On Computing Enclosing Isosceles Triangles and Related Problems

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Abstract

Given a set of n points in the plane, we show how to compute various enclosing isosceles triangles where different parameters such as area or perimeter are optimized. We then study a 3-dimensional version of the problem where we enclose a point set with a cone of fixed aperture α.

1 Introduction

Given a set of points in 2 or 3 dimensions, the problem of computing a geometric structure enclosing the point set while optimizing some criteria of the enclosing structure such as area, perimeter, surface area or volume has been widely studied in the literature [1, 2, 9, 7, 11, 16, 3, 5, 4, 6, 17, 14, 18]. In this paper, we are particularly interested in the 2 dimensional setting where the enclosing structure is a triangle. The two natural parameters to optimize in this setting are the area or the perimeter of the enclosing triangle. Both problems are well-studied in the literature. For the former, Klee and Laskowski [11] presented an $O(n \log^2 n)$ time algorithm to compute the enclosing triangle of minimum area and O'Rourke et al. [16] improved this to linear, which is optimal. For the latter, De Pano [9] proposed an $O(n^3)$ algorithm for computing the enclosing triangle of minimum perimeter. This was subsequently improved by Chang and Yap [7] as well as Aggarwal and Park [2] culminating in the optimal linear time algorithm of Bhattacharya and Mukhopadhyay [3]. In this paper, we study several variants of the problem where the enclosing structure is not an arbitrary triangle, but an isosceles triangle. As we shall see, this constraint changes the problem. For convenience, we will denote the apex of an isosceles triangle by a, the angle at the apex by α, the edge opposite the apex referred to as the base by b and the distance from the apex to the base which is the height by h (see Figure 1).

The natural criteria to optimize when computing an enclosing isosceles triangle are the angle at the apex α, the height h, the perimeter or the area. Given an isosceles triangle of fixed angle α, one can observe the following.

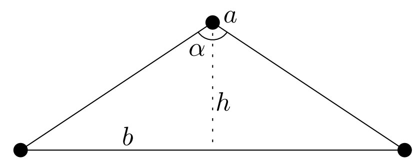

Figure 1: Notation for Isosceles Triangle.

Observation 1 *The enclosing isosceles triangle with fixed apex angle α of minimum height also has minimum area and minimum perimeter.*

This is a key difference between the optimization problem with isosceles triangles versus arbitrary triangles. This leads the to first optimization problem studied:

Problem 1 *Given a set of n points in the plane, and a positive constant $\alpha < \pi$, compute the enclosing isosceles triangle with apex angle α and minimum height.*

If the height of the enclosing isosceles triangle is fixed, one can observe:

Observation 2 *The enclosing isosceles triangle with fixed height h of minimum apex angle also has minimum area and minimum perimeter.*

This motivates the second problem studied:

Problem 2 *Given a set of n points in the plane, and a fixed height h, compute the enclosing isosceles triangle of minimum apex angle.*

We end with a limited exploration of a 3D variant of this problem. In this setting, the enclosing structure is a cone with angle α at the apex and a circular base, which we call an α-cone. We solve 3D versions of Problems 1 and 2 with the additional constraint that the orientation of the α-cone is fixed. The general 3D problem is more difficult since there are more degrees of freedom for the problem. We elucidate some of these difficulties and conclude with a few open problems.

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2 Enclosing Isosceles Triangle

Prior to solving Problems 1 and 2, we first investigate a simpler problem which will prove to be useful in the solutions of the other two problems. We assume that all point sets are in general position.

2.1 Simpler Subproblem

Given a set of n points in the plane, we wish to compute an enclosing isosceles triangle with fixed apex angle and a horizontal base that has minimum height. Since the apex angle is fixed and the base is horizontal, the two edges of the enclosing triangle adjacent to the apex have a fixed direction. In order to minimize the height, both these edges must be tangent to the convex hull of the set. Both these tangents can be computed in $O(n)$ time by computing the supporting lines forming the corresponding angles with the horizontal line. If the input set is a convex polygon, then both the supporting lines can be computed in $O(\log n)$ time [15].

Proposition 1 *Given a set of n points in the plane, the enclosing isosceles triangle with fixed apex angle and horizontal base that has minimum height can be computed in $O(n)$ time. If the input set is a convex polygon, then the enclosing triangle can be computed in $O(\log n)$ time.*

This simple observation immediately gives rise to an approximation scheme.

2.2 Problem 1: PTAS for Minimum Height

In the following we describe a Polynomial Time Approximation Scheme for Problem 1. Given a set of n points in the plane and a fixed positive angle $\alpha < \pi$, our algorithm finds an enclosing isosceles triangle with apex α whose height is within a multiplicative factor of $(1+\epsilon)$ of the optimal height. The running time is $O(n)$ with a constant that depends on $\csc\alpha$. Hence as α tends to 0, the running time tends to infinity. We outline the algorithm below.

