THE GEOMETRY OF E-MANIFOLDS

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ABSTRACT. Motivated by the study of symplectic Lie algebroids, we study a describe a type of algebroid (called an E-tangent bundle) which is particularly well-suited to study of singular differential forms and their cohomology. This setting generalizes the study of b-symplectic manifolds, foliated manifolds, and a wide class of Poisson manifolds. We generalize Moser's theorem to this setting, and use it to construct symplectomorphisms between singular symplectic forms. We give applications of this machinery (including the study of Poisson cohomology), and study specific examples of a few of them in depth.

1. INTRODUCTION

Symplectic Lie algebroids were studied by Nest and Tsygan in [NT2]. In this paper we review some of their properties and give a catalogue of examples. Our main motivation for writing this paper is on the one hand to export Moser's path method to this realm but also to give explicit computations in terms of their algebroid cohomology and Poisson cohomology which are not isomorphic in general.

Definition 1. Let E be a locally free submodule of the C^∞ module $Vect(M)$ of vector fields on M. By the Serre-Swan theorem, there is an E-**tangent bundle** ^ETM whose sections (locally) are sections of E, and an E-**cotangent bundle** $^ET^*M := (^ETM)^*$. We will call the global sections of $\wedge^p(^ET^*M)$ E-**forms of degree** p, and denote the space of all such sections by $^E\Omega^p(M)$. If E satisfies the involutivity condition $[E, E] \subseteq E$, there is a differential $d : {}^E\Omega^p(M) \to {}^E\Omega^{p+1}(M)$ given by

$$d\omega(V_0, \ldots, V_p) = \sum_i (-1)^i V_i \left(\omega\left(V_0, \ldots, \hat{V}_i, \ldots, V_p\right)\right)$$
$$+ \sum_{i<j} (-1)^{i+j} \omega\left([V_i, V_j], V_0, \ldots, \hat{V}_i, \ldots, \hat{V}_j, \ldots, V_p\right).$$

The cohomology of this complex is the E-**cohomology** $^E H^*(M)$. We call a closed nondegenerate E-form of degree 2 an E-**symplectic form**, and the triple (M, E, ω) an E-**symplectic manifold**.

The bundle $^E TM \to M$ is naturally a Lie algebroid, whose bracket is given by the standard bracket of vector fields, and whose anchor map $^E TM \to TM$ is induced by the inclusion $E \subseteq Vect(M)$. The E-cohomology is the same as the Lie algebroid cohomology, and an E-**symplectic manifold** is an example of a symplectic Lie algebroid. When $E = \text{Vect}(M)$, then the E-(co)tangent bundle is the same as the standard (co)tangent bundle, and an E-symplectic form is a standard symplectic form. Throughout this paper, we will denote by $EVect(M)$ the set of all locally free submodules E of $Vect(M)$ satisfying the involutivity condition $[E, E] \subseteq [E]$, and the *rank* of an $E \in EVect(M)$ means the rank of the vector bundle $^E TM$. In this paper, we study examples of E-symplectic manifolds.

b-**manifolds,** b^k-**manifolds:** For any closed embedded hypersurface $Z \subseteq M$, we can let E be the submodule of vector fields tangent to Z. In this case, the geometry of E-symplectic forms has been well-studied under the name of b-symplectic geometry (see for instance [GMP1], [GMP2], [S], [KM], [DKM], [KMS]). In the presence of the additional structure of a k-jet of a defining function along Z, the submodule E of vector fields with "order k tangency" to Z defines an element of $EVect(M)$, and the resulting geometry has been studied under the name of b^k-geometry.

c-**manifolds:** A c-symplectic manifold is a generalization of a b-symplectic manifold where the hypersurface Z is allowed to have transverse self-intersection. We prove a Mazzeo-Melrose type theorem for the E-cohomology, and also give an explicit example of an E-symplectic structure of this type on S^4, which demonstrates that certain obstruction theorems in b-symplectic geometry proven in [MO] do not generalize to the c-symplectic setting.

Elliptic type singularities: The submodule $E \subseteq Vect(\mathbb{R}^2)$ generated (as a C^∞-module) by the vector fields

$$v = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \qquad \text{and} \qquad w = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

is involutive, hence $E \in EVect(\mathbb{R}^2)$. This is the prototypical example of an *elliptic type* E-structure, which in general consists of a codimension-2 submanifold together with a E-structure on its normal bundle. These have been used in [CG] to study stable generalized complex structures. We calculate the E-cohomology of such a manifold.

Regular foliations: To any regular foliation of rank r, the submodule $E \subseteq Vect(M)$ of vector fields tangent to the leaves of the foliation is involutive by the Frobenius theorem, hence $E \in EVect(M)$. The

complex $\left({}^E\Omega^*(M), d\right)$ is the familiar complex of foliated differential forms whose cohomology is foliated cohomology. In this context, Moser's lemma was proven in [HM].

In addition, it is possible to construct examples by describing the generators of the submodule explictly. For example, the vector fields on $\mathbb{R}^2$

$$v = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \qquad w = y\frac{\partial}{\partial x}.$$

generate an $E \in EVect(\mathbb{R}^2)$.

Organization of this paper: The paper consists on 4 different sections. In section 2, we prove a Moser theorem for E-symplectic forms and discuss applications to the specific examples of E-symplectic geometries listed in the introduction. In section 3, we compute the E-cohomology of a c-manifold and prove a Mazzeo-Melrose formula. We also give an example of two c-symplectic forms on $\mathbb{R}^2$ which are not symplectomorphic, showing that the Darboux theorem of symplectic geometry does not generalize to this context. In the last section of this paper we give an explicit example of E-manifold and compute its E-cohomology proving that it is not isomorphic to its Poisson cohomology.

2. Moser Path Method for E-symplectic Manifold

In this section, we generalize Moser's theorem to the setting of E-symplectic manifolds. Towards this goal, we first generalize some basic operations on differential forms to the context of E-forms. A diffeomorphism $\phi : M \to M$ acts on a submodule $E \in EVect(M)$ by the usual pushforward, which induces a dual map $^{\phi_*(E)}\Omega(M) \to {}^E\Omega(M)$. Notice that unless $\phi_*(E) = E$, an E-form ω and its pullback $\phi^*(\omega)$ are not sections of the same bundle. For diffeomorphisms given as the flows of sections of E, the condition $\phi_*(E) = E$ is satisfied.

