on it, of course) not touching anything else. This shows that applying rules $r_{\underline{n}}$ in a recurrent way, one finally removes all arcs where some abstraction operation is defined.

As in Example 10, if $\mathbf{G}$ is not hierarchic then it may happen that there does not exist any closed homomorphism $m : \mathbf{G}(\underline{n}) \to \mathbf{G}$ for any $\underline{n}$.

To close this section, notice that arguing as in Proposition 13 we obtain the following result, which we shall use in the next section.

Proposition 29 *Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two closed homomorphisms of partial Γ-algebras, and let*

$$\begin{tikzcd} \mathbf{K} \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{A} \arrow[d, "\tilde{g}"] \\ \mathbf{B} \arrow[r, "\tilde{f}"'] & \mathbf{H} \end{tikzcd}$$

be their pushout in CDCQ-Alg$_\Gamma$. *Then it is also their pushout in* C-Alg$_\Gamma$. □

6 HLR conditions

Contrary to the case of double-pushout algebraic transformation, where many HLR conditions for many results have been introduced (see [17] for a survey), the field of HLR conditions in single-pushout algebraic transformation is at its beginnings. In particular, and to our knowledge, only four HLR conditions, that entail the Local Church-Rosser Theorem, the Parallelism Closure and the Parallelism Theorem, respectively (cf. [9]), have been introduced so far in the literature; see §6.1 below.

In this section, and in order to motivate our claim that single-pushout transformation in CF-Alg$_\Gamma$, CQ-Alg$_\Gamma$ and in CDCQ-Alg$_\Gamma$ has the same good properties as single-pushout transformation in P-TAlg$_\Gamma$ [12], on the one hand we prove that the aforementioned HLR conditions introduced in [9] are satisfied in all three cases taking total morphisms as occurrences. And, on the other hand, we introduce several HLR conditions that are proved to entail some basic amalgamation properties, and we show that they are also satisfied in CF-Alg$_\Gamma$, CQ-Alg$_\Gamma$ and CDCQ-Alg$_\Gamma$, again taking total morphisms as occurrences. We plan to study elsewhere other features of the single-pushout transformation approaches introduced in this paper.

Throughout this section we deal with a general category $\mathcal{C}$, whose morphisms shall be called *rules*, and a subcategory $\mathcal{O}$ of $\mathcal{C}$, whose morphisms shall be called *occurrences*, such that $\mathcal{C}$ and $\mathcal{O}$ have the same class of objects. In applications, $\mathcal{C}$ will be CF-Alg$_\Gamma$, CQ-Alg$_\Gamma$ or CDCQ-Alg$_\Gamma$ (and in some remarks P-TAlg$_\Gamma$), and the occurrences will be the corresponding total morphisms.

Before proceeding with the different subsections, let us recall the important concept of clean occurrences.

Definition 30 *An occurrence $m : \mathbf{L} \to \mathbf{G}$ is said to be* clean *for a rule $r : \mathbf{L} \to \mathbf{R}$ if whenever r is factorized into two rules, $r = q \circ p$ with $\mathbf{L} \xrightarrow{p} \mathbf{S} \xrightarrow{q} \mathbf{R}$, and if the square (1) in*

$$\begin{tikzcd} \mathbf{L} \arrow[r, "p"] \arrow[d, "m"'] \arrow[rr, bend left, "r"] \arrow[dr, phantom, "(1)"] & \boldsymbol{S} \arrow[r, "q"] \arrow[d, "m'"] & \mathbf{R} \\ \mathbf{G} \arrow[r, "p'"'] & \mathbf{H} & \end{tikzcd}$$

is a pushout square in $\mathcal{C}$ then the morphism m' is an occurrence.

6.1 HLR conditions for parallelism

Let us recall the HLR conditions introduced in [9] (they are called there "SPO-conditions for parallelism of HLR-systems of type SPO").

Definition 31 *([9, Def. 3.13]) We say that the pair $(\mathcal{C}, \mathcal{O})$* satisfies the HLR conditions for parallelism *when it satisfies the following four conditions:*

(HLRP1) *There exists a pushout in $\mathcal{C}$ of every rule $r : \mathbf{L} \to \mathbf{R}$ and every occurrence $m : \mathbf{L} \to \mathbf{G}$.*

(HLRP2) *$\mathcal{O}$ has all finite coproducts, which are preserved by the inclusion functor $\mathcal{O} \hookrightarrow \mathcal{C}$.*

(HLRP3) *If $f, g \in \mathcal{C}$ and $f \circ g \in \mathcal{O}$ then $g \in \mathcal{O}$.*

(HLRP4) *Let $f : \mathbf{A}_1 \to \mathbf{A}_2$ and $g : \mathbf{B}_1 \to \mathbf{B}_2$ be two rules in $\mathcal{C}$, let $f + g : \mathbf{A}_1 + \mathbf{B}_1 \to \mathbf{A}_2 + \mathbf{B}_2$ be their coproduct morphism (their* parallel rule*), let $i : \mathbf{A}_1 \to \mathbf{A}_1 + \mathbf{B}_1$ be the corresponding coproduct embedding, and let $p : \mathbf{A}_1 + \mathbf{B}_1 \to \mathbf{C}$ be any rule in $\mathcal{C}$. If there exists some $q : \mathbf{C} \to \mathbf{A}_2 + \mathbf{B}_2$ such that $q \circ p = f + g$ then there exists some $q' : \mathbf{C} \to \mathbf{A}_2$ such that*

$q' \circ (p \circ i) = f$.

In [9] it is proved that single-pushout algebraic transformation in a category $\mathcal{C}$ through occurrences in $\mathcal{O}$ satisfies the *Local Church-Rosser Theorem*, the *Parallelism Closure* and the *Parallelism Theorem* (respectively, point (a), point (c) and points (b) to (d) in Theorem 33 below) whenever $(\mathcal{C}, \mathcal{O})$ satisfies these HLR conditions for parallelism.

Definition 32 *Two direct derivations* $\mathbf{G} \overset{p}{\Longrightarrow} \mathbf{H}$ *and* $\mathbf{G} \overset{q}{\Longrightarrow} \mathbf{H}'$*, given by the pushout squares*

$$\begin{array}{ccc} \mathbf{L} & \xrightarrow{p} & \mathbf{R} \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ \mathbf{G} & \xrightarrow[p']{} & \mathbf{H} \end{array} \qquad \begin{array}{ccc} \mathbf{L}' & \xrightarrow{q} & \mathbf{R}' \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle n'} \\ \mathbf{G} & \xrightarrow[q']{} & \mathbf{H}' \end{array}$$

are said to be parallel independent *when the occurrences of both productions are not destroyed by the other derivation: i.e. when* $q' \circ m, p' \circ n \in \mathcal{O}$.

Theorem 33 *([9, Th. 3.5, Cor. 3.11, and Th. 3.12]) Let* $\mathcal{C}$ *and* $\mathcal{O}$ *be as in Definition 31, and assume that* $(\mathcal{C}, \mathcal{O})$ *satisfies the HLR conditions for parallelism.*

a) For every pair of parallel independent direct derivations $\mathbf{G} \overset{p}{\Longrightarrow} \mathbf{H}$ *and* $\mathbf{G} \overset{q}{\Longrightarrow} \mathbf{H}'$ *there exists an object* $\mathbf{X}$ *and direct derivations* $\mathbf{H} \overset{q}{\Longrightarrow} \mathbf{X}$ *and* $\mathbf{H}' \overset{p}{\Longrightarrow} \mathbf{X}$.

b) If $\mathbf{G} \overset{p}{\Longrightarrow} \mathbf{H}$ *and* $\mathbf{G} \overset{q}{\Longrightarrow} \mathbf{H}'$ *are two parallel independent direct derivations, such that we have* $\mathbf{H} \overset{q}{\Longrightarrow} \mathbf{X}$ *and* $\mathbf{H}' \overset{p}{\Longrightarrow} \mathbf{X}$ *by (a), then there exists a parallel derivation* $\mathbf{G} \overset{p+q}{\Longrightarrow} \mathbf{X}$.

c) Assume that all derivations are made through clean occurrences. For every pair of rules p, q *and for every parallel direct derivation* $\mathbf{G} \overset{p+q}{\Longrightarrow} \mathbf{X}$ *there exist derivation sequences* $\mathbf{G} \overset{p}{\Longrightarrow} \mathbf{H} \overset{q}{\Longrightarrow} \mathbf{X}$ *and* $\mathbf{G} \overset{q}{\Longrightarrow} \mathbf{H}' \overset{p}{\Longrightarrow} \mathbf{X}$ *such that* $\mathbf{G} \overset{p}{\Longrightarrow} \mathbf{H}$ *and* $\mathbf{G} \overset{q}{\Longrightarrow} \mathbf{H}'$ *are parallel independent.*

d) Assume again that all derivations are made through clean occurrences. Then the constructions given by points (b) and (c) are in bijective correspondence.

