Discrete-Time Static Output-Feedback H_∞ Controller Design for Vehicle Suspensions

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Abstract—This paper provides a direct and practical presentation of a novel methodology for static output-feedback controller design. The proposed design strategy has been successfully applied in the fields of control systems for seismic protection of large buildings and multi-building structures, control of offshore wind turbines, and active control of vehicle suspensions. The positive results obtained in these initial applications clearly indicate that this approach could be an effective tool in a large variety of control problems, for which an LMI formulation of the state-feedback version of the problem is available. The main objective of the paper is to facilitate a brief and friendly presentation of the main ideas involved in the new design methodology. To this end, a discrete-time static output-feedback H_∞ controller is designed for a simplified quarter-car suspension system. Numerical simulations indicate that the proposed controller exhibits a remarkably good behavior when compared with the corresponding state-feedback H_∞ controller.

I. INTRODUCTION

When designing a feedback control system, the amount of information available for feedback purposes is an element of particular relevance. In the ideal (and uncommon) case that the entire state vector is available, many advanced state-feedback controller designs can be formulated as Linear Matrix Inequality (LMI) optimization problems, and efficiently computed using standard computational tools as those provided by the *MATLAB Robust Control Toolbox* [1]. In a more realistic scenario, however, the complete state vector is rarely accessible and the available feedback information consists only in a reduced set of linear combinations of the states. In this context, static output-feedback control strategies are an excellent option that can facilitate a simpler implementation in practice.

The main difficulty of this alternative approach is the numerical complexity of the non-convex problems associated to static output-feedback controller designs [2], [3]. To deal with these challenging problems, a number of multi-step algorithms have been proposed [4]–[7]. Typically, these multi-step methods require solving complex matrix equations or LMI optimization problems at each step, which can be a critical issue in large-scale designs. Some single-step methods, based on a proper transformation of the state variables, are also available [8]–[10]. In these single-step methods, the static output-feedback controller design is formulated in terms of a single LMI optimization problem. The main drawback of this second kind of methods is that they are highly problem-dependent, in the sense that a complete derivation of the LMI optimization problem needs to be carried out for most controller designs.

A new design strategy has been recently proposed [11], [12], which can be applied to control problems that admit an LMI-based state-feedback controller design. In this case, an LMI formulation to compute the output-feedback control gain matrix can be easily derived by introducing a simple change of variables in the LMI state-feedback formulation. This design methodology is computationally effective, conceptually simple and easy to implement. Moreover, it makes possible to obtain static output-feedback controllers for a wide variety of problems by taking advantage of the rich literature on LMI formulations for state-feedback controller design.

The new approach was initially motivated by large-scale control problems associated to vibration control of large structures, and has been successfully applied in designing static output-feedback controllers for seismic protection of large buildings [13], [14] and multi-building systems [15]. Other successful applications include optimal design of passive-damping systems for large structures [16], control of offshore wind turbines [17], [18], and active control of vehicle suspensions [19].

The main objective of the paper is to provide a summarized, direct and practical presentation of this new design methodology, which we believe can be of general interest for researchers and control engineers in different fields. For clarity and simplicity, a small-scale control problem and the H_∞ control approach have been selected to introduce and illustrate the fundamental ideas. Specifically, a discrete-time static output-feedback H_∞ controller is designed for a simplified quarter-car suspension system. A discrete-time state-feedback H_∞ controller is also designed to be used as a reference in the performance assessment. Moreover, the LMI formulation of the state-feedback controller serves as a natural starting point to derive the LMI formulation for the static output-feedback controller design.

The paper is organized as follows: In Section II, a mathematical model for a simplified quarter-car suspension system is
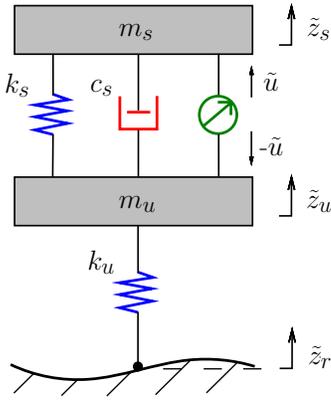
Fig. 1. Quarter-car model with an active suspension system.

presented. In Section III, a minimal theoretical background of the new strategy for static output-feedback controller design is provided. In Section IV, a discrete-time static output-feedback H_∞ controller is designed for a particular quarter-car suspension system, and the time response to a road bump disturbance is computed to demonstrate the effectiveness of the proposed controller. Finally, in Section V, some conclusions and future lines of research are briefly discussed.