Proposition 2. *(Proposition 1.6 of* [AS]*) Let $E \in EVect(M)$, and let $X \in \Gamma(M, E)$. The time-t flow ρ_t of X satisfies $(\rho_t)_*(E) = E$.*

We denote by $Aut(M, E)$ the set of all diffeomorphisms $\phi : M \to M$ such that $\phi_*(E) = E$.

Definition 3. Let $E \in EVect(M)$, let $\omega \in {}^E\Omega^p(M)$, and let $X \in \Gamma(M, E)$. Define $\iota_X : {}^E\Omega^p(M) \to {}^E\Omega^{p-1}(M)$ by

$$\iota_X(\omega)(X_1, \ldots, X_{p-1}) = \omega(X, X_1, \ldots, X_{p-1})$$

for $p \geq 1$ and by the zero map for $p = 0$. Define $\mathcal{L}_X(\omega) := \frac{d}{dt}\big|_{t=0}\rho_t^*(\omega)$, where ρ_t is the time-t flow of the time-dependent vector field X_t.

These operations satisfy

$$\mathcal{L}_X(\omega) = (d\iota_X + \iota_X d)(\omega) \qquad \text{and} \qquad \frac{d}{dt}\rho_t^*\omega_t = \rho_t^*(\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t).$$

The proof of this is the same as for usual differential forms (for example, see Theorem 5.2.2 of [Ma] and Proposition 6.4 of [CdS])

Lemma 4. *Let $E \in EVect(M)$, and let v_t be a time dependent E-vector field with flow φ_t defined for $0 \le t \le 1$. Then the map ${}^E H^p(M) \to {}^E H^p(M)$ induced by φ_1^* is the identity.*
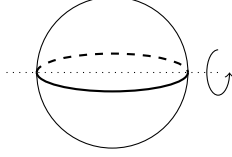
Proof. Let $\omega \in {}^E \Omega^p(M)$, and $\omega_1 = \varphi_1^*(\omega)$, and consider the homotopy operator $Q : {}^E \Omega^p(M) \to {}^E \Omega^{p-1}(M)$ given by

$$Q\omega = \int_0^1 \varphi_t^*(\iota_{v_t}\omega)dt$$

For any closed $\omega \in {}^E \Omega^p(M)$, This homotopy operator satisfies $\varphi_1^*(\omega) - \omega = dQ(\omega)$, so $[\varphi_1^*(\omega)] = [\omega]$. □

Beware that not every diffeomorphism in $Aut(M, E)$ isotopic to the identity is isotopic to the identity *through a path in $Aut(M, E)$.*

Example 5. Consider the submodule E of TS^2 consisting of vector fields tangent to the equator. Let ρ_t denote a rotation of $t\pi$ along the axis shown below.

If $\omega = dh/h \wedge d\theta$, then $\rho_1^*(\omega) = -dh/h \wedge d\theta$, so $[\rho_1^*\omega] \neq [\omega]$. Although the map ρ_1 is isotopic to $\rho_0 = \text{id}$, it is not isotopic through a path in $Aut(M, E)$.

Theorem 6. *Let M be a compact manifold, $E \in EVect(M)$, and let $\omega_0, \omega_1 \in {}^E \Omega^2(M)$. If $\omega_t := (1 - t)\omega_0 + t\omega_1$ is an E-symplectic form for all $t \in [0, 1]$, and $[\omega_0] = [\omega_1] \in {}^E H^2(M)$, then there is a time-dependent E-vector field X_t whose flow ρ_t satisfies $\rho_t^*\omega_t = \omega_0$ for all $t \in [0, 1]$.*

Proof. Let X_t be a time-dependent E-vector field, and let ρ_t be its flow. Then $\frac{d}{dt}\rho_t^*\omega_t = \rho_t^*(\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t)$, so the condition that $\rho_t^*\omega_t$ is constant is equivalent to the condition that $\mathcal{L}_{X_t}\omega_t = -\frac{d}{dt}\omega_t$. Because ω_t is closed, the Cartan formula simplifies this condition to $d(\iota_{X_t}\omega_t) = -\frac{d}{dt}\omega_t$. Therefore, the construction of the desired isotopy ρ_t entails finding a X_t satisfying this equation. Using that $\frac{d\omega_t}{dt} = \omega_1 - \omega_0$ is cohomologically trivial, pick $\mu \in {}^E \Omega^1(M)$ such that $d\mu = \omega_1 - \omega_0$ and define X_t to be the E-vector field satisfying $\iota_{X_t}\omega_t = -\mu$. Then, $d\iota_{X_t}\omega_t = -\frac{d}{dt}\omega_t$ as required. □

We can recover the b-Moser theorem from [GMP2] and the b^m-Moser theorem from [GMW] as a special case of Theorem 6, when the E-submodule in question consists of the vector fields tangent to a hypersurface of M (or, in the b^m case, those with a higher-order tangency to a hypersurface).

Corollary 7 (b-**Moser theorem**). *Suppose that M is compact and let ω_0 and ω_1 be two cohomologous b-symplectic forms on (M, Z). If $\omega_t = (1 - t)\omega_0 + t\omega_1$ is b-symplectic for $0 \leq t \leq 1$, there is a family of diffeomorphisms $\gamma_t : M \to M$, for $0 \leq t \leq 1$ such that γ_t leaves Z invariant and $\gamma_t^* \omega_t = \omega_0$.*

Theorem 8. *Let M be a compact manifold, and let $E \in EVect(M)$. If ^{E}TM has rank 2, and ω_0, ω_1 are two E-symplectic forms such that $[\omega_0] = [\omega_1]$, and ω_0, ω_1 induce the same orientation on ^{E}TM, then there is a $\phi \in Aut(M, E)$ such that $\phi^*(\omega_1) = \omega_0$.*

Proof. Let $\omega_t = t\omega_1 + (1 - t)\omega_0$. Then ω_t is a path of E-symplectic forms such that $[\omega_t]$ is constant. By Theorem 6, there is a time-dependent E-vector field X_t whose flow ρ_t satisfies $\rho_t^* \omega_t = \omega_0$. Then ρ_1 is the desired diffeomorphism.