More specifically, HLR condition (HLRP1) entails (a); conditions (HLRP1) and (HLRP2) entail (b) and (c); and (HLRP1) to (HLRP4) entail (d). □

It turns out that the HLR conditions for parallelism listed above are satisfied by every pair of the form $(\mathcal{C}, \mathcal{O})$ when $\mathcal{C}$ is any one of the categories CF-Alg$_\Gamma$ or CDCQ-Alg$_\Gamma$ and $\mathcal{O}$ is the corresponding subcategory given by the total morphisms, TCF-Alg$_\Gamma$ and C-Alg$_\Gamma$ respectively. Therefore, single-pushout algebraic transformation of partial unary algebras using conformisms or cdc-quomorphisms (taking in both cases totally defined morphisms as occurrences) satisfy points (a) to (d) in the last theorem.

Proposition 34 *The three pairs* $(\text{CF-Alg}_\Gamma, \text{TCF-Alg}_\Gamma)$*,* $(\text{CQ-Alg}_\Gamma, \text{TAlg}_\Gamma)$ *and* $(\text{CDCQ-Alg}_\Gamma, \text{C-Alg}_\Gamma)$ *satisfy the HLR conditions for parallelism, for every graph structure* Γ*.*

PROOF. The categories CF-Alg_Γ, CQ-Alg_Γ and CDCQ-Alg_Γ have all pushouts. Moreover, the coproduct of two partial Γ-algebras $\mathbf{A}$ and $\mathbf{B}$ is given by the usual disjoint union $\mathbf{A}+\mathbf{B}$ in C-Alg_Γ, CQ-Alg_Γ, CF-Alg_Γ, CDCQ-Alg_Γ and TCF-Alg_Γ (it can be seen, for instance, noticing that the coproduct of $\mathbf{A}$ and $\mathbf{B}$ in any one of these categories is the pushout of the empty morphisms $\emptyset \to \mathbf{A}$ and $\emptyset \to \mathbf{B}$, and using also Corollaries 13, 17 and 29; see also [2,8,16]).

Condition (HLRP3) for all three pairs is trivial for set-theoretical reasons.

Finally, as far as condition (HLRP4) goes, it is an easy consequence of the fact that the partial mapping $\pi : \mathbf{A}_2 + \mathbf{B}_2 \to \mathbf{A}_2$ given by the identity on $\mathbf{A}_2$ is a cdc-quomorphism, and therefore if $f+g = q \circ p$ then

$$f = \pi \circ (f+g) \circ i = \pi \circ (q \circ p) \circ i = (\pi \circ q) \circ (p \circ i)$$

and if q is a conformism (resp. a c-quomorphism, a cdc-quomorphism) then $\pi \circ q$ is also a conformism (resp. a c-quomorphism, a cdc-quomorphism). $\square$

In order to give full content to the Parallelism Closure and the Parallelism Theorem for single-pushout transformation in CF-Alg_Γ, C-Alg_Γ and CDCQ-Alg_Γ, we have to identify in each category which occurrences are clean for a given rule.

Definition 35 *Let* $\mathbf{L}$*,* $\mathbf{G}$ *and* $\mathbf{R}$ *be partial* Γ*-algebras, with carriers* L*,* G *and* R *respectively, and let* $m : L \to G$ *and* $r : L \to R$ *be respectively a total and a partial mapping of* S*-sets.*

i) m *is* d-injective *w.r.t.* r *(see for instance [12, Def. 3.10.(2)]) when for every* $s \in S$ *and* $x, y \in L_s$*, if* $m_s(x) = m_s(y)$ *then* $x, y \in \operatorname{Dom} r_s$ *or* $x = y$*.*

ii) m *is* d-closed-injective *w.r.t.* r *when for every* $s \in S$ *and* $x, y \in L_s$*, if* $m_s(x) = m_s(y)$ *then* $x, y \in \operatorname{Dom} r_s$ *or* $C_{\mathbf{L}}(\{x\}) = C_{\mathbf{L}}(\{y\})$ [8]*.*

Notice so that m is d-injective w.r.t. r when $m_s(x) = m_s(y)$ implies that either $x, y \in \operatorname{Dom} r_s$ or x and y "generate" the same relative subalgebra of $\mathbf{L}$,

[8] As in Lemma 19, by $C_{\mathbf{L}}(\{z\})$ we denote the closed subset of $\mathbf{L}$ generated by the S-set with all carriers of all sorts empty, except the carrier of sort s, which is $\{z\}$.

while m is d-closed-injective w.r.t. r when $m_s(x) = m_s(y)$ implies that either $x, y \in \operatorname{Dom} r_s$ or x and y generate the same closed subalgebra of $\mathbf{L}$.

Proposition 36 *Given a rule* $r : \mathbf{L} \to \mathbf{R}$ *and an occurrence* $m : \mathbf{L} \to \mathbf{G}$ *in* CF-Alg_Γ, *m is clean for r iff it is d-injective w.r.t. r.*

PROOF. Assume that m is a clean occurrence for r and let $x, y \in L_s$ such that $m_s(x) = m_s(y)$ and $x \neq y$. Consider the subset D of L with $D_s = \operatorname{Dom} r_s \cup \{y\}$ and $D_t = \operatorname{Dom} r_t$ for every $t \neq s$. Let $\mathbf{D}$ be the relative subalgebra of $\mathbf{L}$ supported on D and consider the factorization $r|_D \circ \mathrm{Id}_D$ of r with $\mathrm{Id}_D : \mathbf{L} \to \mathbf{D}$ the c-quomorphism given by the identity on D and $r|_D : \mathbf{D} \to \mathbf{R}$ the conformism $r : \mathbf{L} \to \mathbf{R}$ taken with source algebra $\mathbf{D}$. Consider the diagram

$$\begin{tikzcd}
\mathbf{L} \arrow[r, "\mathrm{Id}_D"] \arrow[d, "m"'] \arrow[rr, bend left, "r"] \arrow[dr, phantom, "(1)"] & \mathbf{D} \arrow[r, "r|_D"] \arrow[d, "m'"] & \mathbf{R} \\
\mathbf{G} \arrow[r, "p'"'] & \mathbf{H} &
\end{tikzcd}$$

where (1) is a pushout square in CF-Alg_Γ. Since m is clean, m' has to be total and therefore $m^{-1}(m(D)) = D$ by Corollary 4.(ii). And since $x \neq y$ and $y \in D_s$, $m^{-1}(m_s(x)) \subseteq D_s$ and then $x \in \operatorname{Dom} r_s$. Therefore, m is d-injective w.r.t. r.

Conversely, let $r : \mathbf{L} \to \mathbf{R}$ be a conformism of partial Γ-algebras and let $m : \mathbf{L} \to \mathbf{G}$ be a total conformism, d-injective w.r.t. r. Let $r = q \circ p$ be any factorization of r, with $\mathbf{L} \xrightarrow{p} \mathbf{S} \xrightarrow{q} \mathbf{R}$, and consider the diagram

$$\begin{tikzcd}
\mathbf{L} \arrow[r, "p"] \arrow[d, "m"'] \arrow[rr, bend left, "r"] \arrow[dr, phantom, "(1)"] & \mathbf{S} \arrow[r, "q"] \arrow[d, "m'"] & \mathbf{R} \\
\mathbf{G} \arrow[r, "p'"'] & \mathbf{H} &
\end{tikzcd}$$

where (1) is a pushout square in CF-Alg_Γ. Let $x \in m_s^{-1}(m_s(\operatorname{Dom} p_s))$, say $m_s(x) = m_s(y)$ with $y \in \operatorname{Dom} p_s$. Then d-injectivity implies that either $x = y \in \operatorname{Dom} p_s$ or $x, y \in \operatorname{Dom} r_s \subseteq \operatorname{Dom} p_s$. Since m is total and $s \in S$ any sort, this implies that $m^{-1}(m(\operatorname{Dom} p)) = \operatorname{Dom} p$ and therefore that m' is total, by Corollary 4.(ii). Therefore, m is clean for r. $\square$

The proof of this proposition applies *mutatis mutandi* to c-quomorphisms, yielding:

Proposition 37 *Given a rule* $r : \mathbf{L} \to \mathbf{R}$ *and an occurrence* $m : \mathbf{L} \to \mathbf{G}$ *in* CQ-Alg$_\Gamma$*,* m *is clean for* r *iff it is d-injective w.r.t.* r*.* $\square$

As far as cdc-quomorphisms, we need the following lemma.

Lemma 38 *Let*

$$\begin{tikzcd} \mathbf{K} \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{A} \arrow[d, "\hat{g}"] \\ \mathbf{B} \arrow[r, "\hat{f}"'] & \mathbf{H}^c \end{tikzcd}$$

be a pushout square in CDCQ-Alg$_\Gamma$*. Then* $\hat{g}$ *is total iff* $g^{-1}(g(\mathrm{Dom}\, f)) = \mathrm{Dom}\, f$*.*

PROOF. Since $\mathrm{Dom}\,\hat{g}$ is the greatest closed subset of $\mathbf{A}$ contained in A', we have that $\hat{g}$ is total iff $A = A'$. And by Corollary 4.(ii) the latter is equivalent to $g^{-1}(g(\mathrm{Dom}\, f)) = \mathrm{Dom}\, f$. $\square$

Proposition 39 *Given a rule* $r : \mathbf{L} \to \mathbf{R}$ *and an occurrence* $m : \mathbf{L} \to \mathbf{G}$ *in* CDCQ-Alg$_\Gamma$*,* m *is clean for* r *iff it is d-closed-injective w.r.t.* r*.*

PROOF. Assume that m is clean for r, and let $x, y \in L_s$ such that $m_s(x) = m_s(y)$. If $y \in \mathrm{Dom}\, r_s$ but $x \notin \mathrm{Dom}\, r_s$ then the cdc-quomorphism m' in the pushout square in CDCQ-Alg$_\Gamma$

$$\begin{tikzcd} \mathbf{L} \arrow[r, "r"] \arrow[d, "m"'] & \mathbf{R} \arrow[d, "m'"] \\ \mathbf{G} \arrow[r, "r'"'] & \mathbf{H} \end{tikzcd}$$

is not total, because $m^{-1}(m(\mathrm{Dom}\, r)) \neq \mathrm{Dom}\, r$. Therefore, if m is clean and $x \in \mathrm{Dom}\, r_s$ or $y \in \mathrm{Dom}\, r_s$ then $x, y \in \mathrm{Dom}\, r_s$.