II.   QUARTER-CAR SUSPENSION MODEL

Let us consider the quarter-car suspension model schematically depicted in Fig. 1. The quarter-car motion can be described by the second-order model

$$m_s\ddot{\tilde{z}}_s(t) = -c_s[\dot{\tilde{z}}_s(t) - \dot{\tilde{z}}_u(t)] - k_s[\tilde{z}_s(t) - \tilde{z}_u(t)] + \tilde{u}(t),$$
$$m_u\ddot{\tilde{z}}_u(t) = c_s[\dot{\tilde{z}}_s(t) - \dot{\tilde{z}}_u(t)] + k_s[\tilde{z}_s(t) - \tilde{z}_u(t)] - \tilde{u}(t)$$
$$- k_u[\tilde{z}_u(t) - \tilde{z}_r(t)],$$

(1)

where m_s and m_u are the sprung and unsprung masses, respectively; k_s and k_u are, respectively, the suspension stiffness and the tire stiffness; c_s is the damping of the suspension system; $\tilde{z}_s(t)$ and $\tilde{z}_u(t)$ represent the vertical displacements of the sprung and unsprung masses, respectively; $\tilde{z}_r(t)$ is the vertical road displacement; and $\tilde{u}(t)$ is the active input of the suspension system. By defining the state vector

$$\tilde{x}(t) = [\tilde{z}_s(t), \tilde{z}_u(t), \dot{\tilde{z}}_s(t), \dot{\tilde{z}}_u(t)]^T, \tag{2}$$

a first-order state-space model in the form

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t) + \tilde{B}_w\tilde{w}(t) \tag{3}$$

can be obtained, where $\tilde{w}(t) = \tilde{z}_r(t)$ is the road disturbance input, and the matrices $\tilde{A}, \tilde{B}$ and $\tilde{B}_w$ are given by

$$\tilde{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_s/m_s & k_s/m_s & -c_s/m_s & c_s/m_s \\ k_s/m_u & -(k_s+k_u)/m_u & c_s/m_u & -c_s/m_u \end{bmatrix},$$

(4)

$$\tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 1/m_s \\ -1/m_u \end{bmatrix}, \quad \tilde{B}_w = \begin{bmatrix} 0 \\ 0 \\ 0 \\ k_u/m_u \end{bmatrix}. \tag{5}$$

Next, by using zero-order hold equivalents with a sampling period τ, we obtain the discrete-time state-space model

$$x(k+1) = Ax(k) + Bu(k) + B_ww(k), \tag{6}$$

where $x(k) = \tilde{x}(\tau k)$, $u(k) = \tilde{u}(\tau k)$, $w(k) = \tilde{w}(\tau k)$; the matrices A, B and B_w are given by

$$A = e^{\tilde{A}\tau}, \quad B = \int_0^\tau e^{\tilde{A}t}\tilde{B}dt, \quad B_w = \int_0^\tau e^{\tilde{A}t}\tilde{B}_wdt; \tag{7}$$

and the index $k \geq 0$ represents the discretized time $t = \tau k$. In the following, notations like $z_s(k), \dot{z}_s(k), \ddot{z}_s(k)$ will be used instead of $\tilde{z}_s(\tau k), \dot{\tilde{z}}_s(\tau k), \ddot{\tilde{z}}_s(\tau k)$, respectively.