$\square$

We can recover the classification of oriented symplectic surfaces, Radko's classification of b-symplectic surfaces, and the classification of b^k-symplectic surfaces in [S] as special cases of Theorem 8.

Corollary 9 (**Classification of symplectic surfaces,** [Mo]). *Let S be a compact oriented surface, and and let ω_0 and ω_1 be two symplectic forms on (M, Z) with $[\omega_0] = [\omega_1]$. Then there exists a diffeomorphism $\phi : M \to M$ such that $\phi^* \omega_1 = \omega_0$.*

Corollary 10 (**Classification of** b-**symplectic surface,** [GMP2]). *Let S be a compact orientable surface and and let ω_0 and ω_1 be two b-symplectic forms on (M, Z) defining the same b-cohomology class (i.e.,$[\omega_0] = [\omega_1]$). Then there exists a diffeomorphism $\phi : M \to M$ such that $\phi^* \omega_1 = \omega_0$.*

Let M be a manifold with a regular foliation, and E the submodule of $Vect(M)$ consisting of vector fields tangent to the foliation. In this case, the complex $^{E}\Omega^*(M)$ is the usual complex for foliated cohomology, $H^*(\mathcal{F})$.

So in particular by rephrasing theorem 8 we obtain the following Moser's theorem for symplectic foliations.

Theorem 11. *Let M be a compact manifold, and let $\mathcal{F}$ be a regular foliation by 2-dimensional leaves. If ω_0, ω_1 are two leafwise-symplectic forms representing the same foliated cohomology class and inducing the same orientation on $\mathcal{F}$, then there is a ϕ preserving the foliation such that $\phi^*(\omega_1) = \omega_0$.*

When the symplectic foliation is the symplectic foliation of a regular Poisson manifolds, it would be desirable to obtain a path method that encodes the classification in terms of Poisson cohomology. The following result enables to do so for unimodular Poisson manifolds. We recall from [MM],

Theorem 12 (Martinez-Torres and Miranda, [MM]). *Let (M, π) be a compact orientable unimodular regular Poisson manifold of dimension m and rank $2n$. Then then there is an isomorphism of cohomology groups:*

$$H_\pi^m(M) \to H^{2n}(\mathcal{F})$$

In particular by applying theorem 11 we obtain,

Theorem 13. *Let M^3 be a regular compact Poisson manifold of corank 1, and let $\mathcal{F}$ be its symplectic foliation. If Π_0, Π_1 are two Poisson structures with the same class in Poisson cohomology and the same orientation, then there is a ϕ preserving the symplectic foliation such that $\phi_*(\Pi_0) = \Pi_1$.*

This global classification of E-symplectic objects in dimension 2 using a Moser theorem can be used to classify Nambu structures of top degree for manifolds in any dimension as it was done in [MP] to obtain,

Theorem 14 (Miranda and Planas, [MP])**.** *Let Θ_0 and Θ_1 be two b^m-Nambu forms of degree n on a compact orientable manifold M^n. If $[\Theta_0] = [\Theta_1]$ in b^m-cohomology then there exists a diffeomorphism ϕ such that $\phi^*\Theta_1 = \Theta_0$.*

This classification theorem generalizes that in [Mt].

3. A case-study: c-manifolds

In the definition of a b-manifold (M, Z), the hypersurface Z is assumed to be smoothly embedded. In this section, we generalize this theory by allowing the hypersurface to have transverse self-intersections as considered in [GMPS] (see v1 in arxiv) and [GLPR].

For these E-manifolds we can compute explicitly its E-cohomology and prove a Mazzeo-Melrose theorem.

3.1. Self-transverse immersions and c-geometry.

Definition 15. Let Z and M be smooth manifolds. An immersion $i : Z \to M$ is **self-transverse** if whenever $p_1, \ldots, p_n \in Z$ are distinct points mapping to the same point of M, the planes $\{i_*(T_{p_i}M)\}$ are in general position.

Definition 16. A self-transverse immersion $i : Z \to M$ has **multiple point manifolds**, for each $k \geq 1$, given by

$$Z_k = \{(p_1, \ldots, p_k) \in Z^k \mid p_i \text{ distinct}, \ i(p_i) = i(p_j)\}$$
$$\overline{Z_k} = Z_k/S_k$$

where the symmetric group S_k acts on Z_k by permuting the p_i's. For $k > \ell$, there are also self-transverse immersions

$$Z_k \to Z_\ell \qquad\qquad \text{and} \qquad\qquad Z_k \to M$$
$$(p_1, \ldots, p_k) \mapsto (p_1, \ldots, p_\ell) \qquad\qquad (p_1, \ldots, p_k) \mapsto i(p_1)$$

These maps are equivariant with respect to the actions of the symmetric groups, hence define self-transverse immersions $\overline{Z_k} \to \overline{Z_\ell}$ and $\overline{Z_k} \to M$. To simplify notation, we set $Z_0 := M$.

Example 17. If S^1 is mapped into $\mathbb{R}^2$ as a figure eight, then there are two points $p_1, p_2 \in S^1$ that are identified under this map. Then Z_2 consists of two points, $(p_1, p_2), (p_2, p_1) \in S^1 \times S^1$, while $\overline{Z_2}$ consists of just one point.

Definition 18. A c-manifold is a manifold M together with a self-transverse immersion i of a hypersurface Z into M. A vector field v on c-manifold M is a c-**vector field** if for each $p \in Z$, $v_{i(p)} \in i_*(T_p Z)$.

The set of c-vector fields forms an element of $EVect(M)$, and we denote the corresponding E-tangent and E-cotangent bundles by ${}^c TM$ and ${}^c T^* M$, respectively. Around every point of M in the image of Z_k but not Z_{k+1}, there are coordinates $(x_1, \ldots, x_n)$ for which the image of Z is $\cup_{i \leq k}\{x_i = 0\}$. In these coordinates, ${}^c TM$ is generated as a C^∞ module by

$$\left\{ x_1 \frac{\partial}{\partial x_1}, \ldots, x_k \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_n} \right\}$$

and ${}^c T^* M$ is generated by the dual basis

$$\left\{ \frac{dx_1}{x_1}, \ldots, \frac{dx_k}{x_k}, dx_{k+1}, \ldots, dx_n \right\}.$$

In this setting, we call an E-symplectic form a c-**symplectic form**.