Assume now that $x, y \notin \mathrm{Dom}\, r_s$, and set $D = \mathrm{Dom}\, r \cup C_{\mathbf{L}}(\{y\})$, which is a closed subset of $\mathbf{L}$. Let $\mathbf{D}$ be the closed subalgebra of $\mathbf{L}$ supported on D and consider the factorization $r|_D \circ \mathrm{Id}_D$ of r with $\mathrm{Id}_D : \mathbf{L} \to \mathbf{D}$ the cdc-quomorphism given by the identity on D and $r|_D : \mathbf{D} \to \mathbf{R}$ the cdc-quomorphism $r : \mathbf{L} \to \mathbf{R}$ taken with source algebra $\mathbf{D}$. Consider the diagram

$$\begin{tikzcd} \mathbf{L} \arrow[r, "\mathrm{Id}_D"] \arrow[rr, bend left, "r"] \arrow[d, "m"'] \arrow[dr, phantom, "(1)"] & \mathbf{D} \arrow[r, "r|_D"] \arrow[d, "m'"] & \mathbf{R} \\ \mathbf{G} \arrow[r, "p'"'] & \mathbf{H} & \end{tikzcd}$$

where (1) is a pushout square in CDCQ-Alg$_\Gamma$. Since m is clean, m' is total. Therefore $m^{-1}(m(D)) = D$, and since $x \in m_s^{-1}(m_s(D_s))$ but $x \notin \mathrm{Dom}\, r_s$, we conclude that $x \in C_{\mathbf{L}}(\{y\})_s$ and therefore $C_{\mathbf{L}}(\{x\}) \subseteq C_{\mathbf{L}}(\{y\})$. By symmetry, we also have that $C_{\mathbf{L}}(\{y\}) \subseteq C_{\mathbf{L}}(\{x\})$.

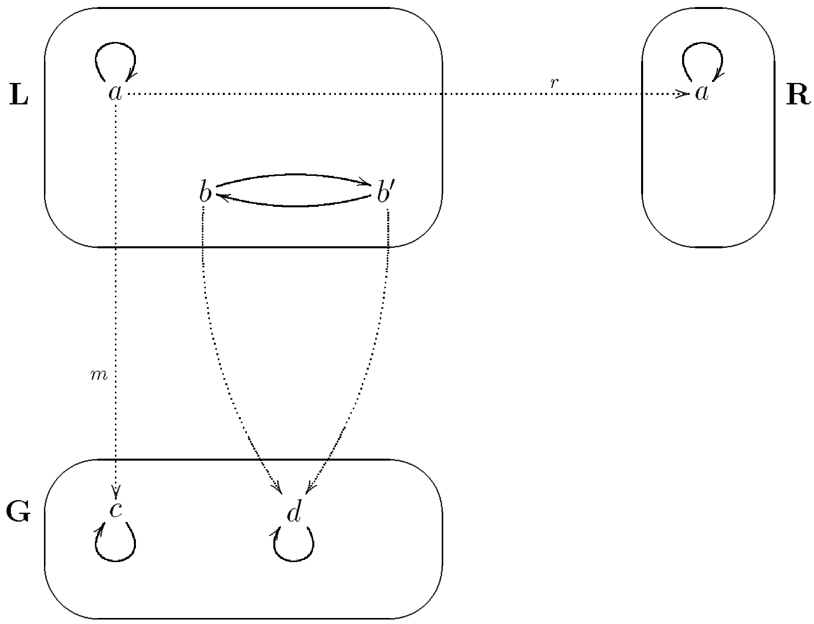
Therefore, if m is clean for r and if $m_s(x) = m_s(y)$ then $x, y \in \mathrm{Dom}\, r_s$ or $C_{\mathbf{L}}(\{x\}) = C_{\mathbf{L}}(\{y\})$.

Conversely, let $r : \mathbf{L} \to \mathbf{R}$ be a cdc-quomorphism of partial Γ-algebras and let $m : \mathbf{L} \to \mathbf{G}$ a closed homomorphism d-closed-injective w.r.t. r. Let $r = q \circ p$ be any factorization of r, with $\mathbf{L} \xrightarrow{p} \mathbf{S} \xrightarrow{q} \mathbf{R}$, and consider the diagram

$$\begin{tikzcd}
\mathbf{L} \arrow[r, "p"] \arrow[rr, bend left, "r"] \arrow[d, "m"'] \arrow[dr, phantom, "(1)"] & \mathbf{S} \arrow[r, "q"] \arrow[d, "m'"] & \mathbf{R} \\
\mathbf{G} \arrow[r, "p'"'] & \mathbf{H} &
\end{tikzcd}$$

where (1) is a pushout square in CDCQ-Alg$_\Gamma$. Let $x \in m_s^{-1}(m_s(\mathrm{Dom}\, p_s))$, say $m_s(x) = m_s(y)$ with $y \in \mathrm{Dom}\, p_s$. Then d-closed-injectivity implies that either $x, y \in \mathrm{Dom}\, r_s \subseteq \mathrm{Dom}\, p_s$ or $C_{\mathbf{L}}(\{x\})_s = C_{\mathbf{L}}(\{y\})_s \subseteq \mathrm{Dom}\, p_s$ (because $\mathrm{Dom}\, p$ is closed), so that in both cases $x \in \mathrm{Dom}\, p_s$. Since m is total and $s \in S$ any sort, this implies that $m^{-1}(m(\mathrm{Dom}\, p)) = \mathrm{Dom}\, p$ and thus, by the previous lemma, that m' is total. Therefore, m is clean for r. $\Box$

Remark. It is stated in [9, p. 44] that the clean occurrences in P-TAlg$_\Gamma$ are exactly the d-injective (total) homomorphisms. It is not true in general, as the following example shows. Let Γ be a graph structure with a single sort and a single unary operation, let $\mathbf{L}$ be a Γ-algebra with carrier $L = \{a, b, b'\}$ and with operation $\varphi^{\mathbf{L}}$ given by $\varphi^{\mathbf{L}}(a) = a$, $\varphi^{\mathbf{L}}(b) = b'$ and $\varphi^{\mathbf{L}}(b') = b$, let $\mathbf{R}$ be its (closed) subalgebra supported on $R = \{a\}$, let $\mathbf{G}$ be a Γ-algebra with carrier $G = \{c, d\}$ and with operation $\varphi^{\mathbf{G}}$ given by $\varphi^{\mathbf{G}}(c) = c$ and $\varphi^{\mathbf{G}}(d) = d$. Let $r : \mathbf{L} \to \mathbf{R}$ be the partial homomorphism given by the identity on $\mathbf{R}$, and let $m : \mathbf{L} \to \mathbf{G}$ be the (closed) homomorphism given by $m(a) = c$ and $m(b) = m(b') = d$.



It is clear that m is not d-injective w.r.t. r, and yet it is clean for it. Indeed, if $r = q \circ p$ then either $\mathrm{Dom}\, p = \{a\}$ or $\mathrm{Dom}\, p = L$ (because $\mathrm{Dom}\, p$ has to be a closed subset of $\mathbf{L}$ containing $\mathrm{Dom}\, r = \{a\}$), and in both cases $m^{-1}(m(\mathrm{Dom}\, p)) = \mathrm{Dom}\, p$. This entails that, if

$$\begin{tikzcd} \mathbf{L} \arrow[r, "p"] \arrow[d, "m"'] & \mathbf{S} \arrow[d, "m'"] \\ \mathbf{G} \arrow[r, "p'"'] & \mathbf{H} \end{tikzcd}$$

is a pushout in P-TAlg$_\Gamma$ (which is also a pushout in CDCQ-Alg$_\Gamma$) then m' is total by Lemma 38.

Actually, the proof of Proposition 39 also applies to partial homomorphisms of total Γ-algebras, showing that the clean occurrences in P-TAlg$_\Gamma$ are exactly the d-closed-injective homomorphisms. Notice that in the case of an acyclic graph structure (i.e., where there does not exist any sequence of operations $\varphi_1, \ldots, \varphi_n$, $n \geq 1$, such that $\eta(\varphi_i) = (s_i, s_{i+1})$ for every $i = 1, \ldots, n$ and $s_{n+1} = s_1$), as for instance the signature corresponding to hypergraphs [6, Ex 4], "d-closed-injective" is equivalent to "d-injective." But this equivalence does not hold for instance for higher-order hypergraphs.