Following the usual approach, together with the state-space model in (6), a vector of controlled outputs $z(k)$ needs to be defined to compute a state-feedback H_∞ controller

$$u(k) = G\,x(k). \tag{8}$$

The vertical body acceleration is probably the most popular measure for quantifying the ride comfort. Consequently, our main interest is focused on minimizing the sprung mass acceleration $\ddot{z}_s(k)$. Additionally, we are obviously interested in minimizing the required control effort $u(k)$. Moreover, in order to preserve the suspension deflection limits and the road holding ability, it will be also convenient to reduce the suspension deflection $z_s(k) - z_u(k)$ and the tire deflection $z_u(k) - z_r(k)$ as much as possible. Accordingly, the following vector of controlled outputs is selected:

$$z(k) = \begin{bmatrix} \ddot{z}_s(k) \\ \alpha(z_s(k) - z_u(k)) \\ \beta(z_u(k) - z_r(k)) \\ \eta u(k) \end{bmatrix}, \tag{9}$$

where α, β and η are adjustable parameters that facilitate the tradeoff between the conflicting design objectives. By considering the upper equation in (1), the discretized controlled output $z(k)$ in (9) can be written as

$$z(k) = Cx(k) + Du(k) + D_ww(k), \tag{10}$$

with

$$C = \begin{bmatrix} -k_s/m_s & k_s/m_s & -c_s/m_s & c_s/m_s \\ \alpha & -\alpha & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{11}$$

and

$$D = \begin{bmatrix} 1/m_s \\ 0 \\ 0 \\ \eta \end{bmatrix}, \quad D_w = \begin{bmatrix} 0 \\ 0 \\ -\beta \\ 0 \end{bmatrix}. \tag{12}$$

III.   CONTROLLER DESIGN

A. State-feedback H_∞ controller

Let us consider a discrete-time linear system of the form

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + B_ww(k), \\ \quad z(k) = Cx(k) + Du(k) + D_ww(k), \end{cases} \tag{13}$$

where $k \geq 0$ is the time, $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control input, $w(k) \in \mathbb{R}^r$ is the disturbance input, $z(k) \in \mathbb{R}^d$ is the controlled output, and A, B, B_w, C, D, D_w are real constant matrices with appropriate dimensions. For a given state-feedback controller $u(k)$ of the form

$$u(k) = Gx(k), \tag{14}$$

we obtain the closed-loop system

$$\begin{cases} x(k+1) = A_G\, x(k) + B_w\, w(k), \\ \quad\quad z(k) = C_G\, x(k) + D_w\, w(k), \end{cases} \tag{15}$$

where A_G and C_G are the closed-loop matrices

$$A_G = A + BG, \quad C_G = C + DG. \tag{16}$$

The H_∞-norm of the system (15) is given by

$$\gamma_G = \|T_G\|_\infty = \sup_{0 \leq \theta \leq 2\pi} \sigma_{\max}\left[T_G(e^{j\theta})\right], \tag{17}$$

where $\sigma_{\max}[\cdot]$ is the maximum singular value, and T_G is the transfer function from the disturbance input to the controlled output

$$T_G(s) = C_G(sI - A_G)^{-1}B_w + D_w. \tag{18}$$

Broadly speaking, the H_∞ controller design approach consists in computing a suitable state-feedback gain matrix G that produces an asymptotically stable matrix A_G and, simultaneously, minimizes the H_∞-norm γ_G. According to the discrete version of the *Bounded Real Lemma* (see [20]; or [21] with a slight modification), for a prescribed scalar $\gamma > 0$, the matrix A_G is asymptotically stable and $\gamma_G < \gamma$, if and only if there exists a symmetric positive-definite matrix $X \in \mathbb{R}^{n \times n}$ satisfying the following matrix inequality:

$$\begin{bmatrix} X & 0 & A_G X & B_w \\ * & \gamma I & C_G X & D_w \\ * & * & X & 0 \\ * & * & * & \gamma I \end{bmatrix} > 0, \tag{19}$$

where $(*)$ denotes the transpose of the symmetric entry. By using the values of the closed-loop matrices A_G and C_G given in (16), and by introducing the new variable $Y = GX$, the matrix inequality (19) can be transformed into the following linear matrix inequality:

$$M(X, Y, \gamma) = \begin{bmatrix} X & 0 & AX + BY & B_w \\ * & \gamma I & CX + DY & D_w \\ * & * & X & 0 \\ * & * & * & \gamma I \end{bmatrix} > 0. \tag{20}$$