Example 19. In this example, we describe a c-symplectic structure on S^4. For $1 \leq i \leq 5$, consider the following open sets of the unit sphere $S^4 \subseteq \mathbb{R}^5$:

$$U_i^+ = \{(y_1, \ldots, y_5) \in S^4 \mid y_i > 0\}$$
$$U_i^- = \{(y_1, \ldots, y_5) \in S^4 \mid y_i < 0\}$$

A line passing through the origin in $\mathbb{R}^5$ and fixed point in U_i^+ intersects the plane $\mathbb{R}^4 \cong \{y_i = 1\} \subseteq \mathbb{R}^5$ in exactly one point; the projection defined in this way gives local coordinates on U_i^+. Likewise, the projection through the origin onto the plane $\mathbb{R}^4 \cong \{y_i = -1\} \subseteq \mathbb{R}^5$ gives local coordinates on each U_i^-. These coordinates are just the pullback of the standard coordinates on $\mathbb{R}P^4$ by the double covering $S^4 \to \mathbb{R}P^4$. Below are examples of two change of coordinate maps between these charts.

$$\left\{ (x_2, x_3, x_4, x_5) \in U_1^+ \mid x_2 > 0 \right\} \qquad \left\{ (x_1, x_3, x_4, x_5) \in U_2^+ \mid x_1 > 0 \right\}$$
$$(x_2, x_3, x_4, x_5) \mapsto x_2^{-1}(1, x_3, x_4, x_5)$$
$$x_1^{-1}(1, x_3, x_4, x_5) \leftmapsto (x_1, x_3, x_4, x_5)$$

$$\left\{ (x_2, x_3, x_4, x_5) \in U_1^+ \mid x_2 < 0 \right\} \qquad \left\{ (x_1, x_3, x_4, x_5) \in U_2^- \mid x_1 > 0 \right\}$$
$$(x_2, x_3, x_4, x_5) \mapsto -x_2^{-1}(1, x_3, x_4, x_5)$$
$$x_1^{-1}(-1, x_3, x_4, x_5) \leftmapsto (x_1, x_3, x_4, x_5)$$

Let v_j for $1 \leq j \leq 5$ be the vector fields on S^4 defined as follows on these coordinate charts.

$$v_j = \begin{cases} x_j \frac{\partial}{\partial x_j} & \text{on } U_i^+ \text{ and } U_i^- \text{ for } i \neq j \\ \sum_{i \neq j} -x_i \frac{\partial}{\partial x_i} & \text{on } U_j^+ \text{ and } U_j^- \end{cases}$$

Consider the bivector $\Pi = (v_1 + v_2) \wedge (v_2 + v_3) + (v_3 + v_4) \wedge (v_4 + v_5)$ on S^4. Because the v_i vector fields commute with one another, $[\Pi, \Pi] = 0$. Also, $\Pi \wedge \Pi = 2(\sum_{i=1}^{5} v_1 \wedge \cdots \wedge \hat{v}_i \wedge \ldots v_5$. Together with the fact that $v_i = -\sum_{j \neq i} v_j$, one can verify that Π is dual to a c-symplectic form on each coordinate chart. For example, on U_1^+ and U_1^-, it is given by

$$\text{On } U_1^+ \text{ and } U_1^- : \; x_2 x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + x_2 x_4 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_4}$$
$$+ 2 x_3 x_4 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} + x_2 x_5 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_5}$$
$$+ 2 x_3 x_5 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_5} + x_4 x_5 \frac{\partial}{\partial x_4} \wedge \frac{\partial}{\partial x_5}$$

This illustrates that S^4 admits a c-symplectic structure.

The main theorem of [MO] implies that for $n \geq 4$, any compact manifold of dimension n admitting a b-symplectic structure must have a nontrivial class in $H^2(M)$. Example 19 shows that a manifold may admit a c-symplectic structure even if it does not admit a b-symplectic structure.

To better understand $^c H^*(M)$, we hope for a result similar to the canonical Mazzeo-Melrose isomorphism in b-geometry, in which case Z is embedded and

$$(1) \qquad\qquad {}^b H^p(M) \cong H^p(M) \oplus H^{p-1}(Z).$$

Towards this goal, in the following two sections, we will generalize the definition of a *Liouville volume* and the *residue* of a b-form.

3.2. **The Liouville volume of a c-form.** Recall that the *Liouville volume* of a b-form ω of top degree is defined as

$$\lim_{\epsilon \to 0} \int_{M \setminus \{|y| \leq \epsilon\}} \omega$$

where y is a local defining function for Z. This number is well-defined and independent of the choice of y. Because the hypersurface of a c-manifold might not be embedded and might not be coorientable, it may have no local defining function in the traditional sense.

Definition 20. Let W either be Z_k for some $k \geq 1$, or a connected component of Z_k, and let $i : W \to M$ be its inclusion into M. The **normal**

bundle of W is the bundle $\pi : N_W \to W$ whose fiber over $p \in W$ is given by $T_{i(p)}M/i_*(T_pW)$.

Remark 21. Let W be as in Definition 20, and let $p = (p_1, \ldots, p_k) \in W$. For each j, $i_*(T_{p_j}Z)$ is a hyperplane in $T_{i(p)}M$ containing $i_*(T_pW)$, hence defines a hyperplane in $(N_W)_p$. We will denote by H_j the codimension-1 subbundle of N_W defined by the hyperplanes $i_*(T_{p_j}Z) \subseteq T_{i(p)}M$. Because $i : Z \to M$ is self-transverse, these H_j are transverse subbundles of N_W.

Definition 22. Let W be as in Definition 20. A **tubular neighborhood** of W is a map $\phi : N_W \to M$ such that

- Around every $p \in W$, there is an open neighborhood $U \subseteq W$ such that the map $\phi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to M$ is a usual tubular neighborhood of $i|_U : U \to M$
- For each j, the map $\phi|_{H_j} : H_j \to M$ factors through a map $H_j \to Z$ that sends the zero section over $p = (p_1, \ldots, p_k)$ to $p_j \in Z$.