6.2 HLR conditions for amalgamation

Non parallel independent derivations can still be "merged" or amalgamated, by means of a construction known as *amalgamated sum*.

Recall that $\mathcal{C}$ is a category (whose morphisms are called rules) and $\mathcal{O}$ is a subcategory of $\mathcal{C}$ (whose morphisms are called occurrences) such that $\mathcal{C}$ and $\mathcal{O}$ have the same class of objects. Let us assume for the moment that the pair $(\mathcal{C}, \mathcal{O})$ satisfies the following two conditions (that we shall later impose as HLR conditions).

(HLRA1) $\mathcal{O}$ has all binary pushouts, which are preserved by the inclusion functor $\mathcal{O} \hookrightarrow \mathcal{C}$.

(HLRA2) There exists a pushout in $\mathcal{C}$ of every rule $r : \mathbf{L} \to \mathbf{R}$ and occurrence $m : \mathbf{L} \to \mathbf{G}$.

The clean occurrences satisfy, under these assumptions, the following properties that shall be used in the sequel.

Lemma 40 *Let $(\mathcal{C}, \mathcal{O})$ satisfy conditions (HLRA1) and (HLRA2). Then the following properties hold.*

i) For every pair of composable rules $r_1 : \mathbf{L} \to \mathbf{R}$ and $r_2 : \mathbf{R} \to \mathbf{S}$, every clean occurrence for $r_2 \circ r_1$ is also a clean occurrence for r_1.

ii) For every pair of composable rules $r_1 : \mathbf{L} \to \mathbf{R}$ and $r_2 : \mathbf{R} \to \mathbf{S}$, if $m : \mathbf{L} \to \mathbf{G}$ is a clean occurrence for $r_2 \circ r_1$ and if (1) in the diagram

$$\begin{tikzcd}
\mathbf{L} \arrow[r, "r_1"] \arrow[d, "m"'] \arrow[dr, phantom, "(1)"] & \mathbf{R} \arrow[r, "r_2"] \arrow[d, "m'"] & S \\
\mathbf{G} \arrow[r, "r_1'"'] & \mathbf{H} &
\end{tikzcd}$$

is a pushout of r_1 and m in $\mathcal{C}$, then m' is a clean occurrence for r_2.

iii) Let $r : \mathbf{L} \to \mathbf{R}$ and $r_1 : \mathbf{L}_1 \to \mathbf{R}_1$ be two rules for which there exist a clean occurrence $i : \mathbf{L} \to \mathbf{L}_1$ for r and an occurrence $j : \mathbf{R} \to \mathbf{R}_1$ such that $r_1 \circ i = j \circ r$. If m is a clean occurrence for r_1 then $m \circ i$ is a clean occurrence for r.

PROOF. As far as points (i) and (ii) go, let $m : \mathbf{L} \to \mathbf{G}$ be a clean occurrence for $r_2 \circ r_1$, let $r_1 = q_1 \circ p_1$ be any factorization of r_1, with $\mathbf{L} \stackrel{p_1}{\to} \mathbf{T}_1 \stackrel{q_1}{\to} \mathbf{R}$, and let $r_2 = q_2 \circ p_2$ be any factorization of r_2, with $\mathbf{R} \stackrel{p_2}{\to} \mathbf{T}_2 \stackrel{q_2}{\to} \mathbf{S}$. Consider the diagram

$$\begin{tikzcd}
\mathbf{L} \arrow[r, "p_1"] \arrow[rr, bend left, "r_1"] \arrow[d, "m"'] \arrow[dr, phantom, "(1)"] & \mathbf{T}_1 \arrow[r, "q_1"] \arrow[d, "m_1"] \arrow[dr, phantom, "(2)"] & \mathbf{R} \arrow[r, "p_2"] \arrow[rr, bend left, "r_2"] \arrow[d, "m'"] \arrow[dr, phantom, "(3)"] & \mathbf{T}_2 \arrow[r, "q_2"] \arrow[d, "m_2"] & \mathbf{S} \\
\mathbf{G} \arrow[r, "p_1'"'] & \mathbf{H}_1 \arrow[r, "q_1'"'] & \mathbf{H} \arrow[r, "p_2'"'] & \mathbf{H}_2 &
\end{tikzcd}$$

where squares (1), (2) and (3) are pushout squares in $\mathcal{C}$.

Since m is clean for $r_2 \circ r_1 = (q_2 \circ p_2 \circ q_1) \circ p_1$, we have that m_1 is an occurrence. This proves that m is also clean for r_1, as it is stated in point (i).

On the other hand, since $r_2 \circ r_1 = q_2 \circ (p_2 \circ q_1 \circ p_1)$ and (1)+(2)+(3) is a pushout square, we have that m_2 is an occurrence. This proves that m' is clean for r_2, as it is stated in point (ii).

As far as point (iii) goes, let $r = q \circ p$ be any factorization of r, with $\mathbf{L} \xrightarrow{p} \mathbf{S} \xrightarrow{q} \mathbf{R}$, and consider the commutative diagram

$$\begin{tikzcd}
\mathbf{L} \arrow[r, "p"] \arrow[rr, bend left, "r"] \arrow[d, "i"'] \arrow[dr, phantom, "(1)"] & \mathbf{S} \arrow[r, "q"] \arrow[d, "i'"] \arrow[dr, phantom, "(3)"] & \mathbf{R} \arrow[d, "j"] \\
\mathbf{L}_1 \arrow[r, "p'"] \arrow[d, "m"'] \arrow[dr, phantom, "(2)"] & \mathbf{S}_1 \arrow[r, "h"'] \arrow[d, "m'"] & \mathbf{R}_1 \\
\mathbf{G} \arrow[r, "p''"'] & \mathbf{H} &
\end{tikzcd}$$

where (1) and (2) are pushout squares, so that (1)+(2) yields a pushout square for p and $m \circ i$, and h is the unique morphism $h : \mathbf{S}_1 \to \mathbf{R}_1$ such that $h \circ i' = j \circ q$ and $h \circ p' = r_1$. Since m is a clean occurrence for r_1, we have that $m' \in \mathcal{O}$, and since i is a clean occurrence for r, we have that $i' \in \mathcal{O}$. Therefore $m' \circ i' \in \mathcal{O}$, which shows that $m \circ i$ is a clean occurrence for r. $\square$

Under the hypothesis of point (iii) in this lemma, it is said that r is a subrule of r_1. Subrules and remainders, together with amalgamated sums to be defined later, are the key concepts in amalgamation.

Definition 41 *A rule $r : \mathbf{L} \to \mathbf{R}$ is a* subrule *of a rule $r_1 : \mathbf{L}_1 \to \mathbf{R}_1$ when there exist a clean occurrence $i : \mathbf{L} \to \mathbf{L}_1$ for r and an occurrence $j : \mathbf{R} \to \mathbf{R}_1$ such that $j \circ r = r_1 \circ i$*

$$\begin{tikzcd}
\mathbf{L} \arrow[r, "r"] \arrow[d, "i"'] & \mathbf{R} \arrow[d, "j"] \\
\mathbf{L}_1 \arrow[r, "r_1"'] & \mathbf{R}_1
\end{tikzcd}$$

We shall denote it $r \leq_{(i,j)} r_1$, or simply $r \leq r_1$, when the pair (i,j) is irrelevant or understood.

*If $r \leq_{(i,j)} r_1$ then let $\mathbf{S}$ together with $i' : \mathbf{R} \to \mathbf{S}$ and $r' : \mathbf{L}_1 \to \mathbf{S}$ be a pushout of r and i in $\mathcal{C}$. The (i,j)-*remainder *of r_1 w.r.t. r, which shall be denoted $r_1 -_{(i,j)} r : \mathbf{S} \to \mathbf{R}_1$ or simply $r_1 - r : \mathbf{S} \to \mathbf{R}_1$ when the pair (i,j) is irrelevant or understood, is the unique rule $\mathbf{S} \to \mathbf{R}_1$ such that $(r_1 - r) \circ i' = j$ and*

$(r_1 - r) \circ r' = r_1$.

$$\begin{tikzcd}
\mathbf{L} \arrow[r, "r"] \arrow[d, "i"'] & \mathbf{R} \arrow[d, "i'"] \arrow[dr, "j"] & \\
\mathbf{L}_1 \arrow[r, "r'"'] \arrow[rr, bend right, "r_1"'] & \boldsymbol{S} \arrow[r, dotted, "r_1 - r"'] & \mathbf{R}_1
\end{tikzcd}$$

Subrules allow to decompose derivations.

Proposition 42 *Assume that all derivations are made through clean occurrences and that $(\mathcal{C}, \mathcal{O})$ satisfies conditions (HLRA1) and (HLRA2) above, and let $r : \mathbf{L} \to \mathbf{R}$ and $r_1 : \mathbf{L}_1 \to \mathbf{R}_1$ be two rules in $\mathcal{C}$ such that $r \leq r_1$. If $\mathbf{G} \stackrel{r_1}{\Longrightarrow} \mathbf{H}$ then there exists an object $\mathbf{H}'$ such that $\mathbf{G} \stackrel{r}{\Longrightarrow} \mathbf{H}' \stackrel{r_1 - r}{\Longrightarrow} \mathbf{H}$.*

PROOF. Let $i : \mathbf{L} \to \mathbf{L}_1$ and $j : \mathbf{R} \to \mathbf{R}_1$ be occurrences (the first one, clean for r) making r to be a subrule of r_1. Assume that $\mathbf{G} \stackrel{r_1}{\Longrightarrow} \mathbf{H}$ and let $m : \mathbf{L}_1 \to \mathbf{G}$ be a clean occurrence for r_1 used to derive $\mathbf{H}$ from $\mathbf{G}$ by the application of rule r_1.