The state-feedback H_∞ controller can now be computed by solving the LMI optimization problem

$$\begin{cases} \text{minimize } \gamma \\ \text{subject to } M(X, Y, \gamma) > 0, \gamma > 0, X > 0, \end{cases} \tag{21}$$

where the matrices X and Y are the optimization variables. If the optimization problem (21) attains an optimal value $\tilde{\gamma}_{\text{sf}}$ for the matrices $\tilde{X}$ and $\tilde{Y}$, then the control gain matrix

$$G = \tilde{Y}\tilde{X}^{-1} \tag{22}$$

defines a state-feedback controller $u(k) = Gx(k)$ with asymptotically stable matrix A_G and optimal H_∞-norm $\gamma_G = \tilde{\gamma}_{\text{sf}}$.

B. Static output-feedback H_∞ controller

Let us now consider an output-feedback controller

$$u(k) = Ky(k), \tag{23}$$

where $K \in \mathbb{R}^{m \times q}$ is the output-feedback control gain matrix,

$$y(k) = C_y\, x(k) \tag{24}$$

is the observed output, and $C_y \in \mathbb{R}^{q \times n}$ is the observed-output matrix, which is assumed to be a full row-rank matrix with $q < n$. Considering (23) and (24), the output-feedback controller can be seen as a state-feedback controller with state gain matrix

$$G_{\text{of}} = KC_y, \tag{25}$$

and the output-feedback H_∞ controller design can be formulated in terms of the following non-convex optimization problem:

$$\begin{cases} \text{minimize } \gamma \\ \text{subject to } M(X, Y, \gamma) > 0, \gamma > 0, X > 0, (X, Y) \in \mathcal{M}, \end{cases} \tag{26}$$

where $\mathcal{M}$ is the set of all pairs of matrices (X, Y) for which there exists an $m \times q$ matrix K satisfying the constraint

$$YX^{-1} = KC_y. \tag{27}$$

Following the ideas presented in [11], we take a matrix $Q \in \mathbb{R}^{n \times (n-q)}$, whose columns are a basis of the nullspace of C_y, and a matrix

$$R = C_y^\dagger + QL, \tag{28}$$

where $C_y^\dagger = C_y^T(C_y C_y^T)^{-1}$ is the *Moore-Penrose pseudoinverse* of C_y, and $L \in \mathbb{R}^{(n-q) \times q}$ is a given matrix. Next, by introducing the transformations

$$X = QX_Q Q^T + RX_R R^T, \quad Y = Y_R R^T, \tag{29}$$

we obtain the LMI optimization problem

$$\begin{cases} \text{minimize } \gamma \\ \text{subject to } M\left(X_Q, X_R, Y_R, \gamma\right) > 0, \gamma > 0, X_Q > 0, X_R > 0, \end{cases} \tag{30}$$

where $M\left(X_Q, X_R, Y_R, \gamma\right) > 0$ denotes the LMI in Fig. 2, and the matrices $X_Q \in \mathbb{R}^{(n-q) \times (n-q)}$, $X_R \in \mathbb{R}^{q \times q}$, $Y_R \in \mathbb{R}^{m \times q}$ are the new optimization variables. According to the results in [11], if the LMI optimization problem (30) attains an optimal value $\tilde{\gamma}_{\text{of}}$ for the matrices $\tilde{X}_Q, \tilde{X}_R$ and $\tilde{Y}_R$, then the pair of matrices $\left(\hat{X}, \hat{Y}\right)$ with

$$\hat{X} = Q\tilde{X}_Q Q^T + R\tilde{X}_R R^T, \quad \hat{Y} = \tilde{Y}_R R^T, \tag{31}$$

provides a feasible solution to the optimization problem (26). Moreover, the constraint (27) holds for the pair $(\hat{X}, \hat{Y})$ and the output gain matrix

$$K = \tilde{Y}_R \tilde{X}_R^{-1}. \tag{32}$$

Consequently, the static output-feedback controller (23) defined by the gain matrix (32) leads to an associated state gain matrix $G_{\text{of}} = KC_y$ with: (i) an asymptotically stable matrix $A_{G_{\text{of}}}$, and (ii) an H_∞-norm satisfying $\gamma_{G_{\text{of}}} \leq \tilde{\gamma}_{\text{of}}$.

