Evidence of a Noise Induced Transition in Fluid Neural Networks

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Abstract

Focusing on a phenomenon observed in the temporal patterns of activity behavior of *Leptothorax* ants, we introduce the notion of *Noise Induced Computation*. We study, from a Noise Induced Transition perspective, the emergent property of autosynchronization in ant colonies, using to model this phenomenon the formalism of Fluid Neural Networks. It is known that through this autosynchronization, ants attain, as a global computational property, a mechanism for information exchange known from computer science as mutual exclusion. We have numerical evidence that this happens at the critical point of the noise induced transition.

KEYWORDS: Selforganization, Noise Induced Transitions, Fluid Neural Networks

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1 Introduction

The importance of noise in swarm dynamics has been stated by some researchers during the past decade; to cite a few examples, Deneubourg et al. [1] showed that random behavior at both, individual and collective levels could be an efficient way to cope with unpredictable environments, Millonas [2] has pointed out the means by which the internal fluctuations of the swarm can modify its behavior, and the algorithmic study of Tofts [3] uses a probabilistic process algebra to demonstrate how an age structure emerges over ants performing various tasks (temporal polyethism). However, as far as we know only Millonas [4] makes an explicit reference to *Noise Induced Transitions* [5] as the mechanism that could be behind the influence of noise in swarm dynamics.

It is well known that in some systems, the presence of noise, appearing in a multiplicative way, can induce ordered and structured behavior. Generally, this means that there is a qualitative change in the shape of a probability density, although recently Van den Broeck et al. [6] have discovered a spatially distributed system that has a nonequilibrium phase transition (in the sense of [7]) induced by external noise. Noise induced transitions, in the former sense, have been studied extensively, theoretically and experimentally, in systems of zero spatial dimension [5], and the study of noise induced transitions in systems at greater spatial dimension (see, for example, Mikhailov et al. [8]) is today a subject of intensive research. The work that we present in this paper seeks to continue this research line, but perhaps in an unorthodox way: swarms in nature, like honey combs or ant colonies, can undoubtedly be considered as spatially distributed systems of agents with simple computational capabilities, that show in their global behavior properties unattainables by individual elements (this is roughly what is known by *emergence*); what we will begin to point out is the fact that noise induced transitions could be the mechanism by which global computational capabilities can emerge from interactions between individuals.

To make our ideas concrete, we will focus on a particular phenomenon which has been observed in the temporal patterns of colony behavioral activity of *Leptothorax* ants. Experimental work with ants of the species *Leptothorax acervorum* [9] shows that these ants exhibit cyclical bursts of activity within the nest. There have been several attempts at providing an explanation of how these cycles arise (for a survey, see Tofts et al. [10], see also Solé et al. [11]), all of them emphasizing the self-organizing character of these oscillations, which emerge through local and individual interactions. Another point, maybe of more interest to us, has been the work of Hatcher et al. [12], where these authors show how the phenomenon of mutual exclusion, a global computational property of the nest, may operate as a mechanism for information exchange. Mutual exclusion is an important general problem in concurrent programming, an instance of which is, for example, resource allocation. As defined in [13], we say that activity A_1 of process P_1 and activity A_2 of process P_2 must exclude each other if the execution of A_1 may not overlap the execution of A_2 In our case, mutual exclusion is attained with the combined effects of autosynchronization and the spatial arrangement of brood in the nests of *L. acervorum*, which appears to limit the number of workers that can access brood simulaneously [12].

The aim of this paper is to make clear enough that this selforganized activity cycles could be a consequence of the subtle interplay between the nonlinearities of the system and the external noise that causes the spontaneous activation of ants. The formalism that will be used is that of Fluid Neural Networks (FNN for short, see [11,14,15,16]), a spatially distributed system of mobile formal neurons, based on the thought-provoking analogies between ant colonies and neural collectives [17], that has been successful at providing an explanation of the above mentioned cyclical bursts. FNN will be described in Section 2. In Section 3 we introduce numerical results that point to the existence of a noise induced transition. This allows us to define in section 4 an order parameter for the transition, extending previous results on FNN [16]. Finally, in section 5, we discuss the emergence of mutual exclusion through the process of autosynchronization, and its relation with noise, introducing the new concept of *Noise Induced Computation.* We will address also in this section the possible relation between noise induced transitions and the so called *Computation at the Edge of Chaos* [18], which has been observed in FNN [16].