Consider the following commutative diagram

$$\begin{tikzcd}
\mathbf{L} \arrow[r, "r"] \arrow[d, "i"'] \arrow[dr, phantom, "(1)"] & \mathbf{R} \arrow[d, "i'"] & \\
\mathbf{L}_1 \arrow[r, "r'"] \arrow[d, "m"'] \arrow[dr, phantom, "(2)"] & \mathbf{S} \arrow[r, "r_1 - r"] \arrow[d, "m'"] \arrow[dr, phantom, "(3)"] & \mathbf{R}_1 \arrow[d, "m''"] \\
\mathbf{G} \arrow[r, "r''"'] & \mathbf{H}' \arrow[r, "p"'] & \mathbf{H}''
\end{tikzcd}$$

where all three squares (1), (2) and (3) are pushout squares in $\mathcal{C}$.

Then on the one hand (1)+(2) is a pushout square for r and $m \circ i$, and since $m \circ i$ is a clean occurrence for r by Lemma 40.(iii), we have that $\mathbf{G} \stackrel{r}{\Longrightarrow} \mathbf{H}'$. And on the other hand, since (2)+(3) is a pushout square for $r_1 = (r_1 - r) \circ r'$ and m, and since m is a clean occurrence for r_1, we have that $\mathbf{H}''$ is the derived object of $\mathbf{G}$ by the application of rule r_1 through occurrence m. Therefore $\mathbf{H}'' = \mathbf{H}$ (up to isomorphism). And since m' is a clean occurrence for $r_1 - r$ by Lemma 40(ii), we have that $\mathbf{H}' \stackrel{r_1 - r}{\Longrightarrow} \mathbf{H}$. $\square$

Definition 43 *Let $r : \mathbf{L} \to \mathbf{R}$, $r_1 : \mathbf{L}_1 \to \mathbf{R}_1$ and $r_2 : \mathbf{L}_2 \to \mathbf{R}_2$ be three rules* *in $\mathcal{C}$ such that $r \leq_{(i_1,j_1)} r_1$ and $r \leq_{(i_2,j_2)} r_2$. Let*

$$\begin{tikzcd} \mathbf{L} \arrow[r, "i_1"] \arrow[d, "i_2"'] & \mathbf{L}_1 \arrow[d, "i_2'"] \\ \mathbf{L}_2 \arrow[r, "i_1'"'] & \mathbf{L}' \end{tikzcd} \qquad \begin{tikzcd} \mathbf{R} \arrow[r, "j_1"] \arrow[d, "j_2"'] & \mathbf{R}_1 \arrow[d, "j_2'"] \\ \mathbf{R}_2 \arrow[r, "j_1'"'] & \mathbf{R}' \end{tikzcd}$$

be the pushout squares in $\mathcal{O}$ (and also in $\mathcal{C}$) of i_1 and i_2, and of j_1 and j_2 respectively. Then the amalgamated sum rule $r_1 +_r r_2 : \mathbf{L}' \to \mathbf{R}'$ *is defined as the unique morphism in $\mathcal{C}$ (which exists by the universal property of pushouts) making the following cube commutative:*

$$\begin{tikzcd} \mathbf{L} \arrow[rr, "r"] \arrow[dr, "i_1"] \arrow[ddd, "i_2"'] & & \mathbf{R} \arrow[dr, "j_1"] \arrow[ddd, "j_2"] & \\ & \mathbf{L}_1 \arrow[rr, "r_1"] \arrow[ddd, "i_2'"] & & \mathbf{R}_1 \arrow[ddd, "j_2'"] \\ & & & \\ \mathbf{L}_2 \arrow[rr, "r_2"'] \arrow[dr, "i_1'"'] & & \mathbf{R}_2 \arrow[dr, "j_1'"] & \\ & \mathbf{L}' \arrow[rr, dotted, "r_1 +_r r_2"'] & & \mathbf{R}' \end{tikzcd}$$

This is time now to introduce the HLR conditions for amalgamation, which, as we have said, will include the conditions introduced at the beginning of this subsection.

Definition 44 *We say that the pair $(\mathcal{C}, \mathcal{O})$ satisfies the* HLR conditions for amalgamation *when it satisfies the following properties:*

(HLRA1) *$\mathcal{O}$ has all binary pushouts, which are preserved by the inclusion functor $\mathcal{O} \hookrightarrow \mathcal{C}$.*
(HLRA2) *There exists a pushout in $\mathcal{C}$ of every rule $r : \mathbf{L} \to \mathbf{R}$ and occurrence $m : \mathbf{L} \to \mathbf{G}$.*
(HLRA3) *If $f, g \in \mathcal{C}$ and $f \circ g \in \mathcal{O}$ then $g \in \mathcal{O}$.*
(HLRA4) *If m is a clean occurrence for $r : \mathbf{L} \to \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{S}$ is an occurrence, then m is also clean for $f \circ r$.*

Lemma 45 *Assume that $(\mathcal{C}, \mathcal{O})$ satisfies the HLR conditions for amalgamation. If r is a subrule of r_1 and r_2 then r_1 and r_2 are subrules of $r_1 +_r r_2$.*

PROOF. Assume $r \leq_{(i_1,j_1)} r_1$ and $r \leq_{(i_2,j_2)} r_2$, and consider the following commutative diagram

$$
\begin{tikzcd}
\mathbf{L} \arrow[rrr, "r"] \arrow[dr, "i_1"] \arrow[dd, "i_2"'] & & & \mathbf{R} \arrow[dr, "j_1"] \arrow[dd, "j_2"] & \\
& \mathbf{L}_1 \arrow[r, "p"] \arrow[dd, "i_2'"] \arrow[rrr, bend right, "r_1"'] & \mathbf{K}_1 \arrow[rr, "q"] \arrow[dd, "\tilde{\imath}_2"] & & \mathbf{R}_1 \arrow[dd, "j_2'"] \\
\mathbf{L}_2 \arrow[rrr, "r_2"'] \arrow[dr, "i_1'"'] & & & \mathbf{R}_2 \arrow[dr, "j_1'"'] & \\
& \mathbf{L}' \arrow[r, "p'"'] \arrow[rrr, bend right, "r_1 +_r r_2"'] & \mathbf{K}' \arrow[rr, "q'"'] & & \mathbf{R}'
\end{tikzcd}
$$

where $r_1 = q \circ p$, $r_1 +_r r_2 = q' \circ p'$, the left-hand side rectangle in the front face (the one involving $\mathbf{L}_1$, $\mathbf{K}_1$, $\mathbf{L}'$ and $\mathbf{K}'$) is a pushout square in $\mathcal{C}$, and both the left-hand side and the right-hand side faces of the cube are pushout squares in $\mathcal{O}$ (and therefore also in $\mathcal{C}$). Since i_2 is a clean occurrence for r, it is also clean for $j_1 \circ r$ by condition (HLRA4). Then from $j_1 \circ r = r_1 \circ i_1 = q \circ (p \circ i_1)$ and the composition of pushout squares

$$
\begin{tikzcd}
\mathbf{L} \arrow[r, "i_1"] \arrow[d, "i_2"'] & \mathbf{L}_1 \arrow[r, "p"] \arrow[d, "i_2'"] & \mathbf{K}_1 \arrow[d, "\tilde{\imath}_2"] \\
\mathbf{L}_2 \arrow[r, "i_1'"'] & \mathbf{L}' \arrow[r, "p'"'] & \mathbf{K}'
\end{tikzcd}
$$

we obtain that $\tilde{\imath}_2$ is an occurrence. This shows that $i_2' : \mathbf{L}_1 \to \mathbf{L}'$ is a clean occurrence for r_1, and therefore, since $j_2' : \mathbf{R}_1 \to \mathbf{R}'$ is an occurrence by construction, that $r_1 \leq_{(i_2', j_2')} r_1 +_r r_2$. By symmetry, $r_2 \leq_{(i_1', j_1')} r_1 +_r r_2$. $\square$

Common subrules also induce a synchronization mechanism for derivations. When two rules share a common subrule, the effect of the shared subrule is produced only once and the whole derivation is obtained by derivations through rule remainders.