Remark 1. It should be highlighted that the optimal value $\tilde{\gamma}_{\text{of}}$ provided by the optimization problem (30) is just an upper

$$\begin{bmatrix} QX_QQ^T + RX_RR^T & 0 & AQX_QQ^T + ARX_RR^T + BY_RR^T & B_w \\ * & \gamma I & CQX_QQ^T + CRX_RR^T + DY_RR^T & D_w \\ * & * & QX_QQ^T + RX_RR^T & 0 \\ * & * & * & \gamma I \end{bmatrix} > 0$$

Fig. 2.   Linear matrix inequality $M\left(X_Q, X_R, Y_R, \gamma\right) > 0$ for the discrete-time static output-feedback H_∞ controller design.

bound of the H_∞-norm $\gamma_{G_{\text{of}}}$, which satisfies $\tilde{\gamma}_{\text{sf}} \le \gamma_{G_{\text{of}}} \le \tilde{\gamma}_{\text{of}}$. The actual value of $\gamma_{G_{\text{of}}}$ can be computed by maximizing the maximum singular value of the matrix $T_{G_{\text{of}}}(j\theta)$, as indicated in (17). Alternatively, $\tilde{\gamma}_{\text{of}}$ can also be computed by setting $G = G_{\text{of}}$ and solving the LMI optimization problem

$$\begin{cases} \text{minimize } \gamma \\ \text{subject to LMI (19), } \gamma > 0, \, X > 0. \end{cases} \tag{33}$$

IV. NUMERICAL RESULTS

In this section, the methodology described in Section III-B is applied to design a static output-feedback H_∞ controller for the quarter-car suspension model introduced in Section II. A state-feedback H_∞ controller is also designed, which is taken as a natural reference in the performance assessment. To illustrate the behavior of the proposed static output-feedback controller, the time response of the quarter-car suspension system to a bump disturbance is presented in Section IV-B. Some closing remarks are provided in Section IV-C. In the controller designs and numerical simulations, the following particular values:

$$m_s = 504.5 \text{ kg}, \quad m_u = 62 \text{ kg}, \quad k_s = 13100 \text{ N/m},$$
$$k_u = 252000 \text{ N/m}, \quad c_s = 400 \text{ Ns/m}, \tag{34}$$

have been taken for the parameters of the quarter-car suspension model [22], [23]. All the computations have been carried out with *MATLAB*, and the *MATLAB Robust Control Toolbox* [1] has been used to solve the LMI optimization problems.

A. Controllers design

Let us consider the quarter-car state-space model (6) corresponding to the sampling time $\tau = 0.01$, and the parameter values given in (34). Let us also take the controlled output (10) defined by the following values of the weighting coefficients:

$$\alpha = 4, \quad \beta = 5, \quad \eta = 2 \times 10^{-3}. \tag{35}$$

As it is done in [23], let us assume that the available feedback information consists only in the suspension deflection and the sprung mass velocity. In this case, we have the vector of observed outputs

$$y(k) = [z_s(k) - z_u(k), \dot{z}_s(k)]^T, \tag{36}$$

which can be written as

$$y(k) = C_y\, x(k) \tag{37}$$

with

$$C_y = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{38}$$

Following the methodology presented in Section III-B, we begin the design of a static output-feedback H_∞ controller

$$u(k) = Ky(k), \tag{39}$$

by computing the matrix

$$Q = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{40}$$

whose columns are a basis of $\text{ker}(C_y)$, and the matrix

$$R = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{41}$$

which can be obtained from (28) for the particular choice $L = 0$. Next, we solve the LMI optimization problem defined in (30) with the matrices Q and R given in (40), (41), and the matrices A, B, B_w, C, D, D_w given in (7), (11), (12). As a result, we obtain the output control gain matrix