2 Fluid Neural Networks

Here we will define our model, the FNN. As a theoretical framework, the standard approach of neural networks is used, but a new set of rules defining local movement and individual activation are also introduced.

A set of N automata is used. The state of each automaton (say the i-th one) is described through a continuous state variable $S_i(t) \in \mathbf{R}$, at each time step $t \in \mathbf{N}$.

Each element can move on a $L \times L$ two-dimensional lattice.

A set of rules is defined:

(a) Neural network structure: the automata are considered as some kind of "neurons" (ants) and interactions are described as in neural networks [19] by means of a sigmoidal function $\Phi(x)$. If $S_j(t)$ is a given automaton (the spatial dependence is omitted for simplicity), the new states are updated following:

$$S_i(t+\Delta t) = \Phi\left[g\left\{ J_{ii}S_i(t) + \sum_{i\neq j\in B(i)} J_{ij}S_j(t) \ - \Theta_i \right\} \right] \tag{1}$$

where $B(i)$ are the nearest automata, located in the neighborhood defined by the eight nearest lattice sites, and $J_{ii} \neq 0$. For simplicity we use the threshold $\Theta_i = 0$, and we take $\Phi(z) = tanh(gz)$ with $g = 0.1$.

(b) Spontaneous activation: each automaton can be either *active* or *inactive* and, if active, it moves randomly to one of the eight nearest cells (if no space is available, no movement takes place). In our model a given automaton will be active if $S_i(t) > \theta$ and inactive otherwise. Here this threshold is $\theta = 10^{-16}$. Once an automaton becomes inactive, it can return to the active state (with an *spontaneous activity level* S_a; here we use $S_a = 0.1$) with some probability p_a.

(c) Coupling matrix: the coupling matrix $\mathbf{J}$ is not fixed. Connections are local and changing over time as a consequence of movement. They are also state-dependent i.e. J_{ij} will be a simple function of the states of the actually interacting pair (i,j) of automata, i.e. $J_{ij} = f(S_i^t, S_j^t)$. In our case, where two basic states are defined, i.e. "active" and "inactive", the connection matrix reduces to the following 2×2 table:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}$$

At a given time step, the interaction J_{ij} between the i-th and the j-th elements is equal to $\lambda_{kl} \in \mathbf{\Lambda}$ by depending of the activity states of the given elements. More precisely, J_{ij} will be equal to: λ_{11} when both ants are active, to $\lambda_{12}, \lambda_{21}$ when one is active and the other inactive and to λ_{22} if both automata are inactive. In this paper we take for simplicity $\lambda_{ij} = 1$. Our choice is based in the observation of ant colonies. Self-interaction and positive feed-back (with local excitability) play an essential role in colony dynamics [20].

When a given individual is in isolation, the previous equation (1) reads

$$S_i(t + \Delta t) = \tanh\Big[\ \mu S_i(t)\ \Big] \tag{2}$$

(here $\mu = gJ_{ii}$, and $J_{ii} > 0$). In the linear approximation, i.e. $S(t + \Delta t) = [\partial_s \Phi(0)]S(t)$, we obtain $S(t + \Delta t) = \mu S(t)$. If we define activity as previously, starting from $S(0) = S_a$ the state after τ iterations will be $S(\tau) = S_a \mu^\tau$, so the time for a given element to inactivate ($S(\tau) \leq \theta$) will be

$$\tau = \frac{log(\Gamma)}{log(gJ_{ii})} \tag{3}$$

where $\Gamma = (\theta/S_a)$. If we consider an isolated individual, we can compute the probability of having n spontaneous activations in a time interval of length T, provided T is long enough, and Δt is very small. We will exploit the analogy between our randomly active "ant" and a model neuron with absolute refractoriness τ, subject to a Poisson sequence of excitatory zero-width point inputs with arrival rate p_a. In short, following [21], to compute the above mentioned probability $P_n(T, \tau)$ we start from a a well known similar result for $\tau = 0$, i.e. the Poisson process, where we have

$$P_n(T) = \frac{(p_a T)^n e^{-p_a T}}{n!}$$

for $n = 0, 1, 2, \ldots$ Then, to calculate $P_n(T, \tau)$ we replace each zero width pulse occurring at, say, t_i by a pulse of duration τ beginning at t_i, we consider different cases depending on the relation between T, n and τ, and, after some tedious calculations, we finally obtain:

$$P_n(T, \tau) = \Theta(T - (n-1)\tau)\Bigg\{1 - e^{-p_a[T-(n-1)\tau]} \sum_{k=0}^{n-1} \frac{p_a^k \left[T - (n-1)\tau\right]^k}{k!} -$$

$$-\Theta(T - n\tau)\left[1 - e^{-p_a[T-n\tau]} \sum_{k=0}^{n} \frac{p_a^k \left[T - n\tau\right]^k}{k!}\right]\Bigg\} \tag{4}$$

where $\Theta(x)$ is the Heaviside step function ($\Theta(x) = 0$ if $x \leq 0$, and $\Theta(x) = 1$ if $x > 0$).