Proposition 46 *Assume that all derivations are made through clean occurrences and that $(\mathcal{C}, \mathcal{O})$ satisfies the HLR conditions for amalgamation, and let $r : \mathbf{L} \to \mathbf{R}$, $r_1 : \mathbf{L}_1 \to \mathbf{R}_1$ and $r_2 : \mathbf{L}_2 \to \mathbf{R}_2$ be three rules in $\mathcal{C}$ such that $r \leq r_1$ and $r \leq r_2$. If $\mathbf{G} \overset{r_1 +_r r_2}{\Longrightarrow} \mathbf{H}$ then there exist $\mathbf{H}'$ and $\mathbf{H}''$ such that*

$$\mathbf{G} \overset{r}{\Longrightarrow} \mathbf{H}' \overset{r_1 - r}{\Longrightarrow} \mathbf{H}'' \overset{r_2 - r}{\Longrightarrow} \mathbf{H}.$$

PROOF. Since by the previous lemma we have that $r_1 \leq r_1 +_r r_2$, by Proposition 42 there exists an object $\mathbf{H}''$ such that $\mathbf{G} \overset{r_1}{\Longrightarrow} \mathbf{H}'' \overset{(r_1 +_r r_2) - r_1}{\Longrightarrow} \mathbf{H}$. And since $r \leq r_1$, again by Proposition 42 we have that there exists an object $\mathbf{H}'$ such that $\mathbf{G} \overset{r}{\Longrightarrow} \mathbf{H}' \overset{r_1 - r}{\Longrightarrow} \mathbf{H}''$.

It is enough to show that $\mathbf{H}'' \overset{r_2 - r}{\Longrightarrow} \mathbf{H}$, and to do that it is enough to find a pushout square in $\mathcal{C}$

$$
\begin{tikzcd}
\mathbf{K}_2 \arrow[r, "r_2 - r"] \arrow[d, "f"'] & \mathbf{R}_2 \arrow[d, "\tilde{f}"] \\
\mathbf{N}' \arrow[r, "(r_1 +_r r_2) - r_1"'] & \mathbf{R}'
\end{tikzcd}
$$

with f a clean occurrence for $r_2 - r$.

Consider the following diagram (referred to henceforth as *the main diagram*)

$$
\begin{tikzcd}
\mathbf{L} \arrow[rr, "r"] \arrow[dr, "i_1"] \arrow[dd, "i_2"'] & & \mathbf{R} \arrow[dr, "k_1"] \arrow[drrr, "j_1"] \arrow[ddrr, bend left, "j_2"] \arrow[dd, "k_2"] & & & & \\
& \mathbf{L}_1 \arrow[rr, "r_1'"] \arrow[dd, "i_2'"'] & & \mathbf{K}_1 \arrow[rr, "r_1 - r"] \arrow[dd, "k_2'"] & & \mathbf{R}_1 \arrow[dd, "n_1"] \arrow[ddr, bend left, "j_2'"] & \\
\mathbf{L}_2 \arrow[rr, "r_2'"'] \arrow[dr, "i_1'"'] & & \mathbf{K}_2 \arrow[rr, "r_2 - r"'] \arrow[dr, "k_1'"'] & & \mathbf{R}_2 \arrow[drr, "j_1'"] & & \\
& \mathbf{L}' \arrow[rr, "r_3'"'] & & \mathbf{K}' \arrow[rr, "n_2"'] & & \mathbf{N}' \arrow[r, "(r_1 +_r r_2) - r_1"'] & \mathbf{R}'
\end{tikzcd}
$$

which is obtained as follows:

- The squares $\mathbf{L}, \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}'$ and $\mathbf{R}, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}'$ are pushout squares in $\mathcal{O}$, and therefore also in $\mathcal{C}$.
- The squares $\mathbf{L}, \mathbf{R}, \mathbf{L}_1, \mathbf{K}_1$, $\mathbf{L}, \mathbf{R}, \mathbf{L}_2, \mathbf{K}_2$ and $\mathbf{L}_2, \mathbf{K}_2, \mathbf{L}', \mathbf{K}'$ are pushout squares in $\mathcal{C}$. Let $r_1 - r : \mathbf{K}_1 \to \mathbf{R}_1$ and $r_2 - r : \mathbf{K}_2 \to \mathbf{R}_2$ be respectively the (i_1, j_1)-remainder of r_1 w.r.t. r and the (i_2, j_2)-remainder of r_2 w.r.t. r (Definition 41).
- $k_2' : \mathbf{K}_1 \to \mathbf{K}'$ is the only rule such that $k_2' \circ k_1 = k_1' \circ k_2$ and $k_2' \circ r_1' = r_3' \circ i_2'$. Then, we have the commutative cube (extracted from the left-hand side of

the main diagram)

$$\begin{tikzcd}
\mathbf{L} \arrow[rr, "r"] \arrow[dr, "i_1"] \arrow[dd, "i_2"'] & & \mathbf{R} \arrow[dr, "k_1"] \arrow[dd, "k_2"] & \\
& \mathbf{L}_1 \arrow[rr, "r'_1"] \arrow[dd, "i'_2"'] & & \mathbf{K}_1 \arrow[dd, "k'_2"] \\
\mathbf{L}_2 \arrow[rr, "r'_2"'] \arrow[dr, "i'_1"'] & & \mathbf{K}_2 \arrow[dr, "k'_1"'] & \\
& \mathbf{L}' \arrow[rr, "r'_3"'] & & \mathbf{K}'
\end{tikzcd}$$

whose top, bottom, back and left-hand side faces are pushout squares. Standard properties of pushouts entail that the remaining two faces, corresponding to squares $\mathbf{R}, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}'$ and $\mathbf{L}_1, \mathbf{K}_1, \mathbf{L}', \mathbf{K}'$, are also pushout squares.

– The square $\mathbf{K}_1, \mathbf{R}_1, \mathbf{K}', \mathbf{N}'$ is a pushout square. Then, the composition of pushout squares

$$\begin{tikzcd}
\mathbf{L}_1 \arrow[r, "r'_1"] \arrow[d, "i'_2"'] & \mathbf{K}_1 \arrow[r, "r_1 - r"] \arrow[d, "k'_2"] & \mathbf{R}_1 \arrow[d, "n_1"] \\
\mathbf{L}' \arrow[r, "r'_3"'] & \mathbf{K}' \arrow[r, "n_2"'] & \mathbf{N}'
\end{tikzcd}$$

yields a pushout

$$\begin{tikzcd}
\mathbf{L}_1 \arrow[r, "r_1"] \arrow[d, "i'_2"'] & \mathbf{R}_1 \arrow[d, "n_1"] \\
\mathbf{L}' \arrow[r, "n_2 \circ r'_3"'] & \mathbf{N}'
\end{tikzcd}$$

Since i'_2 is a clean occurrence for r_1 by the proof of Lemma 45, we have that k'_2 and n_1 are occurrences.

– Let $(r_1 +_r r_2) - r_1 : \mathbf{N}' \to \mathbf{R}'$ be the (i'_2, j'_2)-remainder of $r_1 +_r r_2$ w.r.t. r_1; in particular it satisfies that $((r_1 +_r r_2) - r_1) \circ n_1 = j'_2$ and $((r_1 +_r r_2) - r_1) \circ (n_2 \circ r'_3) = r_1 +_r r_2$.

We shall show that the square

$$\begin{tikzcd}
\mathbf{K}_2 \arrow[r, "r_2 - r"] \arrow[d, "f := n_2 \circ k'_1"'] & \mathbf{R}_2 \arrow[d, "j'_1"] \\
\mathbf{N}' \arrow[r, "(r_1 +_r r_2) - r_1"'] & \mathbf{R}'
\end{tikzcd}$$

is a pushout square in $\mathcal{C}$ and that $n_2 \circ k'_1$ is a clean occurrence for $r_2 - r$. As we have mentioned at the beginning of this proof, this will finish the proof.

To do that, consider the following commutative prism, extracted from the right-hand side of the main diagram:

$$
\begin{tikzcd}
\mathbf{R} \arrow[rr, "j_1"] \arrow[d, "k_2"'] \arrow[dr, "j_2"] & & \mathbf{R}_1 \arrow[dd, "n_1"'] \arrow[ddr, "j_2'"] & \\
\mathbf{K}_2 \arrow[r, "r_2-r"] \arrow[drr, "f"'] & \mathbf{R}_2 \arrow[drr, "j_1'"] & & \\
& & \mathbf{N}' \arrow[r, "(r_1+_r r_2)-r_1"'] & \mathbf{R}'
\end{tikzcd}
$$

In this commutative diagram, the right-hand side face $\mathbf{R}, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}'$ is a pushout square by construction. And the left-hand side face $\mathbf{R}, \mathbf{R}_1, \mathbf{K}_2, \mathbf{N}'$ is a pushout square, because it is the composition of pushout squares

$$
\begin{tikzcd}
\mathbf{R} \arrow[r, "k_1"] \arrow[d, "k_2"'] & \mathbf{K}_1 \arrow[r, "r_1-r"] \arrow[d, "k_2'"] & \mathbf{R}_1 \arrow[d, "n_1"] \\
\mathbf{K}_2 \arrow[r, "k_1'"'] & \mathbf{K}' \arrow[r, "n_2"'] & \mathbf{N}'
\end{tikzcd}
=
\begin{tikzcd}
\mathbf{R} \arrow[r, "j_1"] \arrow[d, "k_2"'] & \mathbf{R}_1 \arrow[d, "n_1"] \\
\mathbf{K}_2 \arrow[r, "f"'] & \mathbf{N}'
\end{tikzcd}
$$

This has two consequences. On the one hand, standard properties of pushouts entail that the bottom square, which is the one we are interested in, is also a pushout square. And on the other hand, since j_1 and k_2 are occurrences (the latter, because i_2 is a clean occurrence for r), it turns out that $f : \mathbf{K}_2 \to \mathbf{N}'$ and $n_1 : \mathbf{R}_1 \to \mathbf{N}'$ are occurrences.