$$K = 10^3 \times [\, 1.9322 \quad -7.0451 \,], \tag{42}$$

and the optimal γ-value

$$\tilde{\gamma}_{\text{of}} = 494.933. \tag{43}$$

To design a state-feedback H_∞ controller

$$u(k) = G\, x(k) \tag{44}$$

that uses the full discrete-time state

$$x(k) = [z_s(k), z_u(k), \dot{z}_s(k), \dot{z}_u(k)]^T \tag{45}$$

as feedback information, we take the same matrices A, B, B_w, C, D, D_w, and solve the LMI optimization problem (21). In this second case, we obtain the state gain matrix

$$G = 10^3 \times [\, 2.9642 \quad -8.8399 \quad -1.7828 \quad -0.0050 \,], \tag{46}$$

with an associated H_∞-norm

$$\tilde{\gamma}_{\text{sf}} = 475.664. \tag{47}$$

According to Remark 1 and the γ-values in (43), (47), the H_∞-norm $\gamma_{G_{\text{of}}}$ corresponding to the state gain matrix $G_{\text{of}} = KC_y$ associated to the observed-output gain matrix K must satisfy

$$475.664 \le \gamma_{G_{\text{of}}} \le 494.933. \tag{48}$$

By setting $G = G_{\text{of}}$ in (19) and solving the optimization problem (33), we obtain the actual γ-value

$$\gamma_{G_{\text{of}}} = 479.128, \tag{49}$$

which is only a 0.73% greater than the optimal value $\tilde{\gamma}_{\mathrm{sf}}$ attained by the state-feedback H_∞ controller.

Remark 2. The output-feedback controller design presented in this section has been carried out by taking the L matrix in (28) as a zero matrix. This choice, which leads to the simplified R-matrix $R = C_y^\dagger$, has been used recently in the field of vibration control of large structures with positive results [13]–[16]. However, it is worth pointing out that certain feasibility problems typically appear when applying the proposed methodology to the design of static output-feedback controllers for structural vibration control. A detailed study of the L-matrix properties and the role that it plays in solving these feasibility issues can be found in [12].

B. Bump response

To provide a more complete picture of the performance achieved by the proposed output-feedback controller, in this section we present the time response of the quarter-car suspension system to an isolated bump of the form

$$\tilde{z}_r(t) = \begin{cases} \dfrac{H}{2}\left[1 - \cos\left(\dfrac{2\pi V}{L}t\right)\right] & \text{if } 0 \le t \le \dfrac{L}{V}, \\ \\ 0 & \text{otherwise}, \end{cases} \quad (50)$$

where H and L are the bump height and length, respectively, and V is the vehicle forward velocity. More precisely, we consider three different control configurations of the quarter-car suspension model: (i) uncontrolled system, (ii) controlled system using the state-feedback H_∞ controller defined by $(44), (45), (46)$, and (iii) controlled system using the static output-feedback H_∞ controller defined by $(36), (39)$ and (42). As a road disturbance, we take the particular bump corresponding to the parameter values

$$H = 0.08 \text{ m}, \quad L = 5 \text{ m}, \quad V = 20 \text{ m/s}. \quad (51)$$

The time histories of the sprung mass acceleration $\ddot{z}_s(k)$ obtained for this road disturbance are presented in Fig. 3, where the black line corresponds to the uncontrolled configuration (denoted as *Uncontrolled* in the legend), the blue line describes the response of the state-feedback controlled configuration (denoted as *State-feedback* in the legend), and the red line displays the response of the output-feedback controlled configuration (denoted as *Output-feedback* in the legend). In the figure, it can be clearly appreciated that a remarkable attenuation of the bump response is provided by both controlled configurations. It becomes also apparent that, despite the reduced feedback information, the bump response of the output-feedback controlled configuration is certainly very close to the response exhibited by the state-feedback configuration. Moreover, the graphics in Fig. 4 point out that there are no significant differences between the levels of control effort required to operate the active controllers.