Algorithm

1. Choose a set D of directions on the unit circle so that any direction d on the unit circle has some direction $d^* \in D$ where the angle between d and d^* is less than δ, with $\sin\delta = \frac{\epsilon \sin 2\alpha}{2\sin^2\alpha+1}$. This can be accomplished using $O(\delta^{-1})$ uniformly spread directions.

2. For each direction in D, compute the enclosing isosceles triangle of minimum height using Proposition 1. Return the triangle of minimum height among all computed triangles.

Lemma 1 *Let T be an isosceles triangle with apex α and height h. Let T' be the isosceles triangle of minimum height h' containing T, with apex angle α, and base edge tilted by an angle of δ. Then, we have that $h' \le h(\frac{\sin\delta(2\sin^2\alpha+1)}{\sin 2\alpha} + \cos\delta)$.*

Theorem 1 *The algorithm returns an enclosing isosceles triangle with apex angle α whose height is at most $1+\epsilon$ times optimal. The running time of the algorithm is $O(n)$ for any fixed α, such that $0 < \alpha < \pi$.*

This algorithm can be extended to the 3D problem. We continue by developing an exact algorithm whose running time increases by a log factor. However, we will show that this is optimal by providing a lower bound.

2.3 Problem 1: Minimum Height

We are now in a position to solve the first problem exactly. Recall that the input is a set P of n points in the plane and we want to compute the enclosing isosceles triangle with fixed apex angle α that has minimum height. Observe that the optimal solution must have at least one point of P on each edge of the triangle. This reduces the search space for the optimal solution. In 2 dimensions, we define an α-wedge, for $0 < \alpha < \pi$, to be the set of points contained between two rays emanating from the same point called the apex, with the angle between the two rays being α. We say that an α-wedge is *minimal* provided that $P \in \alpha$-wedge and each of the rays contains at least one point of P. As a first step, we compute the locus of points L with the property that for each point x of L, a minimal α-wedge exists with apex at x.

To compute L, we first compute the convex hull $CH(P)$ of P. Note that for each orientation of an α-wedge, there is exactly one position for which it is minimal. Given a fixed orientation, if the minimal α-wedge for P only contains $p_i \in CH(P)$ on one ray and $p_j \in CH(P)$ on the other ray, as we rotate the orientation slightly in a clockwise or counter-clockwise fashion, these two points of contact do not change for minimal α-wedges in those directions. It is not until a second point of the convex hull touches one of the two rays does the contact point change. Therefore, for a fixed pair of contact points p_i and p_j, the locus of apices of minimal α-wedges forms an arc of a circle since α is fixed. By modifying the rotating calipers algorithm [19, 15], we can rotate the α-wedge around the convex hull and compute the sequence of (at most linear number of) circular arcs, which we will call an α-*cloud* [18], representing the set L (See Figure 2).

Theorem 2 *The locus of apices of minimal α-wedges for P can be computed in $O(n\log n)$ time. If P is a convex (or simple) polygon the locus can be computed in $O(n)$ time.*

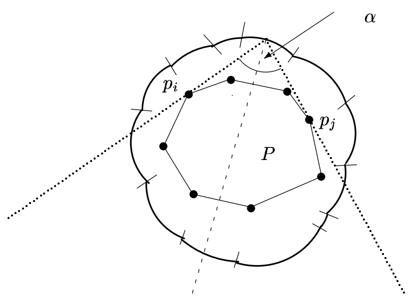

Figure 2: Cloud.

The α-cloud consists of a sequence of at most a linear number of circular arcs. We next show how to compute an enclosing isosceles triangle of minimum height where the apex is constrained to lie on one circular arc of the α-cloud. If the apex lies on one arc, then each of the sides of the enclosing triangle adjacent to the apex is in contact with at least one point of $CH(P)$. Let p_i be in contact with one side and p_j the other. As the apex slides over the circular arc, the base of the triangle is also in contact with a point. This third point may change multiple times. Let us first restrict our attention to the situation where the point p_k on the base of the triangle is also fixed.

Lemma 2 *Let $[a, b]$ be the interval of a circular arc of the α-cloud where the contact points of each of the edges of the enclosing triangle are $p_i, p_j, p_k \in P$ respectively. The height of the enclosing isosceles triangle is a continuous, monotone and unimodal function over the interval $[a, b]$. Therefore this function has at most one maximum and at most two minima. The minima can only be located at the extremes of the interval.*

A counting argument shows that the total number of triples of contact points examined over the whole α-cloud is linear. Since for each triple, the solution can be found in $O(\log n)$ time, the overall complexity of the algorithm is $O(n \log n)$. We conclude with the following.