Although the individual automaton is easily analysed, the collective system, i.e. N automata with interactions, is much more difficult. We will characterize the dynamical behavior of FNN by:

$$\rho_t^+ = \frac{1}{N}\sum_{j=1}^{N} \Theta\left[S_j(t) - \theta\right] \tag{5}$$

i.e. the density of active elements at each time step, where $\rho_t^+ \in [0,1]$. We define also the total density of automata as $\rho = N/L^2$. See Fig.1 for a sample of the temporal behavior of ρ_t^+ for different ρ. At low ρ we have a very disordered pattern, which becomes more and more ordered as ρ is increased. Despite we are focusing on a quantity that apparently does not include the spatial dependence, this dependence is partially responsible of the dynamics of ρ_t^+ (see below).

In a previous paper [16] it was shown that a first approach to a characterization of a transition in the dynamical behavior of the FNN was to measure the Shannon-Kolmogorov entropy (SKE) of the following probability distribution $\{p(j,\rho)\}$: let T_j the number of time steps where exactly j elements were active ($j = 0, 1, ..., N$); now, if T is the total number of time steps, the relative frequency of j simultaneously active elements will be $p(j,\rho) = T_j/T$. The SKE is then defined as:

$$S(\rho) = -\sum_{i=0}^{N} p(i,\rho)\ log\ p(i,\rho) \tag{6}$$

It is expected that SKE will provide us with a measure of the diversity of macroscopic states. $S(\rho)$ has an upper and a lower bound: $S(\rho) \in [0, log(N+1)]$. At low densities, $S(\rho)$ will grow until the onset of collective oscillations, where it will decrease due to the highly correlated temporal patterns (Fig.2). See [16] for details on the SKE of FNN.

3 Noise Induced Transition

The time evolution of ρ_t^+ is a stochastic process because of the spontaneous activation of inactive automata. The process ρ_t^+ is non-Markovian due to the existence of the time interval in which the automaton is active. In the absence of interactions this time interval is the above mentioned τ: the number of active automata at time t will be those active at $t-1$ plus those inactive at $t-1$ that spontaneously become active, minus those that were active at $t-\tau$; however, if we consider the interactions between automata, this time interval is altered by the patterns of interaction. It is the interplay between this $\tau'(\rho, p_a)$ and the “velocity” of activity propagation (where we have the "hidden" spatial dependence of ρ_t^+) that will allow us to make some analytical work, in spite of the difficulties implied by the non-Markovian character of FNN (this work will be published elsewhere).

In this paper we will focus on the numerical evidence of a noise induced transition at the point of maximum SKE. Noise induced transitions are phenomena observed in systems coupled to a fluctuating environment, where fluctuations depend on the state of the system. Roughly, the effect of this system-randomness coupling is that noise can induce macroscopic ordered behavior on the system, which becomes manifest through a qualitative change on the stationary probability density of the process (in Horsthemke and Lefever's outstanding book [5], noise induced transitions for differential systems of one variable are treated in depth). We have computed, for various parameters sets, an estimation of the stationary probability density (histogram) for ρ_t^+: $P(\rho^+)$. The stationarity has been assured using the results of [16] for the transient length. After the transient has finished, we have used $T = 2 \times 10^4$ time steps to compute the above mentioned probability density. The final result has been obtained averaging over different initial conditions (see Fig.3).