Figure 1 below on the right shows a self-transverse map of a cylinder and a plane into $\mathbb{R}^3$. The double point manifold Z_2 consists of two circles and two lines. One if these circles is shown below, together with an illustration of the codimension-1 subbundles of N_M.
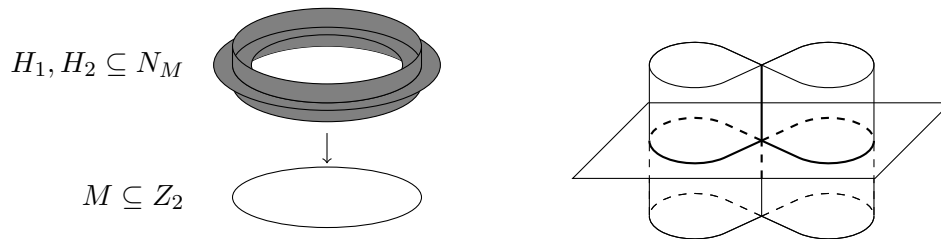

$H_1, H_2 \subseteq N_M$

$M \subseteq Z_2$

FIGURE 1. A normal bundle of a component of Z_2 with two codimension-1 subbundles

Definition 23. Let (M, Z) be a c-manifold. Pick a tubular neighborhood $\phi : N_Z \to M$ of Z, pick an auxillary metric on N_Z, and let $U_\epsilon := \phi(\{v \in N_Z \mid |v| < \epsilon\})$. The **Liouville volume** of a compactly supported c-form ω of top degree is

$$(2) \qquad \int_M^c \omega = \lim_{\epsilon \to 0} \int_{M \setminus U_\epsilon} \omega$$

Proposition 24. *The Liouville volume of a compactly supported c-form is finite and independent of the choice of tubular neighborhood and auxillary metric used to define U_ϵ*

Proof. By passing to a partition of unity, it suffices to prove the claim when ω is supported in $M = [-1, 1]^n \subseteq \mathbb{R}^n$ where $i(Z) = \cup_{i \leq r}\{x_i = 0\}$. If $r = 0$, ω is a smooth form and there is nothing to prove. If $r > 0$, fix a tubular

neighborhood of Z and auxillary metric on N_Z; after a diffeomorphism of $\mathbb{R}^n$ preserving Z we may assume that $U_\epsilon = \cup_{i=1}^r \{|x_i| < \epsilon\}$ for small ϵ. There is some $f \in C^\infty(\mathbb{R}^n)$ such that

$$\omega = f\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_r}{r} \wedge dx_{r+1} \wedge \cdots \wedge dx_n$$

By Taylor's theorem, there are smooth functions f_i such that

$$f = f(0, \ldots, 0, x_{r+1}, \ldots, x_n) + \sum_{i=1}^r x_i f_i.$$

Then

$$\int_{M\backslash U_\epsilon} f\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_r}{r} \wedge dx_{r+1} \wedge \cdots \wedge dx_n$$

$$= \int_{M\backslash U_\epsilon} f(0, \ldots, 0, x_{r+1}, \ldots, x_n)\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_r}{r} \wedge dx_{r+1} \wedge \cdots \wedge dx_n$$

$$+ \sum_{i=1}^r \int_{M\backslash U_\epsilon} f_i\frac{dx_1}{x_1} \wedge \cdots \wedge dx_i \wedge \cdots \wedge \frac{dx_r}{r} \wedge dx_{r+1} \wedge \cdots \wedge dx_n$$

The first term is finite by Fubini's theorem; the second is finite by induction on the order of the singularity of the differential form. It remains to show that this does not depend on the choice of tubular neighborhood or metric. Suppose we change the tubular neighborhood or metric just for the hyperplane $\{x_1 = 0\}$, and denote by $\widetilde{U_\epsilon}$ the neighborhood U_ϵ obtained using these new choices. The formula for $\widetilde{U_\epsilon}$ can be written

$$\widetilde{U_\epsilon} = \{|h(x_1, \ldots, x_n)| < \epsilon\} \cup \bigcup_{i=2}^r \{|x_i| < \epsilon\}$$

for h a defining function of $\{x_1 = 0\}$. Then

$$\lim_{\epsilon \to 0} \int_{M\backslash\widetilde{U_\epsilon}} \omega = \lim_{\epsilon_2 \to 0} \lim_{\epsilon_1 \to 0} \int_{M\backslash\{|h|<\epsilon_1\}\cup\bigcup_{i=2}^r\{|x_i|<\epsilon_2\}} \omega$$

$$= \lim_{\epsilon_2 \to 0} \lim_{\epsilon_1 \to 0} \int_{M\backslash\{|x_1|<\epsilon_1\}\cup\bigcup_{i=2}^r\{|x_i|<\epsilon_2\}} \omega$$

$$= \lim_{\epsilon \to 0} \int_{M\backslash U_\epsilon} \omega$$

where the second equality is given by the independence of the Liouville volume of a b-form on the choice of defining function for the hypersurface. Because every change of local defining function can be written as a sequence of changes that affect only one hyperplane, this computation proves that the Liouville volume of a c-form is independent of the choices in its definition. $\square$

Now we can define a map sm : $^cH^k(M) \to H^k(M)$ as follows. First, consider the case when M is orientable. For $[\omega] \in {}^cH^k(M)$, let $\mathrm{sm}([\omega]) \in H^k(M)$ be defined as the cohomology class that corresponds via Poincare duality to the element of $H_c^{n-k}(M)^*$ given by

$$(3) \qquad\qquad [\eta] \to \int_M^b \omega \wedge \eta.$$

For the non-orientable case, we use the non-orientable Poincare duality (see Bott and Tu, Theorem 7.8), $H^k(M) \cong H_c^{n-k}(M, \mathrm{or}_M)^*$, where or_M is the orientation bundle of M. We also define an E-density on an n-dimensional E-manifold to be a section of $\Lambda^n({}^ET^*M) \otimes \mathrm{or}_M$. Just as a c-form restricted to the complement of Z is a smooth form, a c-density restricted to the complement of Z is a smooth density. The description of the Liouville volume of a c-form generalizes without any changes to the case of c-densities: because the proof of Proposition 24 is local in nature, it applies to c-densities as well. So for a closed form ω on a non-orientable manifold, $\mathrm{sm}([\omega]) \in H^k(M)$ is defined as the cohomology class that corresponds to the element of $H_c^{n-k}(M, \mathrm{or}_M)^*$ given by equation 3