Theorem 3 *Given a set P of n points in the plane and a positive angle $\alpha < \pi$, the enclosing isosceles triangle with apex α with minimum height, minimum area and minimum perimeter can be computed in $O(n \log n)$ time. If the input P is a convex (or simple) polygon, the running time is reduced to $O(n)$.*

2.4 Problem 2: Minimum Angle

In this setting, we wish to compute an enclosing isosceles triangle of fixed height h, and minimize the apex angle. The intuition behind the solution is similar to the solution where we minimize the height, except the details are a little more tedious. The complete details of the algorithm are available in the full version of the paper. We outline the main ideas. First, similar to the former problem, we note that if the orientation of the enclosing triangle is fixed, minimizing the angle can be done in linear time.

Proposition 2 *Given a set of n points in the plane, the enclosing isosceles triangle with fixed height and horizontal base that has minimum apex angle can be computed in $O(n)$ time. If the input set is a convex polygon, then the enclosing triangle can be computed in $O(\log n)$ time.*

Notice that if the height h is less than the width of P, then the problem has no solution. Therefore, h is at least the width $W(P)$ of P. In addition to the property outlined in Lemma 2, we exploit the following two additional properties of α-clouds.

- *Given two different angles α_1 and α_2, if $\alpha_1 \leq \alpha_2$ then the α_2-cloud is contained in the α_1-cloud.*
- *The minimum height of the enclosing isosceles triangles with apex angle α_2 and apex on the α_2-cloud is at most the minimum height of all the enclosing isosceles triangles with apex α_1 and apex on the α_1-cloud.*

With these two properties, we can see that the minimum angle is the value α such that the minimum height of all the enclosing isosceles triangles with apices on the α-cloud is exactly h. The value h can be achieved at most a linear number of times on the α-cloud. To compute this value α, we use Proposition 2 to compute an initial solution, α_0. We compute the cloud for α_0 and concentrate on the pieces of circular arc where the height is at most h. By exploiting the properties of clouds outlined above, we are able to extract the minimum angle for all these orientations.

Theorem 4 *Given a set of point P in the plane and a fixed value $h \geq |W(P)|$, the enclosing isosceles triangle with height h and minimum apex angle can be computed in $O(n \log n)$ time. If P is a convex (or simple) polygon, the running time is reduced to $O(n)$.*

Lower bound

Using a reduction to Max-Gap problem [12], we can show an $\Omega(n \log n)$ time lower bound for the problem of computing the minimum height of an enclosing isosceles triangle with fixed apex angle α.

Theorem 5 *Given a set of n points P and a positive angle $\alpha < \pi$, computing the enclosing isosceles triangle containing P with minimum height (perimeter, area) requires $\Omega(n \log n)$ operations in the algebraic decision tree model.*

3 α-cones in 3D

We consider a 3D variant of the problems defined in the introduction. Let $P = \{p_1, \cdots, p_n\}$ be a set of n points in $\mathbb{R}^3$. We assume that the points are in general position. Recall, that the enclosing structure is a cone with angle α at the apex and a circular base. We will call this an α-cone. We wish to compute an enclosing α-cone whose base is contained in a horizontal plane with minimum height. This is the 3D equivalent to the 2D problem addressed in Subsection 2.1.

In the optimal solution, excluding the base of the cone, only the vertices of the upper hull of $CH(P)$ can be in contact with the cone. We proceed as follows.

1. Let H be a horizontal plane such that P is above H. For each $p_i \in P$ compute a vertical α-cone with apex p_i. Compute the intersection of these α-cones with H, resulting in n circles with different radii.

2. Compute the smallest enclosing circle C' for this set of circles. C' has at least three (in some cases two) contact points. Knowing the center and radius of C', we can compute the apex of the vertical α-cone of minimum height.

Since computing the smallest enclosing circle can be done in linear time [13], we conclude with the following.

Theorem 6 *Given a set of n points in R^3 and a positive angle $\alpha < \pi$, the α-cone enclosing P of minimum height with horizontal base can be computed in linear time.*

Notice that for any given direction there always exists an α-cone in that direction. The α-cone with minimum height has at least one point on its base.

Lemma 3 *An α-cone that minimizes the height, or an h-height cone which minimizes α, both must touch at least four points of the convex hull. In particular, if the base of the cone is touching only one point of the convex hull, then the conic body must touch three points of the convex hull.*

The above lemma provides an inefficient method for computing the α-cone when the base is not fixed to be horizontal by enumerating all possible 4-tuples of P. Therefore, we conclude with the following two open problems.

Open problem. *Given a set P of n points in 3D and an aperture angle α, efficiently compute the apex and the direction of the α-cone for P with minimum height.*

Open problem. *Given a set P of n points in 3D and a height h, efficiently compute the apex and the direction of the α-cone with height h and minimum aperture angle.*

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