Finally, in order to show that f is clean for $r_2 - r$, let $r_2 - r = q \circ p$ be any decomposition of $r_2 - r$, with $\mathbf{K}_2 \xrightarrow{p} \mathbf{S}_2 \xrightarrow{q} \mathbf{R}_2$, and let

$$
\begin{tikzcd}
\mathbf{K}_2 \arrow[r, "p"] \arrow[d, "f"'] & \mathbf{S}_2 \arrow[d, "f'"] \\
\mathbf{N}' \arrow[r, "p'"'] & \mathbf{S}_3
\end{tikzcd}
$$

be a pushout square in $\mathcal{C}$. We have to prove that f' is an occurrence.

Notice now that we have the composition of pushouts

$$
\begin{tikzcd}
\mathbf{R} \arrow[r, "k_2"] \arrow[d, "j_1"'] & \mathbf{K}_2 \arrow[r, "p"] \arrow[d, "f"] & \mathbf{S}_2 \arrow[d, "f'"] \\
\mathbf{R}_1 \arrow[r, "n_1"'] & \mathbf{N}' \arrow[r, "p'"'] & \mathbf{S}_3
\end{tikzcd}
=
\begin{tikzcd}
\mathbf{R} \arrow[r, "p \circ k_2"] \arrow[d, "j_1"'] & \mathbf{S}_2 \arrow[d, "f'"] \\
\mathbf{R}_1 \arrow[r, "p' \circ n_1"'] & \mathbf{S}_3
\end{tikzcd}
$$

where j_1 is an occurrence by definition and $p \circ k_2$ is an occurrence because $q \circ (p \circ k_2) = j_2$ is an occurrence (and then condition (HLRA3) applies). Therefore f' is an occurrence, because of (HLRA1). $\square$

Proposition 47 *Assume that all derivations are made through clean occurrences and that* $(\mathcal{C}, \mathcal{O})$ *satisfies the HLR conditions for amalgamation, and let* $r : \mathbf{L} \to \mathbf{R}$, $r_1 : \mathbf{L}_1 \to \mathbf{R}_1$ *and* $r_2 : \mathbf{L}_2 \to \mathbf{R}_2$ *be three rules in* $\mathcal{C}$ *such that* $r \leq r_1$ *and* $r \leq r_2$. *If* $\mathbf{G} \overset{r_1 +_r r_2}{\Longrightarrow} \mathbf{H}$ *then there exists* $\mathbf{H}'$ *and two parallel independent derivations* $\mathbf{H}' \overset{r_1 - r}{\Longrightarrow} \mathbf{H}''$ *and* $\mathbf{H}' \overset{r_2 - r}{\Longrightarrow} \mathbf{H}'''$ *such that* $\mathbf{G} \overset{r}{\Longrightarrow} \mathbf{H}' \overset{(r_1 - r) + (r_2 - r)}{\Longrightarrow} \mathbf{H}$.

The situation dealt with in Prop. 47 is illustrated by the derivation diagram

$$\begin{tikzcd}
 & & \mathbf{H}'' \arrow[dr, Rightarrow, "r_2 - r"] & \\
\mathbf{G} \arrow[r, Rightarrow, "r"] & \mathbf{H}' \arrow[ur, Rightarrow, "r_1 - r"] \arrow[rr, Rightarrow, "(r_1 - r) + (r_2 - r)"] \arrow[dr, Rightarrow, "r_2 - r"'] & & \mathbf{H} \\
 & & \mathbf{H}''' \arrow[ur, Rightarrow, "r_1 - r"'] &
\end{tikzcd}$$

where the upper path corresponds to Prop. 46.

PROOF. If $(\mathcal{C}, \mathcal{O})$ satisfies the HLR conditions for amalgamation then points (a) and (b) in Theorem 33 hold. Therefore, this proposition will be implied by the previous one if, with the notations therein, we can find the two parallel independent derivations $\mathbf{H}' \overset{r_1 - r}{\Longrightarrow} \mathbf{H}''$ and $\mathbf{H}' \overset{r_2 - r}{\Longrightarrow} \mathbf{H}'''$.

Now consider the main diagram in the proof of the previous proposition, enlarged with the pushout square corresponding to the direct derivation $\mathbf{G} \overset{r_1 +_r r_2}{\Longrightarrow} \mathbf{H}$ through a clean occurrence $m : \mathbf{L}' \to \mathbf{G}$, decomposed into several pushout squares that correspond to the direct derivations $\mathbf{G} \overset{r}{\Longrightarrow} \mathbf{H}'$, $\mathbf{H}' \overset{r_1 - r}{\Longrightarrow} \mathbf{H}''$ and $\mathbf{H}'' \overset{r_2 - r}{\Longrightarrow} \mathbf{H}$:

$$\begin{tikzcd}
\mathbf{L} \arrow[rr, "r"] \arrow[dr, "i_1"] \arrow[dd, "i_2"'] & & \mathbf{R} \arrow[dr, "k_1"] \arrow[dd, "k_2"] \arrow[drrr, "j_1"] \arrow[ddrr, bend left, "j_2"] & & & & \\
 & \mathbf{L}_1 \arrow[rr, "r'_1"] \arrow[dd, "i'_2"] & & \mathbf{K}_1 \arrow[rr, "r_1 - r"] \arrow[dd, "k'_2"] & & \mathbf{R}_1 \arrow[dd, "n_1"] \arrow[ddr, bend left, "j'_2"] & \\
\mathbf{L}_2 \arrow[rr, "r'_2"'] \arrow[dr, "i'_1"'] & & \mathbf{K}_2 \arrow[rr, "r_2 - r"'] \arrow[dr, "k'_1"'] & & \mathbf{R}_2 \arrow[drr, "j'_1"] & & \\
 & \mathbf{L}' \arrow[rr, "r'_3"'] \arrow[d, "m"'] & & \mathbf{K}' \arrow[rr, "n_2"] \arrow[d, "m'"] & & \mathbf{N}' \arrow[r, "(r_1 +_r r_2) - r_1"'] \arrow[d, "m''"] & \mathbf{R}' \arrow[d] \\
 & \mathbf{G} \arrow[rr, "r'"'] & & \mathbf{H}' \arrow[rr, "n'_2"'] & & \mathbf{H}'' \arrow[r] & \mathbf{H}
\end{tikzcd}$$

In this diagram, we have that $n'_2 \circ (m' \circ k'_1)$ is an occurrence, because $n'_2 \circ (m' \circ k'_1) = m'' \circ n_2 \circ k'_1$ where $n_2 \circ k'_1$ is an occurrence, as it has been seen in the proof of the previous proposition, and m'' is an occurrence, because m is a clean occurrence for $r_1 +_r r_2$ by assumption.

By symmetry, if

$$\begin{tikzcd}
\mathbf{K}_2 \arrow[r, "r_2 - r"] \arrow[d, "k'_1"'] & \mathbf{R}_2 \arrow[d, "m_1"] \\
\mathbf{K}' \arrow[r, "m_2"'] \arrow[d, "m'"'] & \mathbf{M}' \arrow[d, "m'''"] \\
\mathbf{H}' \arrow[r, "m'_2"'] & \mathbf{H}'''
\end{tikzcd}$$

are two pushout squares in $\mathcal{C}$ then $m'_2 \circ (m' \circ k'_1)$ is an occurrence. Then the pushout squares

$$\begin{tikzcd}
\mathbf{K}_1 \arrow[r, "r_1 - r"] \arrow[d, "m' \circ k'_2"'] & \mathbf{R}_1 \arrow[d, "m'' \circ n_1"] \\
\mathbf{H}' \arrow[r, "n'_2"'] & \mathbf{H}''
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{K}_2 \arrow[r, "r_2 - r"] \arrow[d, "m' \circ k'_1"'] & \mathbf{R}_2 \arrow[d, "m''' \circ m_1"] \\
\mathbf{H}' \arrow[r, "m'_2"'] & \mathbf{H}'''
\end{tikzcd}$$

correspond to parallel independent derivations $\mathbf{H}' \overset{r_1 - r}{\Longrightarrow} \mathbf{H}''$ and $\mathbf{H}' \overset{r_2 - r}{\Longrightarrow} \mathbf{H}'''$. As we mentioned earlier, this finishes the proof. $\square$

It remains to show that the pairs (CF-Alg$_\Gamma$, TCF-Alg$_\Gamma$), (CQ-Alg$_\Gamma$, C-Alg$_\Gamma$) and (CDCQ-Alg$_\Gamma$, C-Alg$_\Gamma$) satisfy the HLR conditions for amalgamation introduced in Definition 44, when Γ is a graph structure.

Proposition 48 *If Γ is a graph structure then the pairs* (CF-Alg$_\Gamma$, TCF-Alg$_\Gamma$), (CQ-Alg$_\Gamma$, C-Alg$_\Gamma$) *and* (CDCQ-Alg$_\Gamma$, C-Alg$_\Gamma$) *satisfy the HLR conditions for amalgamation.*

PROOF. Conditions (HLRA2) and (HLRA3) have already been considered in Proposition 34. Condition (HLRA1) is given by Propositions 13, 17 and 29. And condition (HLRA4) is straightforward from the fact that if $r : L \to R$ is a partial mapping and $f : R \to S$ is a total mapping then $\mathrm{Dom}\,(f \circ r) = \mathrm{Dom}\, r$. $\square$

Therefore, single-pushout transformation in CF-Alg$_\Gamma$, CQ-Alg$_\Gamma$ and CDCQ-Alg$_\Gamma$ through d-injective total conformisms, d-injective closed homomorphisms and d-closed-injective closed homomorphisms, respectively, satisfies the basic properties of amalgamation given in Propositions 42, 46 and 47.