3.3. **The residue of a c-form.** We describe, for any c-manifold (M, Z) and any $k \geq 0$, a *residue* map

$$\mathrm{res} : {}^c\Omega^p(Z_k) \to {}^c\Omega^{p-1}(Z_{k-1})$$

that generalizes the residue map $^b\Omega^p(M) \to \Omega^{p-1}(Z)$ for b-forms. Let $(x_1, \ldots, x_n)$ be coordinates on Z_k in which $\{x_1 = 0\}$ is the image of an open set $U \subseteq Z_{k+1}$ under the induced inclusion $Z_{k+1} \to Z_k$. In these coordinates, ω can be written as

$$(4) \qquad\qquad \omega = \alpha \wedge \frac{dx_1}{x_1} + \beta$$

where α, β are c-forms of degree $p-1$ and p, respectively, whose coordinate expressions do not contain $\frac{dx_1}{x_1}$. On $U \subseteq Z_{k+1}$, the form $\mathrm{res}(\omega)$ is defined as the pullback of α to U. Equivalently, let

$$\mathbb{L} = (-1)^{p-1} x_1 \frac{\partial}{\partial x_1}.$$

Although $\mathbb{L}$ depends on the choice of coordinates, one can use the same proof in [GMP2] for the b-case to show that $\mathbb{L}\big|_{i(Z_{k+1})}$ does not depend on the coordinates, and that the pullback $i^*(\iota_{\mathbb{L}}\omega)$ to Z_k is a well-defined c-form on Z_k which also equals $\mathrm{res}(\omega)$. Also note that the residue map can be composed with itself; if $\omega \in {}^c\Omega^p(M)$, then $\mathrm{res}^k(\omega) \in {}^c\Omega^{p-k}(Z_k)$.

Example 25. The form $\omega = \frac{dx}{x} \wedge \frac{dy}{y} \wedge \frac{dz}{z}$ is a c-form on $\mathbb{R}^3$, where Z consists of the inclusion of the three coordinate hyperplanes. Its residue is written in Figure 3.
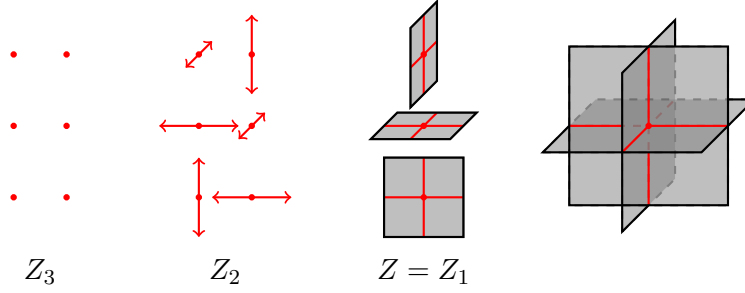
$$Z_3 \qquad Z_2 \qquad Z = Z_1$$

FIGURE 2. The components of Z_1 are $\{x = 0\}, \{z = 0\}$, and $\{y = 0\}$, drawn from top to bottom. The diagram of Z_2 (respectively, Z_3) is arranged so that the canonical maps $Z_2 \to Z_1$ send components of Z_2 to components of Z_1 drawn to their right.

1	-1	$\frac{dy}{y}$	$-\frac{dz}{z}$	$\frac{dy}{y} \wedge \frac{dz}{z}$	
1	-1	$\frac{dx}{x}$	$-\frac{dy}{y}$	$\frac{dx}{x} \wedge \frac{dy}{y}$	$\frac{dx}{x} \wedge \frac{dy}{y} \wedge \frac{dz}{z}$
1	-1	$\frac{dz}{z}$	$-\frac{dx}{x}$	$\frac{dz}{z} \wedge \frac{dx}{x}$	

FIGURE 3. The residue of ω, graphically arranged to match Figure 2.

Definition 26. Let $\omega \in {}^c\Omega^*(Z_k)$. We say that ω is **compatible** if the following two conditions hold

- The action of any $\sigma \in S_k$ satisfies $\sigma^*(\omega') = \text{sign}(\sigma)\omega'$, where $\text{sign}(\sigma)$ is 1 if σ corresponds to an even permutation, and -1 otherwise.
- Either ω is smooth, or $\text{res}(\omega)$ is compatible.

Because taking the repeated residue of a form will eventually give a smooth form, this recursive definition makes sense.

By the nature of the residue map, any c-form on a manifold M will be compatible, but for a c-form on Z_k to be compatible, it must "agree" at the points where Z_k is identified in M, as illustrated in the example below.

Example 27. Let $i : \mathbb{R} \sqcup \mathbb{R} \to \mathbb{R}^2$ be given by $x \mapsto (x, 0)$ on the first copy of $\mathbb{R}$ and $y \mapsto (0, y)$ on the second. Then an element of ${}^c\Omega^1(Z_1)$ which is described by $f\frac{dx}{x}$ and $g\frac{dy}{y}$ is *compatible* if $f(0) = -g(0)$.

Let $\widehat{\Omega}^*(Z_i)$ (and ${}^c\widehat{\Omega}^*(Z_i)$) denote the space of compatible smooth forms (and c-forms). The differential preserves the property of compatibility; we write $\widehat{H}^*(Z_i)$ (and ${}^c\widehat{H}^*(Z_i)$) for the cohomology of this subcomplex. For the remainder of this section, we will use the notation Z_0 to denote M.

Proposition 28. *Let (M, Z) be a c-manifold. Then for all $i \geq 0$,*

$$(5) \qquad 0 \to \widehat{\Omega}^p(Z_i) \to {}^c\widehat{\Omega}^p(Z_i) \xrightarrow{\text{res}} {}^c\widehat{\Omega}^{p-1}(Z_{i+1}) \to 0$$

is a short exact sequence of chain complexes. Moreover, any closed form in ${}^c\widehat{\Omega}^{p-1}(Z_{i+1})$ has a closed form in its preimage in ${}^c\widehat{\Omega}^p(Z_i)$.
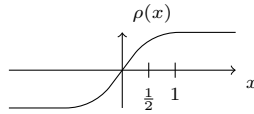
Proof. In verifying that Equation 5 is a short exact sequence of chain maps, the nontrivial parts are to verify that res commutes with the differentials, and is surjective. First, let $\omega \in {}^c\widehat{\Omega}^p(Z_i)$. Around a point of $i(Z_{i+1}) \subseteq Z_i$, we can pick local coordinates and use the decomposition of ω described in Equation 4,

$$\omega = \alpha \wedge \frac{dx_1}{x_1} + \beta.$$

Then $d\omega = d\alpha \wedge \frac{dx_1}{x_1} + d\beta$, whose residue on Z_{i+1} is given by the pullback of $d\alpha$. The fact that res commutes with d then follows from the fact that differentials commute with pullbacks.