The computed $P(\rho^+)$ for different values of (p_a, ρ) makes clear the relation between the shape of $P(\rho^+)$ and the dynamical behavior of ρ_t^+. If we have a low density ρ or a low activation probability p_a, ρ_t^+ will be zero, except some irregular bursts of activity that cannot propagate, so the histogram will have a clear maximum at $\rho^+ = 0$. At high enough values of ρ, the rapid propagation of activity because of interaction between automata will make them to remain active almost all the time (if we have not a very low p_a). But at intermediate values of both p_a and ρ, the interplay between the spontaneous activation and the propagation of the activity will make $\rho_t^+ \in (0, 1)$. As we can see in Fig.3, the qualitative change of shape of $P(\rho^+)$, where the transition occurs, consists of a displacement of the maximum of $P(\rho^+)$ toward the boundary $\rho^+ = 1$, until there is not any extrema in the interval $(0, 1)$. In the examples of Fig.3 we can observe that the above mentioned qualitative change takes place between $\rho = 0.15$ and $\rho = 0.25$ for $p_a = 0.01$ (figures 3 (A)(c) and 3 (A)(d)) and between $\rho = 0.25$ and $\rho = 0.35$ for $p_a = 0.001$ (figures 3 (B)(c) and 3 (B)(d)), which was already suggested by the SKE in Fig.2. These phenomena is precisely the type of phenomena for which noise induced transitions theory [5] is well suited, since it shows us how to define an *order parameter* to accurately determine the nature and location (in parameter space) of the transition.

4 Order Parameter

The numerical evidence of a noise induced transition introduced in section 3 suggests us an approach to the definition of an order parameter [1]. An appropriate indicator of a noise induced transition is, following standard theory [5], the extrema of the stationary probability law, which correspond to the macroscopic phases of the system. Then, in FNN we can define ρ_m^+ such that:

$$P(\rho_m^+) = \max_{\rho^+ \in [0,1]} P(\rho^+) \tag{7}$$

As we have checked extensively in numerical simulations, this quantity will be equivalent to find ρ_m^+ equating to zero the first derivative of $P(\rho^+)$ if $\rho_m^+ \in (0,1)$, and will give us the correct value of either $\rho_m^+ = 0$ or $\rho_m^+ = 1$ otherwise. Finally we will define the order parameter (Fig.4):

$$\Gamma(\rho, p_a) = 1 - \rho_m^+ \tag{8}$$

where obviously the (ρ, p_a) dependence comes from $P(\rho^+)$. In figure 4 we can see the order parameter for the FNN whose histograms $P(\rho^+)$ we have seen in Fig.3. The points where takes place the transition are, for $p_a = 0.01$, i.e. Fig.4 (A), $\rho_c \simeq 0.215$, and for $p_a = 0.001$, i.e. Fig.4 (B), $\rho_c \simeq 0.295$, which agrees accurately with the SKE maxima.

Once we have an order parameter $\Gamma(\rho, p_a)$ we can explore the parameter space (p_a, ρ) to locate the transition boundaries and to construct a qualitative *phase diagram*, which is shown in figure 5. There are three clear areas in the diagram, corresponding to the three shapes that can have $P(\rho^+)$, with the associated three values of $\Gamma(\rho, p_a)$: $\Gamma(\rho, p_a) = 0$ in region II, where we have selforganized oscillatory behavior. This is so because once all the automata are active, they will remain in this state until each automaton, approximately $\tau'(\rho, p_a)$ steps after, becomes inactive again. Since the propagation is very fast, any spontaneous activation will initiate the process again (see figures 1(D), 1(E), and 1(F)). $\Gamma(\rho, p_a) = 1$ in region

[1] In a previous paper, Solé et al. [16] put forward an order parameter: $\Gamma(\rho) = log(N+1) - S(\rho)$ based on the assumption that at low ρ the SKE will be maximum due to the practical absence of interactions between the automata. This has been found to be not as general as was supposed, because of at certain values of p_a and in spite of the low ρ, interactions are frequent enough to make $S(\rho)$ have values below its maximum.

III where we have almost total inactivity, due to either a very low p_a, where spontaneous activation is a very rare event, or very low ρ, where activity cannot propagate. $\Gamma(\rho, p_a) \in (0, 1)$ in region I. In region I we have an intermediate activity level almost all the time, but with a very irregular behavior (see figures 1(A), 1(B) and 1(C)).