Remark. The pair (P-TAlg$_\Gamma$, TAlg$_\Gamma$), where TAlg$_\Gamma$ denotes the category of all total Γ-algebras with usual (total) homomorphisms as morphisms, also satisfies the HLR conditions for amalgamation (the proof is similar to the one for $(\text{CDCQ-Alg}_\Gamma, \text{C-Alg}_\Gamma)$). Therefore, Propositions 42, 46 and 47 are satisfied by single-pushout transformation in P-TAlg_Γ through d-closed-injective homomorphisms. This generalizes Theorem 6.3 and the "only if" implication in Theorem 6.7 of [12], where these properties are proved for single-pushout transformation through d-injective homomorphisms. This contributes to the solution of the problem proposed in Footnote 35, p. 210, of the aforementioned paper, where it is asked to investigate conditions more general than d-injectivity under which the results on amalgamation stated therein still hold. Since d-closed-injective homomorphisms (and not d-injective homomorphisms) are the clean occurrences in P-TAlg_Γ, we believe that this is the right setting where to prove those results. And, indeed, it is not difficult to prove directly that the remaining "if" implication in Theorem 6.7 of *loc. cit.* also holds replacing d-injectivity by d-closed-injectivity.

7 Conclusion

The single-pushout approach to graph transformation is extended in this article to the algebraic transformation of partial many-sorted unary algebras; i.e., for Γ any graph structure. The algebraic characterization is developed in the category CF-Alg_Γ of all conformisms of partial Γ-algebras, in the category CQ-Alg_Γ of all closed quomorphisms of partial Γ-algebras, and in the category CDCQ-Alg_Γ of all closed-domain closed quomorphisms of partial Γ-algebras. Such an algebraic characterization is accompanied by an operational characterization, which may serve as a basis for implementation. We have the following relations between these new single-pushout approaches to algebraic transformation, and between them and previous approaches:

- Single-pushout transformation in CQ-Alg_Γ turns out to be a particular case of transformation in CF-Alg_Γ.
- Single-pushout transformation in CQ-Alg_Γ and in CDCQ-Alg_Γ are independent, yielding two completely different approaches to transformation of unary partial algebras.
- The single-pushout transformation of unary total algebras introduced by M. Löwe in [12] turns out to be a special case of single-pushout transformation in CDCQ-Alg_Γ.
- Single-pushout transformation in CF-Alg_Γ has no relation with any double-pushout approach to transformation introduced in [6], and its results fit better into one's intuitive expectations than those of double-pushout transformation in Alg_Γ.
- Single-pushout transformation in CDCQ-Alg_Γ generalizes double-pushout transformation in C-Alg_Γ, exactly in the same way as single-pushout transformation of unary total algebras (or hypergraphs) generalizes double-push-

out transformation.

The categories of partial many-sorted unary algebras considered in this paper are also shown to satisfy all of the HLR conditions for parallelism introduced in [9]. Similar to the case of the double-pushout approach [6], the HLR conditions satisfied by a pair $(\mathcal{C}, \mathcal{O})$ entail the satisfaction of different rewriting theorems by single-pushout transformation systems in $(\mathcal{C}, \mathcal{O})$ formed by production rules in $\mathcal{C}$ and occurrences in $\mathcal{O}$. In particular, the HLR conditions for parallelism entail the Local Church-Rosser Theorem, the Parallelism Closure and the Parallelism Theorem, which are therefore satisfied by the three single-pushout transformation systems of unary partial algebras introduced here.

HLR conditions for amalgamation are also defined which allow to decompose non parallel-independent derivations sharing a common subrule into a common derivation followed by parallel-independent derivations. The categories CF-Alg_Γ, CQ-Alg_Γ and CDCQ-Alg_Γ are also shown to satisfy all of the HLR conditions for amalgamation, taking the corresponding total morphisms as occurrences.

Let us finally mention that our results, together with the work in [14,15], motivate further study of general asymmetric transformations, where occurrences are not necessarily total morphisms but some class of morphisms determined by additional HLR conditions.

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References

[1] R. Alberich, P. Burmeister, F. Rosselló, G. Valiente and B. Wojdyło, A Partial Algebras Approach to Graph Transformation. To appear in *Proc. Fifth Int. Workshop on Graph Grammars and their Application to Computer Science*, Lect. Notes in Comp. Sc. (1996).

[2] R. Alberich, M. Moyà, F. Rosselló and L. Sastre, A note on a series of papers by Burmeister and Wojdyło. To appear in *Demonstratio Mathematica* (1997).

[3] P. Burmeister, A Model Theoretic Oriented Approach to Partial Algebras (Part I of: Introduction to the Theory and Applications of Partial Algebras). *Mathematical Research* vol. 32, Akademie-Verlag (1986).

[4] P. Burmeister, Partial Algebras. An Introductory Survey. In: Algebras and Orders: *Proc. NATO Advanced Study Institute and Séminaire de Mathématiques Supérieures on Algebras and Orders*, Montréal, Canada, July 29–August 9, 1991 (Eds.: I. G. Rosenberg and G. Sabidussi), NATO ASI Series C, Vol. 389, Kluwer Academic Publ. (1993), 1–70.

[5] P. Burmeister, Properties of homomorphisms and quomorphisms between partial algebras II. Some equivalences and some new factorization systems. Preliminary Report (1992).

[6] P. Burmeister, F. Rosselló, J. Torrens and G. Valiente, Algebraic Transformation of Unary Partial Algebras I: Double-Pushout Approach. To appear in *Theoretical Computer Science* (1996).

[7] P. Burmeister and B. Wojdyło, Properties of homomorphisms and quomorphisms between partial algebras. *Contributions to General Algebra* 5 (Proc. Salzburg Conf. 1986; eds. J. Czermak, G. Eigenthaler, H. K. Kaiser, W. B. Müller, W. Nöbauer), Verlag Hölder-Pichler-Tempski and Verlag B. G. Teubner (1987), 69–90.

[8] P. Burmeister and B. Wojdyło, The meaning of Basic Category Theoretical Notions in Some Categories of Partial Algebras I, II, III. Part I, *Demonstratio Mathematica* XXV (1992), 583–602; Part II, *ibidem*, 973–994; Part III to appear in *ibidem* (1997).

[9] H. Ehrig and M. Löwe, Categorical Principles, Techniques and Results for High-Level-Replacement Systems in Computer Science. *Appl. Cat. Struct.* 1 (1993), 21–50.

[10] H. J. Hoehnke, Superposition partieller Funktionen. *Studien zur Algebra und ihre Anwendungen* (Hoehnke, ed.), Akademie Verlag (1972), 7–26.

[11] R. Kennaway, Graph Rewriting in Some Categories of Partial Morphisms. In: *Proc. Fourth Int. Workshop on Graph Grammars and their Application to Computer Science*, Lect. Notes in Comp. Sc. 532 (1991), 490–504.

[12] M. Löwe, Algebraic approach to single-pushout graph transformation. *Theoretical Computer Science* 109 (1993), 181–224.

[13] M. Monserrat, F. Rosselló and J. Torrens, When is a Category of Many-Sorted Partial Algebras Cartesian Closed? *International Journal of Foundations of Computer Science* 6 (1995), 51–66.

[14] M. Monserrat, F. Rosselló, J. Torrens and G. Valiente, Hypergraph Rewriting using Conformisms. *Electronic Notes in Theoretical Computer Science* 2 (1995), 105–112.

[15] M. Monserrat, F. Rosselló, J. Torrens and G. Valiente, Rewriting in Categories of Spans. Research Report LSI-96-17-R, Department of Software, Technical University of Catalonia (1996).

[16] M.-R. Motthagi, *Kategorientheoretische Eigenschaften der totaldefinierten Konformismen und der Quomorphismen auf initialen Segmenten.* Diploma Thesis, Fachbereich Mathematik, Technische Hochschule Darmstadt, 1993.

[17] J. Padberg, Survey of High-Level Replacement Systems. Tech. Report 8/93, Tech. Univ. Berlin (March 1993).

[18] V. Poythress, Partial morphisms of partial algebras. *Algebra Universalis* 3 (1973) 182–202.

[19] J. C. Raoult, On Graph Rewritings. *Theoretical Computer Science* 32 (1984), 1–24.

[20] G. Valiente, Verification of Knowledge Base Redundancy and Subsumption using Graph Transformations. *International Journal of Expert Systems* 6 (1993), 341–355.

[21] G. Valiente, *Knowledge Base Verification using Algebraic Graph Transformations.* PhD Thesis, University of the Balearic Islands (1994).

[22] A. Wagner and M. Gogolla, Defining Operational Behavior of Object Specifications by Attributed Graph Transformations. To appear in *Fundamenta Informaticæ* (1996).