To verify that res is surjective, let $\alpha \in {}^c\hat{\Omega}^{p-1}(Z_{i+1})$. We will construct an $\omega \in {}^c\Omega^p(Z_i)$ whose residue is α. In the case that α is closed, this construction will produce a closed ω, proving the final part of the theorem.

The S_{i+1} action on Z_{i+1} acts on the components of Z_{i+1}. Let W be a union of components of Z_{i+1}, one from each orbit of this action. Let $\pi : N_W \to W$ be the tubular neighborhood of W in Z_i. If $W \subseteq Z_i$ is coorientable, let $x_1 \in C^\infty(N_W)$ be a defining function for the zero section of N_W, and let $\tilde{x}_i = \rho \circ x$, where $\rho : \mathbb{R} \to \mathbb{R}$ be an odd function which is locally constant outside $[-1, 1]$ and equal to $\rho(x) = x$ on $[\frac{1}{2}, \frac{1}{2}]$.

Then, let $\eta = \pi^*(\alpha_k) \wedge \frac{d\tilde{x}_1}{\tilde{x}_1}$. In the case when $W \subseteq Z_i$ is not coorientable, then a defining function x_1 can only be defined up to sign. But because ρ is an odd function, $d\tilde{x}_1/\tilde{x}_1$ does not change if x_1 is replaced by $-x_1$, so the construction of η is well-defined even when $W \subseteq Z_i$ is not coorientable.

Then η is a c-form on N_M whose restriction to each fiber is compactly supported, $\text{res}(\eta) = \alpha$, and η is closed if α is. Next, let $\phi : N_W \to M$ be a tubular neighborhood of W. If ϕ is an embedding, it is a diffeomorphism onto its image, and we could construct a form ω' in a neighborhood of $W \subseteq Z_i$ simply by identifying N_M with its image and taking ω' to be η. If ϕ is not an embedding (for example, if $W = S^1$ is immersed as a figure-eight), then for every open set $U \subseteq Z_i$ for which $\phi|_{\phi^{-1}(U)}$ is a k-to-one cover, then

define $\omega'\big|_U$ to be the *sum* of the k different forms obtained by identifying U with each of the k different preimages, and taking η in each preimage.

Finally, let ω equal ω' summed over the action of S_k. By construction, ω is a compatible form with $\mathrm{res}(\omega) = \alpha$. Because η is closed whenever α is closed, the same is true for ω' and for ω.

$\square$

Theorem 29. *Let (M, Z) be a c-manifold. Then*

$$^c H^p(M) \cong H^p(M) \oplus \widehat{H}^{p-1}(Z_1) \oplus \widehat{H}^{p-2}(Z_2) \oplus \cdots \oplus \widehat{H}^0(Z_p)$$

$$[\omega] \mapsto (sm(\omega), sm(res(\omega)), sm(res^2(\omega)), \ldots, sm(res^p(\omega)))$$

Proof. For each i, the short exact sequence in Equation 5 induces short exact sequences in cohomology

$$0 \to \widehat{H}^p(Z_i) \to {}^c\widehat{H}^p(Z_i) \xrightarrow{-\mathrm{res}} {}^c\widehat{H}^{p-1}(Z_{i+1}) \to 0.$$

which is split by the map

$$sm : {}^c\widehat{H}^p(Z_i) \to \widehat{H}^p(Z_i).$$

Applying induction on i, and the fact that ${}^c\widehat{\Omega}^p(Z_0) = {}^c\widehat{\Omega}^p(M) = {}^c\Omega^p(M)$ yields the result. $\square$

Notice that each $\widehat{H}^p(Z_i)$ is isomorphic to $H^p(\overline{Z}_i)$, but that this isomorphism is only canonical up to a choice of sign.

Theorem 29 implies that unlike usual symplectic forms (for which there is a Darboux theorem), c-symplectic forms have local invariants.

Example 30. Consider the family $k\frac{dx}{x} \wedge \frac{dy}{y}$ of c-symplectic forms on $\mathbb{R}^2$, parametrized by $k > 0$. The terms of the residue decomposition of $[k\frac{dx}{x} \wedge \frac{dy}{y}]$ from Theorem 29 all vanish except for the part on Z_2. On Z_2, the residue is given by the locally constant function equal to k on one of the components of Z_2, and $-k$ on the other component. Therefore, each c-symplectic form in this family represents a different c-cohomology class, and hence the c-symplectic manifolds in this family are pairwise non-symplectomorphic.

Example 30 shows that there may be cohomological obstructions to constructing local symplectomorphisms between c-symplectic forms; the following theorem shows that these are the *only* such obstructions.

Theorem 31 (c-Darboux theorem)**.** *Let ω_0 and ω_1 be two c-symplectic forms for which the three elements under the isomorphism in theorem 29 above coincide, then ω_0 and ω_1 are symplectomorphic.*

Proof. From theorem 29, the three classes determine the cohomology class in $^c H^p(M)$ thus from Moser's theorem [Mo] we obtain the desired result. $\square$

Remark 32. As a consequence of the theorem above, there is not a unique model for c-symplectic structures due to the fact that the residue in theorem 29 is not constant.

4. E-COHOMOLOGY AND POISSON COHOMOLOGY

In this section we focus our attention on comparing E-cohomology with Poisson cohomology associated to Poisson structures whose associated algebroid is E. In [GMP2] it was proven that for b-symplectic manifolds, b-cohomology is isomorphic to Poisson cohomology.

In this section we give an example proving that this is not true in general for E-symplectic manifolds. In this section, E refers to the locally free involutive submodule of $Vect(\mathbb{R}^2)$ generated by

$$v_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \qquad \text{and} \qquad v_2 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

Then $^{E}T^*M$ is generated by the sections

$$v_1^* = \frac{xdx + ydy}{x^2 + y^2} \qquad \text{and} \qquad v_2^* = \frac{-ydx + xdy}{x^2 + y^2}$$

We can naturally associate a Poisson structure to E, $\Pi_E = v_1 \wedge v_2 = (x^2 + y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$.