5 Discussion

Fluid Neural Networks is a successful mathematical formalism [11,14,15,16] to describe and explain the autosynchronization of activity in *Leptothorax* ant colonies [9]. Previous work on FNN [16] has emphasized the density dependent transition from disorder for low densities to ordered oscillating behavior for higher densities, focusing the study on certain measures, as the SKE, that are good enough to suggest the existence of a transition, but fails at linking the numerical work with any well developed theory of phase transitions. In this paper we have shown that the phenomena under study can be characterized numerically in such a way that the transition is equally well detected, but also provides us with the possibility of a best understanding, relating the phenomenology of FNN to the well understood phenomena of noise induced transitions [5]. This finding paves the way to the possibility of analytic work, which is being accomplished through the work on the two factors that are mainly influenced by ρ and p_a, i.e. the mean time an automaton is active and the velocity of activity propagation, which are responsible of the dynamic behavior of ρ_t^+. Moreover, if the time an automaton is active is much less than the time the activation needs to reach the whole system (this will occur if we have a growing L), we will not have any transition at all, so we think what we have is a much more interesting phenomenon than a "simple" noise induced transition, we have what has been called a *finite size* (noise) induced transition [22]. These results will be published elsewhere.

It was shown [12] that the collective synchronization of activity in ant colonies has as a side effect a mutual exclusion mechanism for information exchange, a global computational property of the colony. On the other hand, we have introduced clear evidence that the phenomena of autosynchronization is due to a state dependent external noise, in the form of both a spontaneous activation and a density dependent activity propagation, giving raise to a noise induced transition. Moreover, for a biologically plausible set of parameters (with which we have worked throughout in this paper, see section 2 and [16]) we have the transition at the same critical density, $\rho_c \simeq 0.2$, experimentally observed [23]. These facts point to the possibility of a noise induced computational capability in the colony, for which we have coined the notion of *Noise Induced Computation*. Nevertheless, if we consider previous works on the influence of noise in general swarm dynamics with computational side effects, such as search (food location or recruiting [1]) or optimization (choice of the best food source [2]) we could postulate the general validity of these phenomena, and the theoretical usefulness of the term "Noise Induced Computation".

Finally we would like to emphasize the equivalence of the noise induced transition found in this paper and the maximization of information transfer, that was one of the main results of [16], also called information at the edge of chaos. The postulated hypothesis that complex adaptive systems are "at the edge of chaos", i.e., roughly, a complex adaptive system displays enough ordered behaviour to propagate signals and store information, but is also disordered so that it does not "freeze" in the process, has been a powerful metaphor to understand some mathematical models, as cellular automata [18], but, as far as we know, the first time it was verified in a real system was [16]. In FNN, the same transition has been characterized as a noise induced transition and as a "edge of chaos" phenomenon, so that we could postulate a connection between both phenomena, and the possibility of, through noise induced transition theory, a deeper understanding of "edge of chaos" behavior.

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8 Figures

[**1**] Temporal behavior of ρ_t^+, with parameters $L = 50$, $S_a = 0.1$, $g = 0.1$, $\theta = 10^{-16}$, $p_a = 0.01$ and (A) $\rho = 0.10$; (B) $\rho = 0.15$; (C) $\rho = 0.20$; (D) $\rho = 0.25$; (E) $\rho = 0.30$; (F) $\rho = 0.35$;

[**2**] Entropy (SKE) vs. Density, with parameters $L = 50$, $S_a = 0.1$, $g = 0.1$, $\theta = 10^{-16}$ and (A) $p_a = 0.001$; (B) $p_a = 0.01$. Here $T = 2 \times 10^4$ after 2×10^3 transients were discarded.

[**3**] Probability densities of ρ^+ with parameters $L = 50$, $S_a = 0.1$, $g = 0.1$, $\theta = 10^{-16}$. (A) $p_a = 0.01$ and (a)$\rho = 0.05$, (b)$\rho = 0.1$, (c)$\rho = 0.15$, (d)$\rho = 0.25$. (B) $p_a = 0.001$ and (a)$\rho = 0.1$, (b)$\rho = 0.2$, (c)$\rho = 0.25$, (d)$\rho = 0.35$.

[**4**] Order Parameter $\Gamma(\rho, p_a)$ with parameters $L = 50$, $S_a = 0.1$, $g = 0.1$, $\theta = 10^{-16}$ and (A) $p_a = 0.01$, (B) $p_a = 0.001$.

[**5**] Phase diagram computed from $\Gamma(\rho, p_a)$. Parameters are $L = 32$, $S_a = 0.1$, $g = 0.1$, $\theta = 10^{-16}$. Region I corresponds to $\Gamma(\rho, p_a) \in (0, 1)$. Regions II and III correspond to $\Gamma(\rho, p_a) = 0$ and $\Gamma(\rho, p_a) = 1$ respectively.