C. Closing remarks

The numerical results obtained in Section IV-B confirm the excellent performance of the proposed static output-feedback H_∞ controller when compared with the corresponding state-feedback H_∞ controller. In fact, from the point of view of H_∞ controller design, the values of the H_∞-norms in (47) and (49)
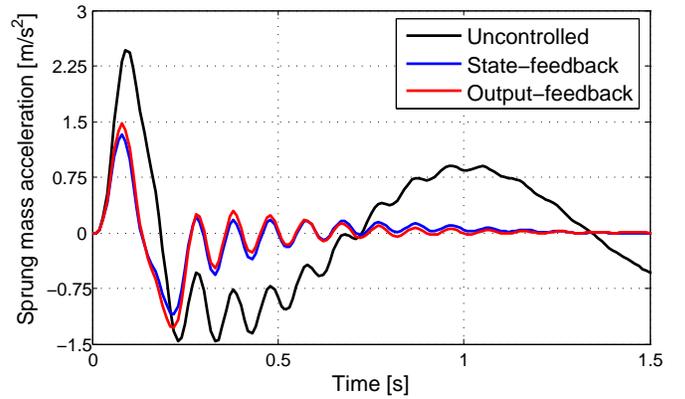

Fig. 3. Time response to a bump disturbance. Sprung mass acceleration corresponding to the uncontrolled system (black line), the static state-feedback controller (blue line), and the static output-feedback (red line) configurations.
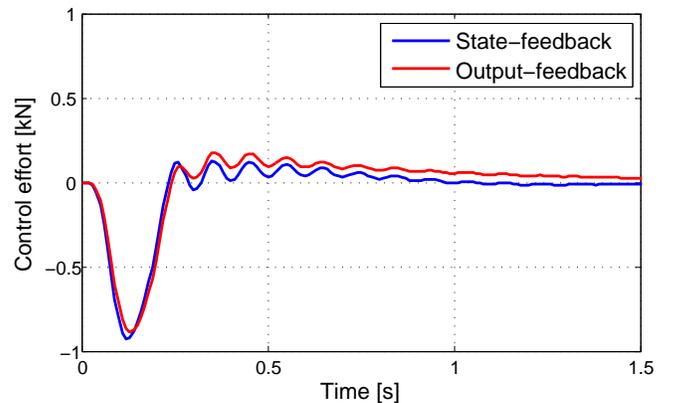

Fig. 4. Time response to a bump disturbance. Control effort corresponding to the static state-feedback controller (blue line) and the static output-feedback controller (red line).

show that the static output-feedback controller is practically optimal. These remarkable numerical results are even more meaningful when considering the following features of the proposed design methodology [19]:

1) *Conceptual simplicity.* The ideas involved in the proposed change of variables are simple and transparent. As shown in Section III-B, new LMI formulations for static output-feedback controller designs can be easily derived from existing state-feedback LMI formulations through a simple change of the LMI variables.

2) *Ease of implementation.* The static output-feedback controller design is formulated in terms of LMI optimization problems, which can be directly solved using standard computational tools, as those provided by the *MATLAB Robust Control Toolbox*.

3) *Computational efficiency.* Static output-feedback gain matrices have been traditionally computed by means of multi-step optimization algorithms that require solving complex matrix equations or LMI problems at each step. In contrast, in the new design methodology, the output-feedback gain matrix is computed by solving a single LMI optimization problem.

4) *Generality.* As indicated in the introduction, the pro-

posed design methodology can be applied to a wide variety of control problems, with the only requirement that the state-feedback version of the problem admits a standard LMI formulation.

V. CONCLUSIONS AND FUTURE DIRECTIONS

In this work, a new strategy to design static output-feedback controllers has been presented. The main focus has been placed on providing a summarized, direct and practical presentation of the new design methodology. To provide a practical illustration of the principal elements of the new approach, a discrete-time static output-feedback H_∞ controller has been designed for a simplified quarter-car suspension system. Numerical simulations indicate that the proposed static output-feedback H_∞ controller exhibits a remarkably good behavior when compared with the corresponding state-feedback H_∞ controller. To date, the new design strategy has been successfully applied in the fields of control systems for seismic protection of large buildings and multi-building structures, control of offshore wind turbines, and active control of vehicle suspensions. The positive results obtained in these initial applications clearly indicate that this approach can be an interesting tool in a large variety of control problems, for which an LMI formulation of the state-feedback version of the problem is available. Consequently, further research effort should be invested in exploring the applicability of this novel design methodology, specially to control problems involving more complex physical models and more sophisticated control strategies.

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