We start by computing E-cohomology.

4.1. A computation of E-cohomology.
Recall that a function $f \in C^\infty(\mathbb{R}^n)$ is called **analytically flat** at a point p if f and all its derivatives vanish at p.

Lemma 33. *Let $f(x,y) \in C^\infty(\mathbb{R}^2)$ be analytically flat at 0 and $g(x,y) \in C^\infty(\mathbb{R}^2)$ be a nonnegative function. For any positive even numbers p, q, the function*

$$h(x,y) = \frac{f(x,y)}{x^p + y^q + g(x,y)}$$

is smooth and analytically flat.

Proof. Because $x^p + y^q + g(x,y)$ is nonzero away from 0, it suffices to show that h is defined and smooth at 0. To show that it is defined, assume without loss of generality that $p \geq q$. By Taylor's theorem, we can write

$$f = \sum_{\substack{i,j \geq 0 \\ i+j=2p}} f_{ij}x^i y^j$$

for flat functions f_{ij}. Notice that

$$|h| = \left| \sum_{i+j=2p} \frac{f_{ij}x^i y^j}{x^p + y^q + g} \right|$$

$$= \left| \sum_{\substack{i+j=2p \\ i \geq p}} f_{ij}x^{i-p}y^j \frac{x^p}{x^p + y^q + g} \right| + \left| \sum_{\substack{i+j=2p \\ i<p}} f_{ij}x^i y^{j-q} \frac{y^q}{x^p + y^q + g} \right|$$

$$\leq \left| \sum_{\substack{i+j=2p \\ i \geq p}} f_{ij}x^{i-p}y^j \right| + \left| \sum_{\substack{i+j=2p \\ i<p}} f_{ij}x^i y^{j-q} \right|$$

By the sandwich theorem, the limit exists. To prove that h is smooth, observe that the first partial derivatives of h are again a quotient of a flat function by a function of the form $(x^{2p} + y^{2q} + \tilde{g})$ and apply induction. $\quad\square$

Theorem 34. *Let $E \subseteq Vect(M)$, v_1, v_2 be as described above.*

$$^E H^k(M) = \begin{cases} \mathbb{R} & k = 0 \\ \mathbb{R}^2 & k = 1 \\ \mathbb{R} & k = 2 \\ 0 & otherwise \end{cases}$$

Proof. The degree 0 cohomology is generated by the classes of the constant functions, and hence is isomorphic to $\mathbb{R}$. Let K be the closed E-forms of degree 1, and consider the map

$$K \to \mathbb{R}^2 \qquad\qquad gv_1^* + hv_2^* \mapsto (g(0), h(0)).$$

Exact 1-forms $v_2(f)v_1^* + v_1(f)v_2^*$ are in the kernel of this map, so this induces $^E H^1(\mathbb{R}^2) \to \mathbb{R}^2$. The map is surjective, since $c_1 v_1^* + c_2 v_2^* \mapsto (c_1, c_2)$ for any c_1, c_2. To show that it is surjective, let $\eta = gv_1^* + hv_2^*$ be in the kernel of this map; we will show that η is exact. Let $\sum_{i,j \geq 0} g_{ij}x^i y^j$ be the analytic jet of g. Then

$$f = \sum_{\substack{i,j \geq 0 \\ (i,j) \neq (0,0)}} (i+j)^{-1} x^i y^j$$

has the property that $v_1(f) - g$ is flat. Because η and $df = v_1(f)v_1^* + v_2(f)v_2^*$ are closed,

$$0 = d\eta - d^2 f = (v_1(h) - v_2(g) - v_1 v_2(f) + v_2 v_1(f))v_1^* \wedge v_2^*$$
$$= (v_1(h - v_2(f)) - v_2(g - v_1(f)))v_1^* \wedge v_2^*$$

Because $v_1(f) - g$ is flat, so is $v_2(g - v_1(f))$, and it therefore follows that $v_1(h - v_2(f))$ is flat and therefore $h - v_2(f)$ is analytically a constant function. Since $h(0) = 0$, $h - v_2(f)$ is flat. Notice that

$$(g - v_2(f))v_1^* + (h - v_1(f))v_2^* = \frac{g - v_2(f)}{x^2 + y^2}(xdx + ydy) + \frac{h - v_1(f)}{x^2 + y^2}(-ydx + xdy)$$

is smooth by the lemma. It is also closed, so by Poincare's lemma has a primitive F. Then $f + F$ is a primitive for η. To calculate ${}^E H^2(\mathbb{R}^2)$, let K be the closed E-forms of degree 2, and consider

$$K \to \mathbb{R} \qquad\qquad gv_1^* \wedge v_2^* \mapsto g(0).$$

By a similar argument as before, this map is surjective and contains all exact forms in its kernel. Let $\eta = gv_1^* \wedge v_2^*$ be a general element of the kernel, and define f in the same way as above so that $v_1(f) - g$ is flat. Then $\eta - df$ is a closed smooth form, and hence has a primitive F. Then $f + F$ is a primitive for η. $\qquad\square$

4.2. A Poisson cohomology computation for E.

The Poisson cohomology associated to the Poisson structure $\Pi_E = v_1 \wedge v_2 = (x^2 + y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ on $\mathbb{R}^2$ was computed by Abreu and Ginzburg and an explicit computation is available at [G] (Proposition 2.16).

Theorem 35 (Abreu-Ginzburg, [G]). *The Poisson cohomology groups of $(\mathbb{R}^2, \Pi_E)$ are as follows:*

(1) $H^0_{\Pi_E}(\mathbb{R}^2) = \mathbb{R}$ *is given by constant functions.*

(2) $H^1_{\Pi_E}(\mathbb{R}^2) = \mathbb{R}^2$ *is generated, by the rotation $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$, and by the dilation $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.*

(3) $H^2_{\Pi_E}(\mathbb{R}^2) = \mathbb{R}^2$ *generated by $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ and Π_E.*

In particular this proves (due to the second cohomology group computation) that E-cohomology is not isomorphic to Poisson cohomology.

Corollary 36. *For the particular choice of E as above and $M = \mathbb{R}^2$, E-cohomology is not isomorphic to Poisson cohomology.*

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